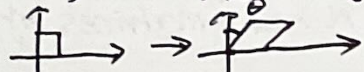
Stretching transformation: $S^{\alpha}_{\mathbf{u}}: \mathbf{v} \to \mathbf{v} + (\alpha-1)(\mathbf{u}\cdot\mathbf{v})\mathbf{u}$. Eigenvalues: α, 1.

Shearing transformation: $\begin{bmatrix} 1 & \tan\theta \\ 0 & 1 \end{bmatrix}$ 

Rotating transformation: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. θ anti-clockwise.

The order of rotations matter.

$A_{parallelogram} = |\mathbf{u}\times\mathbf{v}|$, $A_{parallelopiped} = |\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}|$

$(T\mathbf{i})\times(T\mathbf{j}) = \det(T)\mathbf{k}$ given T is a 2D transformation.

$\mathbf{i}\times\mathbf{i} = 0$, $\mathbf{i}\times\mathbf{j} \neq \mathbf{j}\times\mathbf{i}$.

$(T\mathbf{i})\times(T\mathbf{j})\cdot(T\mathbf{k}) = \det(T)$ given T is a 3D transformation.

$\det(TS) = \det(ST)$, $\det(M^T) = \det(M)$, $\det(cM) = (\det(M))^c$.

Orthogonal Matrix $MM^T = I$. $\Rightarrow \det(M) = \pm 1$

A singular LT: $\det(T) = 0$, maps 2 different vectors to one.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Steps to find inv of M:

1. Work out cofactor matrix C.
2. Do piece-wise multiplication $\begin{matrix} + & - & + & \cdots \\ - & + & - & \\ + & - & + & \\ \vdots & & & \end{matrix}$ on C.
3. Take the transpose.
4. Divide by $\det(M)$.

For a linear system $M\mathbf{r} = \mathbf{a}$,

1. If $\det(M) \neq 0$, then there is exactly one solution.
2. If $\det(M) = 0$, then there is exactly zero solutions or many.

Eigenvalues and eigenvectors can be complex. How to find eigenvectors:

Solve $(A - \lambda I)\mathbf{u} = \mathbf{0}$. There are many solutions

Let $\mathbf{u} = \begin{pmatrix} P_{11} \\ P_{21} \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}$, $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$. Two interpretations.

1. P tells the new bases corresponding to $\mathbf{i}$ and $\mathbf{j}$ if we right multiply $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
2. P tells the representation in $\mathbf{i}$ and $\mathbf{j}$ if we right multiply the representation /components in $\mathbf{u}$ and $\mathbf{v}$.

Under P, $\begin{bmatrix} a \\ b \end{bmatrix}_{\mathbf{i}\mathbf{j}} = P\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{\mathbf{u},\mathbf{v}}$, therefore, $\mathbf{c}$ changes to $P^{-1}\mathbf{c}$ if $\mathbf{c}$ is a column
$\mathbf{r}$ changes $\mathbf{r}P$ if $\mathbf{r}$ is a row vector.

$T\mathbf{o}_{ij} = \mathbf{n}_{ij} \Rightarrow P^{-1}TPP^{-1}\mathbf{o}_{ij} = P^{-1}\mathbf{n}_{ij} = P^{-1}TP\mathbf{o}_{uv} = \mathbf{n}_{uv}$. $\Rightarrow T$ changes to $P^{-1}TP$

Let $D = \begin{pmatrix} \lambda_1 & \cdots \\ \cdots & \lambda_2 \end{pmatrix}$, $T\mathbf{o}_{ij} = \mathbf{n}_{ij} \Rightarrow D\mathbf{o}_{uv} = \mathbf{n}_{uv}$ given $\mathbf{u}$ and $\mathbf{v}$ are bases $\Rightarrow D = P^{-1}TP$

$Tr(NM) = Tr(MN)$ given that NM and MN are valid. $\Rightarrow Tr(T) = Tr(D) = \sum_i \lambda_i$. even if T is not diagonalizable. Similarly, $\det(T) = \prod_i \lambda_i$

Examples of vectors (spaces) (things that can be added/subtracted and multiplied by a scalar: $\mathbb{R}$, $\mathbb{C}$, {matrices of the same size}, {polynomials of the same degree},

Definition of a vector space: Commutative addition, associative addition, additive identity, additive inverse, multiplicative identity, multiplication is distributive in both ways. These depends on the choice of $\mathbb{F}$.

$\{\underset{\sim}{0}\}$ is a subspace

If U_1 and U_2 are subsets of V, then their sum is U_1+U_2. If they are subspaces of V, and $U_1 \cap U_2 = \{0\}$, then their direct sum is $U_1 \oplus U_2$.

The expression for any $\underset{\sim}{u} \in U_1 \oplus U_2$ is unique. But $\underset{\sim}{u}$ may be in neither of U_1, U_2.

In $\mathbb{F}^n$, $\mathbb{F}^n = U \oplus U^\perp$ where U is a subspace and $U^\perp$ is the set of all vectors that are perp. to every $\underset{\sim}{u} \in U$

Let U and V be vector spaces, $\Phi: U \to V$ is a vector homomorphism if it satisfies $\forall \underset{\sim}{u}, \underset{\sim}{v} \in U, a \in \mathbb{F}$, $\Phi(\underset{\sim}{u}+\underset{\sim}{v}) = \Phi(\underset{\sim}{u}) + \Phi(\underset{\sim}{v})$ and $\Phi(a\underset{\sim}{u}) = a\Phi(\underset{\sim}{u})$.

Φ is an isomorphism if it is bijective.

$\mathbb{F}^n \oplus \mathbb{F}^m \leftrightarrow \mathbb{F}^{n+m}$.

A vector space is isomorphic to itself in many ways if dim of it is ≥ 1.

A vector space is finite-dimensional if there is an isomorphism from it to $\mathbb{F}^n$ for some finite integer n.

E.g. $\mathbb{C}$ is one-dim over complex numbers, but two-dim over real numbers.

The span of some set of vectors is all of their linear combinations $a^i \underset{\sim}{v}_i$.

Let V be a vector space, if $|V|>1$, then V has many bases.

Tricks:

$A+A^T$ is symmetric. $A-A^T$ is antisymmetric ($A=-A^T$)

Tr is also a linear transformation.

$e^B = \sum_{i=0}^{\infty} \frac{B^i}{i!}$, $e^{\theta A} = I\cos\theta + A\sin\theta$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\tan\theta} = \begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$.

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

If $T\underset{\sim}{x} = S\underset{\sim}{x}$ for all $\underset{\sim}{x}$, then $T=S$.

$\det(T)=0 \Leftrightarrow \operatorname{rank}(T), \dim(\operatorname{col}(T)), \dim(\operatorname{row}(T)) <$ size of T.

To check if $\underset{\sim}{u}$ lies in a plane spanned by $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$, check $\underset{\sim}{u} \cdot (\underset{\sim}{v}_1 \times \underset{\sim}{v}_2)$.

Check what a LT does to canonical bases.

If you need a scalar, set up $\mathbb{R}$ $\mathbb{C}$.

$\det(e^B) = \det(e^D) = \prod_i e^{\lambda_i} = e^{\sum_i \lambda_i} = e^{\operatorname{Tr}(B)}$. even if B is not diagonalizable

$e^D = \begin{pmatrix} e^{D_{11}} & \cdots \\ \cdots & e^{D_{22}} \end{pmatrix}$

To prove if 2 subspaces are different, find a vector that is in V but not in U. Usually have sth to do with perp./etc.

$\int_0^\pi \sin(nx)\sin(mx)\,dx = 0$ for $n \neq m$

A basis is an isomorphism that sends a vector in $\mathbb{F}^n$ to V.

Spanning $\Leftrightarrow$ Surjective.
Linearly Dependent $\Leftrightarrow$ Injective.

If U_1 and $U_2 \in V$, $U_1 \cap U_2 = \{0\}$, then
$\dim(U_1 \oplus U_2) = \dim(U_1) + \dim(U_2)$.

If $U \in V$, then $\exists W \in V$ s.t. $V = U \oplus W$

Let $U \in V$, then $\mathbb{F}^n = z^{-1}(U) \oplus X$ and
$V = z(\mathbb{F}^n) = U \oplus z(X)$.

A mapping T is linear if:
① $T(u+v) = T(u) + T(v) \quad \forall u, v \in V$.
② $T(\alpha u) = \alpha T(u) \quad \forall \alpha \in \mathbb{F}, \forall u \in V$

$\mathcal{L}(V, W)$ contains non-isomorphisms from V to W

Change of Basis
If z and y are bases of V, let $P = z^{-1} \circ y$. Then all other bases can be expressed in the form $z \circ P$, where P is an isomorphism from $\mathbb{F}^n$ to itself.

Dual Space
$\mathcal{L}(V, \mathbb{F})$ is the dual space to V

$u \to \Delta u \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$

Consider V, the linear mapping that sends every $x \in V$ to $0 \in V$ is the zero vector in $\mathcal{L}(V, V)$.

The mapping that sends every $v \in V$ to itself is the (additive) identity vector of $\mathcal{L}(V,V)$.

Consider T:

Nullity of $T = \mathrm{Null}(T) = \dim(\mathrm{Ker}(T)) = \dim\{v \in V : T(v) = 0\}$

Rank of $T = \mathrm{Rank}(T) = \dim(\mathrm{Rang}(T)) = \dim(T(V))$

T is injective $\iff \mathrm{Null}(T) = 0 \iff \mathrm{Ker}(T) = \{0\}$.

T is surjective $\iff \mathrm{Rank}(T) = \dim(W) \iff \mathrm{Rang}(T) = W$

The Fundamental Theorem of Linear Transformations.

$\dim(V) = \mathrm{Null}(T) + \mathrm{Rank}(T)$.

does not depend on T

$\therefore \mathrm{Rank}(T) \leq \dim(V), \dim(W)$

Consider $T: V \to V$

Bijection $\iff$ Injection $\iff$ Surjection.

$I^i_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{~~otherwise~~} \end{cases}$.

Dual Basis

Given a basis $z_i \in V$, ζ^i is a dual basis of z_i if $\zeta^i z_j = I^i_j$

Dual Vector:

$p_j \zeta^j$

Showing Dual Basis is a Basis.

Linear Independence:

Let $p_i \zeta^i = 0 \Rightarrow p_i \zeta^i(z_j) = p_i I^i_j = p_j = 0$.

Spanning:

$\beta = (\alpha(z_i))(\zeta^i)$

$\beta(z_j) = (\alpha z_i)(\zeta^i(z_j)) = \alpha z_i I^i_j = \alpha(z_j) \Rightarrow \alpha = \beta$.

$\dim(\hat{V}) = \dim(V)$.

There is only one dual basis to a basis of V.

Extracting Components.

$\zeta^i(a^j z_j) = a^i$

$p_i \zeta^i(z_j) = p_j$.

If $z: F^n \xrightarrow{iso} V$ $\quad \zeta: \hat{F}^n \xrightarrow{iso} \hat{V}$

$a \in F^n$ $\quad p \in \hat{F}^n$, then

$\zeta p(z a) = p a$.

Proof: $a = a^i e_i$, $p = p_j \varepsilon^j$.

$\therefore \zeta(p_j \varepsilon^j)(z(a^i e_i))$

$= p_j \zeta^j (a^i z_i)$

$= a^i p_j \zeta^j z_i = a^i p_j I^i_j = p a$.

Transpose.

Let $T: V \to W$

Then $\hat{T}: \hat{W} \to \hat{V}$ s.t. $\forall \alpha \in \hat{W}, v \in V$

$\alpha(T(v)) = \hat{T}(\alpha)(v)$.

Tensor Products

$(w \otimes \alpha)(v) = \alpha(v) w$, $\quad \alpha \in \hat{V}, v \in V, w \in W$.

Not Commutative! Linear in both slots.

$\beta \to \beta \circ (w \otimes \alpha)$ is a mapping from $\hat{W}$ to $\hat{V}$

and is the transpose of $w \otimes \alpha$.

$\beta \circ (w \otimes \alpha) = \beta(w) \alpha$.

$\therefore w \otimes \alpha \in \mathcal{L}(V, W), \mathcal{L}(\hat{W}, \hat{V})$.

The full set of $z_j \otimes \zeta^i$ is a basis for $\mathcal{L}(V, V)$.

Spanning:

Let $S = T^j_i z_j \otimes \zeta^i$

Then $S(z_k) = T^j_i z_j \cdot \zeta^i(z_k) = T^j_i z_j I^i_k = T^j_k z_j = T(z_k)$

Linearly Indenpendence:

$T^j_i z_j \otimes \zeta^i = 0 \Rightarrow \zeta^h(T^j_i z_j \otimes \zeta^i(z_k)) = T^k_h = 0$.

When defining a LT from V to W, the dual basis should be relative to the source space.

$\dim(\mathcal{L}(V,V)) = (\dim(V))^2$ $\quad$ $\dim(\mathcal{L}(V,W)) = \dim(V)\dim(W)$

Extracting Components.

T^j_i = the component of z_j that one z_i will turn into for each j.

$\therefore T(z_i) = \sum_j T^j_i (z_j)$

Matrix Multiplication.

$MN = M^i_j N^j_k$ (first down, then up).

$\alpha M = \alpha_j M^j_i$ if α is a row vector.

$Mv = M^j_i v^i$ if v is a column vector.

Associative Algebra.

Products of $n\times n$ square matrices are again $n \times n$ matrices. Square matrices form an associative algebra: a vector space with a multiplication that takes pairs of vectors back into the vector space.

If T is $\in \mathcal{L}(V,V)$, and it can be expressed as $T^j_i z_j \otimes \zeta^i$ for some basis, then T^j_i is the matrix of T relative to the basis z_j. It is possible that T can have different matrices relative to different bases.

$\zeta^i(T(z_j))$ is the (i,j) entry of T.

Let $S: U \to V$, $T: V \to W$. x: basis of U, $y \in V$, $z \in W$.
Then:

$M_{x,z}(TS) = M_{y,z}(T) M_{x,y}(S)$ order matters.

Change of Basis II.

Fix a basis $z: F^n \to V$. $\forall v \in V$, $\exists v^* = z^{-1}(v) \in F^n$ basically the comp.s

$\forall \alpha \in \hat{V}$, $\exists \alpha^* = \alpha \circ z \in \hat{F}^n$

$\forall T \in \mathcal{L}(V,V)$, $\exists T^* = T^i_j z^*_i \otimes \alpha^{*j} = z^{-1} \circ T \circ z \in \mathcal{L}(F^n, F^n)$

$(P^{-1})^{j}_{i} P^{i}_{k} = I^{j}_{k}$.

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T is the matrix relative to both the canonical basis in F^n and a basis z in V.

If z changes to $y = z \circ P$, then.
v^* changes to $P^{-1} \circ z^{-1} v = P^{-1} v^*$.
α^* changes to $\alpha^* P$.
T^* changes to $P^{-1} T^* P$.
} basically the components.

$T z_k = \sum_j T^j_k z_j$, $\hat{T} \zeta^k = \sum_k \hat{T}^k_i \zeta^k = \sum_k T^i_k \zeta^k$

$\therefore z_j \hat{T} \zeta^i = \sum_k \hat{T}^k_i \zeta^k z_j = \hat{T}^j_i$
$\quad \|$
$\zeta^i T z_j = T^i_j$

Trace II.
① $\sum M^i_i$
② characterised by being zero when evaluated to all commutators and being equal to n when evaluated to I_n.
③ $\mathcal{I} : v \otimes \alpha \mapsto \alpha(v)$

When prooving two things are the same or there is only one thing that does something:
Consider the behaviour.

Compositions of LTs are still linear.
To prove an isomorphism: ① find its inverse ② Inj. & Sur.
E.g. $Z: M \mapsto z \circ M \circ z^{-1}$ and $X: T \mapsto z^{-1} \circ T \circ z$ are inverse.

Change of Basis III.
$y_i = P^j_i z_j$.
$v = a^i z_i$

$z_j \to y_j$
$v \to (P^{-1})^j_i a^i y_j$
$\zeta^i \to \sum_j (P^{-1})^j_i \zeta^i = \eta^j$
$T^i_j \to (P^{-1})^h_i T^i_j P^j_k$.

$$Tz_i = \sum_j T_i^j z_j.$$

$$\hat{T}\xi^i = \sum_j \hat{T}_j^i \xi^j$$

MA2101 Notes II.

Chap 4: Operators: $T: V \to V$.

A subspace $U \subseteq V$ is an <u>Invariant Subspace</u> with respect to $T \in \mathcal{L}(V, V)$ iff $Tu \in U$ $\forall u \in U$

The kernel and the range are always invariant w.r.t T.

Eigen-

Eigenray: one-dimensional invariant subspace.

Eigenvector: non-zero elements of an eigenray.

- Only one eigenvalue for one eigenray.

Fact 4.17 "Never parallel"

- If 2 eigenrays have different eigenvalues, vectors from them are never parallel.

Fact 4.19 "Always linearly independent"

- If any number of eigenrays have all different eigenvalues then a set of eigenvectors, one from each eigenray, is always linearly independent.

Fundamental Theorem of Algebra.

① Any polynomial with complex coefficients can be completely factorised, that is, it can be written in the form $a(x-a_1)(x-a_2)\dots$ where a, a_i are complex numbers.

This factorisation is unique, though up to ordering.

② Any polynomial with real coefficients can be expressed in the form $a(x-a_1)(x-a_2)\cdots(x^2+b_1x+c_1)(x^2+b_2x+c_2)\dots$ where a, a_i, b_i, c_i are real numbers, satisfying $b_i^2 < 4c_i$.

The factorisation is unique.

Fact 4.21 "At least one eigenvalue"

- Every operator on a finite dimensional vector space over the complex numbers has at least one eigenvalue. →

Does not have to be real eigenvalue.

- Use $A_0 v + \dots + A_n T^n v = 0$ and injectivity to prove.

※ Any polynomial of an upper-triangular matrix is also upper triangular.
※ The inverse of a UTM is also a UTM.

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Upper-Triangular Matrix.

$M^i_j = 0$ if $i > j$.

Prelude of Schur's Theorem

For every vector spaces over the complex numbers, every operator has an upper-triangular matrix.

"Similar"

- A and B are similar if $\exists P$ s.t. $A = P^{-1}BP$.
- They satisfy the same set of polynomials.

Any complex operators are similar to a UTM.

Proof: ① Find a basis such that the first vector is an eigenvector. $\therefore N = \begin{pmatrix} \lambda_1 & \vec{\alpha} \\ \vec{0} & M \end{pmatrix}$

② Do the same to M s.t., $Q^{-1}MQ = \begin{pmatrix} \lambda_2 & \vec{\beta} \\ \vec{0} & L \end{pmatrix}$, this time λ_2 is M's eigenvalue.

③ Now the proof goes to show that

$$\begin{pmatrix} 1 & 0 \\ \vec{0} & Q \end{pmatrix}^{-1}\begin{pmatrix} \lambda_1 & \alpha \\ \vec{0} & M \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \vec{0} & Q \end{pmatrix} = \begin{pmatrix} \lambda_1 & \vec{\alpha}Q \\ \vec{0} & Q^{-1}MQ \end{pmatrix}.$$

$\therefore$ Doing a $\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ similarity is the same as taking $Q^{-1}MQ$ at the lower-right corner.

"Invertible Complex Matrix"

- A complex operator is invertible iff every entry down the diagonal is non-zero when it is represented in a UTM form.
- Proof: $\lambda_1 \neq 0$ otherwise N is not injective.
  $\lambda_2 \neq 0$ otherwise $Ne_2 = \underbrace{\alpha_1}_{\text{cannot be zero}} e_1$ and $Ne_1 = \lambda_1 e_1$
  $\Rightarrow Ne_1 = \frac{\lambda_1}{\alpha_1} Ne_2 \Rightarrow e_1 = \frac{\lambda_1}{\alpha_1} e_2$.

The diagonal entries in a UTM are its eigenvalues.

- "Diagonalizable"
  - An operator is diagonalizable if there is a basis with respect to which its matrix is a diagonal matrix.
  - ⇒ An operator is diagonalizable iff there is a basis consisting of eigenvectors (only!).
  - ⇒ An operator on $V \Leftrightarrow F^n$ has n distinct eigenvalues ⇒ diagonalizable.

Eigenspace.

$E(\lambda, T) = \underline{\ker}(T - \lambda I)$.

Intuitively, this is the space (including 0) from which vectors make the eigenvalue λ hold.

It can be $\{0\}$ if λ is not an eigenvalue.

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$E(\lambda_1, T) \perp E(\lambda_2, T)$ if $\lambda_1 \neq \lambda_2$. $\Rightarrow$ They are orthogonal and can form a direct sum.

Remark 4.39 "Diagonalizibility"

$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T) \iff T$ is diagonalizable.

Jordan Block $\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \Rightarrow$ only one eigenray.

Jordan Basis.

- A Jordan basis is one such that the matrix of T consists of Jordan Blocks. $\begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ & \cdots & \end{pmatrix}$ up to ordering of the basis. J_1 and J_2 can have the same λ.
- Every operator on a <u>complex</u> vector space has a unique Jordan basis
- This is called Jordan Canonical Forms (JCF).

Multiplicity_i = # times λ_i occurs down along the diagonal

Characteristic Polynomial of T:

$\chi_T(\underline{x}) = (x - \lambda_1)^{m_1} \times (x - \lambda_2)^{m_2} \times \cdots \rightarrow$ multiplicity.

↑ the variable.

Cayley-Hamilton Theorem. $\chi_T(T) = \vec{0}$. regardless of the choice of basis. Change 0 to $\vec{0}$, 1 to I.

Proof: For a Jordan block of size m, λ's multiplicity is at least m. $\Rightarrow (T - \lambda I)^m$ is a factor of $\chi_T(T)$.

- $T - \lambda I$ has at least m 0's along the diagonal.
- It sends z_i to z_{i-1}. After powering by m, $z_i, \ldots, z_{i+m-1}$ vanish to 0. This transformation is "nilpotent"
- Now we have each block vanishing to 0. When they get multiplied together, everything becomes 0.

∴ We can know other powers (including the <u>inverse</u>) by the polynomial.

↑ If $\lambda_i \neq 0 \; \forall i$.

Geometric meaning of JCF:

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix} \rightarrow \frac{1}{\lambda_1}\begin{pmatrix} 1 & \frac{1}{\lambda} & 0 \\ 0 & 1 & \frac{1}{\lambda} \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{\lambda_1}\begin{pmatrix} 1 & \tan\phi & 0 \\ 0 & 1 & \tan\phi \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \text{Shearing}$$

∴ It stretches/compresses/shears in the (1,2) plane as well as the (2,3) plane. stretch/compress by $\frac{1}{\lambda_1}$

Combination.

	Stretch/Compress	Shear	Project.	Rotate.
Non-zero real λ_i $\forall i$	✓	✓		
Real λ_i $\forall i$	✓	✓	✓	
Complex λ_i $\forall i$	✓	✓	✓	✓

Chap 5.

Use dot product to extract components: $a^i = e_i \cdot (a^i z_i)$.

Bilinear Forms.

- $g(u,v): V \times V \rightarrow \mathbb{R}$ (has to be $\mathbb{R}$). that is linear at both slots.
- The set of all bilinear forms on V is $B(V)$, which is another vector space with a basis, the full set $\xi^i \otimes \xi^j$.
- $\alpha \otimes \beta(u,v) = \alpha(u)\beta(v)$ (Take care of the order!)
- b_{ij} are the components of b relative to z_i.
- $\text{dot} = I_{ij}\, \varepsilon^i \otimes \varepsilon^j$.
- Under a change of basis $y_i = P^j_i z_j$, $\eta^j = (P^{-1})^j_i \xi^i$
  ∴ $b = P^i_k b_{ij} P^j_h\, \eta^k \otimes \eta^h$.

Inner Products.

An inner product space is a pair (V,g) consisting of a vector space V and a bilinear form g, the inner product, with the following properties:

1. Positivity: $g(u,u) \geq 0$ $\forall u \in V$. $\Rightarrow g_{ij}u^i u^j \geq 0$
2. Definiteness: $g(u,u) = 0 \Leftrightarrow u = \underline{0}$ $\forall u$. $\Rightarrow g_{ij}u^i u^j = 0 \Leftrightarrow u^i = 0$.
3. Symmetry: $g(u,v) = g(v,u) \Rightarrow g_{ij} = g_{ji}$.

- The set of inner products does not constitute a <u>subspace</u> of $B(V)$.
- The matrix g_{ij} is not singular.

Length/Norm: $|v| := \sqrt{g(v,v)}$. Angle: $\cos\theta = \frac{g(u,v)}{|u|\cdot|v|}$.

Orthogonality: $u \perp u \Leftrightarrow g(u,v) = 0$.

∴ Two vectors may not be orthogonal in another inner space.

"Orthogonal Decomposition"

$$u = \underbrace{\frac{g(u,v)}{|v|^2} v}_{u_{\parallel v}} + \underbrace{\left(u - \frac{g(u,v)}{|v|^2} v\right)}_{u_{\perp v}}.$$

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Cauchy-Schwarz Inequality.

$|g(u,v)| \leq |u||v|. \quad \forall u, v \in V.$

Proof: ① we know $|u|^2 = |u_{\parallel v}|^2 + |u_{\perp v}|^2$.

② $|u_{\parallel v}|^2 = \left(\frac{g(u,v)}{|v|^2}\right)^2 g(v,v) = \left(\frac{g(u,v)}{|v|}\right)^2$.

③ $\therefore |u|^2 \geq |u_{\parallel v}|^2 = \left(\frac{g(u,v)}{|v|}\right)^2$

- Take strict equality only when $u \parallel v$.

Triangular Inequality.

$|u+v| \leq |u| + |v|.$

- Take strict equality only when u is a non-negative multiple of u.
- $|u_{\parallel v}| = |u| \cos\theta$. given $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
  $|u_{\perp v}| = |u| \sin\theta$.

Orthonormal Bases.

- z_i, a basis on (V, g), is orthonormal $\Leftrightarrow$.
  $g(z_i, z_j) = I^i_j \quad \forall i, j$.
- g_{ij} is the identity matrix.
- let $a, b \in \mathbb{R}^n$. $g(za, zb) = g_{ij} a^i b^j = \mathrm{dot}(a,b)$ if z is an orthonormal basis.
  $\therefore |a^i z_i| = \sqrt{(a^i)^2}$ (summation of squares).
- Any list of orthonormal vectors is linearly independent. If it has $n = \dim(V)$ vectors, it becomes a basis.

Unit vector

- $\hat{v} = \frac{v}{|v|}$
- If v is a unit vector, $u_{\parallel v} = g(u,v)\, v$.

Gram-Schmidt Process.

- Given z_i, let $\begin{cases} y_i = z_i - \sum_{j=1}^{i-1} g(\hat{z}_j, z_i)\, \hat{z}_j \\ \hat{y}_i = \frac{y_i}{|y_i|} \end{cases}$ is an orthonormal basis.

- The change-of-basis matrix is a UTM!

Riesz Representation.

$\Gamma_{\underline{u}}(v) = g(v, \underline{u})$, $\forall v \in V$.

$\Gamma: u \mapsto \Gamma_u$ is linear $\in \mathcal{L}(V, \hat{V})$ which is an isomorphism

$\Downarrow$

Riesz Representation Theorem.

- Given (V, g), any dual vector α can be expressed as $T(u)$ for some unique $u \in V$ in the real case.
- A generalized way to extract components of v under a certain choice of g: Given orthonormal basis, $T(z_j)v = I_{ij}a^i = a^j$
- Let $\alpha = p_i \xi^i = \Gamma_u = \Gamma_{a^j z_j} \Rightarrow p_i b^k \xi^i(z_k) =$
  $\alpha(v) = \alpha(b^k z_k) = g_{kj} a^j b^k$, since v is arbitrary
  $\Rightarrow p_k = g_{kj} a^j$, this means that the coefficients of α are determined by g and u.
  $\Rightarrow a^j = (g)^{-1}{}_{kj} p_k$.

Dot in $\mathbb{C}^n$

$a \cdot b = a^i \bar{b}^i$, $(cu) \cdot v = c(u \cdot v)$, $u \cdot (cv) = \bar{c}(u \cdot v)$

Sesquilinear Forms.

- $s(u, v): V \times V \to \mathbb{C}$, linear in the first slot, "conjugate linear" in the second slot.
- $s(cu, v) = c\,s(u, v)$, $s(u, cv) = \bar{c}\,s(u, v)$.

Complex Inner Product.

- g satisfies Positivity, Definiteness as in the real case.
- g satisfies the Conjugate-Symmetry property.
- Pythagoras, triangular inequality, Cauchy-Schwarz, orthogonality are the same.

※ Calculations Tricks:

① $c\bar{c} = (\text{Real}(c))^2$ ② $\bar{\alpha}(v) = \overline{\alpha(v)} = \alpha(\bar{v})$

③ $c = |c| \operatorname{cis}\theta = |c| e^{i\theta} = |c|(\cos\theta + i\sin\theta)$.

④ $|cv| = |c||v|$ where $|c|$ is in the complex sense.

Riesz Representation in Complex cases.

$\Gamma: u \mapsto \Gamma_u$ is a conjugate isomorphism. s.t.

$\Gamma_{au+bv} = \bar{a}\Gamma_u + \bar{b}\Gamma_v$ $\forall u, v \in V$, $a, b \in \mathbb{C}$.

Schur's Theorem

- Given an operator T on a finite-dimensional complex inner product space, there exists an orthonormal basis w.r.t. which T has an upper triangular matrix.
- Follows from the prelude and Gram-Schmidt.

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$\boxed{\tau}$

Given T on (V, g), define
$\tau(u, v) := g(u, Tv)\ \forall u, v \in V.$
$\Rightarrow \tau_{ij} = g_{ik} \overline{T}^k_j \Rightarrow g_{ij}$ is invertible, τ and T are one-to-one

"Hermitian/Symmetric"

- A sesquilinear/bilinear form is Hermitian/Symmetric in the complex/real case if $\tau(u, v) = \overline{\tau(v, u)}\ \forall u, v \in V$.
- A linear transformation is Hermitian/Symmetric if its corresponding sesquilinear/bilinear form is Hermitian/symmetric. $\Rightarrow g(u, Tv) = \overline{g(v, Tu)} = g(Tu, v)$.
- Symmetric: $\tau_{ij} = \tau_{ji}$ Hermitian: $\tau_{ij} = \overline{\tau_{ji}}$

"τ and T"

- If we use an orthonormal basis, then the matrix of τ and T are conjugates of each other.
- If we have an orthonormal basis, $g_{ij} = I^i_j$.

"Real Eigenvalues"

- The eigenvalues of a symmetric/Hermitian LT are real.

"Complex Hermitian $\to$ Real Diagonal" Spectral Theorem☆

Any Hermitian LT has a real diagonal matrix relative to some orthonormal matrix.

$\Rightarrow$ This orthonormal basis consists of eigenvectors only.

Spectral Decomposition.

S: Hermitian sesquilinear form T: the corresponding LT.
z_i: an orthonormal basis ξ^i: the dual.
$\Rightarrow S = \lambda_1 \xi^1 \otimes \overline{\xi}^1 + \cdots + \lambda_n \xi^n \otimes \overline{\xi}^n$.

"How τ responds to a change of basis"

note the order! this is different from normal case

Given symmetric bilinear form S, we have an orthonormal basis z_j s.t. $\underline{e_i = P^j_i z_j}$ and S is diagonal under z_j.

$\therefore S = D^h_k \xi^h \otimes \xi^k = P^h_i D_{hk} P^k_j \varepsilon^i \otimes \varepsilon^j = P^T D P$

We know e_i and z_i are orthonormal $\Rightarrow S(e_i, e_j) = I^i_j = S(z_i, z_j)$

(cont.). $\Rightarrow S(P^h_i z_h, P^k_j z_k) = P^h_i I_{hk} P^k_j = P^T I P = P^T P = I$.

Orthogonal Matrix

P is orthogonal if $P^T P = I$. If P changes an orthonormal basis to another, then $P^T P = I$. If $M^T M = N^T N = I$, then $MN(MN)^T = I$

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"Properties of an orthogonal matrix"

The rows of an orthogonal matrix have length 1 and are orthogonal to each other, under dot. So applies to cols.

"$O^T D O$ and $\bar{U}^T D \bar{U}$"

- A symmetric matrix can be expressed as $O^T D O$.
- A Hermitian matrix can be expressed as $U^T D \bar{U}$ where U is a complex unitary matrix, satisfying $\bar{U}^T U = I$. Remember to apply the complex dot when checking lengths and orthogonality.
  $\Rightarrow$ Called "unitarily diagonalizable"

Conjugate Dual Vectors.

- Define $\gamma : V \to C$ which is conjugate linear. γ is a conjugate dual vector.
- $\alpha \longmapsto \bar{\alpha}$ s.t. $\bar{\alpha}(v) = \overline{\alpha(v)}$ is a mapping that eats a dual vector and produces a conjugate dual vector.
- Now $\alpha \otimes \gamma$ becomes sesquilinear.
- Any CDV can be written as $p_i \bar{\xi}^i$
  Any sesquilinear form can be written as $S_{ij} \xi^i \otimes \bar{\xi}^j$
- Under a change of basis. $\xi^i = P^i_k \eta^k$,
  $s = s_{ij} \xi^i \otimes \bar{\xi}^j = (P^i_k s_{ij} \bar{P}^j_h) \eta^k \otimes \bar{\eta}^h$
  $\therefore S \to U^T D \bar{U}$

Chap 6.

Multilinear Form.

$\underbrace{V \times V \times \cdots \times V}_{m} \to F$ which is linear in every slot.

$m \leftarrow$ called degree m.

Two-Forms.

A bilinear form ψ is a two-form if it is antisymmetric

$\psi(u, v) = -\psi(v, u)$

Wedge Product.

$\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$. is a two-form. $\alpha \wedge \alpha = 0$

Any two-forms can be written as $\psi = \psi_{ij}\, \xi^i \wedge \xi^j$

Note that $\psi_{ii}\, \xi^i \wedge \xi^j = 0$. $\psi_{ij} = \psi_{ji}$

Three-forms

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- Can be written as $\psi_{ijk}\, \xi^i \wedge \xi^j \wedge \xi^k$.
- If there are odd number switches, put a minus sign.

m-forms.

- In general, if $\dim(V) = n$, then the space of m-forms on V is C^n_m, ie, n choose m.
- The space of n-forms is one-dimensional $\Rightarrow$ they are scalar multiples of each other (except the 0 n-form).

"$\hat{T}$"

- Define $\hat{T}(\psi)(u, v, w, \dots) = \psi(Tu, Tv, Tw, \dots)$, which is a map from m-forms to m-forms.
- If Ω is an n-form, then $\hat{T}\Omega$ is a scalar multiple of Ω. $\Rightarrow$ Define $\Delta(T)\Omega = \hat{T}\Omega$, $\Delta(T)$ is the eigenvalue of $\hat{T}$ since $\hat{T}$ is one-dimensional.

※ $J = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow Jz_1 = \lambda z_1,\ Jz_2 = z_1 + \lambda z_2$.

※ Any repetition in an m-form will yield 0. For example, $\Omega(u, v, v) = 0 \Rightarrow \Omega(u, v, v+w) = \Omega(u, v, w)$.

Determinant

- $\Delta(TS) = \Delta(T)\Delta(S)$, $\Delta(T) = \Delta(M(T))$. regardless of the basis
- $\Delta(M) = \lambda_1^{m_1} \times \lambda_2^{m_2} \times \dots$ ↖ so we can use a Jordan basis.
- For a Hermitian matrix, $\Delta(M)$ is real.
- $\Delta(T) \neq 0 \Leftrightarrow T$ is bijective $\Leftrightarrow M(T)$ is invertible.
- $\Delta(T - \lambda I) = 0$ for every eigenvalue λ.
- ↖ Solve this to get λ, pick a basis to get M, then $M' = M - \lambda I$, is explicit, so use $\Delta(M')$ to work out λ_i.
- $\Delta(M^T) = \Delta(M)$
- Δ switches sign when two rows/columns swapped $\Rightarrow$ antisymmetry

"Orthonormal Basis and n-Forms" → need to have a g in place.

- If we use an <u>orthonormal basis</u> to construct an n-form, then we only have two choices ($\pm$), called the orientation, of $\pm \xi^1 \wedge \xi^2 \wedge \dots \wedge \xi^n$. Then we have an <u>oriented</u> inner product space

Volume.

- Given an oriented inner product space, a list of vectors, $u, v, w \ldots$ defines a generalized parallelopiped with volume $\omega(u, v, w \ldots)$
- In particular, if $n=3$, $\omega(u, v, w) = u \cdot v \times w$
- The volume changes by a factor given by the determinant
- The eigenvectors define a generalized parallelopiped which will change under an operator.

Tensors

Multilinear forms that produce other <u>linear</u> objects (LTs, other multilinear forms) are called tensors.

Curvature Tensors.

$$R(z_i, z_j) = R^m_{kij}\, z_m \otimes \zeta^k \in \mathcal{L}(V, V)$$

It is a two form that produces an operator.

Tricks:

$\psi(z_1, z_2, \ldots, z_n) = 1$ + $\psi(z_2, z_1, \ldots, z_k) = \pm 1$ depending on # switches.

Change of basis

$y = z \circ P \Leftrightarrow y_i = P^j_i z_j$, then

V^* changes to $P^{-1} V^*$

α^* changes to $\alpha^* P$.

T^* changes to $P^{-1} T^* P$.

ζ^i changes to $(P^{-1})^i_j \zeta^j = \eta^i$

z_i changes to y_i.

$V = \alpha^i z_i$ changes to $(P^{-1})^j_i a^i y_j$.

$\bar{\zeta}^i \to \bar{P}^i_j \bar{\zeta}^j$

write the new basis here

$Pe = \begin{pmatrix} 0 & 0 \end{pmatrix} e : P: V \to W$

$\varepsilon P^{-1} = \varepsilon \begin{pmatrix} 8 \end{pmatrix} \to$ write the vector here.

$\therefore P^{-1}: \hat{V} \to \hat{W}$

$\therefore P: V \to W$

$\hat{W} \to \hat{V}$

$g \to P^T g P$

UTM: (complex) matrices → schur's ← Orthonormal basis: All LTs.

JCF: (complex) matrices

Real Eigenvalues: Hermitian.

Diagonal Form: Hermitian

$O^T D O$: Real symmetric + ortho basis.

$U^T D \bar{U}$: Complex Hermitian + ortho basis.

Tricks:

① Find an eigenvector $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ x \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ x \end{pmatrix}$ solve for x.

② Find a change of basis → let a basis absorb coefficients in $z_i \otimes \zeta^j$.

③ ~~Find the~~ Prove CH theorem for non-diagonalizable by taking the limit of a diagonalizable.

④ $A+A^T$ is symmetric; $A-A^T$ is anti-symmetric

⑤ $e^B = \sum_{i=0}^{\infty} \frac{B^i}{i!}$ $\quad e^{\theta A} = I\cos\theta + A\sin\theta$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\tan\theta} = \begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$.

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ $\quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

⑥ If $Tx = Sx$ $\forall x$, then $T = S$.

⑦ $\dim(T) = \mathrm{Null}(T) + \mathrm{Rank}(T)$.

⑧ $\det(e^B) = \det(e^D) = \prod_i e^{\lambda_i} = e^{\mathrm{Tr}(B)}$.

⑨ $e^D = \begin{pmatrix} D_{11} & \cdots \\ \cdots & D_{nn} \end{pmatrix}$

⑩ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -I$.

⑪ Express higher powers of A in terms of lower ones.

⑫ Riesz representation: $p^j = g^{jk} a_k$ to construct a dual.

⑬ Decompose a vector w.r.t. U and $U_\perp$ or $/\!/ z_i$ and $\perp z_i$

⑭ Given an ortho basis z_i, Γ_{z_i} is the dual basis.

⑮ For a symmetric/Hermitian matrix, eigenvectors from distinct eigenvalues are ortho.

⑯ $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $Te_1 = ae_1 + ce_2$; $Te_2 = be_1 + de_2$.

$\Rightarrow \Omega(e_1, e_2) = (ad-bc)\,\Omega(e_1, e_2)$

⑰ If ω_1 and ω_2 are n-forms constructed from ortho bases, then $\omega_1 = \pm\omega_2$.

⑱ Every complex matrix is similar to its transpose.

$\Leftarrow$ J is similar to J^T by reversing order of basis.

⑲ A change-of-basis matrix from an orth to an orth is orth.

⑳ Γ_u can be expressed as $\Gamma_u(z_i)\,\xi^i = g(z_i, u)\,\xi^i$

㉑ For an Hermitian LT, the eigenvectors from different eigenvalues are orthogonal.

e.

Isometry Group.

[1] $(f\cdot g)\cdot h = f\cdot(g\cdot h)$ [2] $\exists e$, $f\cdot e = e\cdot f = f$. [3] $\forall f$, $\exists f^{-1}$, $ff^{-1} = f^{-1}f = e$

$\{T\}$ s.t. $g(Tu, Tv) = g(u, v)$.