Examples to Lie Theory

This note is aimed for who wants to enter Lie theory. It consists of two beginning examples most part of which can be done using only elementary algebra, and some basic definitions with remarks giving a glance at general cases. We work over $\mathbb{C}$ unless point out specifically.

1 Type A_{n-1}

This example should be borne in mind during the study of Lie theory.

Lie algebra

Consider $\mathfrak{g} = \mathfrak{gl}_n := \{n \times n \text{ matrices over } \mathbb{C}\}$ and $\mathfrak{t} := \{\text{diagonal matrices}\} \subseteq \mathfrak{gl}_n$. ($\mathfrak{t}$ is called a **maximal torus** of $\mathfrak{g}$, isomorphic as a linear space to $\mathbb{C}^n$.) For $\forall x \in \mathfrak{gl}_n$, define

$$\mathrm{ad}x : \mathfrak{gl}_n \to \mathfrak{gl}_n, y \mapsto \mathrm{ad}x(y) = [x, y] := xy - yx.$$

Exercise 1 *Under such an definition,* ad *defines a linear map from* $\mathfrak{gl}_n$ *to* $\mathfrak{gl}(\mathfrak{gl}_n)$*, i.e. the linear space of all linear maps on* $\mathfrak{gl}_n$*, by* $x \mapsto \mathrm{ad}x$*, satisfying*

$$\mathrm{ad}[x, y] = [\mathrm{ad}x, \mathrm{ad}y].$$

With the bracket $[-, -]$ defined above, $\mathfrak{gl}_n$ can be viewed as a Lie algebra in the following sense.

Definition 2 *A vector space* $\mathfrak{g}$ *over* $\mathbb{C}$*, with a binary operation denoted by* $(x, y) \mapsto [x, y]$*, is called a* ***Lie algebra*** *if the following axioms are satisfied,*

(L1) The operation is bilinear.

(L2) $[x, x] = 0$, $\forall x \in \mathfrak{g}$.

(L3) ***(Jacobi identity)*** $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, $\forall x, y, z \in \mathfrak{g}$.

We can call the operation as ***Lie bracket****.*

Exercise 3

1. *Verify* $\mathfrak{gl}_n$ *is a Lie algebra (and hence so is* $\mathfrak{gl}(\mathfrak{gl}_n) \simeq \mathfrak{gl}_{n^2}$*).*
2. *Since* $\mathrm{Char}\mathbb{C} \neq 2$*, (L2) can be replaced by: (L2)′* $[x, y] = -[y, x], \forall x, y \in \mathfrak{g}$.
3. *How should we define a Lie subalgebra? How about a homomorphism between Lie algebras?*
4. *If you have the correct definition of a homomorphism,* $\mathrm{ad} : \mathfrak{gl}_n \to \mathfrak{gl}(\mathfrak{gl}_n)$ *is such a map, while* $\mathrm{ad}x : \mathfrak{gl}_n \to \mathfrak{gl}_n$ *is not. We call* $\mathrm{ad} : \mathfrak{gl}_n \to \mathfrak{gl}(\mathfrak{gl}_n)$ *as the* ***adjoint representation*** *of* $\mathfrak{gl}_n$*.*

5.* *How should we define in general a representation of a Lie algebra?*

In the exercise below, we study the structure of $\mathfrak{gl}_n$ as a Lie algebra.

Exercise 4 *Consider* $\mathrm{ad}|_{\mathfrak{t}}$*, then* $\mathfrak{gl}_n$ *admits a so-called "root spaces decomposition":*

$$\mathfrak{gl}_n \xlongequal{as\ v.s.} \mathfrak{t} \oplus \Big(\bigoplus_{1 \leq i, j \leq n, i \neq j} \mathfrak{g}_{ij} \Big)$$

satisfying:

1. $\dim_{\mathbb{C}} \mathfrak{g}_{\mathrm{ij}} = 1, \forall i, j$;

2. $\forall i,j, \exists \alpha_{ij} \in \mathfrak{t}^*$, *s.t.* $\forall h \in \mathfrak{t}, x \in \mathfrak{g}_{ij}$, $\mathrm{ad}h(x) = \alpha_{ij}(h) \cdot x$.

(Hint: consider elementary matrices E_{ij}*.)*

Determine all α_{ij}*, and find a basis for each* $\mathfrak{g}_{ij}$*.* ($\Phi := \{\alpha_{ij}\}$ *is called the* ***root system*** *of* $\mathfrak{gl}_n$*,* α_{ij}*'s are called* ***roots****,* $\mathfrak{g}_{ij}$*'s are called* ***root spaces****.) One can rewrite the decomposition as*

$$\mathfrak{gl}_n = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha$$

where $\dim \mathfrak{g}_\alpha \leq 1$ *unless* $\alpha = 0$*, and* $\mathfrak{g}_0 = \mathfrak{t}$*.*

Such a decomposition exists in any complex semisimple (more generally, reductive) Lie algebra, see the remark after exercise **5**. For another example, type C_2, see section 2, exercises **10**, **11**.

Root system and Weyl group

Exercises **5** to **6** are devoted to study the root system and its Weyl group for this example.

Exercise 5 *1. Choose the standard basis* $\{\varepsilon_i\}_{1 \leq i \leq n}$ *of* $\mathbb{R}^n$*, Identify* $\mathbb{R}^n$ *with a real subspace of* $\mathfrak{t}^*$ *by*

$$\varepsilon_i : \mathfrak{t} \to \mathbb{C}, \mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i,$$

then $\Phi \subseteq \mathbb{R}^n$*. Find all roots in* $\mathbb{R}^n$ *using this standard basis.*

2. For any $\alpha_{ij} \in \Phi \subseteq \mathbb{R}^n$*, let* σ_{ij} *be the reflection of* $\mathbb{R}^n$ *with respect to* α_{ij}*. Let* W *be the reflection group generated by all* σ_{ij}*'s, then* $W \simeq S_n$*. With this isomorphism, determine the image of* α_{ij} *under* $\sigma \in S_n$*. Moreover,* W *can be generated only by* ***simple reflections*** $\{\sigma_{i,i+1}\}_{1 \leq i \leq n-1}$*. (*$W$ *is called the* ***Weyl group*** *of* $\mathfrak{gl}_n$*, or of root system* Φ*.)*

3. Φ *satisfies the following axioms for root systems,*

(R1) Φ *is finite, and doesn't contain* 0*.*

(R2) If $\alpha \in \Phi$*, the only multiples of* α *in* Φ *are* $\pm\alpha$*.*

(R3) If $\alpha \in \Phi$*, the reflection* σ_α *leaves* Φ *invariant. In another words, since* Φ *is finite,* σ_α *induces a permutation on all elements in* Φ*.*

(R4) If $\alpha, \beta \in \Phi$*, then* $2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle \in \mathbb{Z}$*, where* $\langle -,- \rangle$ *is the standard inner product in* $\mathbb{R}^n$*. (The reason of considering such a number is that it appears in* $\sigma_\alpha(\beta) = \beta - 2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle \cdot \alpha$*.)*

In general, if a subset $\Phi \subseteq \mathbb{R}^n$ satisfies (R1) to (R4), we call it as a(n) (abstract) **root system**. The dimension of the real subspace spanned by Φ is called the **rank** of this root system. In our example, the rank is $n-1$, that is why we say "type A_{n-1}" rather than "type A_n".

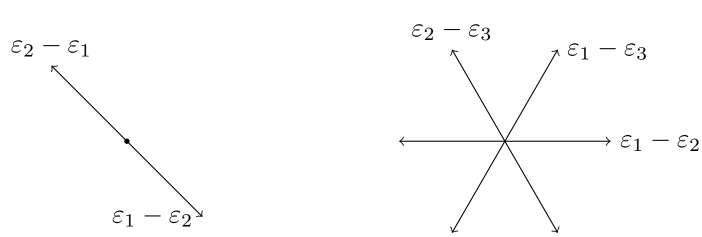

Figure 1: Root systems in type A_1 (left) and A_2 (right)

For any root system, it is associated with a unique semisimple Lie algebra $\mathfrak{g}$ which admits a similar notion "maximal torus". Fix a maximal torus $\mathfrak{t} \subseteq \mathfrak{g}$, the root spaces decomposition writes as

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{t}^*} \mathfrak{g}_\alpha = \mathfrak{t} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha), \tag{1}$$

where $\mathfrak{t} = \mathfrak{g}_0$, $\dim \mathfrak{g}_\alpha \leq 1$ for any $\alpha \neq 0$, $\mathfrak{t}$ acts via ad on $\mathfrak{g}_\alpha$ by α, and we call $\Phi = \{\alpha \in \mathfrak{t}^* \backslash \{0\}, \mathfrak{g}_\alpha \neq 0\}$ as the **root system** of $\mathfrak{g}$. It should be pointed out that the choice of a maximal torus is not unique.

Let's return to our example $\mathfrak{gl}_n$.

Exercise 6 *1. If you got the correct answer for exercise **5**.1, $\Phi = \{\varepsilon_i - \varepsilon_j\}_{i \neq j}$, then Φ can be partitioned into two disjoint subsets, $\Phi = \Phi^+ \sqcup \Phi^-$ where $\Phi^+ = \{\varepsilon_i - \varepsilon_j\}_{i<j}$ and $\Phi^- = -\Phi^+$. Let $\Delta = \{\varepsilon_i - \varepsilon_{i+1}\}_{1 \leq i \leq n-1} \subseteq \Phi^+$, then Δ is a basis of $\mathbb{R}\Phi$. (Δ is called a **base** of Φ.)*

*2. Moreover, any $\alpha \in \Phi^+$ is a linear combination of elements in Δ where all coefficients are all nonnegative integers. Similar for elements in Φ^-. (Elements in Φ^+, Φ^- are called **positive roots**, **negative roots** resp., those in Δ are called **simple roots** with respect to Φ^+.)*

*3. $\mathfrak{b} := \mathfrak{t} \oplus (\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha)$ is a Lie subalgebra (called a **Borel subalgebra**). What is it in form of matrix? How about $\mathfrak{b}^- := \mathfrak{t} \oplus (\bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha)$ (another Borel)?*

4. Such a partition and choice of Δ is not unique, can you find another partition and Δ satisfying the same property?*

In general, for any root system Φ we can non uniquely find a **base** $\Delta \subseteq \Phi$, elements in which are called **simple roots**, such that it forms a basis for $\mathbb{R}\Phi$, and any root is an integral combination of simple roots with all coefficients nonnegative or nonpositive. In this way Φ is partitioned into **positive roots** and **negative roots**, and they are related by multiplying -1. The Lie subalgebras $\mathfrak{b} := \mathfrak{t} \oplus (\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha)$ and $\mathfrak{b}^-$ are called **Borel subalgebras**. The **Weyl group** W of Φ, which is defined to be the reflection group generated by $\{\sigma_\alpha, \alpha \in \Phi\}$, can be generated only by **simple reflections**, namely those reflections with respect to simple roots. A useful fact is, W acts simply transitively on the set of all choices of Δ, as well as all choices of Φ^+ (so now you may be able to do exercise **6**.4). In particular, there is a **longest element** in W, swapping Φ^+ and Φ^- (for example, see exercise **8**.7, **12**.5). You will see the proof, which is beyond our examples, when study Lie theory in detail.

Exercise 7 *Modify $\mathfrak{g}$ to be $\mathfrak{sl}_n = \{n \times n$ matrices with trace $0\}$ and $\mathfrak{t}$ to be the subspace consisting of traceless diagonal matrices. Then $\mathfrak{gl}_n = \mathbb{C} \cdot \mathrm{Id}_n \oplus \mathfrak{sl}_n$, everything above stays unchanged. (In fact, $\mathfrak{gl}_n$ is just reductive but not semisimple, the (semi)simple Lie algebra corresponding to type A_{n-1} is $\mathfrak{sl}_n$ instead.)*

For root system in type C_2, see exercises **11** in section 2.

Algebraic group

Next we go into the corresponding **algebraic group** $G = GL_n := \{$invertible $n \times n$ matrices over $\mathbb{C}\}$. (One can also consider $SL_n := \{$matrices with determinant $1\}$. The rest of this section are also applied to SL_n with a natural modification.) Let:

$$T = \{\text{diagonal matrices in } GL_n\} \text{ (a \textbf{maximal torus} of } GL_n),$$
$$B = \{\text{upper triangular matrices in } GL_n\} \text{ (a \textbf{Borel subgroup} of } GL_n),$$
$$B^- = \{\text{lower triangular matrices in } GL_n\} \text{ (another Borel, an \textbf{opposite Borel} to } B),$$
$$U = \{\text{matrices in } B \text{ with 1's in diagonal positions}\} \text{ (a \textbf{unipotent subgroup})},$$
$$U^- \text{ similarly},$$
$$\text{For } i \neq j, U_{ij} = \{u_{ij}(x) := \mathrm{Id}_n + x \cdot E_{ij}, x \in \mathbb{C}\} \text{ (a \textbf{root subgroup})}.$$

We say U, U^- is the unipotent part of B, B^- resp., here unipotent means being nilpotent after subtracting identity matrix.

The next exercise, some problems in which may be hard, studies the structure of $T \subseteq B \subseteq GL_n$, and its relation to Weyl group.

Exercise 8 *1. $B = U \rtimes T$, $B^- = U^- \rtimes T$, $B \cap B^- = T \simeq (\mathbb{C}^*)^n$.*

2. *As groups, as well as varieties (or manifolds), $(U_{ij}, \cdot) \simeq (\mathbb{C}, +)$. (In fact, they are isomorphic as algebraic groups.)*

3. *By notation abuse, we also use α_{ij} to indicate the homomorphism $T \to \mathbb{C}^*$ determined by $tu_{ij}(x)t^{-1} = u_{ij}(\alpha_{ij}(t) \cdot x), \forall t \in T, x \in \mathbb{C}$. For $t = \mathrm{diag}(t_1, \ldots, t_n) \in T$, express $\alpha_{ij}(t)$ in terms of t_i's. (In previews setting, a root α_{ij} is a linear function on $\mathfrak{t}$, now it becomes a homomorphism of multiplicative groups from T to $\mathbb{C}^*$.)*

4. *Compute the normalizer $N_G(T)$ and the centralizer $Z_G(T)$.*

5. *Define the **Weyl group** W to be $N_G(T)/Z_G(T)$, then $W \simeq S_n$. In particular, it is finite. (Hint: choose representatives of elements in W to be permutation matrices, and identify S_n with the subgroup of permutation matrices. A matrix is called permutation matrix if in each row and column all entries are 0 except an 1.)*

   <u>*In what follows, fix the set of representatives above,*</u> *i.e. identify S_n with permutation matrices.*

6. *For $\sigma \in S_n$, $\sigma u_{ij}(x)\sigma^{-1} = u_{\sigma(i)\sigma(j)}(x)$. (This problem, as well as 3., explains the name "root subgroup".)*

7. *Let $w_0 = \begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix} \in S_n$ (called the **longest element**), then $w_0^{-1} = w_0$ and $w_0 B w_0^{-1} = B^-$.*

8. *(**Bruhat decomposition**) Compute the double coset decomposition of G with respect to subgroup B (and B^-):*
$$GL_n = \bigsqcup_{w \in S_n} BwB = \bigsqcup_{w \in S_n} B^- wB$$

9. *Arrange $\{(i,j), i<j\}$ in <u>any</u> fixed order (for example, $(1,n), (2,n), \ldots, (n-1,n), (1,n-1), \ldots, (n-2, n-1), \ldots, (1,3), (2,3), \overline{(1,2)}$), then*
$$U = \prod_{i<j} U_{ij}$$
in that order. (For example, $U = U_{1,n} \ldots U_{n-1,n} U_{1,n-1} \ldots U_{n-2,n-1} \ldots U_{1,2}$)

10. *In 8., $BwB = UwB = \left(\prod_{\substack{i<j \\ w^{-1}(i) > w^{-1}(j)}} U_{ij} \right) wB$, using 6. and 9..*

11. *In language of exercise **6**, $i < j$ can be rewritten as $\alpha_{ij} = \varepsilon_i - \varepsilon_j \in \Phi^+$, and $w^{-1}(i) > w^{-1}(j)$ as $w^{-1}(\alpha_{ij}) \in \Phi^-$.*

12. *Redo 7. using 1. 6. and 9.*

13. *B is a (maximal) solvable subgroup of GL_n. (In general, a **Borel subgroup** of an algebraic group is defined to be a maximal connected solvable closed subgroup. All Borels are conjugated, and forms the flag variety of G. See the remark in exercise **9**.4.)*

In Grothendieck's vision, the data $B \cap B^- = T \subset G$ look like a butterfly: the body is a maximal torus T, the wings are two opposite Borel subgroups B and B^-.

Flag variety

Let's introduce the **flag variety** of GL_n, i.e. $\mathscr{B} := GL_n/B$. (It has a "quotient" variety structure, which we don't introduce here.)

By definition, a (complete) **flag** F is a sequence of subspaces of $\mathbb{C}^n$, say $0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{C}^n$, satisfying $\dim F_i = i$. Choosing the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$, define the standard flag to be $F^{\text{std}} = (F_i^{\text{std}})$ where $F_i^{\text{std}} = \langle e_1, \ldots, e_i \rangle$. GL_n acts from the left on the set of all flags naturally.

Exercise 9 1. *Under such a left action, the stabilizer of F^{std} is exactly B.*

2. *The action is transitive, i.e. for any two flags, there exists an invertible matrix sending one flag to the other.*

3. *For two distinct flags F and $g \cdot F$, $g \in GL_n$, their stabilizers are conjugated by g.*

4. *We have bijections*

$$\mathscr{B} \xrightarrow{\sim} \{\text{all flags}\} \xrightarrow{\sim} \{\text{subgroups conjugate to } B\}$$
$$gB \mapsto gF^{std} \mapsto gBg^{-1} = Stab_G(gF^{std}).$$

(Lie-Kolchin theorem says, any connected solvable closed subgroup of GL_n leaves a flag invariant, thus contained in a conjugate of B. That means all Borels are conjugated to B, the flag variety is also the variety formed by all Borel subgroups.)

5. *(Bruhat decomposition) $\mathscr{B} = \bigsqcup_{w \in S_n} BwB/B$. ($\mathscr{B}_w := BwB/B$ is called a **Schubert cell**.)*

6. *For $w \in S_n, 1 \le i, j \le n$, define*

$$r_w(i,j) = \#\{1 \le k \le i, w(k) \le j\}.$$

Viewing w as a permutation matrix, $r_w(i,j)$ is the number of 1's in the submatrix intersected by the first i columns and the first j rows. Then

$$\mathscr{B}_w = \{F \text{ a flag}, \dim(F_i \cap F_j^{std}) = r_w(i,j) \text{ for all } i, j\}.$$

(Hint: consider wF^{std} first, then show that the action of B doesn't change this property.)

7. *By exercise **8**.10 and **8**.2, a Schubert cell $\mathscr{B}_w$ is isomorphic to affine space $\mathbb{C}^{\ell(w)}$ as a variety (or manifold), where*

$$\ell(w) := \#\{(i,j), i < j, w^{-1}(i) > w^{-1}(j)\}$$

*(called **length** of w).*

The next two problems explain why we call $\ell(w)$ "length".

8. *Let $S = \{s_i := (i, i+1) \in S_n, 1 \le i \le n-1\}$ be the set of simple reflections. Then $\forall w \in S_n, s_i \in S$, $\ell(ws_i) = \ell(w) \pm 1, \ell(s_i w) = \ell(w) \pm 1$.*

9. *For any $w \in S_n$, w can be written as a product of a sequence of elements in S, say $w = s_{i_1} \dots s_{i_k}$, such that $k = \ell(w)$. Moreover, $\ell(w)$ is the minimal number to do so. (Such a sequence is called a* ***reduced expression*** *of w, which is not unique.)*

The last one describes Schubert varieties corresponding to simple reflections.

10. *Note that the Weyl group of GL_2 is $S_2 = \{e, s_1\}$, and the flag variety of GL_2 is isomorphic to $\mathbb{CP}^1$. Show that in general, $\mathscr{B}_e = \{F^{std}\}$, $\overline{\mathscr{B}_{s_i}} := \mathscr{B}_{s_i} \sqcup \mathscr{B}_e = \{F, F_j = F_j^{std}, \forall j \ne i\}$, also a $\mathbb{CP}^1$. (In fact, it is the closure of the one dimensional Schubert cell.)*

By exercise **8**.11, $\ell(w) = \#\{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-\} = \ell(w^{-1})$. This holds generally in every Weyl group. Can you prove the second equality?

A **Schubert variety**, by definition, is the closure of some Schubert cell. It is a union of some Schubert cell, indexed by lower elements in Weyl group in the sense of **Bruhat order**. In this example, it turns out that $\overline{\mathscr{B}_w} = \{F, \dim(F_i \cap F_j^{\mathrm{std}}) \ge r_w(i,j), \forall i, j\}$. Bruhat order is define on any Coxeter group, which is also an important object in Lie theory and representation theory. Weyl groups are Coxeter. We don't go into this context in this note.

For this type of example, we stop here.

2 Type C_2

In fact, type C_2 is the same as type B_2, and we usually call it B_2 for some reason. But the symplectic type C_n is different from orthogonal type B_n when $n > 2$.

When doing this example, please compare everything to its counterpart in the previous section. Be aware that notations sharing with section 1 are redefined in this type.

Lie algebra and root system

On $\mathbb{C}^4$, define a symplectic form $\langle -, - \rangle_s$ by the matrix

$$J = \begin{pmatrix} 0 & \dots & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & \dots & 0 \end{pmatrix},$$

i.e. an anti-symmetric bilinear form given by $\langle e_1, e_4 \rangle_s = \langle e_2, e_3 \rangle_s = 1$ and $\langle e_i, e_j \rangle_s = 0$ if $i + j \ne 5$.

Let

$$\mathfrak{g} = \mathfrak{sp}_4 := \{x \in \mathfrak{gl}_4, \langle xv_1, v_2 \rangle_s = -\langle v_1, xv_2 \rangle_s, \forall v_1, v_2 \in \mathbb{C}^4\} = \{x \in \mathfrak{gl}_4, x^t J + Jx = 0\}.$$

By easy computations, any element in $\mathfrak{sp}_4$ has the form

$$\begin{pmatrix} t_1 & a & b & d \\ a' & t_2 & c & b \\ b' & c' & -t_2 & -a \\ d' & b' & -a' & -t_1 \end{pmatrix} \tag{2}$$

in which each variable can be arbitrary in $\mathbb{C}$.

Exercise 10 *Verify all computations above, and that $\mathfrak{sp}_4$ is a Lie algebra equipped with $[-,-]$. If you got the right definition of a Lie subalgebra in exercise 3.3, $\mathfrak{sp}_4$ is in fact a Lie subalgebra of $\mathfrak{gl}_4$ (and $\mathfrak{sl}_4$).*

A maximal torus in $\mathfrak{sp}_4$ is $\mathfrak{t} = \{\mathrm{diag}(t_1, t_2, -t_2, -t_1)\} \simeq \mathbb{C}^2$. Similar to exercise **5**.1, let $\varepsilon_i \in \mathfrak{t}^*, i = 1, 2$ be

$$\varepsilon_i : \mathrm{diag}(t_1, t_2, -t_2, -t_1) \mapsto t_i$$

and span a real space $\mathbb{R}^2 = \mathbb{R}\varepsilon_1 \oplus \mathbb{R}\varepsilon_2$.

Exercise 11 *1. Imitate exercise 4, decompose $\mathfrak{sp}_4$ into root spaces in form of (1). Determine the root system and find bases for root spaces. (The matrix (2) may gives a hint, consider "elementary matrices" corresponding to $a, b, c, d, a', b', c', d'$. The root system should be $\Phi = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 + \varepsilon_2), \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$.)*

2. *Verify root system axioms for Φ.*

3. *Let $\alpha = \varepsilon_1 - \varepsilon_2, \beta = 2\varepsilon_2$, then $\Delta = \{\alpha, \beta\}$ is a base of Φ. Find Φ^+ and Φ^- with respect to this base. What are the corresponding Borel subalgebras?*

4. *Recall the weyl group W is generated by reflections of $\mathbb{R}^2$ with respect to roots in Φ. W is isomorphic to dihedral group of a square. Verify that W can be generated by $s := \sigma_\alpha$ and $t := \sigma_\beta$: W is the quotient of free group on $\{s, t\}$ by the relation $(st)^4 = e$.*

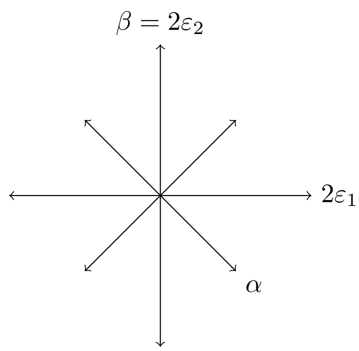

Figure 2: Root system in type C_2

Algebraic group

Let

$$G = Sp_4 := \{g \in GL_4, \langle gv_1, gv_2 \rangle_s = \langle v_1, v_2 \rangle_s, \forall v_1, v_2 \in \mathbb{C}^4\} = \{g \in GL_4, g^t J g = J\}.$$

A maximal torus is

$$T = \{\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}), t_i \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^2,$$

a Borel B being all upper triangular matrices in Sp_4, thus we are in the situation $T \subseteq B \subseteq Sp_4$.

The unipotent part of B, namely U, is those upper triangular unipotent matrices lying in Sp_4. Similarly define B^- and U^-. We still have $B = U \rtimes T$, $B^- = U^- \rtimes T$, and $B \cap B^- = T$.

For any root $\gamma \in \Phi$, there are root subgroups $U_\gamma = \{u_\gamma(x), x \in \mathbb{C}\}$ defined by

$$u_\alpha(x) = \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix}, u_\beta(x) = \begin{pmatrix} 1 & & & \\ & 1 & x & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$u_{\beta+\alpha}(x) = \begin{pmatrix} 1 & & x & \\ & 1 & & x \\ & & 1 & \\ & & & 1 \end{pmatrix}, u_{\beta+2\alpha}(x) = \begin{pmatrix} 1 & & & x \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

with zeros in empty positions, and $u_{-\gamma}(x) = u_\gamma(x)^t$.

$N_G(T)/Z_G(T)$ will be isomorphic to the Weyl group W, and we can choose representatives for $s = \sigma_\alpha$ and $t = \sigma_\beta$ to be

$$\dot{s} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \dot{t} = \begin{pmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{pmatrix} \in N_G(T).$$

Note that $\dot{t} \neq \dot{t}^{-1}$ although $t = t^{-1}$.

Exercise 12 *1. Check that all matrices above indeed lie in Sp_4, and $\dot{s}, \dot{t} \in N_G(T)$.*

2. U_γ is isomorphic to $(\mathbb{C}, +)$ as groups.

*3. As in exercise **8**.3, regard ε_i $(i = 1, 2)$ as a homomorphism from T to $\mathbb{C}^*$ by*

$$\varepsilon_i : \mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1}) \mapsto t_i,$$

thus we have a homomorphism from additive group $\mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ to multiplicative group $\mathrm{Hom}(T, \mathbb{C}^*)$. In this way, compute that $tu_\gamma(x)t^{-1} = u_\gamma(\gamma(t)x)$ for all $t \in T$, $\gamma \in \Phi$, $x \in \mathbb{C}$.

4. Let $w \in \{s, t\}$, then $\dot{w}u_\gamma(x)\dot{w}^{-1} = u_{w(\gamma)}(c_{\gamma,\dot{w}}x)$. Here $c_{\gamma,\dot{w}}$ is a nonzero constant depending on $\gamma \in \Phi$ and $\dot{w}$.

5. $\dot{w}_0 = \dot{s}\dot{t}\dot{s}\dot{t} \in N_G(T)$ is a representative for the longest element $w_0 = stst = tsts$ in W. Compute the last statement for $w = w_0$, and that $\dot{w}_0 B \dot{w}_0 = B^-$.

Choosing a set of representatives of W in $N_G(T)$, we have the Bruhat decomposition written as

$$Sp_4 = \bigsqcup_{w \in W} B\dot{w}B = \bigsqcup_{w \in W} B^- \dot{w}B,$$

and $B\dot{w}B = U\dot{w}B = \left(\prod_{\substack{\gamma \in \Phi^+ \\ w^{-1}(\gamma) \in \Phi^-}} U_\gamma \right) \dot{w}B$. The computation seems to be harder, but you can have a try.

Flag variety

For a subspace $V \subseteq \mathbb{C}^4$, define complementary space with respect to the symplectic form by

$$V^\perp = \{w \in \mathbb{C}^4, \langle w, v \rangle_s = 0, \forall v \in V\}.$$

Let

$$\mathscr{F} = \{F = (0 \subseteq F_1 \subseteq F_2 \subseteq F_3 \subseteq \mathbb{C}^4), \dim F_i = i, F_1 = F_3^\perp\}$$

be a subset of all flags. For such a flag, we have $F_2 = F_2^\perp$ for free. Since Sp_4 preserves the symplectic form, it acts on $\mathscr{F}$ from the left by the usual way.

Exercise 13 *1. F^{std} lies in $\mathscr{F}$, and $Stab_G(F^{std}) = B$*

2. The action is transitive. It is equivalent to show, for any $F \in \mathscr{F}$, we can find a basis of $\mathbb{C}^4$, say $\{v_1, v_2, v_3, v_4\}$, such that $F_i = \langle v_1, \dots, v_i \rangle$ for all $i = 1, 2, 3$, and $\langle v_1, v_4 \rangle = \langle v_2, v_3 \rangle = 1, \langle v_i, v_j \rangle = 0$ for all $i + j \neq 5$.

3. Thus, we can identify $\mathscr{F}$ with $\mathscr{B} = Sp_4/B$, the flag variety, and with the set of all Borels,

$$\mathscr{B} \xrightarrow{\sim} \mathscr{F} \xrightarrow{\sim} \{\textit{subgroups conjugate to } B\}$$
$$gB \mapsto gF^{std} \mapsto gBg^{-1} = Stab_G(gF^{std}).$$

By the Bruhat decomposition of Sp_4, we get $\mathscr{B} = \bigsqcup_{w \in W} B\dot{w}B/B$. The Schubert cell $\mathscr{B}_w = B\dot{w}B/B = \left(\prod_{\substack{\gamma \in \Phi^+ \\ w^{-1}(\gamma) \in \Phi^-}} U_\gamma\right) \dot{w}B/B \simeq \mathbb{C}^{\ell(w)}$.

4. Write down the length of every element in $W = \{e, s, t, st, ts, sts, tst, stst = tsts\}$.

5. In particular, $\mathscr{B}_s = U_\alpha \dot{s}B/B$, $\mathscr{B}_t = U_\beta \dot{s}B/B$, deduce that

$$(\overline{\mathscr{B}_s} =)\mathscr{B}_s \sqcup \mathscr{B}_e = \{F \in \mathscr{F}, F_2 = F_2^{std}\},$$
$$(\overline{\mathscr{B}_t} =)\mathscr{B}_t \sqcup \mathscr{B}_e = \{F \in \mathscr{F}, F_1 = F_1^{std} (\Leftrightarrow F_3 = F_3^{std})\}.$$

Using Bott-Samelson resolution, we can know more about the structure of $\mathscr{B}$. For example, it will be easy to see $\overline{\mathscr{B}_{sts}} = \{F \in \mathscr{F}, \dim(F_2 \cap F_2^{\mathrm{std}}) \geq 1\}$.

Further reading

A standard and elementary textbook for Lie algebras and root systems is

[1] J. E. Humphreys. *Introduction to Lie algebras and representation theory*. GTM 9. Springer. 1972.

For more reference,

[2] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4-6*. Springer. 2002.

[3] N. Jacobson. *Lie algebras*. Dover. 1962.

Systematic knowledge on Coxeter groups and root systems (as well as axiomatic Bruhat decomposition) can be found in [2]. As sequels of [1], one can read

[4] D. H. Collingwood, W. M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold. 1993.

for deeper understanding about structure of semisimple Lie algebras and their nilpotent cones, or read

[5] J. E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*. GSM 94. AMS. 2008.

for their representation theory. Besides [2], the following two materials are also good references for Coxeter groups, Hecke algebras, and Kazhdan-Lusztig theory:

[6] J. E. Humphreys. *Reflection groups and Coxeter groups*. Cambridge. 1990.

[7] G. Lusztig. *Hecke algebras with unequal parameters*. arXiv:math/0208154v2. 2014.

For the structure theory of algebraic groups, standard textbooks are

[8] A. Borel. *Linear algebraic groups*. 2nd ed. GTM 126. Springer. 1991.

[9] J. E. Humphreys. *Linear algebraic groups*. GTM 21. Springer. 1975.

[10] T. A. Springer. *Linear algebraic groups*. 2nd ed. Birkhäuser. 2009.

For their representations, refer to

[11] J. C. Jantzen. *Representations of algebraic groups*. 2nd ed. AMS. 2003.

To be added more.