OPTIMAL TRANSPORT

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1 Week 1

1.1 Introduction

Monge, 1781. Have some soil extracted from earth and want to build a fortification. The cost is $c(x, y) = |x - y|$. The question is who goes where?

Kantorovic, 1940. Have some bakeries at x_i and coffeeshops at y_j. Bakery i produces $\alpha_i \geq 0$ amount of bread and coffee shop j needs $\beta_j \geq 0$. Assume that $\sum \alpha_i = \sum \beta_j = 1$. Note that for Monge, the transport is "deterministic"; x is sent to $T(x)$. Kantorovic looked for matrices $(\gamma_{ij})_{i\in[N],j\in[M]}$ such that

$$\begin{cases} \gamma_{ij} \geq 0, \\ \alpha_i = \sum \gamma_{ij}, \forall i, \\ \beta_j = \sum \gamma_{ij}, \forall j, \\ \gamma_{ij} \text{ minimizes the cost } \sum_{i,j} \gamma_{ij} c(x_i, y_j). \end{cases}$$

Applications. $c(x, y) = |x - y|^2$: Euler equation, isoperimetric/Sobolev inequalities, and PDEs such as $\partial_t u = \Delta u$, $\partial_t u = \Delta(u^m)$, and $\partial_t u = \operatorname{div}(\nabla W \star_u u)$.

For $c(x, y) = |x - y|^p$, there are cases for $p < 1$, $p = 1$, and $p > 1$, the hardest one of which is $p = 1$ (Monge tried to tackle the hardest!). It has applications to probability and kinetic theory.

Also one can have $c(x, y) = d^2(x, y)$ on (M, g). It has applications to curvature.

Remark 1. All my spaces are metric spaces that are separable and complete. Most of the course is about $X = \mathbb{R}^n$. Also, all measures and maps are Borel.

1.2 A Rigorous Definition of Transport

Definition 2 (Image Measure)**.** Given X, Y, let $T : X \to Y$. Let $\mu \in P(X)$, the set of probability measures on X. Define $T\#\mu \in P(Y)$ as follows:

$$(T\#\mu)(A) = \mu(T^{-1}(A)).$$

Lemma 3. *$T\#\mu$ is a probability measure on Y.*

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Proof. First, it is clear that $T\#\mu(\emptyset) = 0$ and $T\#\mu(Y) = 1$. Let $\{A_i\}_{i\in I}$ be a countable family of disjoint sets in Y. Then $\{T^{-1}(A_i)\}$ are disjoint in X. Also $\bigcup T^{-1}(A_i) = T^{-1}(\bigcup A_i)$. Taking measures from both sides results in additivity of $T\#\mu$. $\square$

Remark 4. One may be tempted to do the following: to define a measure on X, *e.g.*, set $(S^{\#}\nu)(E) = \nu(S(E))$. This is NOT a measure. Take the constant map and dirac measure as example.

Lemma 5. *Let $T : X \to Y$ and $\mu \in P(X)$. Then $\nu = T\#\mu$ iff $\forall \phi : Y \to \mathbb{R}$ Borel and bounded, one has*

$$\int_Y \phi(y)\, d\nu(y) = \int_X \phi(T(x))\, d\mu(x).$$

Corollary 6.

$$\int_Y \phi\, d(T\#\mu) = \int_X \phi \circ T\, d\mu.$$

Proof of Lemma 5. Real analysis induction: one can prove for all simple functions that the equality holds. It remains to prove that one can approximate any bounded function with simple functions. Let $\phi : Z \to \mathbb{R}$ be Borel and bounded. Set $M \in \mathbb{N}$ and define

$$A_i := \{\tfrac{i}{M} \le \phi \le \tfrac{i+1}{M}\}, \quad \forall i \in \mathbb{Z},$$

and set $\phi_M = \sum_{i\in\mathbb{Z}} \frac{i}{M} \mathbb{1}_{A_i}$. Note that $A_i = \emptyset$ for large i. Then

$$\|\phi - \phi_M\|_{L^\infty} \le \max_i \|\phi - \phi_M\|_{L^\infty(A_i)} \le \frac{1}{M}.$$

Hence, $\forall \sigma \in P(Z)$,

$$\left|\int_Z (\phi - \phi_M)\, d\sigma\right| \le \|\phi - \phi_M\|_{L^\infty} \int_Z d\sigma \le \frac{1}{M}.$$ $\square$

Lemma 7. $(S \circ T)\#\mu = S\#(T\#\mu)$.

Proof. For all ϕ one has

$$\int \phi\, d(S\circ T)\#\mu = \int \phi\circ(S\circ T)\, d\mu = \int (\phi\circ S)\circ T\, d\mu = \int (\phi\circ S)\, dT\#\mu = \int \phi\, dS\#(T\#\mu).$$

$\square$

1.3 Transport Maps

Given $\mu \in P(X)$ and $\nu \in P(Y)$, $T : X \to Y$ is a *transport map* from μ to ν if $T\#\mu = \nu$.

Remark 8. Given μ, ν the set $\{T : T\#\mu = \nu\}$ can be empty! For example, take $\mu = \delta_{x_0}$ for $x_0 \in X$. Given any $T : X \to Y$, one has $T\#\mu = \delta_{T(x_0)}$ (Prove using Lemma 5).

Definition 9 (Coupling)**.** $\gamma \in P(X \times Y)$ is a coupling of μ and ν if

$$(\Pi_X)\#\gamma = \mu, \quad (\Pi_Y)\#\gamma = \nu,$$

where Π_X is projection on X. This condition is equivalent to having for all $\phi : X \to \mathbb{R}$,

$$\int_{X\times Y} \phi(x)\, d\gamma(x,y) = \int \phi \circ \Pi_X(x,y)\, d\gamma(x,y) = \int_X \phi(x)\, d\mu(x),$$

and for all $\psi : Y \to \mathbb{R}$, $\int_{X\times Y} \psi(y)\, d\gamma(x,y) = \int_Y \psi(y)\, d\nu(y)$. We write $\gamma \in \Gamma(\mu,\nu)$ for a coupling.

Remark 10. Given μ, ν, the set $\{\gamma : \gamma \in \Gamma(\mu,\nu)\}$ is always nonempty. Indeed $\gamma = \mu \otimes \nu$ is a coupling.

Remark 11 (Transport vs. Coupling). Let $T : X \to Y$ be such that $T\#\mu = \nu$. Set $(\mathrm{Id} \times T) : X \to X \times Y$ be $x \mapsto (x, T(x))$. Define $\gamma_T = (\mathrm{Id} \times T)\#\mu \in P(X \times Y)$. We claim that $\gamma_T \in \Gamma(\mu,\nu)$. For proof, observe that

$$(\Pi_X)\#\gamma_T = (\Pi_X)\#(\mathrm{Id} \times T)\#\mu = (\Pi_X \circ (\mathrm{Id} \times T))\#\mu = \mu,$$

and

$$(\Pi_Y)\#\gamma_T = \cdots = \nu.$$

1.4 Examples of Transport

Measurable Transport. One has the following theorem:

Theorem 12. *Let $\mu \in P(X)$ such that μ has no atoms. Then there exists $T : X \to \mathbb{R}$ such that T is injective μ-a.e. and $T\#\mu = dx|_{[0,1]}$ and $T^{-1} : [0,1] \to X$ exists a.e. with $(T^{-1})\#dx = \mu$.*

Monotone Arrangement. Assume μ, ν be measures on $\mathbb{R}$. Let $F(x) := \int_{-\infty}^{x} d\mu(t)$ and $G(y) := \int_{-\infty}^{y} d\nu(t)$. By convention, we assume that $G(y) = \lim_{\epsilon\to 0+} G(y-\epsilon)$. Define $G^{-1}(t) := \inf\{y : G(y) > t\}$ and let $T := G^{-1} \circ F : \mathbb{R} \to \mathbb{R}$.

Theorem 13. *If μ has no atoms, then $T\#\mu = \nu$.*

Proof. It suffices to show that $\nu(A) = \mu(T^{-1}(A))$ for all $A = (-\infty, a)$. □