Homework 10s

Submission instructions.

- Submissions are due on Thursday 11/14 at 7.00pm ET == 4.00pm PT
- Please upload scans of your solution in GradeScope (via Canvas)
- Please ensure that your scans are readable.

Instructions

- Please show all necessary steps to get the final answer. However, there is no need to be overly elaborate. Crisp and complete answers.
- For all MATLAB problems, include all code written to generate solutions.
- Please post all questions on the discussion board on the Piazza course website, rather than email-ing the course staff. This will allow other students with the same question to see the response and any ensuing discussion.
- **If you feel some information is missing, you are welcome to make reasonable assumptions and proceed. Sometimes the omissions are intentional. Needless to say, only reasonable assumptions will be accepted.**

1. *(Group sparsity)* Let $\mathbf{x} \in \mathbb{R}^N$ and let $f(x) = \|\mathbf{x}\|_2$.

   (Part a) Is $f(x)$ differentiable? Justify.

   (Part b) Derive the sub-differential of $f(x)$.

   (Part c) Show that the solution to the optimization problem

   $$\min_{\mathbf{x}} \beta\|\mathbf{x}\|_2 + \frac{1}{2}\|\mathbf{y} - \mathbf{x}\|_2^2,$$

   is

   $$\widehat{\mathbf{x}} = \frac{\mathbf{y}}{\|\mathbf{y}\|} \max(0, \|\mathbf{y}\| - \beta).$$

2. *(Orthogonality in OMP)* Consider the OMP algorithm that we derived in class, and also given below as pseudo-code

   function $x = \mathrm{OMP}(y, A, K)$
   $\Omega \leftarrow \phi$
   $r \leftarrow y$
   for $k = 1, \ldots, K$
       $j \leftarrow \arg\max_i |\langle r, a_j\rangle|$
       $\Omega \leftarrow \Omega \cup \{j\}$
       $x_\Omega \leftarrow \arg\min_\alpha \|y - A_\Omega \alpha\|^2, \qquad x_{\Omega^c} \leftarrow 0$
       $r \leftarrow y - Ax$

Show that, in any iteration, the residue r is orthogonal to the columns of A in the index set Ω.

3. *(Implement OMP)*

   (Deliverable 1) Implement OMP. We expect a function of the form

```
function [x] = OMP(y, A, K)
%Implements OMP to solve: min ||y - Ax||^2  st. ||x||_0 \le K
% Input
%     y - vector of size Mx1
%     A - matrix of size MxN
%     K - sparsity of solution
% Output
%     x  - K-sparse vector of size Nx1
%
```

   Next, in `hw10.mat`, you can find a matrix A, a vector y and a scalar K. Test your code on these. Note that columns of A are not normalized.

   (Deliverable 2) Plot the solution vector x. Also list the estimated support of the solution.

4. **(0.1 points)** How many hours did this homework take?