Four lectures on simple groups and singularities

Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, **11**(1980), ii+64 pp.

Peter Slodowy

Retyped by somebody*, 2022

Contents

*Email: mathongxing@163.com

Introduction

These notes essentially reproduce the material given in a course of four lectures at the Mathematical Institute of the Rijksuniversiteit Utrecht in February-March 1979. One aim of the lectures was to explain the connection between the so called simple singularities, classified by the homogeneous Dynkin diagrams A_r, D_r and E_r, and the corresponding simple Lie groups as established by Grothendieck and Brieskorn. Furthermore we discussed a generalization to Lie groups with inhomogeneous diagrams $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$ and a construction of Weyl group representations as monodromy transformations which is related to a recent different construction by T.A. Springer.

The first three lectures are written up in an informal style following closely the original talks. Many arguments are only sketched or left out, but complete details may be found in the reference [Sl] for which, on the other hand, these notes may serve as a useful guide. In contrast the presentation of the fourth lecture on monodromy representations is systematic and quite complete as we have no other reference for it.

I wish to thank the Mathematical Institute of the Rijksuniversiteit Utrecht for its hospitality and especially D. Siersma and T.A. Springer for the interesting discussions I had with them. My thanks go also to S. Koorn and J. Stalpers for the careful typing of the manuscript.

Lecture 1 Simple singularities and simple groups

1.1 Simple singularities

Let us start with the so called simple singularities. Their first occurrence in history stands in connection with a uniform method for their construction. Consider finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Up to conjugacy these groups are of the following types (cf. [Sp3]):

$$\begin{aligned}
\mathcal{C}_n &- \text{cyclic groups of order } n\\
\mathcal{D}_n &- \text{binary dihedral group of order } 4n\\
\mathcal{T} &- \text{binary tetrahedral group of order 24}\\
\mathcal{O} &- \text{binary octahedral group of order 48}\\
\mathcal{I} &- \text{binary icosahedral group of order 120}
\end{aligned}$$

This classification goes almost back to the ancient Greeks as it is related in the following way to the classification of regular solids. Regard a finite subgroup F of $\mathrm{SL}_2(\mathbb{C})$ as sitting in the compact subgroup $\mathrm{SU}_2(\mathbb{C})$ which is a double cover of $\mathrm{SO}_3(\mathbb{R})$. The image of F in $\mathrm{SO}_3(\mathbb{R})$ corresponds now to the symmetry group of a regular body in $\mathbb{R}^3$, in case $F = \mathcal{T}, \mathcal{O}, \mathcal{I}$ to that of the tetrahedron, octahedron, icosahedron.

The explicit classification of the finite subgroups F of $\mathrm{SL}_2(\mathbb{C})$ appears in the book of Klein "Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade" where he also determines the polynomials on $\mathbb{C}^2$ in variant under F. More precisely he shows these polynomials to be generated by three fundamental ones X, Y, Z subject to only one nontrivial relation as follows (cf. [Sp3], [Kl]):

group F	relation $R(X,Y,Z)=0$
$\mathcal{C}_n$	$X^n + YZ = 0$
$\mathcal{D}_n$	$X^{n+1} + XY^2 + Z^2 = 0$
$\mathcal{T}$	$X^4 + Y^3 + Z^2 = 0$
$\mathcal{O}$	$X^3Y + Y^3 + Z^2 = 0$

$$\mathcal{I} \qquad X^5 + Y^3 + Z^2 = 0$$

Let $q : \mathbb{C}^2 \to \mathbb{C}^3$ be the quotient map

$$v \mapsto (X(v), Y(v), Z(v)), \quad v \in \mathbb{C}^2,$$

given by the fundamental invariants X, Y, Z. The image $q(\mathbb{C}^2)$ of $\mathbb{C}^2$ under q may be identified with the quotient space $\mathbb{C}^2/F$ and is given by the equation $R(X, Y, Z) = 0$ in $\mathbb{C}^3$. One easily notes that F operates freely on $\mathbb{C}^2 \setminus 0$ (equivalently F doesn't contain any reflection). Hence $\mathbb{C}^2/F$ has at most, and in fact, a singular point at the origin $0 \in \mathbb{C}^2/F \subset \mathbb{C}^3$. As a hypersurface, $\mathbb{C}^2/F$ is now a normal variety (one might also argue: as the quotient by F of the normal variety $\mathbb{C}^2$).

By **simple singularity** we will understand from now on one of the quotients $\mathbb{C}^2/F$, or more flexibly, the complex analytic germ of $\mathbb{C}^2/F$ at the singular point.

The simple singularities occurred in the work of many mathematicians (Du Val, Kirby, M. Artin, Brieskorn, Tjurina, Arnol'd), characterized in many different ways. They are also called **rational double points** or **Kleinian singularities**. We follow Arnol'd's notation ([A2]) stressing the connection to the simple Lie groups to be discussed further on.

1.2 Resolution of simple singularities

In the above mentioned studies of simple singularities their resolution plays an important role. Let us recall some definitions:

A **resolution** of a reduced variety X is a morphism $\pi : Y \to X$ from a smooth variety Y to X with the following properties

(i) π is proper;
(ii) Let X^{reg} be the regular points of X, then $\pi^{-1}(X^{\text{reg}})$ is dense in Y and the restriction $\pi^{-1}(X^{\text{reg}}) \to X^{\text{reg}}$ of π is an isomorphism.

The subvariety $E = Y \setminus \pi^{-1}(X^{\text{reg}})$ is called the **exceptional set** of π. A resolution $\pi_0 : Y_0 \to X$ of X is called **minimal** iff any other resolution $Y \to X$ factorizes over π_0

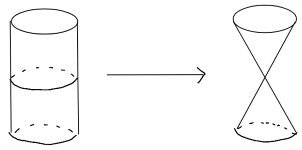
$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, dashed] & & X \\ & Y_0 \arrow[ur, "\pi_0"'] & \end{tikzcd}$$

Similar definitions work for germs of analytic spaces.

The simple singularities may be resolved in a particularly easy way: The **blowing up** $\beta : \widetilde{\mathbb{C}}^3 \to \mathbb{C}^3$ of $\mathbb{C}^3$ at the origin $0 \in \mathbb{C}^3$ is the natural projection from $\widetilde{\mathbb{C}}^3 = \{(x, g) \in \mathbb{C}^3 \times \mathbb{P}^2 \,|\, x \in g\}$ to $\mathbb{C}^3$ (we consider $g \in \mathbb{P}^2$ as a line in $\mathbb{C}^3$). Outside $\beta^{-1}(0)$, which is isomorphic to $\mathbb{P}^2$, the restriction $\beta^{-1}(\mathbb{C}^3 \setminus 0) \to \mathbb{C}^3 \setminus 0$ is an isomorphism. Let $X \subset \mathbb{C}^3$ be a subvariety with an isolated singular point at 0 and $\widetilde{X}$ the closure of $\beta^{-1}(X \setminus 0)$ in $\widetilde{\mathbb{C}}^3$. Then $\widetilde{X}$ is the **blow up** of X in 0. Actually this process doesn't depend on the particular embedding of X into $\mathbb{C}^3$ and may be applied again to points of $\widetilde{X}$. The simple singularities may now even be characterized as the only surface singularities of multiplicity two which can be resolved by iterating a finite number of times this process of blowing up points (cf. [Ki], [Br1]).

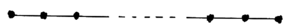
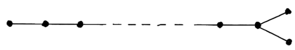
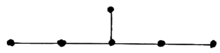
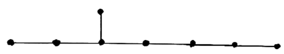
Example Consider the quotient of $\mathbb{C}^2$ by the involution $(z_1, z_2) \mapsto (-z_1, -z_2)$. There are three fundamental invariants $X = z_1 z_2, Y = z_1^2, Z = z_2^2$ obeying the relation $X^2 - YZ = 0$. The corresponding singularity is resolved by one blowing up. The resolving surface is isomorphic to the cotangent bundle $T^*\mathbb{P}^1$ of the projective line. The exceptional set corresponds to the zero-section of $T^*\mathbb{P}^1$

$$T^*\mathbb{P}^1 \to \{X^2 - YZ = 0\}$$



The bundle $T^*\mathbb{P}^1$ may be characterized as the unique line bundle on $\mathbb{P}^1$ with Chern class -2 or, equivalently, whose zero-section has self-intersection -2.

There is a simple description of the exceptional set E in the minimal resolution of the simple singularities. It consists of finitely many components isomorphic to projective lines with self-intersection -2. These components cut each other transversally in at most one point. If one indexes the components of E by the vertices of a graph, connecting vertices by an edge iff the corresponding components intersect, one obtains the following graphs known as the Dynkin diagrams of the simple Lie groups whose root systems have only roots of equal length:

group F	graph of the resolution of $\mathbb{C}^2/F$	type of the Lie group
$\mathcal{C}_n$		A_{n-1}
$\mathcal{D}_n$		D_{n+2}
$\mathcal{T}$		E_6
$\mathcal{O}$		E_7
$\mathcal{I}$		E_8

Equivalently we can say: the intersection matrix for the components of E is the negative of the Cartan matrix of the corresponding root system. Up to analytic isomorphism the simple singularities are characterized by these resolution data.

In the work [Br1] and [Br3] Brieskorn studied the problem of simultaneous resolution of singularities of mappings (see the third lecture) and found more relations of simple singularities to structures connected with the corresponding root systems, which led Grothendieck to certain conjectures relating the singularities and the Lie groups themselves. These conjectures were proved by Brieskorn shortly afterwards (cf. [Br4]). Before stating the results we need some more information on adjoint quotient and deformations.

1.3 Adjoint quotients

Let G be a semisimple Lie group acting on its Lie algebra $\mathfrak{g}$ by the adjoint action and let $\mathfrak{g}/G$ be the variety corresponding to the G-invariant polynomials on $\mathfrak{g}$. The quotient morphism $\chi : \mathfrak{g} \to \mathfrak{g}/G$ was intensively studies b Kostant ([Ko2]). He obtained the following description.

We let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$ and W the corresponding Weyl group.

(i) the space $\mathfrak{g}/G$ may be identified with the set of semisimple G-classes in $\mathfrak{g}$ such that χ maps an element $x \in \mathfrak{g}$ to the class of its semisimple part x_s,
(ii) by a theorem of Chevalley the space $\mathfrak{g}/G$ is isomorphic to $\mathfrak{h}/W$, an affine space of dimension $r = \operatorname{rank} \mathfrak{g}$, the isomorphism being given by the map of a semisimple class to its intersection with $\mathfrak{h}$ (a W-orbit!),

(iii) the morphism χ is flat, i.e. all fibres have the same dimension $\dim \mathfrak{g} - r$, all fibres are normal and consist of finitely many G-orbits.
An element $x \in \mathfrak{g}$ is called **regular** if its G-orbit has maximal possible dimension.

(iv) for x regular we have $\dim Z_G(x) = r$, each fibre of χ contains a unique regular (hence dense) orbit, an element x is regular if and only if it is regular for χ, i.e. the differential $D_x\chi : T_x\mathfrak{g} \to T_{\chi(x)}(\mathfrak{h}/W)$ has maximal rank.

From the above description we obtain that the zero-fibre $\chi^{-1}(\chi(0))$ is exactly the set of nilpotent elements of $\mathfrak{g}$. We call it the **nilpotent variety** $\mathcal{N}(\mathfrak{g})$ **of** $\mathfrak{g}$.

Besides the regular elements we have to consider **subregular** elements defined by the condition $\dim Z_G(x) = r + 2$. It follows from results of Dynkin ([Dy]) that for $\mathfrak{g}$ simple there is exactly one G-class of subregular nilpotent elements. It can be shown that the orbit dimensions occurring in $\mathfrak{g}$ have to be even. Hence subregular orbits are the next smaller ones following the regular orbits.

Example We look at $G = \mathrm{SL}_{r+1}$ with Lie algebra $\mathfrak{g} = \mathfrak{sl}_{r+1}$ given by the trace zero $(r+1) \times (r+1)$-matrices. We may choose the diagonal matrices of $\mathfrak{g}$ as a Cartan subalgebra $\mathfrak{h}$ of dimension $r = \operatorname{rank} \mathfrak{g}$. The Weyl group W is isomorphic to $\mathfrak{S}_{r+1}$, the permutation group of $r+1$ letters. It acts on $\mathfrak{h}$ by permuting the entries. The adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ can be realized by the following map

$$\chi : \mathfrak{sl}_{r+1} \to \mathbb{C}^r$$

which sends an element $x \in \mathfrak{sl}_{r+1}$ to the nontrivial coefficients $\chi_1(x), \dots, \chi_r(x)$ of its characteristic polynomial

$$P(\lambda, x) = \det(\lambda \cdot \mathrm{Id} - x) = \lambda^{r+1} + \chi_1(x)\lambda^{r-1} + \cdots + \chi_r(x).$$

Let $c = (c_1, \dots, c_r) \in \mathbb{C}^r$. The multiplicities $m_1, \dots, m_k$ of the roots of $\lambda^{r+1} + c_1\lambda^{r-1} + \cdots + c_r = 0$ determine a partition $\pi(c) = (m_1, \dots, m_k)$ of $r+1$. Up to conjugacy an element $x \in \chi^{-1}(c)$ is determined in $\chi^{-1}(c)$ by a sequence $\sigma_1(x), \dots, \sigma_k(x)$ of partitions $\sigma_i(x) = (s_{i1}(x), \dots, s_{il_i}(x))$ of m_i, where $s_{ij}(x)$ denotes the size of the j-th block for the i-th eigenvalue (of multiplicity m_i) in the Jordan normal form of x.

An element $x \in \chi^{-1}(c)$ is regular resp. subregular resp. semisimple if and only if $\sigma_i(x) = (m_i)$ for all i resp. $\sigma_i(x) = (m_i)$ for all i but one, say j, for which $\sigma_j(x) = (m_j - 1, 1)$ resp. $\sigma_i(x) = (1, 1, \dots, 1)$ for all i. For $c = 0$ and $x \in \chi^{-1}(0)$ we obtain the characteristic polynomial $P(\lambda, x) = \lambda^{r+1}$, which means that x is nilpotent. Any regular resp. subregular resp. semisimple element of $\chi^{-1}(0)$ is conjugate to

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} & 0 & \end{pmatrix}.$$

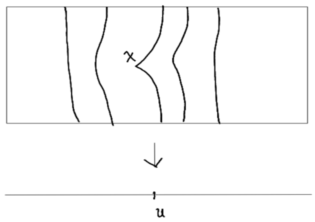
1.4 Deformation theory

Let (X_0, x) be the germ of an analytic variety at the point x. In our applications X_0 will be a hypersurface in $\mathbb{C}^3$ with isolated singularity at the origin $x = 0 \in \mathbb{C}^3$.

A **deformation** of (X_0, x) is a flat morphism of germs of analytic spaces $\chi : (X, x) \to (U, u)$ together with an isomorphism i:

$$i : (X_0, x) \xrightarrow{\sim} (\chi^{-1}(u), x).$$

$$\begin{array}{c} (X,x) \\ \Big\downarrow{\scriptstyle \chi} \\ (U,u) \end{array}$$



The space (U,u) is called the **base** of the deformation (χ, i). An **isomorphism** $\phi : (\chi, i) \to (\chi', i')$ of two deformations $\chi : (X,x) \to (U,u)$ and $\chi' : (X',x) \to (U,u)$ of (X_0,x) over (U,u) is an isomorphism $\phi : (X,x) \to (X',x)$ such that the following diagram commutes

$$\begin{tikzcd} & (X_0,x) \arrow[dl, hook', "i"'] \arrow[dr, hook, "i'"] & \\ (X,x) \arrow[rr, "\phi"] \arrow[dr, "\chi"'] & & (X',x) \arrow[dl, "\chi'"] \\ & (U,u) & \end{tikzcd}$$

If $\chi : (X,x) \to (U,u)$ is a deformation of (X_0,x) and $\psi : (T,t) \to (U,u)$ is any morphism, then the pull-back $\chi_T : (X,x)\times_{(U,u)}(T,t) \to (T,t)$ of χ by ψ is flat again, hence a deformation of (X_0,x) over (T,t), which will be called the deformation **induced by ψ from χ**.

$$\begin{tikzcd} (X,x) \times_{(U,u)} (T,t) \arrow[r] \arrow[d, "\chi_T"'] & (X,x) \arrow[d, "\chi"] \\ (T,t) \arrow[r, "\psi"] & (U,u) \end{tikzcd}$$

A deformation $\chi : (X,x) \to (U,u)$ of (X_0,x) is called **semiuniversal** if any other deformation $\chi' : (X',x) \to (T,t)$ of (X_0,x) is isomorphic to a deformation induced from χ by a base change $\psi : (T,t) \to (U,u)$ whose differential at $t \in T$ is uniquely determined.

It follows immediately that semiuniversal deformations are unique up to isomorphism.

From the general theory of deformations (cf. [K-S], [Tj1]) we obtain the existence of semiuniversal deformations in our case, i.e. let $f \in \mathbb{C}\{x,y,z\}$ be a convergent power series on $(\mathbb{C}^3, 0)$ with an isolated singularity at $0 \in \mathbb{C}^3$. Then

$$T^1 := \mathbb{C}\{x,y,z\}/(f, \tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z}) \cdot \mathbb{C}\{x,y,z\}$$

has finite dimension r over $\mathbb{C}$. If $b_1, \dots, b_r \in \mathbb{C}\{x,y,z\}$ are representatives of a basis of T^1 and $b_r = 1$, a semiuniversal deformation of $(X_0,x) = (f^{-1}(0), 0)$ is given by the morphism

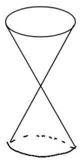
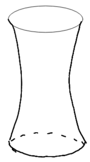
$$\chi : (\mathbb{C}^3 \times \mathbb{C}^{r-1}, 0) \to (\mathbb{C}^r, 0)$$
$$(x,y,z,u_1,\dots,u_{r-1}) \mapsto (f(x,y,z) + \sum_{i=1}^{r-1} u_i b_i(x,y,z), u_1, \dots, u_{r-1}).$$

Example Let $f = x^n + yz \in \mathbb{C}\{x,y,z\}$. Then a basis of T^1 is given by $1, x, x^2, \dots, x^{n-2}$. Hence the semiuniversal deformation is given by

$$\chi : \mathbb{C}^3 \times \mathbb{C}^{n-2} \to \mathbb{C}^{n-1}$$
$$(x,y,z,u_1,\dots,u_{n-2}) \mapsto (yz + x^n + \sum_{i=1}^{n-2} u_i x^i, u_1, \dots, u_{n-2}).$$

For $n = 2$ especially, we obtain

$$\chi : \mathbb{C}^3 \to \mathbb{C}$$
$$(x, y, z) \mapsto x^2 + yz$$

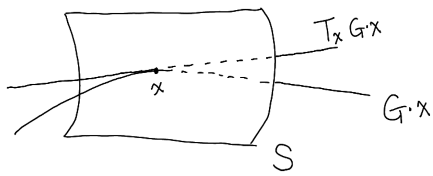
$x^2 + yz = 0$  $x^2 + yz \neq 0$ 

1.5 Brieskorn's theorem

We can now state the theorem of Brieskorn (Nice 1970, [Br4]) conjectured by Grothendieck.

Theorem 1 *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type $\mathsf{A}_r, \mathsf{D}_r, \mathsf{E}_r$ and x a subregular nilpotent element of $\mathfrak{g}$. Let $S \subset \mathfrak{g}$ be a transversal slice in x to the G-orbit of x in $\mathfrak{g}$. Then the restriction $(S, x) \to (\mathfrak{h}/W, \overline{0})$ of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ to S is a semiuniversal deformation of the corresponding simple singularity.*

In the category of algebraic varieties a transversal slice S in x to the orbit of x is a locally closed subvariety $S \subset \mathfrak{g}$, $x \in S$ such that the morphism $G \times S \to \mathfrak{g}$, $(g, s) \mapsto (\operatorname{ad} g)s$, is smooth and $\dim S = \operatorname{codim} Gx$. In the context of deformations we think of S as replaced by its analytic germ at x.



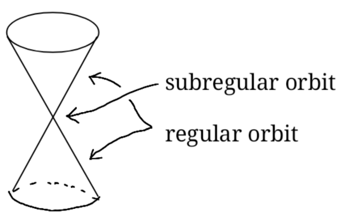
In our situation a transversal slice is easily obtained by choosing an affine subspace in $\mathfrak{g}$ complementary to the affine tangent space at x of the orbit $G \cdot x$ and localizing at x.

The theorem implies as a special case that $(S \cap \mathcal{N}(\mathfrak{g}), x) = (S \cap \chi^{-1}(\chi(0)), x)$ is a simple singularity of the corresponding type.

Let us consider the case of $\mathfrak{g}$ of type A_1 as a first example: there is only one G-invariant (generating) polynomial on $\mathfrak{g} = \mathfrak{sl}_2$ given by the determinant $(\det(x - \lambda \cdot \mathrm{Id}) = \lambda^2 + \det x)$

$$\chi : \mathfrak{sl}_2 = \mathbb{C}^3 \to \mathfrak{h}/W = \mathbb{C}$$
$$\begin{pmatrix} x & y \\ z & -x \end{pmatrix} \mapsto -x^2 - yz$$

The nilpotent variety is now given by the equation $x^2 + yz = 0$. It consists of two orbits, the regular and the subregular, the last one being 0.



A transversal slice to 0 is $\mathfrak{sl}_2$ itself and $(x, y, z) \mapsto x^2 + yz$ is the semiuniversal deformation of the singularity of type A_1 (cf. 1.4).

We now take $\mathfrak{g}$ of type A_2, i.e. $\mathfrak{g} = \mathfrak{sl}_3$. A subregular element x is given by

$$x = \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}. \quad \text{Let } y = \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}.$$

Then by the theory of 2.4 (lecture two) a transversal slice to the orbit of x is given by

$$S = x + \mathfrak{z}_{\mathfrak{g}}(y) = \left\{ \begin{pmatrix} x & 1 & 0 \\ t & x & y \\ z & 0 & -2x \end{pmatrix} \middle| (x, y, z, t) \in \mathbb{C}^4 \right\}.$$

The characteristic polynomial $\det(s - \lambda \cdot \mathrm{Id})$ for $s \in S$ has the form $-\lambda^3 + \lambda(t + 3x^2) + yz - 2x^3 + 2xt$. Hence the restriction of the adjoint quotient $\mathfrak{sl}_3 \to \mathfrak{h}/W$ to S is given by

$$(x, y, z, t) \mapsto (t + 3x^2, yz - 2x^3 + 2xt)$$

or, after replacing t by $u - 3x^2$,

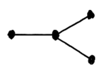
$$(x, y, z, u) \mapsto (u, yz - 8x^3 + 2xu)$$

which is (the) semiuniversal deformation of the simple singularity $yz - 8x^3 = 0$ of type A_2. For the other cases A_r one may proceed similarly (for an economical choice of S cf. [A1]).

As a corollary of the description of the semiuniversal deformation of a simple singularity given by theorem 1 we get complete information on the singularities in the nonspecial fibres in any deformation of a simple singularity. In speaking of nonspecial fibres we silently assume the morphism $\chi : (X, x) \to (U, u)$ of germs replaced by a sufficiently small representative which we denote simply by $\chi : X \to U$.

Definition An **adjacent diagram** Δ' of a Dynkin diagram Δ is obtained by omitting nodes from Δ together with the edges leading to them.

Example We consider Δ of type D_4

$\Delta =$  ,

the adjacent diagrams are then of the following types:

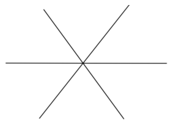
A_3  , A_2  , A_1 , $\mathsf{A}_1 + \mathsf{A}_1$, $\mathsf{A}_1 + \mathsf{A}_1 + \mathsf{A}_1$  and $\emptyset$.

Corollary *Let $\chi : X \to U$ be a deformation of a simple singularity X_0 with corresponding Dynkin diagram Δ, and let $X_\varepsilon \neq X_0$ be a nonspecial fibre of χ. Then there is an adjacent diagram Δ' of Δ (possibly empty) and a type-preserving bijection of the components of Δ' onto the singular points of X_ε, i.e. a connected component is sent to a singular point which is a simple singularity of the corresponding type. If χ is semiuniversal, then all adjacent diagrams of Δ are realized as singular configurations in nonspecial fibres.*

Example Consider X_0 of type D_4 given by $z^2 = x^3 - 3xy^2$ and the deformation

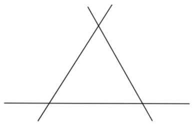
$$\begin{aligned} \mathbb{C}^4 &\to \mathbb{C}^2 \\ (x, y, z, t) &\mapsto (z^2 - x^3 + 3xy^2 + t(x^2 + y^2), t). \end{aligned}$$

It is sufficient to restrict to $z = 0$:

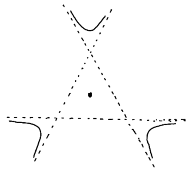


$$x^3 - 3xy^2 = x(x - \sqrt{3}y)(x + \sqrt{3}y) = 0$$

A fibre $\mathbb{C}^4_{(u,t)}$ is either nonsingular or has a singular configuration of type $\mathsf{A}_1 + \mathsf{A}_1 + \mathsf{A}_1$ or A_1:



or



Apparently this deformation is not semiuniversal.

1.6 Inhomogeneous Dynkin diagrams

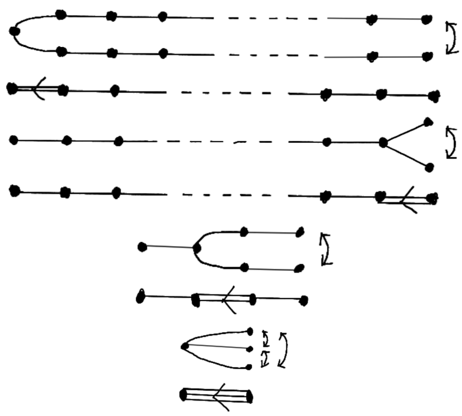
Up to now we have mentioned only the homogeneous Dynkin diagrams of type $\mathsf{A}_r, \mathsf{D}_r, \mathsf{E}_r$. But there are simple Lie groups of types $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$ too for which one can apply the same constructions as before. We will see that one can get quite a similar description as in theorem 1.

Definition A **simple singularity of type** $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$ is a couple (X_0, Γ) of a simple singularity X_0 (in the former sense) and a group Γ of automorphisms of X_0 according to the following list: (in all cases $\Gamma = F'/F$ operates naturally on $X_0 = \mathbb{C}^2/F$)

type (X_0, Γ)	type X_0	F	F'	Γ
B_r	A_{2r-1}	$\mathcal{C}_{2r}$	$\mathcal{D}_r$	$\mathbb{Z}/(2)$
C_r	D_{r+1}	$\mathcal{D}_{r-1}$	$\mathcal{D}_{2(r-1)}$	$\mathbb{Z}/(2)$
F_4	E_6	$\mathcal{T}$	$\mathcal{O}$	$\mathbb{Z}/(2)$
G_2	D_4	$\mathcal{D}_2$	$\mathcal{O}$	$\mathfrak{S}_3$

The connection between the diagram of (X_0, Γ) and that of X_0 can be easily memorized in the following way: the action of Γ lifts to a minimal resolution of X_0 and acts as a permutation group of the components of the exceptional divisor; we obtain the diagram of (X_0, Γ) as a "Γ-quotient" of that of X_0.

(X_0, Γ)	X_0
	A_{2r-1}
B_r	
	D_{r+1}
C_r	
	E_6
F_4	
	D_4
G_2	



In the following we will call the diagram of X_0 the **associated homogeneous** diagram of (X_0, Γ).

1.7 Deformations with group actions

Before we can state our second main theorem we have to define the notion of (semiuniversal) deformation for a couple (X_0, Γ) of a singularity X_0 and an automorphism group Γ of X_0. Though we are considering germs of varieties we will skip the germ notation for simplicity.

Definition A **deformation** of (X_0, Γ) is a deformation $\chi : X \to U$ of X_0 together with an action of Γ on X which induces the given one on X_0 and such that χ is invariant with respect to Γ.

An **isomorphism** of two deformations of (X_0, Γ) is an isomorphism in the previous sense which is in addition equivariant with respect to Γ.

A **semiuniversal** deformation of (X_0, Γ) is then defined in an analogous way as in 1.4.

The construction of a semiuniversal deformation of (X_0, Γ) in the envisaged case of X_0 a hypersurface with isolated singularity and Γ reductive is easy. One can show the existence of a semiuniversal deformation $\chi : X \to U$ of X_0 in the sense of 1.4 with the following additional property (cf. [Sl] 2): there are Γ-actions on X and U such that χ is equivariant and the restricted action of Γ on X_0 is the given one. Moreover, if $Y \to T$ is any Γ-equivariant deformation of X_0, it can be induced from χ by a Γ-equivariant morphism.

A semiuniversal deformation of (X_0, Γ) is then obtained by the deformation induced from χ by the base change $U^\Gamma \to U$, where U^Γ is the fixed point set of Γ in U.

$$\begin{tikzcd} X \times_U U^\Gamma \arrow[r] \arrow[d] & X \arrow[d] \\ U^\Gamma \arrow[r] & U \end{tikzcd}$$

Example The deformation of the singularity D_4 in the example 1.5 is a semiuniversal deformation of $(\mathsf{D}_4, \mathfrak{S}_3)$, where the $\mathfrak{S}_3$-action is given by the apparent symmetry of the singularity D_4.

1.8 The extension of Brieskorn's theorem

Theorem 2 *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of type $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$ and x a subregular nilpotent element of $\mathfrak{g}$. Then there exists a finite subgroup Γ of the centralizer $Z_G(x)$ of x and a Γ-stable transversal slice S in x to the G-orbit of x such that the (Γ-invariant) restriction of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ to S realizes a semiuniversal deformation of a simple singularity of the corresponding type.*

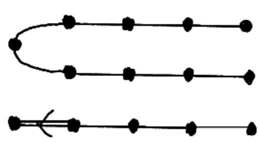
As a special case the theorem says that the intersection $X_0 = S \cap \mathcal{N}(\mathfrak{g})$ of S with the nilpotent variety plus the induced action of Γ on X_0 is a simple singularity (X_0, Γ) of the corresponding type $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$.

As in 1.5 for the types $\mathsf{A}_r, \mathsf{D}_r, \mathsf{E}_r$, we obtain by the above theorem a description of the possible singular configurations in the nonspecial fibres of any deformation of (X_0, Γ).

We define adjacent diagrams for $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$ as for the homogeneous diagrams. Each inhomogeneous diagram Δ may be regraded as a quotient of the associated homogeneous diagram ${}_h\Delta$ by the action of a group of diagram isometries

${}_h\Delta$

Δ

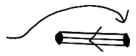


The pre-images in ${}_h\Delta$ of adjacent diagrams of Δ are just the Γ stable adjacent diagrams of ${}_h\Delta$.

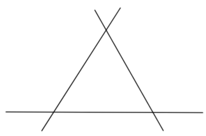
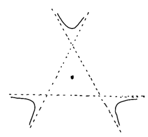
Corollary *Let $\chi : X \to U$ be a deformation of a simple singularity (X_0, Γ) of inhomogeneous type Δ, and let $X_\varepsilon \neq X_0$ be a nonspecial fibre of χ. Then there is a Γ-stable adjacent diagram ${}_h\Delta'$ of ${}_h\Delta$ and a Γ-equivariant type-preserving bijection of the components of ${}_h\Delta'$ onto the singular points of X_ε.*

Example In the Lie algebra G_2 we obtain the semiuniversal deformation of $(\mathsf{D}_4, \mathfrak{S}_3)$. Consider the projection of ${}_h\Delta$ onto Δ

there are only two nontrivial possibilities for adjacent diagrams

1) omit  , gives $= \mathsf{A}_1 + \mathsf{A}_1 + \mathsf{A}_1$ plus permutation of $\mathfrak{S}_3$
2) omit  , gives $\bullet = \mathsf{A}_1$ stable under $\mathfrak{S}_3$.

The corresponding geometric situations were discussed already in 1.5:

1)  and 2) 

1.9 Further remarks

There is an apparent connection of the above mentioned diagram symmetries to the outer automorphisms of the corresponding Lie algebras. In fact one can use the outer automorphisms of a Lie algebra of type $\mathsf{A}_{2r-1}, \mathsf{D}_r, \mathsf{E}_6, \mathsf{D}_4$ to make the deformations $\chi : S \to \mathfrak{h}/W$ of the corresponding singularities in these Lie algebras equivariant with respect to an action of Γ. The deformations $\chi_0 : S_0 \to \mathfrak{h}_0/W_0$ of $\mathsf{B}_r, \mathsf{C}_{r-1}, \mathsf{F}_4, \mathsf{G}_2$ in the inhomogeneous Lie algebras are then induced by immersions $\mathfrak{h}_0/W_0 \xrightarrow{\sim} (\mathfrak{h}/W)^\Gamma \hookrightarrow \mathfrak{h}/W$. It can be shown that the corresponding mapping on the total spaces $S_0 \hookrightarrow S$ cannot be realized via a representation of the corresponding Lie algebras (for details cf. [Sl] 8.8).

Our discussion in the preceding sections was limited to the field $\mathbb{C}$. Under some mild restrictions on the characteristic of k all goes through over arbitrary algebraically closed fields k (see [Sl]) and one can also drop the condition of an algebraically closed field. In this case one has to refine the considerations taking into account "forms" of singularities and groups (unpublished).

Lecture 2 Technique and proofs

The aim of this lecture is to explain the main steps in the proof of theorem 1 and 2 of the first lecture. We need some technical preparation.

2.1 Associated bundles

Let G be an algebraic group and H an algebraic subgroup. The natural morphism $G \to G/H$ is a principal fibre bundle with structure group H. Let F be a variety on which H acts. We can then define the **associated bundle** $G \times^H F$ over G/H (with fibre F) by taking the quotient of $G \times F$ by the H-action

$$h \cdot (g, f) = (gh^{-1}, hf) \text{ for } h \in H, g \in G, f \in F.$$

There are some trivial nonetheless useful properties of associated bundles. We observe that $G \times^H F$ is a G-bundle by the action of G on the "left factor". If the action of H on F is the restriction of a G-action on F we obtain a G-isomorphism $G\times^H F \to G/H \times F$ by mapping the class $g * f$ of (g, f) to (gH, gf). In the case that $E \subset F$ is an H-stable subvariety we may thus prolongate the embedding $G\times^H E \to G\times^H F$ by the isomorphism from $G\times^H F$ to $G/H\times F$. Moreover if H is the normalizer $\{g \in G \mid gE = E\}$ of E in G we may identify G/H with the variety $\mathcal{E}$ of all conjugates of E, and the embedding $G \times^H E \to G/H \times F \simeq \mathcal{E} \times F$ maps $G \times^H E$ onto the variety of all pairs $(E', f) \in \mathcal{E} \times F$ satisfying $f \in E'$. Very often associated bundles will occur in the following way.

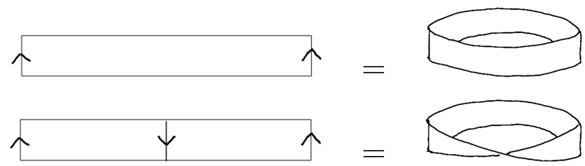
Lemma *Let X be a G-variety and $\pi : X \to G/H$ a G-equivariant morphism. Then π is G-isomorphic to the bundle $G \times^H F \to G/H$ where F is the H-stable fibre $\pi^{-1}(e \cdot H)$.*

For the proof consider the embedding $G \times^H F \to G/H \times X$. Its image is exactly the graph of $X \to G/H$.

Example For intuition we switch to the topological category. Take $G = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, $H = \{\pm 1\}$ and $F = \mathbb{R}$. Let $-1 \in H$ act on $\mathbb{R}$ by -1. Then $G \times^H F$ is the well-known Möbius-strip:

$G \times F =$

$G \times^H F =$



2.2 More on the adjoint fibres

We can now give a more precise description of the fibres of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ (we may assume $\mathfrak{g}$ reductive).

By the general theory of affine quotients (cf. [Mu], [Sp3]) there is exactly one closed orbit in each fibre of χ. It may be shown that this orbit consists of the semisimple elements in that fibre. Moreover if $x = x_s + x_n$ is the Jordan decomposition of an element $x \in \mathfrak{g}$ we have $\chi(x) = \chi(x_s)$ (cf. [Sl], [St2]). Let x_s be conjugate to $h \in \mathfrak{h}$. Denoting the class of h in $\mathfrak{h}/W$ by $\overline{h}$ we have $\chi(x) = \overline{h}$. We may identify the closed orbit in $\chi^{-1}(\overline{h})$ with $G/Z(h)$, where $Z(h)$ is the centralizer of h in G. By mapping $x \in \chi^{-1}(\overline{h})$ to its semisimple part x_s (being conjugate to h) we obtain a G-morphism

$$\sigma : \chi^{-1}(\overline{h}) \to G/Z(h).$$

Hence according to 2.1 the fibre $\chi^{-1}(\overline{h})$ is an associated bundle $G \times^{Z(h)} F$ where F is the fibre of σ over $e \cdot Z(h)$. Hence F is equal to the set of all $h + n$ with n nilpotent and commuting with h, i.e. $F = h + \mathcal{N}(\mathfrak{z}(h))$, where $\mathcal{N}(\mathfrak{z}(h))$ is the nilpotent variety of the reductive Lie algebra $\mathfrak{z}(h) = \mathrm{Lie}(Z(h))$. With respect to the $Z(h)$-action we may forget about h and identify F with $\mathcal{N}(\mathfrak{z}(h))$ and the adjoint $Z(h)$-action on it.

If Σ is the root system of $\mathfrak{h}$ in $\mathfrak{g}$, the root system of $\mathfrak{h}$ in $\mathfrak{z}(h)$ is given by

$$\Sigma_h = \{\alpha \in \Sigma \mid \alpha(h) = 0\},$$

which is rationally closed, i.e. $(\mathbb{Q} \cdot \Sigma_h) \cap \Sigma = \Sigma_h$, and corresponds hence to an adjacent subdiagram of the Dynkin diagram of $\mathfrak{g}$ (cf. 1.5) (in case $\Sigma \neq \Sigma_h$; otherwise h is central in $\mathfrak{g}$ and $\chi^{-1}(\overline{h}) \simeq \mathcal{N}(\mathfrak{g})$). The description above reduces questions about the fibres $\chi^{-1}(\overline{h})$ to questions about the nilpotent variety of smaller Lie algebras.

2.3 The Jacobson-Morozov lemma

In the following section we will make use of the following embedding property (cf. [S-S] III.4 or [LIE] VIII).

Jacobson-Morozov Lemma *Let $x \in \mathfrak{g}$ be a nilpotent element of a reductive Lie algebra $\mathfrak{g}$. Then there exists a homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$ mapping $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to the element x. All such homomorphisms are conjugate under the centralizer $Z_G(x)$ of x in the adjoint group G.*

If such a homomorphism is given, we denote the image of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ by h, that of $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ by y. The following commutation rules hold then

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$

We recall some facts on the representation theory of the Lie algebra $\mathfrak{sl}_2$.

1) each finite-dimensional representation $\mathfrak{sl}_2$ is completely reducible.
2) for each $n \in \mathbb{N}$ there is exactly one irreducible representation $\rho_n : \mathfrak{sl}_2 \to \mathfrak{gl}(V_n)$ of dimension n.
3) the vector space V_n decomposes into a direct sum of one-dimensional eigenspaces $V_n(\lambda)$ with respect to h, where λ runs through the eigenvalues $n-1, n-3, \ldots, -(n-3), -(n-1)$. Furthermore, x (resp. y) induces isomorphism $V_n(\lambda) \xrightarrow{\sim} V_n(\lambda+2)$ if $\lambda \leq n-3$ (resp. $V_n(\lambda) \xrightarrow{\sim} V_n(\lambda-2)$ if $\lambda \geq -(n-3)$) and $V_n(n-1)$ (resp. $V_n(-(n-1))$) is killed by x (resp. y).

$$\begin{tikzcd}
-(n-1) & -(n-3) & -(n-5) & \cdots & n-5 & n-3 & n-1 \\
\bullet \arrow[r, bend left, "x"] & \bullet \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \bullet \arrow[l, bend left, "y"] & \cdots & \bullet \arrow[r, bend left, "x"] & \bullet \arrow[l, bend left, "y"] \arrow[r, bend left, "x"] & \bullet \arrow[l, bend left, "y"]
\end{tikzcd}$$

2.4 The transversal slice

One may show that the theorems of lecture one do not depend on the choice of the transversal slice up to analytic isomorphism. Hence we may choose a technically very suitable slice. Let x be nilpotent (later we will specialize to subreglar x) in the reductive Lie algebra $\mathfrak{g}$ (later $\mathfrak{g}$ will be simple). The tangent space $T_x(G \cdot x)$ to the G-orbit of x in x is given by the affine space

$$T_x(G \cdot x) = x + [\mathfrak{g}, x].$$

A transversal slice has thus the form

$$S = x + \mathfrak{z},$$

where $\mathfrak{z}$ is any linear complement to $[\mathfrak{g}, x]$ in $\mathfrak{g}$.

Let $\mathfrak{sl}_2 \to (x, h, y) \to \mathfrak{g}$ be a Jacobson-Morozov-homomorphism as in 2.3. By composing it with the adjoint representation $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ we may regard $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module. Hence there is a decomposition

$$\mathfrak{g} = \bigoplus_i E_i$$

of $\mathfrak{g}$ into irreducible $\mathfrak{sl}_2$-modules E_i. We have

$$[\mathfrak{g}, x] = (\operatorname{ad} x)(\mathfrak{g}) = \bigoplus_i (\operatorname{ad} x)(E_i).$$

In each E_i, isomorphic to an irreducible representation V_{n_i}, a complement to $(\operatorname{ad} x)E_i$ is given by the lowest weight space of weight $-(n_i - 1)$, i.e. by $\mathfrak{z}_{E_i}(y) = \{e \in E_i \mid [y, e] = 0\}$. Hence $\mathfrak{z}_{\mathfrak{g}}(y) = \bigoplus_i \mathfrak{z}_{E_i}(y)$ is a complement to $[\mathfrak{g}, x]$ in $\mathfrak{g}$, i.e.

$$S = x + \mathfrak{z}_{\mathfrak{g}}(y)$$

is a transversal slice to the orbit of x in x.

2.5 The $\mathbb{G}_m$-action on S

The above choice of S admits an easy construction of an action of the multiplicative group $\mathbb{G}_m$ on it, which behaves well with respect to the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$.

The usual scalar multiplication

$$\begin{aligned} \mathbb{G}_m \times \mathfrak{g} &\to \mathfrak{g} \\ (t, x) &\mapsto \sigma(t)x = tx \end{aligned}$$

does not stabilize the affine subspace $S = x + \mathfrak{z}_{\mathfrak{g}}(y)$. But it may be compensated by a $\mathbb{G}_m$-action induced by the Jacobson-Morozov-homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$. If $\mathfrak{g} = \bigoplus_{i=1}^{r'} V_{n_i}$ is the corresponding decomposition of $\mathfrak{g}$ into simple $\mathfrak{sl}_2$-modules V_{n_i} of dimension n_i and $e_i \in V_{n_i}(-(n_i - 1))$ are non-zero lowest weight vectors, the action of the torus $\mathbb{G}_m \subset \mathrm{SL}_2$, belonging to the Cartan subalgebra $\langle h \rangle \subset \mathfrak{sl}_2$, on elements $x + \sum c_i e_i \in S = x + \mathfrak{z}_{\mathfrak{g}}(y)$ is given by

$$\rho(t)(x + \sum c_i e_i) = t^2 x + \sum_{i=1}^{r'} t^{-n_i+1} c_i e_i, \quad t \in \mathbb{G}_m.$$

The actions ρ and σ commute, hence we may consider the product $\mathfrak{j}(t) := \sigma(t^2)\rho(t^{-1})$ which acts on S by

$$\mathfrak{j}(t)(x + \sum c_i e_i) = x + \sum_{i=1}^{r'} t^{n_i+1} c_i e_i$$

and thus stabilizes S. With respect to the coordinate system (c_i) on S the $\mathfrak{j}$-action is linear with the positive weights $n_i + 1$.

According to a theorem of Chevalley the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ may be realized as a morphism $\chi = (\chi_1, \ldots, \chi_r) : \mathfrak{g} \to \mathbb{C}^r$ where the χ_j are homogeneous generators of degree $d_j = m_j + 1$ for $\mathbb{C}[\mathfrak{g}]^G$ and where the m_j, $j = 1, \ldots, r$ are the exponents of $\mathfrak{g}$.

For the restriction of χ to S we obtain the following behaviour with respect to the $\mathfrak{j}$-action: for all $s \in S$, $t \in \mathbb{G}_m$, $j = 1, \ldots, r$, we have

$$\begin{aligned} \chi_j(\mathfrak{j}(t)s) &= \chi_j(\sigma(t^2)\rho(t^{-1})s) \\ &= \chi_j(\sigma(t^2)s) \qquad \text{(Invariance of } \chi) \\ &= t^{2d_j}\chi_j(s) \qquad \text{(Homogeneity of } \chi). \end{aligned}$$

If we let $\mathbb{G}_m$ act on S by $\mathfrak{j}$ and on $\mathfrak{h}/W \simeq \mathbb{C}^r$ by the weights $2d_j, j = 1, \ldots, r$, the map $\chi : S \to \mathfrak{h}/W$ becomes $\mathbb{G}_m$-equivariant. For short we say that $\chi : S \to \mathfrak{h}/W$ is quasihomogeneous of type $(2d_1, \ldots, 2d_r; n_1 + 1, \ldots, n_{r'} + 1)$. It is possible to calculate the d_j and n_i. The n_i constitute the basic invariants in Dynkin's classification of nilpotent classes in $\mathfrak{g}$. They may be encoded into a valuation of the Dynkin diagram of $\mathfrak{g}$ (see for instance [S-S] III 4). For the subregular elements in simple Lie algebras one obtains the following values for d_j and $w_j = \frac{n_j+1}{2}$:

	d_1,	d_2,	d_3,	$\ldots$	$\ldots$	$\ldots$,	d_{r-1}	d_r	w_r,	w_{r+1},	w_{r+2}
A_r	2	3	4	$\ldots$			r	$r+1$	1	$\frac{r+1}{2}$	$\frac{r+1}{2}$
B_r	2	4	6	$\ldots$			$2r-2$	$2r$	1	r	r
C_r	2	4	6	$\ldots$			$2r-2$	$2r$	2	$r-1$	r
D_r	2	4	6	$\ldots$		$2r-4$	r	$2r-2$	2	$r-2$	$r-1$

E_6	2	5	6	8			9	12	3	4	6
E_7	2	6	8	10	12		14	18	4	6	9
E_8	2	8	12	14	18	20	24	30	6	10	15
F_4	2	6					8	12	3	4	6
G_2	2							6	2	2	3

One has always $d_j = w_j$ for $j = 1, \dots, r-1$, and $r' = r+2$ because $\dim \mathfrak{z}_{\mathfrak{g}}(y) = \dim Z_G(x) = r+2$ for subregular x.

2.6 The centralizer-action on S

Whereas the $\mathbb{G}_m$-action on S will only be of technical use, the action discussed now enters in a decisive way in the discussion of the deformations in the Lie algebras $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$.

Contrarily to the situation of closed orbits, it is not possible to choose a $Z_G(x)$-stable transversal slice to the G-orbit of a nilpotent x mainly because $Z_G(x)$ is not reductive in general. But apparently

$$C(x,y) = Z_G(x) \cap Z_G(y)$$

stabilizes $S = x + \mathfrak{z}_{\mathfrak{g}}(y)$. It can be shown that $C(x,y)$ is the centralizer of (x,h,y), hence commutes with the $\mathfrak{j}$-action of 2.5, and that $C(x,y)$ is a reductive part of $Z_G(x)$ and $Z_G(y)$. By abuse of language it is simply called the **reductive centralizer** of x (cf. [S-S] III 4). As $C(x,y)$ is contained in the adjoint group G, the restriction $\chi : S \to \mathfrak{h}/W$ is invariant with respect to $C(x,y)$. In case x is subregular and $\mathfrak{g}$ is simple it is possible to determine the structure of $C(x,y)$ and the action on S. For $C(x,y)$ we obtain the following groups (for details cf. [Sl] 7.5, 8.5):

type of $\mathfrak{g}$	A_r	B_r	C_r	D_r	E_6	E_7	E_8	F_4	G_2
$C(x,y)$	$\mathbb{G}_m$	$\mathbb{G}_m \times \mathbb{Z}_2$	$\mathbb{Z}_2$	1	1	1	1	$\mathbb{Z}_2$	$\mathfrak{S}_3$

For a general nilpotent element $x \in \mathfrak{g}$ it can be shown that the component groups $C(x,y)/C(x,y)^0$ and $Z_G(x)/Z_G(x)^0$ coincide (cf. [El]).

2.7 Identification of the singularities

We let now x be nilpotent subregular in a simple Lie algebra $\mathfrak{g}$ and $S = x + Z_{\mathfrak{g}}(y)$ a transversal slice to the orbit of x as in 2.4. We want to determine the singularity of $S \cap \mathcal{N}(\mathfrak{g})$ in x. This singularity is isolated as S meets the regular nilpotent orbit transversely too and meets the subregular one in the isolated point x. Let $r = \operatorname{rank} \mathfrak{g}$.

Lemma *The restriction $\chi : S \to \mathfrak{h}/W$ of the adjoint quotient $\mathfrak{g} \to \mathfrak{h}/W$ to S has rank $r-1$ in x.*

For the proof we refer to [Sl] 8.3; it uses the quasihomogeneous structure of χ, its flatness and the isolatedness of the singularity $S \cap \mathcal{N}(\mathfrak{g})$.

As χ is of quasihomogeneous type $(d_1, \dots, d_r; w_1, \dots, w_{r+2})$, where $d_i = w_i$ for $i = 1, \dots, r-1$, the lemma implies that $S \cap \mathcal{N}(\mathfrak{g})$ is defined by a quasihomogeneous polynomial of type $(d_r; w_r, w_{r+1}, w_{r+2})$. It is easy to check that an isolated singularity $\{f = 0\}$ defined by a quasihomogeneous polynomial of type $(d_r; w_r, w_{r+1}, w_{r+2})$, where the $d_r, w_r, w_{r+1}, w_{r+2}$ are as in the list of 2.5, can only be a simple singularity of the right type, i.e.:

type $\mathfrak{g}$	A_r	B_r	C_r	D_r	E_r	F_4	G_2
type $\{f = 0\}$	A_r	A_{2r-1}	D_{r+1}	D_r	E_r	E_6	D_4

Example In case E_8 we get f of type $(30; 6, 10, 15)$. A general f of this type has the form

$$f = aX^5 + bY^3 + cZ^2$$

and $\{f = 0\}$ has an isolated singularity if and only if $abc \neq 0$. By a change of coordinates we may choose $a = b = c = 1$.

Remark The fact that $\{f = 0\}$ is a simple singularity can also be seen by using a criterion of Saito [Sa] stating that the inequality $w_r + w_{r+1} + w_{r+2} > d_r$ and the isolatedness of the singularity $\{f = 0\}$ imply simplicity.

We actually have $w_r + w_{r+1} + w_{r+2} = d_r + 1$ which can be read off from the list in 2.5. But there is an a priori argument for it. The numbers $2d_j - 1$ are the dimensions of the irreducible $\mathfrak{sl}_2$-modules in $\mathfrak{g}$ with respect to a regular homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$ (mapping $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to a regular element). Hence we have

$$\sum_{j=1}^{r} (2d_j - 1) = \dim \mathfrak{g} = \sum_{i=1}^{r+2} n_i = \sum_{i=1}^{r+2} (2w_j - 1)$$

and because of $d_i = w_i$ for $i = 1, \ldots, r-1$ we get

$$2d_r - 1 = \sum_{i=r}^{r+2} (2w_i - 1)$$

or $d_r + 1 = w_r + w_{r+1} + w_{r+2}$. (This answers a question of J.J. Duistermaat).

In the cases $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$ we still have to say what the symmetry group Γ is. In all cases except B_r we may take $\Gamma = C(x, y)$ the reductive centralizer (cf. 2.6). In case B_r any section $\mathbb{Z}/(2) = C(x, y)/C(x, y)^0 \to C(x, y)$ will do. For the details of the proof we refer to [Sl] 8.5.

2.8 Identification of the deformations

In the situation of 2.7 we now want to prove the semiuniversality of the deformation $\chi : S \to \mathfrak{h}/W$. We first deal with the homogeneous diagrams $\mathsf{A}_r, \mathsf{D}_r, \mathsf{E}_r$. In 1.7 we already mentioned the existence of a Γ-equivariant semiuniversal deformation for isolated hypersurface singularities with a reductive group Γ acting on it.

As the simple singularities are defined by quasi-homogeneous polynomials there is an action of the multiplicative group $\mathbb{G}_m$ on them. Their semiuniversal deformation $X \to U$ admits thus a quasi-homogeneous structure too. An explicit verification shows that it is of the same type $(d_1, \ldots, d_r; w_1, \ldots, w_{r+2})$ as the morphism $\chi : S \to \mathfrak{h}/W$ in the corresponding Lie algebra. By the $\mathbb{G}_m$-equivariant semiuniversality we may induce χ from $X \to U$ by a $\mathbb{G}_m$-equivariant morphism:

$$\begin{tikzcd}
 & X_0 = S \cap \mathcal{N}(\mathfrak{g}) \arrow[dl, hook'] \arrow[dr, hook] & \\
S \arrow[rr] \arrow[d] & & X \arrow[d] \\
\mathfrak{h}/W \arrow[rr, "\phi"] & & U
\end{tikzcd}$$

Especially ϕ is of type $(d_1, \ldots, d_r; d_1, \ldots, d_r)$. From the description of the fibres of $\mathfrak{g} \to \mathfrak{h}/W$ in 2.2 one deduces that ϕ is a finite morphism, i.e. the fibres $\chi^{-1}(\overline{h})$ of $S \to \mathfrak{h}/W$ have a simple singularity of the same type as $\mathcal{N}(\mathfrak{g}) \cap S$ only for $h = 0$. This now implies that ϕ is an isomorphism, which proves Theorem 1.

Example We determine the quasihomogeneous structure of the semiuniversal deformation $\mathbb{C}^{10} \to \mathbb{C}^8$ of the simple singularity E_8. The singularity is given by

$$f = X^5 + Y^3 + Z^2 = 0.$$

A basis of $\mathbb{C}\{x, y, z\}/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ is formed by $1, x, x^2, x^3, y, yx, yx^2, yx^3$, which are monomials of degrees $0, 6, 12, 18, 10, 16, 22, 28$. The morphism

$$\begin{aligned} \mathbb{C}^{10} &\to \mathbb{C}^8 \\ (x, y, z, u_1, \ldots, u_7) &\mapsto (f + u_1 x + u_2 x^2 + \cdots + u_7 y x^3, u_1, \ldots, u_7) \end{aligned}$$

is quasihomogeneous of type (30, 24, 18, 12, 20, 14, 8, 2; 6, 10, 15, 24, 18, 12, 20, 14, 8, 2).

We now discuss the inhomogeneous diagrams $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$. Let $(X_0, \Gamma) = (S \cap \mathcal{N}(\mathfrak{g}), \Gamma)$. We may construct again a $\mathbb{G}_m$-equivariant semiuniversal deformation of (X_0, Γ), for example by restricting a $\Gamma \times \mathbb{G}_m$-equivariant semiuniversal deformation to Γ-fixed points of the base. The quasihomogeneous structure may be computed again and it coincides with that of the corresponding morphism $\chi : S \to \mathfrak{h}/W$. The arguments now run as in the previous cases (cf. [Sl] 8.5, 8.7).

To obtain the corollaries on the singularities in the nonspecial fibres of the semiuniversal deformations, one combines the results of 2.2 on the fibres of $\mathfrak{g} \to \mathfrak{h}/W$ with a careful analysis of the position of S inside $\mathfrak{g}$ (cf. [Sl] 6.4, 6.5). It should be mentioned that for these corollaries the full strength of Theorem 1 and 2 is not needed. It is sufficient to know that $\chi : S \to \mathfrak{h}/W$ is obtained from the corresponding semiuniversal deformation by a finite base change, a fact which can be established by the results of 2.2 combined with the identification of the singularity $S \cap \mathcal{N}(\mathfrak{g})$. In the next lecture we will give such an identification without any use of the quasihomogeneous structure of $\chi : S \to \mathfrak{h}/W$.

Lecture 3 Group theoretic resolutions

This lecture deals with the resolution of simple singularities and the simultaneous resolution of their deformations. The last aspect had been studied by Brieskorn ([Br1], [Br3]) and gave rise to Grothendieck's conjectures relating simple singularities and simple groups.

3.1 Simultaneous resolution

Let $\chi : X \to U$ be a flat morphism of (germ of) analytic spaces. A **strong simultaneous resolution** is a commutative diagram of morphisms (of germs of) analytic spaces

$$\begin{tikzcd} Y \arrow[rr, "\psi"] \arrow[dr, "\theta"'] & & X \arrow[dl, "\chi"] \\ & U & \end{tikzcd}$$

such that θ is smooth, ψ is proper and surjective and ψ induces by $\psi_u : \theta^{-1}(u) \to \chi^{-1}(u)$ a resolution of singularities for $\chi^{-1}(u)$ for all $u \in U$.

A **simultaneous resolution** for $\chi : X \to U$ is a strong simultaneous resolution for the pull-back $\chi_V : X \times_U V \to V$ of χ by a finite surjective base change $V \to U$.

We actually consider only the second definition. This weaker notion allows to kill the topological obstruction of monodromy against strong simultaneous resolution. In lecture 4 we will come back to this point.

Let us now consider a flat morphism $\chi : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ whose only singular fibre is $\chi^{-1}(0)$ having only one isolated singularity. In [Br1] and [Br3] Brieskorn obtained the following results for this situation.

Theorem 3 *There is a simultaneous resolution for χ if and only if $\chi^{-1}(0)$ has a simple singularity.*

Let now χ be as in the theorem and $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ a finite base change such that the pull-back of χ by ϕ admits a strong simultaneous resolution.

Theorem 4 *The order of the covering ϕ over 0 is a multiple of the Coxeter number of the corresponding root system.*

We say that two strong simultaneous resolutions

$$\begin{tikzcd} Y_1 \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & U & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y_2 \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & U & \end{tikzcd}$$

are isomorphic if there is an isomorphism $Y_1 \to Y_2$ such that the diagram commutes

$$\begin{tikzcd} Y_1 \arrow[rr] \arrow[dr] \arrow[ddr] & & Y_2 \arrow[dl] \arrow[ddl] \\ & U & \\ & X \arrow[u] & \end{tikzcd}$$

Let $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a base change as for Theorem 4, and let $|W|$ denote the cardinality of a set W.

Theorem 5 *There are exactly $|W|$ nonisomorphic strong simultaneous resolutions for the pull-back of χ by ϕ, where W is the corresponding Weyl group.*

Actually Brieskorn obtained Theorem 5 as corollary of a more detailed description, for which we refer to [Br3].

3.2 Springer's resolution of the nilpotent variety

Besides the above mentioned results of Brieskorn the following one of Springer played a crucial role for Grothendieck's conjectures. First we recall some properties of regular nilpotent elements x in a reductive Lie algebra $\mathfrak{g}$ (cf. [Ko1], [Sp1], [Ve]).

Theorem *The following properties are equivalent for a nilpotent element x of $\mathfrak{g}$:*

(i) *x is regular, i.e.* $\dim Z_G(x) = r$
(ii) *x is contained in exactly one Borel subalgebra of* $\mathfrak{g}$
(iii) *If $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{n}_\alpha$ is the root-decomposition of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and $x \in \mathfrak{b}$, then the projection of x into $\mathfrak{n}_\alpha$ is nontrivial for all simple roots.*

We now fix a Borel group $B \subset G$ and maximal torus $T \subset B$ with Lie algebras $\mathfrak{b} = \operatorname{Lie} B$ and $\mathfrak{h} = \operatorname{Lie} T$. Then $\mathfrak{b}$ decomposes $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{b}$. As $\mathfrak{n}$ is stable under the adjoint action of B, the associated bundle $G \times^B \mathfrak{n}$ is defined. We recall that all Borel subalgebras of $\mathfrak{g}$ are conjugate under G and form a complete variety $\mathcal{B} = G/B$.

Theorem (Springer [Sp2]) *The morphism $\psi_0 = G \times^B \mathfrak{n} \to \mathcal{N}(\mathfrak{g})$ given by $g * n \mapsto \mathrm{Ad}(g)n$ is a resolution of singularities for the nilpotent variety $\mathcal{N}(\mathfrak{g})$.*

Proof: The bundle $G \times^B \mathfrak{n}$ is smooth of the same dimension as $\mathcal{N}(\mathfrak{g})$. We may embed $G \times^B \mathfrak{n}$ onto the closed subvariety $\{(A, x) \in \mathcal{B} \times \mathcal{N}(\mathfrak{g}) \mid x \in A\}$ of $\mathcal{B} \times \mathcal{N}(\mathfrak{g})$. Under this identification the morphism $\psi_0 : G \times^B \mathfrak{n} \to \mathcal{N}(\mathfrak{g})$ goes over into the second projection. From this the properness of ψ_0 already follows. Furthermore ψ_0 is surjective as any nilpotent element is contained in the nilradical of a Borel subalgebra. By the G-equivariance of ψ_0 it suffices to prove that the preimage under ψ_0 of a regular element in $\mathcal{N}(\mathfrak{g})$ consists of one point (the orbit of a regular element is dense in $\mathcal{N}(\mathfrak{g})$). But the fibre of ψ_0 over $x \in \mathcal{N}(\mathfrak{g})$ may be identified with $\mathcal{B}_x = \{A \in \mathcal{B} \mid x \in A\}$ which consists of one element, in case x is regular, by the previous theorem. □

The **exceptional set of** ψ_0 is the preimage of the singular set of $\mathcal{N}(\mathfrak{g})$, which consists of the nonregular elements. Endowed with its reduced structure it is a divisor with normal crossings. This can be seen as follows. The nonregular elements in $\mathfrak{n}$ are given by the vanishing of at least one $\mathfrak{n}_\alpha$-coordinate, where α is simple. Hence they form a divisor with normal crossings in $\mathfrak{n}$. By G-homogeneity the exceptional set in $G \times^B \mathfrak{n}$ is a divisor with normal crossings too.

3.3 Grothendieck's simultaneous resolution for adjoint quotients

Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra in the reductive Lie algebra $\mathfrak{g}$. We consider the following diagram

$$\begin{tikzcd} G \times^B \mathfrak{b} \arrow[r, "\psi"] \arrow[d, "\theta"'] & \mathfrak{g} \arrow[d, "\chi"] \\ \mathfrak{h} \arrow[r, "\phi"] & \mathfrak{h}/W \end{tikzcd}$$

where χ is the adjoint quotient, ϕ the natural map of $\mathfrak{h}$ onto the quotient $\mathfrak{h}/W$ and ψ and θ are defined by

$$\psi(g * b) = \mathrm{Ad}(g)b \text{ and } \theta(g * (h + n)) = h \text{ for } g \in G, b \in \mathfrak{b}, h \in \mathfrak{h}, n \in \mathfrak{n}.$$

Theorem (Grothendieck) *The above diagram is a simultaneous resolution for χ.*

Proof: 1) The diagram is commutative. This follows from the fact that the semisimple part of $\mathrm{Ad}(g)(h + n)$ is conjugate to h. (In general $(h + n)_s \neq h$!)

2) The morphism θ is smooth as follows from the factorization

$$G \times^B \mathfrak{b} \to G \times^B \mathfrak{h} \xrightarrow{\sim} G/B \times \mathfrak{h} \xrightarrow{\pi_2} \mathfrak{h}. \quad ???!!!$$

3) The morphism ψ is proper and surjective for similar reasons as ψ_0 was in 3.2 (each $x \in \mathfrak{g}$ is contained in a Borel subalgebra).

4) It remains to prove that for all $h \in \mathfrak{h}$ the restriction $\psi_h : \theta^{-1}(h) \to \chi^{-1}(\overline{h})$ is a resolution. This will be accomplished by a geometric reduction to Springer's theorem (3.2).

In 2.2 we obtained a description of $\chi^{-1}(\overline{h})$ as an associated bundle $G \times^{Z(h)} \mathcal{N}(\mathfrak{z}(h))$. By composing the G-invariant map ψ_h with the projection onto $G/Z(h)$ we can make an associated bundle out of $\theta^{-1}(h)$ too:

$$\theta^{-1}(h) \to \chi^{-1}(\overline{h}) \simeq G \times^{Z(h)} \mathcal{N}(\mathfrak{z}(h)) \to G/Z(h).$$

Following 2.1 we only need to look at the $Z(h)$-stable fibre of this composition over $e \cdot Z(h) \in G/Z(h)$. This is nothing but the set of all $g * (h + n) \in G \times^B (h + \mathfrak{n})$ such that the semisimple part of $(\operatorname{Ad} g)(h+n)$ is h. Replacing $g*(h+n)$ by an equivalent element $gb^{-1}*\operatorname{Ad}(b)(h+n)$ for a suitable $b \in B$, we may assume already $(\operatorname{Ad}(b)(h+n))_s = h$ and hence $gb^{-1} \in Z(h)$. This shows that the fibre over $e \cdot Z(h)$ is given by $Z(h) * (h + \mathfrak{n}(h))$ where $\mathfrak{n}(h) = \mathfrak{n} \cap \mathfrak{z}(h)$. But this is isomorphic to $Z(h) \times^{B(h)} \mathfrak{n}(h)$, where $B(h) = B \cap Z(h)$ is a Borel subgroup of $Z(h)$, whose Lie algebra has the nilradical $\mathfrak{n}(h)$. Finally $\psi_h : \theta^{-1}(h) \to \chi^{-1}(\overline{h})$ is induced by $\psi_0(h) : Z(h) \times^{B(h)} \mathfrak{n}(h) \to \mathcal{N}(\mathfrak{z}(h))$, i.e. it is isomorphic to

$$G \times^{Z(h)} \psi_0(h) : G \times^{Z(h)} (Z(h) \times^{B(h)} \mathfrak{n}(h)) \to G \times^{Z(h)} \mathcal{N}(\mathfrak{z}(h))$$

which is a resolution because $\psi_0(h)$ is one by Springer's theorem. $\square$

For later applications (cf. 4.2, 4.4) let us note one more property of this simultaneous resolution. Let us denote by $\mathfrak{h}' = \{h \in \mathfrak{h} \mid \mathfrak{z}(h) = \mathfrak{h}\}$ the regular elements in $\mathfrak{h}$. Define

$$\alpha : G/T \times \mathfrak{h}' \to \theta^{-1}(\mathfrak{h}') \text{ by } \alpha(gT, h) = g * h$$
$$\text{and } \beta : G/T \times^W \mathfrak{h}' \to \chi^{-1}(\mathfrak{h}'/W) \text{ by } \beta(gT * h) = (\operatorname{Ad} g)h.$$

Here W acts on G/T as the group of G-automorphisms of the homogeneous space G/T.

Corollary *The following G-equivariant diagram is commutative and α and β are isomorphisms.*

$$\begin{tikzcd}
G/T \times \mathfrak{h}' \arrow[rr, "\alpha"] \arrow[dr, "\mathrm{pr}_2"] \arrow[dddd, "\nu"'] & & \theta^{-1}(\mathfrak{h}') \arrow[dl, "\theta'"'] \arrow[dddd, "\psi'"] \\
& \mathfrak{h}' \arrow[dd, "\phi'"] & \\
& & \\
& \mathfrak{h}'/W & \\
G/T \times^W \mathfrak{h}' \arrow[rr, "\beta"] \arrow[ur, "\mathrm{pr}_2/W"] & & \chi^{-1}(\mathfrak{h}'/W) \arrow[ul, "\chi'"']
\end{tikzcd}$$

(Here ν is the natural map to the quotient by W).

Proof: The commutativity is trivial. The other statement follows from the fact that the fibres of pr_2, pr_2/W, θ' and χ' are all G-isomorphic to G/T. $\square$

3.4 Simultaneous resolution for transversal slices

Let now S be a transversal slice to a nilpotent orbit in $\mathfrak{g}$ (not necessarily subregular). Let $\widetilde{S} = \psi^{-1}(S)$, the preimage of S under ψ in $G \times^B \mathfrak{b}$.

Corollary *The restricted Grothendieck resolution is a simultaneous resolution*

$$\begin{tikzcd}
\widetilde{S} \arrow[r] \arrow[d, hook] & S \arrow[d, hook] \\
G \times^B \mathfrak{b} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\
\mathfrak{h} \arrow[r] & \mathfrak{h}/W
\end{tikzcd}$$

Proof: As S is transversal to the orbits in $\mathfrak{g}$, also $\widetilde{S}$ is transversal to the orbits in $G \times^B \mathfrak{b}$. Hence the composition

$$\begin{array}{ccccc} G \times \widetilde{S} & \to & G \times^B \mathfrak{b} & \to & \mathfrak{h} \\ (g, s) & \mapsto & g \cdot s & \mapsto & \theta(g \cdot s) \end{array}$$

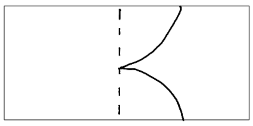
is smooth. As it factorizes in

$$G \times \widetilde{S} \xrightarrow{\pi_2} \widetilde{S} \xrightarrow{\theta|_{\widetilde{S}}} \mathfrak{h}$$

we obtain the smoothness of $\widetilde{\theta} = \theta|_{\widetilde{S}}$. Especially $\widetilde{S}$ itself is smooth. One can easily check the other properties of a simultaneous resolution too (cf. [Sl] 5.3). $\square$

Choosing S transversal to the subregular nilpotent orbit in a simple Lie algebra of type $\mathsf{A}_r, \mathsf{D}_r, \mathsf{E}_r$ we obtain a simultaneous resolution for the semiuniversal deformation of the corresponding simple singularity. By applying a suitable base change this gives a simultaneous resolution for any given deformation of the singularities in question, especially for that considered by Brieskorn. Generalizing Brieskorn's method this general result had also been obtained by Kas (A_r) and Tjurina ($\mathsf{A}_r, \mathsf{D}_r, \mathsf{E}_r$).

Brieskorn's deformation can be obtained by restricting the semiuniversal one to a line transversal to the discriminant in the base $\mathfrak{h}/W$.



$\mathfrak{h}/W$

By using 2.2 it is easy to see that the discriminant of the deformation $S \to \mathfrak{h}/W$, i.e. the set of critical values, coincides with the set of singular semisimple classes or, equivalently, with the discriminant of the finite cover $\mathfrak{h} \to \mathfrak{h}/W$. After identifying fundamental homogeneous generators of $\mathbb{C}[\mathfrak{h}]^W$ with coordinates on $\mathfrak{h}/W$, a line transversal to the discriminant is given by the vanishing of all invariant polynomials of degree smaller than h, the Coxeter number. The Coxeter number is the highest degree occuring for fundamental generators. Let c be a Coxeter element of W and $L \subset \mathfrak{h}$ the one-dimensional eigenspace for the eigenvalue $\mathrm{e}^{2\pi\mathrm{i}/h}$. All W-invariant homogeneous polynomials of degree $< h$ have to vanish on L, hence L is mapped to a line transversal to the discriminant with mapping degree h. Hence by applying the base change $L \to \mathfrak{h}$ we obtain a simultaneous resolution for Brieskorn's deformation. Theorem 4 of 3.1 may be either explained by a monodromy argument as in [Br3] or, more complicated, by the above description and a universality result for the simultaneous resolution

$$\begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & S \arrow[d] \\ \mathfrak{h} \arrow[r] & \mathfrak{h}/W \end{tikzcd}$$

with respect to all deformations with resolutions (cf. [Hui], [Ar3]). The existence of the $|W|$ inequivalent simultaneous resolution in Brieskorn's Theorem 5 (3.1) is reflected in Grothendieck's construction by the $|W|$ possible choices of a Borel group B containing a given maximal torus T of G.

3.5 Identification of the subregular singularities by resolution

Let S be again a transversal slice to the orbit of a subregular nilpotent element x in a simple Lie algebra $\mathfrak{g}$. The simultaneous resolution

$$\begin{tikzcd} \widetilde{S} \arrow[r] \arrow[d] & S \arrow[d] \\ \mathfrak{h} \arrow[r] & \mathfrak{h}/W \end{tikzcd}$$

furnishes especially a resolution

$$\widetilde{S}_0 \to S_0 = \mathcal{N}(\mathfrak{g}) \cap S$$

of the simple singularity $S \cap \mathcal{N}(\mathfrak{g})$. We now want to show that this is the minimal resolution of $S_{\bar{0}}$ without using the previous identification of $S_{\bar{0}}$. As simple singularities are determined by their minimal resolution this gives an alternative method to identify $S \cap \mathcal{N}(\mathfrak{g})$. Historically this method was the firstly envisaged one, but complete information was obtained only recently. As $\widetilde{S}_0$ is transversal to the G-orbits in $G \times^B \mathfrak{n}$ it is transversal to the G-stable exceptional set in $G \times^B \mathfrak{n}$. Hence the exceptional set E of $\widetilde{S}_0 \to S_{\bar{0}}$ is a divisor with normal crossings too, which in our case, $\widetilde{S}_0$ is a smooth surface, means that E consists of a union of smooth curves intersecting transversally. On the other hand, E may be identified with the variety $\mathcal{B}_x = \{A \in \mathcal{B} \,|\, x \in A\}$ of all Borel algebras containing x (cf. 3.2). A result of Tits (cf. [Sp2]) states that any two points of $\mathcal{B}_x$ may be connected by a connected sequence of projective lines. Hence E consists of rational curves. Steinberg and Tits determined the complete configuration of the curves in $\mathcal{B}_x$ and obtained the right ones i.e. those corresponding to the simple singularities of the same type (cf. [St2]). The only missing information was the knowledge of the self-intersection numbers for the exceptional curves inside the surface $\widetilde{S}_0$. This was only recently achieved by H. Esnault ([Es]).

Theorem *Let $C \simeq \mathbb{P}^1$ be an exceptional component of the resolution $\widetilde{S}_0 \to S_{\bar{0}}$. Then C has self-intersection -2 in $\widetilde{S}_0$.*

Proof: For a submanifold X of a manifold Y we denote by $c_1(X, Y)$ the first Chern class of the normal bundle of X in Y. Hence we have to show $c_1(C, \widetilde{S}_0) = -2$. Because of the additivity of c_1 with respect to short exact sequence we obtain

$$c_1(C, \widetilde{S}_0) = c_1(C, G \times^B \mathfrak{n}) - c_1(\widetilde{S}_0, G \times^B \mathfrak{n})|_C.$$

But the normal bundle of $\widetilde{S}_0$ in $G \times^B \mathfrak{n}$ is trivial as it is induced from the normal bundle of S in $\mathfrak{g}$, which by any choice of S is trivial in a neighbourhood of x.

Up to conjugacy of x we may assume the projection of C under $C \subset \widetilde{S}_0 \subset G \times^B \mathfrak{n} \to G/B$ to be P/B where $P = \langle U_{-\alpha}, B \rangle$ is a proper minimal parabolic subgroup corresponding to a simple root α (cf. [St2]). From the cartesian diagram

$$\begin{tikzcd}
P \times^B \mathfrak{n} \arrow[r, hook] \arrow[d] & G \times^B \mathfrak{n} \arrow[d] \\
P/B \arrow[r, hook] \arrow[d] & G/B \arrow[d] \\
P/P \arrow[r, hook] & G/P
\end{tikzcd}$$

and the smoothness of $G \times^B \mathfrak{n} \to G/P$ we obtain

$$c_1(C, G \times^B \mathfrak{n}) = c_1(C, P \times^B \mathfrak{n}).$$

We may regard C as a section of the bundle $P \times^B \mathfrak{n}$. Hence $c_1(C, P \times^B \mathfrak{n})$ is equal to $c_1(P \times^B \mathfrak{n})$, the first Chern class of the bundle $P \times^B \mathfrak{n}$.

Let $\mathfrak{n}_P$ be the nilradical of $\operatorname{Lie} P$. Then the following B-equivariant exact sequence

$$0 \to \mathfrak{n}_P \to \mathfrak{n} \to \mathfrak{n}/\mathfrak{n}_P \to 0$$

induces one of bundles

$$0 \to P \times^B \mathfrak{n}_P \to P \times^B \mathfrak{n} \to P \times^B (\mathfrak{n}/\mathfrak{n}_P) \to 0$$

as the B-module $\mathfrak{n}_P$ is the restriction of the P-module $\mathfrak{n}_P$, the first bundle is trivial. The quotient $\mathfrak{n}/\mathfrak{n}_P$ is one-dimensional and B acts on it via the simple root $\alpha : B \to T \to \mathbb{C}^*$.

Hence the third bundle is isomorphic to $\mathrm{SL}_2 \times^{B_2} \mathfrak{n}_2$, where B_2 is a Borel group of SL_2 acting on the nilradical $\mathfrak{n}_2$ of its Lie algebra by the adjoint action. Via the Killing form this may be identified with the cotangent bundle of $\mathrm{SL}_2/B_2 \simeq \mathbb{P}^1$. Hence $c_1(P \times^B \mathfrak{n}) = c_1(T^*\mathbb{P}^1) = -2$, and the theorem is proved. $\square$

Lecture 4 Representations of Weyl groups

This lecture introduces representations of Weyl groups as monodromy transformations and relates them to previously defined representations of T.A. Springer.

4.1 Monodromy

Let U be a reasonable connected topological space (in our applications a complex manifold) and $\mathcal{F}$ a locally constant sheaf on U. One should think of $\mathcal{F}$ as given by a topological covering of U or as the sheaf of locally constant sections of a vector bundle $V(\mathcal{F})$ over U with discrete structure group. We may then identify stalks of $\mathcal{F}$ with fibres of the corresponding bundle. Let $\gamma : [0,1] \to U$ be a continuous path from the unit interval to U. The induced sheaf $\gamma^*\mathcal{F}$ on $[0,1]$ is then a trivial one and we get an isomorphism of stalks

$$\phi(\gamma) : \mathcal{F}_{\gamma(0)} \xrightarrow{\sim} \mathcal{F}_{\gamma(1)}$$

which depends only on the homotopy class of γ. Especially if we choose a base point $u_0 \in U$ we get an action of the fundamental group $\pi_1(U, u_0)$ of U on the stalk $\mathcal{F}_{u_0}$. We call this representation

$$\pi_1(U, u_0) \to \mathrm{Aut}(\mathcal{F}_{u_0})$$

the **monodromy** of $\mathcal{F}$.

Let $\widetilde{U}$ be the universal covering of U. The inverse image of $\mathcal{F}$ on $\widetilde{U}$ is a trivial sheaf isomorphic to $\widetilde{U} \times \mathcal{F}_{u_0}$. We may obtain $\mathcal{F}$ on U by dividing out the action of $\pi_1(U, u_0)$, i.e. $\mathcal{F}$ is the associated fibre bundle

$$\widetilde{U} \times^{\pi_1(U,u_0)} \mathcal{F}_{u_0}$$

where $\pi_1(U, u_0)$ acts on $\widetilde{U}$ by covering transformations and on $\mathcal{F}_{u_0}$ by monodromy.

The following criterion is obvious.

Lemma *Let $V \to U$ be a normal connected covering of U corresponding to the normal subgroup $\pi_1(V)$ of $\pi_1(U)$. The pullback of $\mathcal{F}$ to V is trivial if and only if $\pi_1(V)$ is contained in the kernel of the monodromy*

$$\pi_1(U) \to \mathrm{Aut}(\mathcal{F}_{u_0}).$$

In the following we denote $\pi_1(U, u_0)$ for simplicity by Γ. Let $\mathcal{E}$ be a category of sets equipped with Γ-actions and Γ-equivariant morphisms. Let $\phi : F \to G$ be a morphism in $\mathcal{E}$. Then the Γ-equivariant map $\mathrm{id} \times \phi : \widetilde{U} \times F \to \widetilde{U} \times G$ induces a morphism $\boldsymbol{\phi}$ of the corresponding sheafs on U

$$\boldsymbol{\phi} : \mathcal{F} = \widetilde{U} \times^{\Gamma} F \to \mathcal{G} = \widetilde{U} \times^{\Gamma} G.$$

It follows easily that $\phi \mapsto \boldsymbol{\phi}$ is an equivalence between $\mathcal{E}$ and the category of locally constant sheafs on U with values in $\mathcal{E}$ (in which morphisms are locally constant).

In the applications we have in mind, the locally constant sheaf $\mathcal{F}$ on U will be given as a higher direct image sheaf $R^i\pi_*\mathcal{A}_E$ of the constant sheaf $\mathcal{A}_E$ on the total space E of a differentiable fibre bundle $\pi : E \to U$, where for $\mathcal{A}$ we choose $\mathbb{Z}$ or $\mathbb{Q}$. The stalks of $R^i\pi_*\mathcal{A}_E$ identify then with the cohomology group $H^i(F, \mathcal{A})$ of the fibre F of π. As homology may be expressed functorially in terms of cohomology (also over $\mathbb{Z}$, cf. [Spa] p.248) there is a locally constant "fibre homology sheaf" on U corresponding to the "fibre cohomology sheaf" $R^i\pi_*\mathcal{A}_E$. Still another variation is the higher direct image $R^i\pi_!\mathcal{A}_E$ with compact support whose stalks give the cohomology $H^i_c(F,\mathcal{A})$ with compact support. Hence we may consider monodromy representations in the homology, the ordinary and the compact cohomology of the fibre F of $\pi : E \to U$.

Example (the oldest one): Let $U = \mathbb{C}^* = \mathbb{C}\setminus 0$ and consider $\pi : \mathbb{C}^* \to \mathbb{C}^*$ given by $z \mapsto z^n$ for some $n > 0$. The fibre F of π may be identified with the group μ_n of the n^{th} roots of unity. The action of $\mathbb{Z} = \pi_1(\mathbb{C}^*, 1)$ on $\mu_n \simeq H_0(F,\mathbb{Z})$ is given by the homomorphism

$$\begin{array}{lll} \mathbb{Z} \to \mathbb{Z}/(n) & \to \operatorname{Aut}(\mu_n) \\ 1 \mapsto \overline{1} & \mapsto (x \mapsto \mathrm{e}^{2\pi\mathrm{i}/n}\cdot x). \end{array}$$

4.2 Monodromy representations of Weyl groups

We consider a nilpotent element x of a semisimple Lie algebra $\mathfrak{g}$ and a transversal slice S in x to the orbit of x. As can be seen from the description of the fibres of $\chi : \mathfrak{g} \to \mathfrak{h}/W$ in 2.2, the critical values of $\chi_S = \chi|_S : S \to \mathfrak{h}/W$ must be singular semisimple classes $\overline{h} \in \mathfrak{h}/W$. These classes can also be described as the ramification locus of the ramified covering $\phi : \mathfrak{h} \to \mathfrak{h}/W$. Usually this locus is called the discriminant Q. Its preimage $\phi^{-1}(Q)$ in $\mathfrak{h}$ is the union of the reflection hyperplanes $\mathfrak{h}_\alpha = \{h \in \mathfrak{h} \mid \alpha(h) = 0\}$, $\alpha \in \Sigma$ (Σ denoting the root system of $\mathfrak{h}$ in $\mathfrak{g}$). The covering

$$\mathfrak{h}' := \mathfrak{h} \setminus \phi^{-1}(Q) \to \mathfrak{h}'/W = (\mathfrak{h}/W)\setminus Q$$

is now unramified and Galois with covering group W.

In the following we will see that the restriction

$$\chi|_{S\setminus\chi^{-1}(Q)} : S\setminus \chi^{-1}(Q) \to \mathfrak{h}'/W$$

(or at least some localization of it) is a differentiable fibre bundle with fibre a manifold F. Moreover this bundle will be trivial after the pull back by $\mathfrak{h}' \to \mathfrak{h}'/W$. Hence we obtain a representation of $\pi_1(\mathfrak{h}'/W)$ on the (co-)homology of F which factorizes over the quotient

$$\pi_1(\mathfrak{h}'/W)/\pi_1(\mathfrak{h}') = W.$$

We call the corresponding representation of W a **monodromy** representation of W. We now come to the technical details. First we note that the simultaneous resolution for χ_S (cf. 3.4)

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\psi_S"] \arrow[d, "\theta_S"'] & S \arrow[d, "\chi_S"] \\ \mathfrak{h} \arrow[r, "\phi"] & \mathfrak{h}/W \end{tikzcd}$$

is $\mathbb{G}_m$-equivariant with respect to natural actions on S, $\widetilde{S}$, $\mathfrak{h}$ and $\mathfrak{h}/W$. In case that S is of the form as in 2.4 these actions are actually global. For a general slice S to the orbit of x we know that it is locally (i.e. near the orbit) analytically isomorphic over $\mathfrak{h}/W$ to such a special one (cf. [Sl] 5.1). When speaking of a $\mathbb{G}_m$-action in general we shall tacitly understand by that the germ of a $\mathbb{G}_m$-action.

In 2.5 we already considered the quasihomogeneous structure of χ_S, the action on $\mathfrak{h}/W$ being given by

$$t\cdot\phi(h) = \phi(\sigma(t^2)h), \quad t \in \mathbb{G}_m, h \in \mathfrak{h}$$

and that on $S \subset \mathfrak{g}$ by the restriction to S of

$$t\cdot y = \rho(t^{-1})\sigma(t^2)y, \quad t \in \mathbb{G}_m, y \in \mathfrak{g},$$

where σ is the scalar action of $\mathbb{G}_m$ on the corresponding spaces and ρ the action coming from a maximal torus of a Jacobson-Morozov-SL_2 for x.

If we let $\mathbb{G}_m$ act on $\mathfrak{h}$ by

$$t \cdot h = \sigma(t^2)h, \quad t \in \mathbb{G}_m, h \in \mathfrak{h}$$

and on $G \times^B \mathfrak{b}$ by

$$t \cdot (g * b) = (\rho(t^{-1})g) * (\sigma(t^2)b), \quad t \in \mathbb{G}_m, g \in G, b \in \mathfrak{b},$$

all arrows in the diagram

$$\begin{tikzcd} G \times^B \mathfrak{b} \arrow[r, "\psi"] \arrow[d, "\theta"'] & \mathfrak{g} \arrow[d, "\chi"] \\ \mathfrak{h} \arrow[r, "\phi"] & \mathfrak{h}/W \end{tikzcd}$$

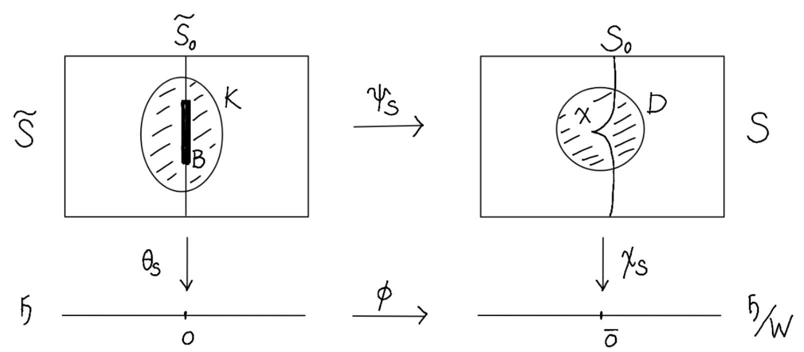
will be $\mathbb{G}_m$-equivariant as well as in the $\mathbb{G}_m$-stable substitution

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\psi_S"] \arrow[d, "\theta_S"'] & S \arrow[d, "\chi_S"] \\ \mathfrak{h} \arrow[r, "\phi"] & \mathfrak{h}/W \end{tikzcd}$$

We will now proceed similarly as in the local study of isolated singularities (cf. [La], [Mi], [Si]) and switch to the category of differential topology.

We recall (cf. 3.2, 3.5) that we may identify the fibre $\psi_S^{-1}(x)$ in $\widetilde{S}$ with the compact variety $\mathcal{B}_x$ of Borel subalgebras of $\mathfrak{g}$ containing x.

Lemma *For any sufficiently small closed ball D around x in S (in some local chart) the preimage $K = \psi_S^{-1}(D)$ is a compact neighbourhood of $\mathcal{B}_x$ whose boundary ∂K is a manifold cutting $\widetilde{S}_0 = \theta_S^{-1}(0)$ transversally.*



Proof: Near the point x the $\mathbb{G}_m$-action on S may be looked at as a linear one with positive weights only (cf. 2.5). A sufficiently small ball D around x will thus cut the $\mathbb{R}^*$-orbits (induced by $\mathbb{R}^* \subset \mathbb{C}^*$) transversally. As ψ_S is proper, $K = \psi_S^{-1}(D)$ is compact. To prove that ∂K is a manifold we need to know that ψ_S is transversal to ∂D. But this follows from the $\mathbb{R}^*$-equivariance of ψ_S and the transversality to ∂D of the $\mathbb{R}^*$-orbits in S. By the same argument ∂K must be transversal to the $\mathbb{R}^*$-orbits in $\widetilde{S}$, especially to the $\mathbb{R}^* \subset \mathbb{G}_m$-stable submanifold $\widetilde{S}_0$. $\square$

Proposition *Let $K = \psi_S^{-1}(D)$ be a neighbourhood of $\mathcal{B}_x$ in S as in the preceding lemma. Then there exists an open W-stable ball H around $0 \in \mathfrak{h}$ such that the restriction*

$$\theta|_{K\cap\theta^{-1}(H)} : K \cap \theta^{-1}(H) \to H$$

is a trivial differentiable fibre bundle.

Proof: We know that ∂K intersects $\widetilde{S}_0$ transversally. By continuity and the compactness of K the boundary ∂K will intersect transversally also the fibres $\widetilde{S}_h = \theta_S^{-1}(h)$ for h in a small neighbourhood H of 0 in $\mathfrak{h}$, which may be chosen W-stable. It follows that θ restricted to $\partial K \cap \theta^{-1}(H)$ as well as to $\mathring{K} \cap \theta^{-1}(H)$ (here $\mathring{K}$ denotes the interior of K) has maximal rank $\dim \mathfrak{h}$. By the Ehresmann-fibration-theorem (cf. [E]) we obtain that

$$\theta|_{K\cap\theta^{-1}(H)} : K \cap \theta^{-1}(H) \to H$$

is a differentiable fibre bundle whose fibre is a manifold with boundary. The triviality follows from the contractibility of H. □

Corollary *Let D be a sufficiently small closed ball around x in S and $H \subset \mathfrak{h}$ a sufficiently small open W-stable ball around $0 \in \mathfrak{h}$ and let $U = (H \cap \mathfrak{h}')/W$. Then the restriction*

$$\chi|_{D\cap\chi^{-1}(U)} : D \cap \chi^{-1}(U) \to U$$

is a differentiable fibre bundle which becomes trivial after pull back by $H \cap \mathfrak{h}' \to U$.

Proof: Because of the preceding proposition it suffices to show that the following diagram is cartesian.

$$\begin{tikzcd} K \cap \theta^{-1}(H \cap \mathfrak{h}') \arrow[r] \arrow[d] & D \cap \chi^{-1}(U) \arrow[d] \\ H \cap \mathfrak{h}' \arrow[r] & U \end{tikzcd}$$

But this follows from corollary 3.3, which implies that

$$\begin{tikzcd} \theta^{-1}(\mathfrak{h}') \arrow[r, "\psi"] \arrow[d, "\theta"'] & \chi^{-1}(\mathfrak{h}'/W) \arrow[d, "\chi"] \\ \mathfrak{h}' \arrow[r, "\phi"] & \mathfrak{h}'/W \end{tikzcd}$$

is cartesian. □

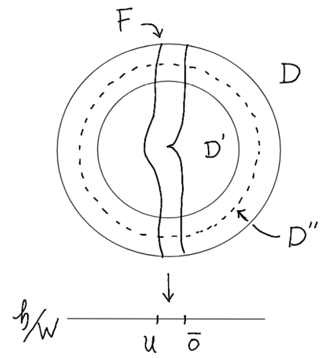
By this we have prove all the claims needed for the definition of the monodromy representation of W on the (co-)homology of the fibre F of the restricted χ_S.

We now wish to see that this definition depends only on the conjugacy class of the nilpotent element x, the essential point being to show that the other choices made do not effect the (co-)homology groups in question (or stronger: the homotopy type of the fibre F).

1) The choice of the size of H and the scalar product, in which it is defined, are irrelevant as we consider a fibre bundle over H (resp. U).

2) The independence of the size of D can be seen as follows. Let $D' \subset D$ be a smaller ball around x and F a fibre in D over $u \in U$ so close to the origin $\overline{0} \in \mathfrak{h}/W$ that it meets all intermediate balls $D' \subset D'' \subset D$ transversally at the boundary $\partial D''$.

The gradient of the distance function $\mathrm{d}(x, f)$ on F to x defines a nowhere vanishing vector field on $F \cap (D \setminus \mathring{D}')$ which may be used to isotop F onto $F' = F \cap D'$.



3) If we choose different scalar products (equivalently: different local charts) to determine a ball D around $x \in S$ the corresponding fundamental systems of ball-neighbourhoods around x will refine each other. One may now argue similarly as in 2).

4) The different transversal slices at possibly different point of the orbit of x are locally analytically equivalent over $\mathfrak{h}/W$. Hence the problem reduces to that of different local charts 3).

Remark If in the above constructions we start with a transversal slice S whose local $\mathbb{G}_m$-action is actually a global one (as in 2.4), we may replace the local statements by global ones. For example $\theta_S : \widetilde{S} \to \mathfrak{h}$ will itself be a trivial bundle. To see this we use the local result of the proposition above for sufficiently small K and H. By expanding K with elements $t \in [1, \infty) \subset \mathbb{R}^*$ we obtain that the fibres $\theta_S^{-1}(h), h \in H$, intersect transversally the boundaries $\partial(tK)$ for all $t \geq 1$. Similarly to the independence argument 2) above we may thus isotop $\theta_S^{-1}(H)$ along its fibres (i.e. respecting the map to H) onto $\mathring{K} \cap \theta_S^{-1}(H)$. Once more expanding the whole situation from H to $\mathfrak{h}$ (by $\mathbb{R}^*$) we get the global result. As a byproduct we obtain that the "global fibres" of θ_S are diffeomorphic to the "short" fibres $K \cap \theta_S^{-1}(h), h \in H$.

4.3 The topology of the fibre F

Our aim is now to get some information on the (co)homology groups of the fibre F of the bundles discussed above. Because of the topological independence of the choices made and for notational simplicity we denote the restrictions of χ to $D \cap \chi^{-1}(U)$ and of θ to $K \cap \theta^{-1}(H)$ by χ_x and θ_x.

Proposition 1 *The general fibre F of θ_x (equivalently χ_x) is homotopy equivalent to $\mathcal{B}_x$.*

Proof: As θ_x is a fibre bundle we may take for F the special fibre $\theta_x^{-1}(0) = K \cap \widetilde{S}_0$, which contains $\mathcal{B}_x$ as a compact subvariety. According to [Lo] the pair $(F, \mathcal{B}_x)$ may be triangulated; and by [Spa] p.124 the variety $\mathcal{B}_x$ is a strong deformation retract of a neighbourhood of $\mathcal{B}_x$ in F. As the $\mathbb{R}^*$-action on S (and hence S_0) contracts to the point x, the $\mathbb{R}^*$-action on $\widetilde{S}_0$ contracts towards $\mathcal{B}_x$, i.e. the translates $t \cdot F$, $t \in (0, 1] \subset \mathbb{R}^*$, form a fundamental system of homotopy equivalent neighbourhoods of $\mathcal{B}_x$. Hence $\mathcal{B}_x$ is homotopy equivalent to F itself.$\square$

Remark 1) The $\mathbb{R}^*$-action itself cannot be used for a deformation retraction of F to $\mathcal{B}_x$ as it operates nontrivially on $\mathcal{B}_x$, in general.

2) The equivalence of $\mathcal{B}_x$ and F on the cohomological level can be seen without using the triangulation result. Namely we have $H^*(\mathcal{B}_x, \mathbb{Z}) = \varinjlim H^*(V, \mathbb{Z})$ where V runs through a fundamental system of neighbourhoods of $\mathcal{B}_x$ (cf. [Go] p.193, [Spa] 6.9) which may all be chosen homotopy equivalent to F as in the proof above.

Proposition 2 *The dimension of $\mathring{F}$ as a complex manifold is* $\dim Z_G(x) - r$.

Proof: As the restriction of the adjoint quotient $\chi : \mathfrak{g} \to \mathfrak{h}/W$ to a transversal slice is a flat morphism again (cf. [Sl] 5.2) the dimension of $\overset{\circ}{F}$ is $\dim S - \dim \mathfrak{h}/W = \dim Z_G(x) - r$. $\square$

Theorem ([S2], [St4]) *The components of $\mathcal{B}_x$ have all the same dimension $d_x = \frac{1}{2}(\dim Z_G(x) - r)$.*

From the above theorem and proposition 2 we obtain that F has the homotopy type of a CW-complex of half the (real) dimension of F. As F is essentially a Stein manifold this can also be seen a priori by Morse theory (cf. [A-F]). But we have of course more specific information.

Corollary *The middle homology group $H_{2d_x}(F, \mathbb{Z})$ of F is isomorphic to the top homology group $H_{2d_x}(\mathcal{B}_x, \mathbb{Z})$ of $\mathcal{B}_x$. Moreover the last one is freely generated over $\mathbb{Z}$ by the fundamental classes of the components of $\mathcal{B}_x$.*

As the real dimension of F is $4d_x$, we have a symmetric intersection form $(\ , \ )$ on the middle homology $H_{2d_x}(F, \mathbb{Z})$ of F given by

$$(c, c') = v(p(c))(c')$$

where p is the Poincaré-duality $H_{2d_x}(F, \mathbb{Z}) \to H_c^{2d_x}(F, \mathbb{Z})$ and v the natural map

$$H_c^{2d_x}(F, \mathbb{Z}) \to H^{2d_x}(F, \mathbb{Z}) \to H_{2d_x}(F, \mathbb{Z})^*.$$

If we consider this form varying along the fibres of the "fibre homology sheaf" of χ_x, it is of course locally constant. Hence the monodromy action of W on $H_{2d_x}(F, \mathbb{Z})$ leaves the intersection form invariant.

Conjecture *The intersection form is definite (positive resp. negative if d_x is even resp. odd).*

Remark This conjecture is true for x subregular where we obtain the intersection form for the resolution of a simple singularity (cf. examples below). For an arbitrary normal surface singularity this form must be negative definite too (cf. [Mu2]). In the case of a general nilpotent element x it often happens that the action of W on $H_{2d_x}(F, \mathbb{Z})$ is irreducible, for example if the centralizer of x is connected, which is always true for SL_n (cf. 4.6 later). Then an invariant form must be zero or definite.

There is still another W-invariant structure of interest on the (co-)homology groups of F. Let (x, h, y) be a Jacobson-Morozov-$\mathfrak{sl}_2$ for x. In 2.6 we saw how the reductive centralizer $C(x, y) \subset Z_G(x)$ operates on the transversal slice $S = x + \mathfrak{z}_{\mathfrak{g}}(y)$. As $\chi_S : S \to \mathfrak{h}/W$ is invariant with respect to this action we get an action of $C(x, y)$ on the fibres of χ_S and their (co-)homology groups, the last one factorizing over the component group $C = C(x, y)/C(x, y)^0 = Z_G(x)/Z_G(x)^0$ (for homotopy reasons). The action of C on the fibre (co-)homology sheaf of χ_S is locally constant by continuity. Hence the monodromy action of W on $H_*(F, \mathbb{Z})$ commutes with that of C. The action of $C(x, y)$ on the special fibre $\widetilde{S}_0$ of θ_S restricts to the natural action of $C(x, y) \subset Z_G(x)$ on $\mathcal{B}_x$, and via the isomorphism $H_*(F, \mathbb{Z}) \simeq H_*(\widetilde{S}_0, \mathbb{Z}) \simeq H_*(\mathcal{B}, \mathbb{Z})$ we see that C acts on the middle homology $H_{2d_x}(F, \mathbb{Z})$ as a permutation group of the components of $\mathcal{B}_x$.

Example 1) Let x be a subregular nilpotent element of a simple Lie algebra $\mathfrak{g}$. Then part of the statements above are special cases of more general results for isolated hypersurface singularities (cf. [Mi]). Let n be the dimension of such a singularity F_0. The Milnor fibre F of F_0 is a smooth deformation of F_0 (for example if $F_0 = \{f = 0\}$ then $F = \{f = \varepsilon\}$ for sufficiently small ε). In [Mi] it is proved that F has the homotopy type of a finite bouquet of n-spheres. In our situation (i.e. x subregular) we have $\dim F = \dim F_0 = 2$ and F has the homotopy type of $\mathcal{B}_x$ which is a simply connected union of projective lines $\mathbb{P}^1(\mathbb{C}) \simeq S^2$. With respect to the basis given by the components of $\mathcal{B}_x$ the intersection form on $H_2(F, \mathbb{Z})$ is written as the negative of the Cartan matrix belonging to the (associated homogeneous) Dynkin diagram of $\mathfrak{g}$. If $\mathfrak{g}$ is of type A_r, D_r or E_r the restriction of $\chi : \mathfrak{g} \to \mathfrak{h}/W$ to a transversal slice S realizes a semiuniversal deformation of the corresponding simple singularity. By applying the theory of Picard-Lefschetz transformations and vanishing cycles (cf. [La]) one can show independently that the monodromy action of $\pi_1((\mathfrak{h}/W) \setminus Q)$ on $H_2(F, \mathbb{Q})$ factorizes over one of W and that this is the natural representation of W as a Coxeter group. If $\mathfrak{g}$ is of type $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4$ or G_2, then the centralizer of a subregular nilpotent element is not any more connected and we obtain an action of C on $H_2(F, \mathbb{Z})$ commuting with W. It can be shown (cf. 4.5) that the W-action on the C-fixed part of $H_2(F, \mathbb{Q})$ is the natural representation of W. From the description of the C-action on the components of $\mathcal{B}_x$ (cf. 1.6, 1.7) it follows already that this part has the dimension $\operatorname{rank}\mathfrak{g}$ whereas the dimension of $H_2(F, \mathbb{Q})$ is the rank of the corresponding homogeneous Dynkin diagram.

2) Of the extreme cases $x = 0$ and x regular the first will be extensively discussed in the next chapter whereas the last is left to the reader (use the differential criterion for regularity, cf. 1.3).

Remark As in the case of a subregular x it may be worthwhile also in the general case not only to consider the general fibre F of $\chi_S : S \to \mathfrak{h}/W$ but also the singular special one $\chi_S^{-1}(\overline{0}) = \mathcal{N}(\mathfrak{g}) \cap S$. Because of the flatness of χ_S and the smoothness of S and $\mathfrak{h}/W$, this will be a complete intersection. Its dimension is even and the singular locus has codimension 2. By recent (unpublished) results of Kraft and Procesi the closures of the strata of $\mathcal{N}(\mathfrak{g}) \cap S$, given by the type of the nilpotent orbits, present only nice singularities (mostly simple up to smooth equivalence) in case of the classical groups. Philosophically the singularities $\mathcal{N}(\mathfrak{g}) \cap S$ should also share the strongest rationality properties for singularities in higher dimensions.

4.4 The case $x = 0$

If we choose $x = 0$ for the nilpotent element we may take $\mathfrak{g}$ itself as a transversal slice to the orbit of 0. The morphism χ_S coincides thus with $\chi : \mathfrak{g} \to \mathfrak{h}/W$. In corollary 3.3 we have already observed the G-equivalence of the diagrams

$$\begin{tikzcd} \theta^{-1}(\mathfrak{h}') \arrow[r, "\psi'"] \arrow[d, "\theta'"'] & \chi^{-1}(\mathfrak{h}'/W) \arrow[d, "\chi'"] \\ \mathfrak{h}' \arrow[r, "\phi'"] & \mathfrak{h}'/W \end{tikzcd}$$

and

$$\begin{tikzcd} G/T \times \mathfrak{h}' \arrow[r] \arrow[d] & G/T \times^W \mathfrak{h}' \arrow[d] \\ \mathfrak{h}' \arrow[r] & \mathfrak{h}'/W \end{tikzcd}$$

Thus the monodromy action of W on the (co-)homology of a fibre $F \simeq G/T$ of χ' is induced by the action of W on G/T as the group $N(G)/T$ of its G-automorphisms. Explicitly this action is given by $(w, gT) \mapsto gw^{-1}T$.

If $B \subset G$ is a Borel group of G containing T, the natural projection $G/T \to G/B$ becomes a fibre bundle with fibre the unipotent radical U of B. As variety and hence as a topological space U is an affine space. This implies the homotopy equivalence of $F \simeq G/T$ and $\mathcal{B}_0 \simeq G/B$, now by a direct argument.

The following description of the cohomology of G/B and the W-action on it is mainly due to A. Borel (for recent accounts cf. [B-G-G], [De] (This reference item was left out in the original document)).

Without loss of generality let G be simply connected and $X^*(T)$ the character group of the maximal torus T. Let $V^* := X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. By means of the projection $B \to T$, each character $\lambda \in X^*(T)$ of T induces one of B. We may now map $\lambda \in X^*(T)$ to the first Chern class in $H^2(G/B, \mathbb{Z})$ of the associated line bundle $G \times^B \mathbb{C}$, where B operates on $\mathbb{C}$ by λ. This map extends to a homomorphism of $\mathbb{Q}$-algebras

$$c : S^*(V) \to H^*(G/B, \mathbb{Q})$$

(here $S^*(V)$ is the symmetric algebra on V^*). Let $J \subset S^*(V)$ denote the ideal in $S^*(V)$ generated by the W-invariant elements of strictly positive degree.

Theorem 6 *The map c factorizes over a W-equivariant isomorphism $\bar{c} : S^*(V)/J \xrightarrow{\sim} H^*(G/B, \mathbb{Q})$.*

The following theorem is due to Chevalley (cf. [LIE] V 5.2).

Theorem 7 *The W-action on $S^*(V)/J$ is isomorphic to the regular representation of W on its group ring $\mathbb{Q}[W]$.*

Let us determine some pieces of the monodromy action of W on $H^*(F, \mathbb{Q}) \simeq H^*(G/B, \mathbb{Q})$.

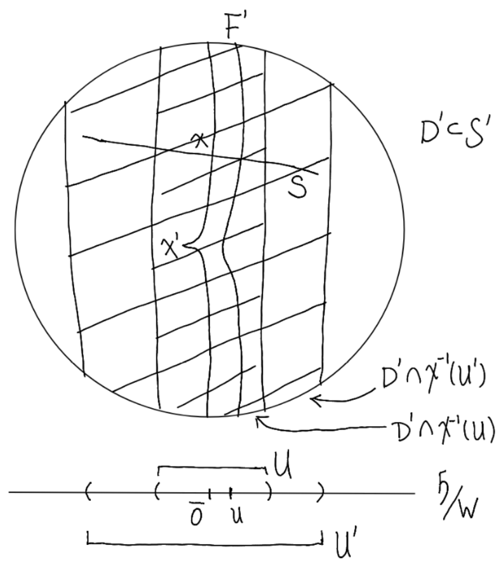
The action on H^0 is of course the trivial one, on H^2 we have an isomorphism to V^*, hence we get the natural representation of W. The highest non zero group is H^{2N}, where $N = \dim G/B = |\Sigma^+|$. It is generated over $\mathbb{Q}$ by the image under c of $\prod_{\alpha \in \Sigma^+} \alpha \in S^N(V)$. Thus we obtain the sign-representation of W, $w \mapsto (-1)^{\ell(w)}$, where ℓ is the length function on W (with respect to B).

4.5 A specialization lemma

Let x and x' be nilpotent elements of $\mathfrak{g}$ such that x' lies in the closure of the G-orbit of x. We wish to relate the monodromy representations for x and x'. Let S' be a transversal slice in x' to the orbit of x' and $D' \subset S'$, $U' \subset \mathfrak{h}/W$ sufficiently small (as in corollary 4.2) such that

$$\chi|_{D' \cap \chi^{-1}(U')} : D' \cap \chi^{-1}(U') \to U'$$

is a fibre bundle with typical fibre F'. As the monodromy representation for x depends only on the conjugacy class of x, we replace x by an element of the intersection $(G \cdot x) \cap D' \cap \chi^{-1}(U')$. We may also choose a transversal slice S in x to the orbit of x which is contained in D'.



For sufficiently small $D \subset S \subset D'$ and $U \subset U' \subset \mathfrak{h}/W$ (as in corollary 4.2) the restriction of χ to $D \cap \chi^{-1}(U)$ is a fibration too. For a fixed point $u \in U$ we get an inclusion

$$j : F := \chi^{-1}(u) \cap D \to \chi^{-1}(u) \cap D' =: F'$$

which induces a homomorphism

$$j^* : H^*(F', \mathbb{Z}) \to H^*(F, \mathbb{Z})$$

on the cohomology.

Lemma *The homomorphism j^* is W-equivariant.*

Proof: The natural restriction from the constant sheaf $\mathcal{C}' = \mathbb{Z}_{D' \cap \chi^{-1}(U')}$ on $D' \cap \chi^{-1}(U')$ to the constant sheaf $\mathcal{C} = \mathbb{Z}_{D \cap \chi^{-1}(U)}$ on $D \cap \chi^{-1}(U)$ induces a homomorphism of the corresponding higher direct images

$$R^* \chi_* \mathcal{C}' \to R^* \chi_* \mathcal{C}.$$

As these are locally constant on U, the homomorphism is given by a W-equivariant homomorphism of the stalks (over u)

$$H^*(F', \mathbb{Z}) \to H^*(F, \mathbb{Z})$$

which is just j^*. □

In the following we fix a Borel subgroup B of G and hence a simultaneous resolution for $\chi : \mathfrak{g} \to \mathfrak{h}/W$. The identification of $H^*(\mathcal{B}_x)$ with $H^*(F)$ (via 4.2 prop. and 4.3 prop.1) is then uniquely determined as well as the monodromy action of W on $H^*(\mathcal{B}_x)$. We regard $\mathcal{B}$ as $\mathcal{B}_0$.

Proposition *Let $i^* : H^*(\mathcal{B}, \mathbb{Z}) \to H^*(\mathcal{B}_x, \mathbb{Z})$ be the homomorphism induced by the inclusion $i : \mathcal{B}_x \to \mathcal{B}$. Then i^* is W-equivariant. Its image lies in the C-fixed part of $H^*(\mathcal{B}_x)$.*

Proof: We will apply the above lemma with $x' = 0$. Then j^* takes the form (we use the previous notations)

$$H^*(\chi^{-1}(u), \mathbb{Z}) \to H^*(\chi^{-1}(u) \cap D, \mathbb{Z})$$

which after using the simultaneous resolution

$$\begin{tikzcd} G \times^B \mathfrak{b} \arrow[r, "\psi"] \arrow[d, "\theta"'] & \mathfrak{g} \arrow[d, "\chi"] \\ \mathfrak{h} \arrow[r, "\phi"] & \mathfrak{h}/W \end{tikzcd}$$

for the fibres over $u \in \mathfrak{h}/W$ and $h \in \phi^{-1}(u) \subset \mathfrak{h}$ may be replaced by

$$H^*(\theta^{-1}(h), \mathbb{Z}) \to H^*(\theta^{-1}(h) \cap \psi^{-1}(D), \mathbb{Z}).$$

As the higher direct images $R^* \theta_* \mathbb{Z}_X$ for $X = G \times^B \mathfrak{b}$ or $\psi^{-1}(D) \cap \theta^{-1}(H)$ are constant on a sufficiently small neighbourhood H of $0 \in \mathfrak{h}$ (cf. 4.2 prop.) we may even replace h by 0, i.e. identify j^* with

$$H^*(G \times^B \mathfrak{n}, \mathbb{Z}) \to H^*((G \times^B \mathfrak{n} \cap \psi^{-1}(D)), \mathbb{Z})$$

which because of the commutative diagram

$$\begin{tikzcd} G \times^B \mathfrak{n} \arrow[d] & (G \times^B \mathfrak{n}) \cap \psi^{-1}(D) \arrow[l, hook'] \\ G/B \arrow[d, dash, "\wr"] & \psi^{-1}(x) \arrow[u, hook] \arrow[d, dash, "\wr"] \\ \mathcal{B} & \mathcal{B}_x \arrow[l, hook'] \end{tikzcd}$$

whose vertical arrows are homotopy equivalence, is nothing else but i^*.

As the centralizer $Z_G(x)$ of x operates on $\mathcal{B}$ as a subgroup of the connected group G, the action of $C = Z_G(x)/Z_G(x)^0$ on $H^*(\mathcal{B})$ is trivial. As i^* is obviously C-equivariant, the image of i^* lies in $H^*(\mathcal{B}_x, \mathbb{Z})^C$. □

Remark In the case $x' \neq 0$ we still may rewrite j^* as $H^*(\mathcal{B}_{x'}) \to H^*(\mathcal{B}_x)$ by homotopy equivalence, but in general it will not be induced by an inclusion, we only have $\lim_{x \to x'} \mathcal{B}_x \subset \mathcal{B}_{x'}$.

Corollary *Let $x \in \mathfrak{g}$ be a subregular nilpotent element of a simple Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is of type $\mathsf{A}_r, \mathsf{D}_r, \mathsf{E}_r$ (resp. $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$), the monodromy representation on $H^2(\mathcal{B}_x, \mathbb{Q})$ (resp. on the C-fixed part $H^2(\mathcal{B}_x, \mathbb{Q})^C$) is the natural representation of W.*

Proof: From the proposition above we obtain a W-equivariant homomorphism

$$i^* : H^2(\mathcal{B}, \mathbb{Q}) \to H^2(\mathcal{B}_x, \mathbb{Q}).$$

We know that $\dim H^2(\mathcal{B}) = \dim H^2(\mathcal{B}_x) = \operatorname{rank} \mathfrak{g} = r$ in the cases $\mathsf{A}_r, \mathsf{D}_r, \mathsf{E}_r$ and $\dim H^2(\mathcal{B}) = \dim H^2(\mathcal{B}_x)^C = \operatorname{rank} \mathfrak{g} = r$ in the cases $\mathsf{B}_r, \mathsf{C}_r, \mathsf{F}_4, \mathsf{G}_2$. Furthermore W acts on $H^2(\mathcal{B})$ irreducibly via the natural representation. Hence we are done if we know that i^* is not zero. But this follows from the general fact that the fundamental class of the closed subvariety $\mathcal{B}_x \subset \mathcal{B}$ of the projective manifold $\mathcal{B}$ is non-zero in $H_2(\mathcal{B})$. □

4.6 A connection to Springer's representations

Weyl group module structures on the ordinary and l-adic cohomology groups of $\mathcal{B}_x$ were recently introduced by T.A. Springer from a completely different point of view (cf. [Sp4], [Sp5]). In [Sp4] they appear in the process of determining the Green functions (equivalently the complex characters) for reductive groups over finite fields. In [Sp5] a characteristic zero definition is given and the independence of the module structure with respect to the base field is shown. Our definition of monodromy representations was actually motivated by the existence of Springer's representations. Though in the present state we are not able to derive a complete equivalence of both definitions, we will at least give an argument (due to G. Lusztig) showing a partial coincidence. For this we do not need the explicit definition of Springer's representations (for which we refer to [Sp4] and [Sp5]). We only use the following two basic properties.

Proposition ([Sp4] 7.2) *Let $x = 0$. Then Springer's representation on $H^*(\mathcal{B}_0, \mathbb{Q}) \simeq H^*(G/T, \mathbb{Q})$ tensorized with the sign representation of W is the same as that induced by the action of W on G/T as the group of G-automorphisms.*

Remark A proof different from that in [Sp4] is obtained by comparing the natural W-actions on $H_c^*(G/T, \mathbb{Q})$ (which coincides with Springer's) and on $H^*(G, T, \mathbb{Q}) \simeq H^{*-2d}(\mathcal{B}_0, \mathbb{Q})$ (where $d = \dim_{\mathbb{C}} \mathcal{B}_0$).

The next result is the counter part of our specialization lemma 4.5.

Theorem ([H-Sp] 1.1) *Let x be an arbitrary nilpotent element and $i : \mathcal{B}_x \to \mathcal{B}$ the natural inclusion. Then the induced map $i^* : H^*(\mathcal{B}, \mathbb{Q}) \to H^*(\mathcal{B}_x, \mathbb{Q})$ is equivariant for Springer's W-representations.*

Combining now the coincidence (up to sign) of both representations on $H^*(\mathcal{B}, \mathbb{Q})$ and the corresponding specialization lemmata we obtain the following partial identification.

Corollary *Up to tensorization by sign the monodromy and Springer's representation coincide on the image $\operatorname{Im} i^* \subset H^*(\mathcal{B}_x)$ of i^*.*

Corollary *Let $\mathfrak{g}$ be of type A_r. Then the monodromy and Springer's representation coincide (up to sign) for all nilpotent elements $x \in \mathfrak{g}$.*

Proof: It is known (cf. [H-Sp] 2.3) that in this case i^* is surjective. One can also show that $i^{2d_x} : H^{2d_x}(\mathcal{B}) \to H^{2d_x}(\mathcal{B}_x)^C$ is always surjective (cf. [H-Sp] 1.3) by using the following important result of Springer concerning his representations. As well as in the monodromy case these representations commute with the natural centralizer action. Denote by $\mathcal{N}$ a set of representatives of the nilpotent classes in $\mathfrak{g}$.

Theorem ([Sp4] 6.10, [Sp5] 1.13) *The C-isotypic components in $H^{2d_x}(\mathcal{B}_x, \mathbb{Q})$ are irreducible for $C \times W$. For any irreducible representation ρ of W there is a unique $x \in \mathcal{N}$ and a unique C-isotypic component of $H^{2d_x}(\mathcal{B}_x, \mathbb{Q})$ in which ρ occurs.*

For more details on the structure of $\mathcal{B}_x$ and the W-representation on $H^{2d_x}(\mathcal{B}_x)$ we refer to [H-Shi], [H-Sp], [Sh2], [Sh3], [S1], [S3], [St4].

We hope to come back to the monodromy representations at another occasion dealing with more aspects like vanishing and diminishing cycles, the intersection form and the W-action on $H^{2d_x}(\mathcal{B}_x, \mathbb{Q})$, the definition in l-adic cohomology and applications to the Frobenius-action. And maybe the following conjecture will then have found a positive answer:

Up to tensorization by sign the monodromy and Springer's representation coincide.

References

[A-F] Andreotti, A.; Frankel, T.: The lefschetz theorem on hyperplane sections, Annals of Math., 69(1959), 713-717.

[A1] Arnol'd, V.I.: On matrices depending on parameters, Russian Math., Surveys 26, 2(1971), 29-43.

[A2] Arnol'd, V.I.: Normal forms for functions near degenerate critical points, the Weyl groups of $\mathsf{A}_k, \mathsf{D}_k, \mathsf{E}_k$ and Lagrangian singularities, Funct. Anal. Appl. 6(1972), 254-275.

[Ar1] Artin, M.: On isolated rational singularities of surfaces, Am. Journ. of Math. 88(1966), 23-58.

[Ar3] Artin, M.: An algebraic construction of Brieskorn's resolutions, Journ. of Algebra 29(1974), 330-348.

[B-G-G] Bernstein, I.N.; Gel'fand, I.M.; Gel'fand, S.I.: Schubert cells and the cohomology of the space G/P, Russian Math. Surveys 28, 3(1973), 1-26.

[Br] Borel, A.: Linear Algebraic Groups, Benjamin, New York, 1969.

[Br1] Brieskorn, E.: Ueber die Auflösung gewisser Singularitäten von holomorphen Abbildungen, Math. Ann. 166(1966), 76-102.

[Br2] Brieskorn, E.: Rationale Singularitäten komplexer Flächen, Invent. Math. 4(1968), 336-356.

[Br3] Brieskorn, E.: Die Auflösung der rationalen Singularitäten holomorpher Abbildungen, Math. Ann. 166(1966), 76-102.

[Br4] Brieskorn, E.: Singular elements of semisimple algebraic groups, in: Actes Congrès Intern. Math. 1970, t.2, 279-284.

[De3] Demazure, M.: Désingularisation des Variétés de Schubert generalisées, Ann. Scient. Ec. Norm. Sup. 7(1974), 53-88.

[Dy] Dynkin, E.B.: Semisimple subalgebras of semisimple Lie algebras, A.M.S. Translations, Ser. 2, 6(1957), 111-245.

[E] Ehresmann, Ch.: Sur les espaces fibrés différentiables, C.R. Acad. Sci. Paris 224(1947), 1611-1612.

[El] Elkington, G.B.: Centralizers of unipotent elements in semisimple algebraic groups, Journ. of Algebra 23(1972), 137-163.

[Es] Esnault, H.: Singularités rationelles et groupes algébriques Thèse de 3ème cycle, Paris VII, 1976.

[Go] Godement, R.: Topologie Algébrique et Théorie des Faisceaux, Hermann, Paris, 1964.

[H-Shi] Hotta, R.; Shimomura, N.: The fixed point subvarieties of unipotent transformations and the Green functions - combinatorial and cohomological treatments centering GL_n, Math. Ann. 241(1979), 193-208.

[H-Sp] Hotta, R.; Springer, T.A.: A specialization theorem for certain Weyl group representations and an application to the Green polynomials of unitary groups, Invent. Math. 41(1977), 113-127.

[Hui] Huikeshoven, F.: On the versal resolutions of deformations of rational double points, Invent. Math. 20(1973), 15-33.

[K-S] Kas, A.; Schlesinger, M.: On the versal deformation of a complex space with an isolated singularity, Math. Ann. 196(1972), 23-29.

[Ki] Kirby, D.: The structure of an isolated multiple point of a surface I, II, III, Proc. Londan Math. Soc. (3), 6(1956), 597-609, 7(1957), 1-28.

[Kl] Klein, F.: Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade, Teubner, Leipzig 1884.

[Ko1] Kostant, B.: The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. of Math. 81(1959), 973-1032.

[Ko2] Kostant, B.: Lie group representations on polynomial rings, Amer. J. of Math. 85(1963), 327-404.

[La] Lamotke, K.: Die Homologie isolierter Singularitäten, Math. Zeitschrift 143(1975), 27-44.

[LIE] Bourbaki, N.: Groupes et algèbres de Lie, I-VIII, Hermann, Paris, 1971, 1972, 1968, 1975.

[Lo] Lojasiewicz, S.: Triangulation of semi-analytic sets, Annali Scu. Norm. Sup. Pisa 3-18, 4(1964), 449-474.

[Mi] Milnor, J.: Singular Points of Complex Hypersurfaces, Annals of Math. Studies No. 61, Princeton, 1968.

[Mu] Mumford, D.: Geometric Invariant Theory, Springer, Berlin, 1965.

[Mu2] Munford, D.: The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. I.H.E.S. No. 9, Paris, 1961.

[Sa] Saito, K.: Einfach-elliptische Singularitäten, Inventiones Math. 23(1974), 289-325.

[Sh2] Shoji, T.: On the Springer representations of the Weyl groups of classical algebraic groups, Preprint, Science University of Tokyo, Noda, Chiba.

[Sh3] Shoji, T.: On the Springer representations of Chevalley groups of type F_4, Preprint, Science University of Tokyo, Noda, Chiba.

[Si] Siersma, D.: Classification and Deformation of Singularities, Thesis, Amsterdam, 1974.

[Sl] Slodowy, P.: Einfache Singularitäten und einfache algebraische Gruppen, Regensburger Math. Schriften 2, 1978.

[S1] Spaltenstein, N.: The fixed point set of a unipotent transformation on the flag manifold, Proc. Kon. Ak. v. Wet. 79(5) (1976), 452-456.

[S2] Spaltenstein, N.: On the fixed point set of a unipotent element on the variety of Borel subgroups, Topology 16(1977), 203-204.

[S3] Spaltenstein, N.: Sous-groupes de Borel conténant un unipotent donné, Preprint, Warwick University.

[Spa] Spanier, E.H.: Algebraic Topology, McGraw-Hill, New York, 1966.

[Sp1] Springer, T.A.: Some arithmetical results on semisimple Lie algebras, Publ. Math. I.H.E.S. 30(1966), 115-141.

[Sp2] Springer, T.A.: The unipotent variety of a semisimple group, Proc. Bombay Colloqu. Alg. Geometry, ed. S. Abhyankar, London, Oxford University Press, 1969, 373-391.

[Sp3] Springer, T.A.: Invariant theory, Lecture Notes in Math. 585, Springer, Berlin-Heidelberg-New York, 1977

[Sp4] Springer, T.A.: Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Invent. Math. 36(1976), 173-207.

[Sp5] Springer, T.A.: A construction of representations of Weyl groups, Invent. Math. 44(1978), 279-293.

[St1] Steinberg, R.: Regular elements of semisimple algebraic groups, Publ. Math. I.H.E.S. 25(1965), 49-80.

[St2] Steinberg, R.: Conjugacy classes in algebraic groups, Lecture Notes in Math. 366, Springer, Berlin-Heidelberg-New York, 1974.

[St4] Steinberg, R.: On the desingularization of the unipotent variety, Invent. Math. 36(1976), 209-224.

[Tj1] Tjurina, G.N.: Locally semiuniversal flat deformations of isolated singularities of complex spaces, Math. USSR Izvestija, Vol. 3, No. 5, (1969), 967-999.

[Tj2] Tjurina, G.N.: Resolutions of singularities of flat deformations of rational double points, Funct. Anal. Appl. 4, (1970), 68-73.

[Ve] Veldkamp, F.D.: The center of the universal enveloping algebra of a Lie algebra in characteristic p, Ann. Scient. Ec. Norm. Sup. 4(1972), 217-240.

To facilitate cross references to [Sl] we have kept the same system of indexing the literature. A revised and enlarged English version of [Sl] will appear in the Springer series Lecture Notes in Mathematics. ([Sl] was later published as [Slodowy, P.: Simple singularities and simple algebraic groups, Lecture Notes in Math. 815, Springer, 1980])

The reference [S-S] was left out in the original document. It should be the following

[S-S] Springer, T.A.; Steinberg, R.: Conjugacy classes, in: Seminar on Algebraic Groups and Related Finite Groups, Borel et al ed., Lecture Notes in Math. 131, Springer-Verlag, Berlin-New York, 1970.