Chapter 2

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Exercise 6

We want to show that $\sum_{i=1}^{n}(y_i - \hat{y}_i)(\hat{y}_i - \bar{y}) = 0$. We need to first show that

a) $(y_i - \hat{y}_i) = (y_i - \bar{y}) - \hat{\beta}_1(x_i - \bar{x})$

b) $(\hat{y}_i - \bar{y}) = \hat{\beta}_1(x_i - \bar{x})$

Proof.

a) Since $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ we have that

$$(y_i - \hat{y}_i) = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \tag{1}$$

substituting $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x}$ into (1) yields

$$(y_i - \hat{y}_i) = y_i - \bar{y} + \hat{\beta}_1\bar{x} - \hat{\beta}_1 x_i = (y_i - \bar{y}) - \hat{\beta}_1(x_i - \bar{x}) \tag{2}$$

b) We have that

$$\begin{aligned}(\hat{y}_i - \bar{y}) = \hat{\beta}_0 + \hat{\beta}_1 x_i - \bar{y} = \hat{\beta}_0 + \hat{\beta}_1 x_i - \frac{1}{n}\left(\sum_{i=1}^{n}\hat{\beta}_0 + \hat{\beta}_1 x_i\right) &= \hat{\beta}_0 + \hat{\beta}_1 x_i - \hat{\beta}_0 - \beta_1\bar{x} \\ &= \beta_1(x_i - \bar{x}) \end{aligned} \tag{3}$$

Finally using these two result we have that

$$\sum_{i=1}^{n}(y_i - \hat{y}_i)(\hat{y}_i - \bar{y}) = \sum_{i=1}^{n}\Big[(y_i - \bar{y}) - \hat{\beta}_1(x_i - \bar{x})\Big] \cdot \hat{\beta}_1(x_i - \bar{x})$$

Using $\hat{\beta}_1 = \frac{SXY}{SXX}$ we get

$$\begin{aligned}\sum_{i=1}^{n}(y_i - \hat{y}_i)(\hat{y}_i - \bar{y}) &= \sum_{i=1}^{n}\left[\frac{SXY}{SXX}(y_i - \bar{y})(x_i - \bar{x}) - \frac{SXY^2}{SXX^2}(x_i - \bar{x})^2\right] \\ &= \frac{SXY}{SXX}\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x}) - \frac{SXY^2}{SXX^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 \\ &= \frac{SXY}{SXX}SXY - \frac{SXY^2}{SXX^2}SXX = \frac{SXY^2}{SXX} - \frac{SXY^2}{SXX} = 0\end{aligned}$$

which is the desired result.