MA 2108 Notes.

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Restrictions of Functions.

$f : A \to B$.

$f_1 : A_1 \to B$, $A_1 \subset A$

$\therefore f_1 = f|A_1$

$\therefore f_1$ is a restriction of f to A_1

1.2.1 **Well-Ordering Property of $\mathbb{N}$**

Every non-empty subset of $\mathbb{N}$ has a least element.

1.2.2 **Principle of Mathematical Induction:**

$S \subset \mathbb{N}$.

① $1 \in S$

② $\forall k \in \mathbb{N}$, $k \in S \Rightarrow k+1 \in S$.

1.3.2 **Uniqueness Theorem:**

S is a finite set $\Rightarrow$ #elements in S is a unique number in $\mathbb{N}$.

1.3.4 (c) **Subset of Infinite Set.**

C is infinite $\wedge$ B is finite $\Rightarrow$ $C \setminus B$ is infinite.

1.3.5 / 1.3.9 **"Balloon: $T \subseteq S$"**

(a) S is finite $\Rightarrow$ T is finite.

(b) T is infinite $\Rightarrow$ S is infinite.

(c) S is countable $\Rightarrow$ T is countable

(d) T is uncountable $\Rightarrow$ S is uncountable.

1.3.10 **"Countability"**

S is countable $\Leftrightarrow$ $\exists$ surjection $\mathbb{N}$ onto S

$\Leftrightarrow$ $\exists$ injection S into $\mathbb{N}$.

1.3.12 **"Union of Countable Sets"**

A_m is countable $\forall m \in \mathbb{N}$ $\Rightarrow$

$A := \bigcup_{m=1}^{\infty} A_m$ is countable.

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Bernoulli's Inequality

$x > -1 \Rightarrow$

$(1+x)^n \geqslant 1+nx \quad , \forall n \in \mathbb{N}.$

2.2.3 Triangle Inequality.

$|a+b| \leq |a| + |b|$

$|a| - |b| \leq \big||a| - |b|\big| \leq |a-b| \leq |a| + |b|$

"Sagh"

$SM \geqslant AM \geqslant GM \geqslant HM.$

Square mean $\geqslant$ arithmetic mean $\geqslant$ geometric mean $\geqslant$ harmonic mean.

2.2.7 ε-Neighbourhood.

$a \in \mathbb{R}, \ \varepsilon > 0 \Rightarrow V_\varepsilon(a) := \{x \in \mathbb{R} : |x-a| < \varepsilon\}$

2.2.8 "Self in ε-Neighbourhood"

$x \in V_\varepsilon(a) \ \forall \varepsilon > 0 \Rightarrow x = a.$

2.2.9 "Transitivity of ε-Neighbourhood"

$x \in V_{\varepsilon_1}(a) \wedge y \in V_{\varepsilon_2}(b) \Rightarrow x+y \in V_{\varepsilon_1+\varepsilon_2}(a+b).$

2.3.2(a) "Supremum"

$S \neq \emptyset, \ S \subset \mathbb{R}.$

$\left.\begin{array}{l} 1.\ u \text{ is an upper bound of } S \\ 2.\ \forall \text{ upper bound } v,\ u \leq v \end{array}\right\} \Rightarrow u \text{ is } \sup S.$

OR:

$\left.\begin{array}{l} 1.\ u \text{ is an upper bound of } S \\ 2.\ \forall v < u,\ \exists s \in S,\ v < s \end{array}\right\} \Rightarrow u = \sup S.$

Inf and sup of a set is either unique or non-existing.

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"Max and Min"

$u = \sup S \land u \in S \Rightarrow \overset{u = \max S.}{u}$ is the maximum of S,

$u = \inf S \land u \in S \Rightarrow \overset{u = \min S.}{u}$ is the minimum of S.

2.1.5 Trichotomy Property:

(i) $a, b \in \mathbb{P} \Rightarrow a+b \in \mathbb{P}$

(ii) $a, b \in \mathbb{P} \Rightarrow ab \in \mathbb{P}$.

(iii) $a \in \mathbb{P}$ or $a = 0$ or $-a \in \mathbb{P} \Rightarrow$ true.

	Always exist?	If exist, only one?	Definition.
Upper Bound	No	No	$\forall s \in S,\ u \geq s$
Supremum	No	Yes	$\forall$upper bound u, $\sup S \leq u$.
Maximum	No	Yes	$\sup S \in S$.

2.3.4 "Slight Insufficiency"

u is an upper bound of $S \subset \mathbb{R}$: $\Rightarrow$

$u = \sup S \Leftrightarrow \forall \varepsilon > 0,\ \exists s_\varepsilon \in S,\ u - \varepsilon < s_\varepsilon$

2.3. The Supremum Property/Axiom of $\mathbb{R}$ (or the Completeness Property/Axiom of $\mathbb{R}$).

$S \subset \mathbb{R} \land S$ has an upper bound $\Rightarrow \sup S$ exists.

2.4 The Archimedean Property.

$x \in \mathbb{R} \Rightarrow \exists\, n_x \in \mathbb{N}$ s.t. $x < n_x$.

Corollary: $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$.

$\inf \{\frac{1}{n} : n \in \mathbb{N}\} = 0$.

$x > 0 \Rightarrow \exists n \in \mathbb{N}$ s.t. $n-1 \leq x < n$.

2.4.8 The Density Theorem of $\mathbb{Q}$

$x, y \in \mathbb{R} \land x < y \Rightarrow \exists r \in \mathbb{Q}$ s.t. $x < r < y$.

The Density Theorem of $\mathbb{I}$

$x, y \in \mathbb{R} \land x < y \Rightarrow \exists z \in \mathbb{I}$, s.t. $x < z < y$.

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Nested Intervals.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots$$

2.5.1 Nested Interval Property.

$I_n = [a_n, b_n]$, $n \in \mathbb{N}$ is a nested sequence of bounded intervals $\Rightarrow \exists\, \xi \in \mathbb{R}$, s.t. $\xi \in I_n\ \forall n$.

2.5.2 "Unique ξ"

Conditions of 2.5.1 $\wedge$ $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$.

$\Rightarrow \xi$ is unique.

3.1.1 Sequence.

A real-valued function X with domain $\mathbb{N}$

$$X : \mathbb{N} \to \mathbb{R}.$$

3.1.2(a) Constant Sequence

$(c) = (c, c, c, \cdots)$

3.1.3 Convergence:

(x_n) is convergent to x $\Leftrightarrow$

$\forall \varepsilon > 0$, $\exists K = K(\varepsilon)$, $\forall n \geq K$, $|x_n - x| < \varepsilon$ $\Leftrightarrow$

$\forall \varepsilon > 0$, $\exists K = K(\varepsilon)$, $\forall n \geq K$, $x_n \in V_\varepsilon(x)$. $\Leftrightarrow$

$\forall \varepsilon$-neighborhood of x, $\exists K \in \mathbb{N}$ s.t. $\forall n \geq K$, $x_n \in V_\varepsilon(x)$

3.2.3 Limit Theorems

$\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $c \in \mathbb{R}$.

- $\lim_{n\to\infty}(x_n + y_n) = x + y$.
- $\lim_{n\to\infty}(x_n - y_n) = x - y$.
- $\lim_{n\to\infty}(x_n y_n) = xy$
- $\lim_{n\to\infty} c x_n = cx$.
- $\lim_{n\to\infty}\left(\frac{x_n}{y_n}\right) = \frac{x}{y}$ if $y \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$.

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3.2.1 Boundedness of Sequence.
Bounded $\Leftrightarrow \exists M > 0 \in \mathbb{R}$, $|x_n| < M$ for some k $\forall n \geq k$.

3.2.2 Convergent sequences are bounded.

3.2.4 $x_n \geq 0$, $\forall n \in \mathbb{N} \land (x_n)$ converges $\Rightarrow \lim_{n\to\infty}(x_n) \geq 0$

3.2.5 $x_n \geq y_n$ $\forall n \in \mathbb{N} \land (x_n)$ and (y_n) converges $\Rightarrow \lim_{n\to\infty}(x_n) \geq \lim_{n\to\infty}(y_n)$

3.2.6 $a \leq x_n \leq b$ $\forall n \in \mathbb{N} \land (x_n)$ converges $\Rightarrow a \leq \lim_{n\to\infty} x_n \leq b$.

3.3.1 Monotone Sequence.
Increasing: $x_n \leq x_{n+1}$ $\forall n \in \mathbb{N}$
Decreasing: $x_n \geq x_{n+1}$ $\forall n \in \mathbb{N}$
} can be both

3.3.2 Monotone Convergence Theorem.
(x_n) is a monotone sequence of real values.
(x_n) converges $\Leftrightarrow$ (x_n) is bounded.
(a) (x_n) is bounded and increasing $\Rightarrow$
$\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.

(b) (x_n) is bounded and decreasing $\Rightarrow$.
$\lim_{n\to\infty} x_n = \inf\{x_n ; n \in \mathbb{N}\}$.

3.4.1 Subsequence
let $n_1 < n_2 < \cdots < n_k < \cdots$ be strictly increasing sequence of natural numbers.
(x_{n_k}) is a subsequence of (x_n).

3.4.2 "Subsequence Convergence"
$\lim_{n\to\infty} x_{n_k} = \lim_{k\to\infty} x_{n_k} = x = \lim_{n\to\infty} x_n$.

for any subsequence (x_{n_k}) of (x_n)

3.4.5 Divergence Criteria.

(x_n) is divergent if:

OR ① Two subsequences of (x_n) have different limits

② (x_n) is unbounded. (its subsequences won't even have limits.).

3.4.7 Monotone Subsequence Theorem.
Every sequence has a monotone subsequence.

3.4.8 Bolzano – Weierstrass Theorem.

Every bounded sequence has a convergent subsequence

3.5.1 Cauchy Sequence

$\Leftrightarrow \forall \varepsilon > 0, \exists H := H(\varepsilon) > 0$ s.t. $|x_n - x_m| < \varepsilon$ for all $n, m > H$.

3.5.4 A Cauchy Sequence of real numbers is bounded.

3.5.5 Cauchy Convergence Criterion.

A real-valued sequence is convergent $\Leftrightarrow$ it is a Cauchy sequence.

How to calculate limits:

1. Definition
2. Limit theorems.
3. Squeeze Theorem
4. Monotone Convergence Theorem
5. Cauchy Criterion. & Contractive Sequence.

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Contractive Sequence.

(x_n) is contractive $\Leftrightarrow$
$\exists C,\ 0<C<1$ s.t. $\forall n \in \mathbb{N}$
$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|$.

A contractive sequence is a Cauchy sequence.

How to Get the Limit of $\frac{f_n}{f_{n+1}}$

① Transform the definition. to $x_{n+1} = \frac{1}{1+x_n}$ where $x_1 = 1$.
② Prove that $|x_{n+2} - x_{n+1}| \leq (\frac{1}{1+c})^2 |x_{n+1} - x_n|$.
③ Find such a lower bound c for (x_n).
④ $x = \frac{1}{1+x}$. $\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2} = \frac{-1+\sqrt{5}}{2}$.

3.6.1 Properly Divergent Sequences.

Properly Divergent
$\lim_{n\to\infty} x_n = +\infty \Leftrightarrow \forall \alpha \in \mathbb{R},\ \exists K(\alpha) \in \mathbb{N},\ \forall n \geq K(\alpha),\ x_n > \alpha$

$\lim_{n\to\infty} x_n = -\infty \Leftrightarrow \forall \beta \in \mathbb{R},\ \exists K(\beta) \in \mathbb{N},\ \forall n \geq K(\beta),\ x_n < \beta$.

$c > 1 \Rightarrow \lim_{n\to\infty} c^n = +\infty$ (use Bernoulli's Inequality)

3.6.3 Unbounded Monotone Sequences

(a) (x_n) is unbounded (above) and increasing $\Rightarrow \lim_{n\to\infty} x_n = +\infty$

(b) (x_n) is unbounded and decreasing $\Rightarrow \lim_{n\to\infty} x_n = -\infty$

3.6.2 "Balloon"

$(x_n), (y_n)$ s.t. $x_n \leq y_n\ \forall n \in \mathbb{N}$

(a) $\lim_{x\to\infty} x_n = +\infty \Rightarrow \lim_{y\to\infty} y_n = +\infty$

(b) $\lim_{y\to\infty} y_n = -\infty \Rightarrow \lim_{x\to\infty} x_n = -\infty$

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3.7.1 (Infinite) Series.

The sequence (s_n) is called the (infinite) series generated by (x_n). Denoted by.

$$\sum_{n=1}^{\infty} x_n \quad \text{or} \quad x_1+x_2+x_3+\cdots$$

The numbers x_n are called the terms.
The numbers s_k are called the partial sums of the series.
If the first term in the series is x_N, define.

$$s_n = \sum_{i=N}^{n} x_n.$$

Geometric Series.

$$\sum_{n=0}^{\infty} r^n = 1+r+r^2+\cdots$$

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$$

$$\text{If } |r|<1 \Rightarrow \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 1.$$

Convergence/Divergence Tests.

1. N-th Term Test
2. Cauchy Criterion Test
3. Partial Sum Bounded Test for Series with non-negative terms
4. Comparison Test
5. Limit Comparison Test.

3.7.3. N-th Term Test.

$$\lim_{n\to\infty} x_n \neq 0 \Rightarrow s_n \text{ diverges.}$$

3.7.4 Cauchy Criterion Test.

$s_n = \sum_n x_n$ converges $\Leftrightarrow$

$\forall \varepsilon > 0\ \exists M(\varepsilon) \in \mathbb{N},\ \forall m, n,\ m > n \geq M(\varepsilon)$

$|s_m - s_n| < \varepsilon.$

That is $|x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon.$

3.7.5 Partial Sum Bounded Test for Series with Non-negative Terms

$$\sum x_n = \lim_{n\to\infty} s_n = \sup\{s_n : n \in \mathbb{N}\}.$$

By Monotone Convergence Theorem.

P-series.

$$\sum_n \frac{1}{n^p}$$

$0 < |p| \leq 1 \Rightarrow$ diverges.

3.7.6 Alternating Harmonic Series. $\sum_{n \geq 1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots$

Converges.

① Grouping : s_{2k} and s_{2k+1}

② Prove $\lim s_{2k} = \lim s_{2k+1}$

③ Use that to facilitate ε, K for s_n.

3.7.7 Comparison Test.

$\exists K \in \mathbb{N},\ 0 \leq x_n \leq y_n \cdot \forall n \geq K.$

(a) $\sum y_n$ converges $\Rightarrow \sum x_n$ converges. } use Cauchy Criterion
(b) $\sum x_n$ diverges $\Rightarrow \sum y_n$ diverges } Test.

3.7.8 Limit Comparison Test

(x_n) and (y_n) are strictly positive sequences

$$r := \lim_{n \to \infty} \left(\frac{x_n}{y_n}\right).$$

(a) $r > 0$ is defined $\Rightarrow (\sum x_n$ converges $\Leftrightarrow \sum y_n$ converges$)$.
(b) $r = 0 \Rightarrow (\sum y_n$ converges $\Rightarrow \sum x_n$ converges$)$.

9.1.1 Absolute Convergence

$\sum x_n$ is absolutely convergent if and only if $\sum |x_n|$ is convergent.
Conditionally / nonabsolutely convergent $\Leftrightarrow$ convergent, but not absolutely convergent.

9.2.1 Limit Comparison Test. II.

$$r := \lim_{n \to \infty} \left(\frac{|x_n|}{|y_n|}\right)$$

(a) $r > 0 \Rightarrow (\sum x_n$ is absolutely convergent $\Leftrightarrow \sum y_n$ is absolutely convergent$)$
(b) $r = 0 \Rightarrow (\sum y_n$ is absolutely convergent $\Rightarrow \sum x_n$ is absolutely convergent$)$

9.2.2 Root Test. (Use comparison and n-th term). $x_n \geq 0$.

(a) $r \in \mathbb{R}$ and $0 \leq r < 1$, $K \in \mathbb{N}$, $|x_n|^{\frac{1}{n}} \leq r$ for $n \geq K$
$\Rightarrow \sum x_n$ is absolutely convergent.
(b) $r \in \mathbb{R}$ and $r > 1$, $K \in \mathbb{N}$, $|x_n|^{\frac{1}{n}} \geq r > 1$ for $n \geq K$.
$\Rightarrow \sum x_n$ is divergent
(c) $r = 1 \Rightarrow$ root test is inconclusive.

9.2.3 Corollary of Root Test. (Use root test to prove).

$r := \lim_{n\to\infty} |x_n|^{\frac{1}{n}}$ exists.

(a) $r > 1 \Rightarrow \sum x_n$ is divergent.

(b) $r < 1 \Rightarrow \sum x_n$ is absolutely convergent.

9.2.4 Ratio Test (Use comparison and n-th term).

$(x_n) :=$ a sequence of non-zero real numbers.

(a) $\exists r$, $0 < r < 1$, $K \in \mathbb{N}$, $\left|\frac{x_{n+1}}{x_n}\right| \le r$ for $n \ge K$.

$\Rightarrow \sum x_n$ is absolutely convergent.

(b) $K \in \mathbb{N}$

$\left|\frac{x_{n+1}}{x_n}\right| \ge 1$ for $n \ge K$

$\Rightarrow \sum x_n$ is divergent.

9.2.5 Corollary of Ratio Test. (Use ratio test).

$(x_n) :=$ sequence of non zero real numbers and $r := \lim \left|\frac{x_{n+1}}{x_n}\right|$ exists.

(a) $r > 1 \Rightarrow \sum x_n$ is divergent.

(b) $r < 1 \Rightarrow \sum x_n$ is absolutely convergent.

(c) $r = 1 \Rightarrow$ Inconclusive.

4.1.1 Definition for Cluster Points.

$A :\subseteq \mathbb{R}$, c is a cluster point of A

$\Leftrightarrow \forall \delta > 0$, $\exists x \in A$, $0 < |x - c| < \delta$

$\Leftrightarrow V_\delta(c) \setminus \{c\} \cap A \neq \emptyset$ for all $\delta > 0$.

4.1.2 Alternative Definition for Cluster Points.

c is a cluster point of A

$\Leftrightarrow \exists$ a sequence $(a_n) \subseteq A$ s.t. $\lim a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

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4.1.4 Definition of Limit of Functions.

$A :\subseteq \mathbb{R}$, $c := $ a cluster point of A. $f: A \to \mathbb{R}$.

A real number L is the limit of f at c

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ (s.t. $x \in A$ and $0 < |x - c| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$.)

$\Leftrightarrow \underbrace{x \in A \cap V_\delta(c) \setminus \{c\}}_{\neq \emptyset} \Rightarrow f(x) \in V_\varepsilon(L)$.

Remark:

c does not necessarily belong to A. We don't require $f(c)$ to be defined.

① Strategy to prove $\lim_{x \to c} f(x) = y$.

① Set up $|f(x) - y| < \varepsilon$

② Get a range for x, (a, b).

③ Find δ s.t. $V_\delta(c) \subseteq (a, b)$.

② Strategy to prove $\lim_{x \to c} f(x) = L$

① Study $|f(x) - L|$ and try to factorize it as $\frac{|f(x) - L|}{|x - c|} \times |x - c|$,

② Find a bound M for $\frac{|f(x) - L|}{|x - c|}$ for $x \in V_1(c)$

Take x_{max} for numerator and x_{min} for deno

③ $|f(x) - L| \leq M|x - c|$.

④ $\delta := \min(\frac{\varepsilon}{M}, 1)$,

$\therefore \forall x \in V_\delta(c)$

$|f(x) - L| \leq M|x - c| < M \cdot \frac{\varepsilon}{M} = \varepsilon$.

4.1.5 Uniqueness of Limit.

$f: A \to \mathbb{R}$, c is a cluster point of A $\Rightarrow$

$\Rightarrow$ f can have only one limit at c. (if any).

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4.2.1 Boundedness of Function on Neighbourhood.

$f: A \to \mathbb{R}$, $c :=$ a cluster point of A.

f is bounded on a neighbourhood of c

$\Leftrightarrow \exists\, V_\delta(c)$ and a constant $M > 0$.

s.t. $|f(x)| \le M$ for all $x \in A \cap V_\delta(c)$.

4.2.2 Existence of Boundness.

$f: A \to \mathbb{R}$, $c :=$ a cluster point of A.

f has a limit at c.

$\Rightarrow f$ is bounded on some neighbourhood of c.

(Be mindful that this time $f(c)$ may be defined so $f(c) >> \lim_{x \to c} f(x)$).

4.1.8 Sequential Criterion of Limits.

$f: A \to \mathbb{R}$ and $a :=$ a cluster point of A.

$\lim_{x \to a} f(x) = L$.

$\Leftrightarrow \forall\, (x_n)$ in A that converges to a and $x_n \neq a$ $\forall n \in \mathbb{N}$, $(f(x_n))$ converges to L.

4.2.3. Notation for Function Operations.

$$f: A \to \mathbb{R},\ g: A \to \mathbb{R}. \quad \begin{cases} (f+g)(x) := f(x) + g(x) \\ (f-g)(x) := f(x) - g(x) \\ (fg)(x) := f(x) \cdot g(x). \\ \left(\frac{f}{g}\right)(x) := \frac{f(x)}{g(x)} \quad \text{if } g(x) \neq 0 \text{ for all } x \in A \end{cases}$$

4.2.4 Limit Theorems. (Use 4.1.8)

$\lim_{x \to c} f(x) = L$, $\lim_{x \to c} g(x) = M$.

(a) $\lim_{x \to c} (f \pm g)(x) = L \pm M$.

(b) $\lim_{x \to c} (fg)(x) = LM$, $\lim_{x \to c} (bf)(x) = bL$.

(c) $\lim_{x \to c} \left(\frac{f}{h}\right)(x) = \frac{L}{H}$, if $h(x) \neq 0$ for all $x \in A$ and $\lim_{x \to c} h(x) = H \neq 0$.

4.2.5 Limit Theorem for polynomials.

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}, \text{ if } q(c) \neq 0.$$

4.2.6 "Squeeze Theorem Prelude".

$f(x) \leq g(x)$ for all $x \in A$. and both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

$$\Rightarrow \lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

4.2.7 Squeeze Theorem (Use 4.1.8 or own methods in mid-term)

$f(x) \leq g(x) \leq h(x)$ for all $x \in A$

$\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$

$$\Rightarrow \lim_{x \to c} g(x) = L.$$

4.2.9 Lower Bound of Limit.

$\begin{cases} \lim_{x \to c} f(x) > 0 \\ \lim_{x \to c} f(x) < 0. \end{cases} \Rightarrow \exists V_\delta(c)$ of c, s.t. $\begin{cases} f(x) > 0 \\ f(x) < 0 \end{cases}$ —/

for all $x \in A \cap V_\delta(c)$, $x \neq c$.

Can be empty $\Rightarrow$ If $A=(a,b)$
$c=b$ cannot result in a ~~right~~ hand limit.

4.3.1 One-Sided Limit.

(i) $c :=$ a cluster point of $A\cap(c,\infty)$

L is the right-hand limit of f at c

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0$ s.t.

$0<x-c<\delta \Rightarrow |f(x)-L|<\varepsilon$.

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0$, s.t.

$x\in(c, c+\delta) \Rightarrow |f(x)-L|<\varepsilon$.

(ii) $c :=$ a cluster point of $A\cap(-\infty, c)$.

L is the left-hand limit of f at c.

$\Leftrightarrow \forall \varepsilon>0, \exists \delta>0$, s.t

$-\delta<x-c<0$ or $0<c-x<\delta \Rightarrow |f(x)-L|<\varepsilon$.

$\Leftrightarrow x\in(c-\delta, c) \Rightarrow |f(x)-L|<\varepsilon$.

4.3.2 Sequential Criterion for One-sided limits.

(i) $\lim_{x\to c^+} f(x) = L$

$\Leftrightarrow \forall (x_n)$ that converges to c and $x_n>c$ ($x_n\neq c$) for all n,
$(f(x_n)) \to L$.

(ii) $\lim_{x\to c^-} f(x) = L$.

$\Leftrightarrow \forall (x_n)$ that converges to c and $x_n<c$ ($x_n\neq c$) for all n,
$(f(x_n)) \to L$.

Remark: limit theorems and the squeeze theorem also hold for one-sided limits. (Use this).

Facts: $e^x > x$ for $x>0 \Rightarrow 0<e^{-x}<\frac{1}{x}$ for $x>0$

$\therefore e^{\frac{1}{t}} > \frac{1}{t}$ for $t>0 \Rightarrow t<e^{-\frac{1}{t}}$ for $t>0$

$\therefore 0<e^{\frac{1}{x}}<-x$ for $x<0$

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4.3.2 "Two sided Limits"

$$\lim_{x \to c} f(x) = L$$

$$\Leftrightarrow \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

Removable Singularity.

Examply:

$$f(x) := \begin{cases} \frac{\sin(x)}{x}, & x \neq 0. \\ 1, & x = 0 \end{cases} \rightarrow \text{Removable Singularity.}$$

How to Define Exponential Functions?

1. $f: x \to 2^x$ from $\mathbb{Q}$ to $\mathbb{R}$ is well-defined.
2. Every real number is a cluster point of $\mathbb{Q}$.
3. $\forall c \in \mathbb{R}$, $\lim_{x \to c} 2^x = \lim_{x \to c^-} 2^x = \sup\{2^x : x \leq c \text{ and } x \in \mathbb{Q}\}$

$$= \inf\{2^x : x \geq c \text{ and } x \in \mathbb{Q}\} = \lim_{x \to c^+} 2^x \text{ exists.}$$

4. Define $f: \mathbb{R} \to \mathbb{R}$.

$$f(c) = 2^c := \lim_{x \to c} 2^x.$$

5.1.1 (ε-δ Definition of Continuity).

$A :\subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $c :\in A$.

f is continuous at c

no "$0<$" May be vacuously true.

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \underbrace{|x - c| < \delta}_{\text{and } x \in A} \Rightarrow |f(x) - f(c)| < \varepsilon$$

Equivalent Definition:

f is continuous at c

$$\Leftrightarrow \underline{f(c)} = \lim_{x \to c} f(x)$$

Defined.

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5.1.2 Equivalent Definition for Continuity.

$A :\subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $c \in A$, $c :=$ a cluster point.

f is continuous at c.

$\Leftrightarrow$ $\forall$ ε-neighbourhood $V_\varepsilon(f(c))$ of $f(c)$

$\exists$ δ-neighbourhood $V_\delta(c)$, s.t

$$x \in A \cap \underbrace{V_\delta(c)}_{\text{no "}\setminus\{c\}\text{"}} \Rightarrow f(x) \in V_\varepsilon(f(c)).$$

Remark:

If f is continuous at every point of A, f is continuous on A.

If $c \in A$ but is not a cluster point, f is vacuously continuous at c. c is called "isolated". In this case, A could be a discrete set.

Facts:

$\sin(x) - \sin(y) = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$.

$|\sin(x)| \le |x|$, $\therefore$ $\frac{\sin(x)}{x} \le 1$.

$\cos(x) - \cos(y) = -\sin\frac{x-y}{2}\sin\frac{x+y}{2}$.

5.1.3 Sequential Criterion for Continuity.

f is continuous at $x = a$.

$\Leftrightarrow$ $\forall$ (x_n) in the domain of f such that $x_n \to a$, possible that $x_k = a$.

$(f(x_n)) \to f(a)$.

5.1.6(g) Dirichlet Discontinuous Function.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Use Sequential criterion.

5.2.1 Arithmetic Operations on Continuous Functions (Use Limit Theorem)

f and g are continuous at $x=c$.

(a) $f \pm g$, $f \cdot g$, bf are also continuous at $x=c$.

(b) $g(c) \neq 0 \Rightarrow \frac{f}{g}$ is continuous at $x=c$.

5.2.2 "Continuous Functions on Set"

f and g are continuous on A

(a) $f \pm g$, $f \cdot g$, bf are continuous on A.

(b) $g(c) \neq 0 \Rightarrow f/g$ is also continuous on A.

Composition Function.

$f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$, $f(A) \subseteq B$.

$\Rightarrow g \circ f : A \to \mathbb{R}$

$(g \circ f)(x) = g(f(x)) \quad \forall x \in A$.

Comes first.

5.2.6 Combination of Continuous Functions aka. Composition Theo.

$f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ and $f(A) \subseteq B$.

f is continuous at c, g is continuous at $b = f(c)$.

$\Rightarrow g \circ f$ is continuous at c.

5.2.7 Combination of Continuous Functions on Set.

$f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$, $f(A) \subseteq B$.

f is continuous on A and g is continuous on B.

$\Rightarrow g \circ f$ is continuous on A.

5.2.5 f is continuous on $A \Rightarrow |f|$ is continuous on A.

5.3.1 Boundedness of Functions.

$f: A \to \mathbb{R}$. is bounded on A.

$\Leftrightarrow \exists M > 0$ s.t. $|f(x)| \le M, \forall x \in A$.

5.3.2 Boundedness Theorem.

closed fully.

f is continuous on $[a, b]$ $\quad a \neq -\infty, \; b \neq \infty$.

$\Rightarrow f$ is bounded on $[a, b]$.

Use contradiction to prove:

f is not bounded $\Rightarrow \exists (x_n) \subseteq [a, b], |f(x_n)| > n, \forall n$

(x_n) has a convergent subsequence, (x_{n_k})

Let $c := \lim_{k \to \infty} x_{n_k}$, $a \le c \le b$.

f is continuous $\Rightarrow f(x_{n_k}) \to f(c)$.

But $n_k \ge n$. $\Rightarrow f(x_{n_k}) > n_k \ge k \to \infty$. Contradiction.

5.3.3 Max & Min.

(i). f has an absolute maximum on A

$\Leftrightarrow \exists x^* \in A$, s.t

$$f(x^*) \ge f(x) \quad \forall x \in A.$$

$\therefore f(x^*) = \sup f(A) = \max f(A)$.

(ii) f has an absolute minimum on A.

$\Leftrightarrow \exists x_* \in A$, s.t.

$$f(x_*) \le f(x), \quad \forall x \in A.$$

$\therefore f(x_*) = \inf f(A) = \min f(A)$.

Remark: bounded $\not\Rightarrow$ max or min exists.

5.3.4 Maximum - Minimum Theorem.

f is continuous on $[a, b]$

$\Rightarrow f$ has abs max and abs min.

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Proof: f is continuous, $\Rightarrow f$ is bounded (5.3.2).
Let $M = \sup\{f(x) : x \in [a,b]\}$.
$\therefore \exists x_n \in [a,b]$, $M - \frac{1}{n} < f(x_n) \leq M$ ($M = \sup$).
By squeeze Theorem, $\lim_{n\to\infty} f(x_n) = M$.
(x_{n_K}) is convergent $\longleftarrow$ Introduce this because (x_n) may not be convergent.
Let $\lim_{n\to\infty} x_{n_K} = x^*$, $a \leq x^* \leq b$.

$\because f$ is continuous at x^*
$\therefore f(x_{n_K}) \to f(x^*)$.
But $(f(x_{n_K}))$ is a subsequence of $(f(x_n))$
by the sequential criterion, $f(x_{n_K}) \to M$.
By uniqueness of limit, $f(x^*) = M$.

5.3.5 Location of Roots Theorem.
f is continuous on $[a,b]$ and $f(a) \cdot f(b) < 0$.
$\Rightarrow \exists c \in (a,b)$ s.t. $f(c) = 0$.

Proof by Bisection Algorithm (Binary Search).
We have a nested family of intervals $I_1, \ldots, I_n$
such that $f(a_K) < 0 < f(b_K)$ for $1 \leq K \leq n$.
$I_K = (a_K, b_K)$. (Suppose $f(a_1) < 0 < f(b_1)$)
Set $p_n = \frac{a_n + b_n}{2}$.

Case ①: $f(p_n) = 0 \Rightarrow$ Terminate.
②: $f(p_n) > 0 \Rightarrow a_{n+1} := a_n$, $b_{n+1} := p_n$.
③: $f(p_n) < 0 \Rightarrow a_{n+1} := p_n$, $b_{n+1} := b_n$.
$\therefore f(a_K) < 0 < f(b_K)$ for $1 \leq K \leq n+1$
$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = \cdots = \frac{b-a}{2^{n-1}}$

$\Rightarrow \lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^{n-1}} = 0$.

$\therefore \lim_{n\to\infty} b_n = \lim_{n\to\infty} a := c$.

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By Nested Intervals property. $\bigcap_n^{\infty}[a_n, b_n] = \{c\} \subseteq [a,b]$

f is continuous $\Rightarrow$

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = \lim_{x\to c} f(x) = f(c).$$

Since $\lim_{n\to\infty} f(a_n) \leq 0$ and $\lim_{n\to\infty} f(b_n) \geq 0$,

by the uniqueness, $f(c) = 0$.

5.3.7 Balzano's Intermediate Theorem.

$I :=$ an interval, f is continuous on I, $a, b \in I$, $f(a) \leq f(b)$.

$\forall k \in [f(a), f(b)]$, $\exists c$ in I s.t. $f(c) = k$.

5.3.10 Preservation of Closed Intervals Theorem.

f is continuous on $[a,b]$.

$\Rightarrow f([a,b]) = \{f(x) : x \in [a,b]\} = [m, M]$.

Use 5.3.3 and 5.3.7

5.6.1 Monotone Functions.

$f : A \to \mathbb{R}$, $\forall x_1, x_2 \in A$.

(a) $(x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)) \Rightarrow$ Increasing.

(b) $(x_1 < x_2 \Rightarrow f(x_1) < f(x_2)) \Rightarrow$ Strictly increasing.

(c) $(x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)) \Rightarrow$ Decreasing.

(d) $(x_1 < x_2 \Rightarrow f(x_1) > f(x_2)) \Rightarrow$ Strictly decreasing.

(e) Increasing $\vee$ decreasing $\Rightarrow$ monotone.

(f) Strictly increasing $\vee$ strictly decreasing $\Rightarrow$ strictly monotone

Inverse Functions.

$f: A \to B$ is a bijection

$\Rightarrow$ the inverse function of f, $f^{-1}: B \to A$, s.t.

$f^{-1}(f(x)) = x,\ \forall x \in A.$

$f(f^{-1}(y)) = y,\ \forall y \in B.$

5.4.1 Uniform Continuity.

$A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$.

f is uniformly continuous on A.

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0$ s.t.

$\forall x, y \in A,\ |x - y| < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon.$

δ only depends on ε.

"Negation of Uniform Continuity".

f is not uniformly continuous on A.

$\Leftrightarrow \exists \varepsilon_0 > 0, \forall \delta > 0,$

$\exists x_\delta, y_\delta \in A,\ |x_\delta - y_\delta| < \delta$ but $|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0.$

Sequential Criterion for Uniform Continuity.

$f: A \to \mathbb{R}$ is uniformly continuously

$\Leftrightarrow (\forall (x_n), (y_n)$ in A s.t. $\lim_{n\to\infty} x_n - y_n = 0) \Rightarrow \lim_{n\to\infty} f(x_n) - f(y_n) = 0.$

$\Rightarrow$ (Use sequential criterion 5.1.3).

$\Leftarrow$ (Use contradiction or contraposition with negation)

5.4.2 Non-uniform Continuity Criteria.

f is not uniformly continuous on A.

$\Leftrightarrow$ (See above) Negative definition.

$\Leftrightarrow$ (Negation of sequential criterion for uniform continuity)

$\exists \varepsilon_0 > 0\ \exists (x_n), (y_n)$ and $\lim_{n\to\infty} x_n - y_n = 0$ and.

$|f(x_n) - f(y_n)| \geq \varepsilon_0.$

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5.4.3 Uniform Continuity Theorem.

f is continuous on a closed bounded interval $[a, b]$.

$\Rightarrow f$ is uniformly continuous on $[a. b]$.

Proof: Suppose not.

That is $\exists \varepsilon_0. \forall \delta = \frac{1}{n} > 0, n \in \mathbb{N}$,

$|x_n - y_n| < \delta$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

$\because (x_n)$ and (y_n) are bounded $\Rightarrow$ convergent subsequent.

$\therefore \lim_{n\to\infty} x_{n_k} = \lim_{n\to\infty} y_{n_k} = C$.

$\therefore \lim_{n\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(y_{n_k}) = f(c)$. Contradicts.

5.4.4 Lipschitz Function.

$f: A \to \mathbb{R}$ is a Lipschitz function

$\Leftrightarrow |f(x) - f(y)| \leq K|x-y| \quad \forall x, y \in A, \ \exists K > 0$.

Equivalent Definition:

$\frac{|f(x) - f(y)|}{|x-y|} \leq K, \ \forall x \neq y. \in A$.

$\therefore f'$ (if exists) is bounded on A.

5.4.5 Lipschitz Condition.

$f: A \to \mathbb{R}$ is a Lipschitz function

$\Rightarrow f$ is uniformly continous on A.

Proof: Let $\delta = \frac{\varepsilon}{K}$.

$|x-y| < \delta \Rightarrow |f(x) - f(y)| \leq K|x-y| < K\delta \doteq \varepsilon$.

5.4.6(c) Example: Prove $f(x)=\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

Proof: $I=[0,2]$, $J=[1,\infty)$.

f is uniformly continuous on I. (by 5.4.3).

$\therefore \delta_I$.

f is uniformly continuous on J (by 5.4.5).

$\therefore \delta_J$.

Let $\delta \leq 1 \Rightarrow \delta := \min\{1, \delta_I, \delta_J\}$.

$\therefore |x-y| < \delta \leq 1$.

If $\min\{x,y\} \geq 1 \Rightarrow x, y \in J$.

If $\min\{x,y\} < 1 \Rightarrow$.

$\max\{x,y\} < 1 + \min\{x,y\} < 2$.

since $|x-y| = \max - \min < 1$.

$\therefore x, y \in I$.

$\therefore x, y$ are either both in I or both in J.

QED.

5.4.7 Uniformly Continuous Functions Preserve Cauchy Sequence.

$f: A \to \mathbb{R}$ is uniformly continuous on A.

$(x_n) \subseteq A$ is a Cauchy sequence.

$\Rightarrow (f(x_n))$ is also a Cauchy sequence.

Remark: Continuous $\not\Rightarrow$ Cauchy sequence preserved.

$f(x) = \frac{1}{x}$ for $x > 0$.

$(\frac{1}{n})$ is Cauchy, but $(f(\frac{1}{n})) = (n)$ diverges, not Cauchy.

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5.4.8 Continuous Extension Theorem.

f is uniformly continuous on (a,b)
$\Leftrightarrow$ f can be defined at a and b s.t. the extended f is uniformly continuous.

Remark:

This means $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist

Define $f(a) := \lim_{x\to a^+} f(x)$ and $f(b) := \lim_{x\to b^-} f(x)$.

$\therefore f$ is continuous on $[a,b]$.

Proof:

Pick $(x_n)\to a \Rightarrow (x_n)$ is Cauchy.
$\Rightarrow (f(x_n))$ is also Cauchy.

Define $f(a) := \lim_{n\to\infty} f(x_n)$

$\forall (y_n) \subseteq (a,b) \to a$, $\lim_{n\to\infty} y_n - x_n = 0$.

By sequential criterion, $\lim_{n\to\infty} f(y_n) - f(x_n) = 0$.

$\therefore \lim_{n\to\infty} f(y_n) = f(a) \Rightarrow (f(y_n)) \to f(a)$

Since (y_n) is arbitrary, by sequential criterion (right-hand version), $\lim_{x\to a^+} f(x) = f(a)$.

Similar arguments for b.
$\therefore$ Extended f is continuous on $[a,b]$.
$\therefore$ Extended f is uniformly continuous on $[a,b]$.

5.6.1 One-Sided Limits for Monotone Functions.

$I :\subseteq \mathbb{R}$, $f: I \to \mathbb{R}$ is increasing on I. $c \in I$ is not an end point of I.

(i) $\lim_{x \to c^-} f(x) = \sup\{f(x) ; x \in I, x < c\}$.

(ii) $\lim_{x \to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}$.

5.6.2 Corollary of 5.6.1.

$I :\subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ is increasing on I. $c \in I$ is not an end point.

(a) f is continuous at c.

$\Leftrightarrow$ (b) $\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$.

$\Leftrightarrow$ (c) $\sup\{f(x) ; x \in I, x < c\} = f(c) = \inf\{f(x) : x \in I, x > c\}$.

If c is the left end point:

f is right continuous at c

$\Leftrightarrow f(c) = \inf\{f(x) : x \in I, x > c\} = \lim_{x \to c^+} f(x)$.

Only two out of three are equal $\Rightarrow$ suffices.

If c is the right end point:

f is left continuous at c

$\Leftrightarrow f(c) = \sup\{f(x) ; x \in I, x < c\} = \lim_{x \to c^-} f(x)$

· Jump.

$f : I \to \mathbb{R}$ is increasing on I and c is not an end point.

$$j_f(c) = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x).$$

$$= \inf\{f(x) ; x \in I, x > c\} - \sup\{f(x) : x \in I, x < c\}.$$

At end points a, b.

$$j_f(c) := \begin{cases} \lim_{x \to a^+} f(x) - f(a), & c = a. \\ \lim_{x \to b^-} f(b) - f(x), & c = b. \end{cases}$$

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5.6.3 "Jump & Continuity"

$f: I \to \mathbb{R}$ is increasing on I.

f is continuous at

$\iff j_f(c) = 0$.

5.6.4 Countable Discontinuous Points of Monotone Functions. not necessarily strictly

$f: I \to \mathbb{R}$ is monotone on I.

The set of points $D \subseteq I$ at which f is discontinuous is a countable set.

Proof: (Increasing). $I = [a, b]$.

Let $D = \{x_1, \dots, x_n\}$.

for $a \le x_1 < \dots < x_n \le b$.

$$f(a) \le f(a) + j_f(x_1) + \dots + j_f(x_n)$$

$$= \underbrace{f(a) - \lim_{x \to x_1^-} f(x)}_{\text{negative}} + \underbrace{\lim_{x \to x_1^+} f(x) + \cdots}_{\text{negative}}$$

$$\underbrace{- \lim_{x \to x_n^-} f(x)}_{\text{negative}} + \lim_{x \to x_n^+} f(x)$$

$$\le \lim_{x \to x_n^+} f(x) \le f(b).$$

$\therefore\ j_f(x_1) + \dots + j_f(x_n) \le f(b) - f(a)$.

There are at most $k \in \mathbb{N}$ points in $[a, b]$ s.t. $j_f \ge \frac{f(b) - f(a)}{k}$.

Otherwise, if there are $k+1$, then $\sum_{i=1}^{k+1} j_f(x_i) > f(b) - f(a)$.

Let $D_k := \{x \in D;\ j_f(x) \ge \frac{f(b) - f(a)}{k}\}$.

$\therefore D = \bigcup_{k=1}^{\infty} D_k$. which is countable. by 1.3.12.

3.6.5 Continuous Inverse Theorem.

$f: I \to \mathbb{R}$ is strictly monotone and continuous on I.

Let $J = f(I)$.

$f^{-1}: J \to \mathbb{R}$ is also strictly monotone and continuous on J.

Proof: (increasing).

$g := f^{-1}$ $\quad x_1 < x_2 \in I$, $y_1 := f(x_1)$, $y_2 := f(x_2)$.

① Strictly monotone.

$x_1 < x_2 \Rightarrow y_1 < y_2$.

If $g(y_1) \geq g(y_2) \Rightarrow x_1 \geq x_2$. Contradiction.

② Continuous.

Suppose g is discontinuous at c.

Then $\lim_{y \to c^-} g(y) < \lim_{y \to c^+} g(y)$

$\exists x$, s.t $\lim_{y \to c^-} g(y) < x < \lim_{y \to c^+} g(y)$ and $x \neq c$.

$\therefore \forall y \in J$, $x \neq g(y) \Rightarrow x \notin I = g(J)$. Contradiction

N-th Root Functions (Even).

$f(x) = x^n$ for some $n = 2k$, $k \in \mathbb{N}$. $f: [0, \infty) \to \mathbb{R}$.

① Prove $g = f^{-1}$ is strictly increasing and continuous

② Prove $g([0, \infty)) = [0, \infty)$ by Archimedean property and intermediate value theorem.

N-th Root Functions (Odd)

$f(x) = x^n$ for some $n = 2k-1$, $k \in \mathbb{N}$, $f: \mathbb{R} \to \mathbb{R}$

① Prove $g = f^{-1}$ is strictly increasing and continuous on $\mathbb{R}$.

5.6.6 Definition of Rational Powers.

(i) $m, n \in \mathbb{N}$ and $x \geq 0$

$\Rightarrow x^{\frac{m}{n}} := (x^{\frac{1}{n}})^m$

(ii) $m, n \in \mathbb{N}$ and $x > 0$

$\Rightarrow x^{-\frac{m}{n}} := (x^{\frac{1}{n}})^{-m}$.

11.1.1 Neighbourhood Set of a Point.

V is a neighbourhood of $x \in \mathbb{R}$

$\Leftrightarrow$ V contains an ε-neighbourhood $V_\varepsilon(x)$ for some, $\varepsilon > 0$.

$\Leftrightarrow \exists \varepsilon > 0, V_\varepsilon(x) \subseteq V$.

11.1.2 Open and Closed Sets

$G :\subseteq \mathbb{R}$.

(i) G is open in $\mathbb{R}$

$\Leftrightarrow \forall x \in G, \exists \varepsilon_x > 0$, s.t. $V_{\varepsilon_x}(x) \subseteq G$. (G is a neighbourhood of all its points)

(ii) G is closed in $\mathbb{R}$

$\Leftrightarrow \forall y \notin G, \exists \varepsilon_0 > 0$, s.t. $V_{\varepsilon_0}(y) \cap G = \emptyset$,

$\Leftrightarrow$ The complement $C(G) = \mathbb{R} \setminus G$ is open in $\mathbb{R}$.

Examples:

Set	Open	Closed
$\mathbb{R}$	✓	✓
$\emptyset$	✓	✓
(a,b)	✓	×
$[a,b]$	×	✓
$[a,b)$	×	×
$[a,\infty)$	×	✓
$\mathbb{Z}$	×	✓
☆ $\mathbb{Q}$	×	×

Use density and 11.1.8

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11.1.4 Open Set Properties.

(a) The $\boxed{\text{union}}$ of an arbitrary collection of open subsets in $\mathbb{R}$ is open.

Equivalently, $\{G_\lambda : \lambda \in \Lambda\} :=$ a family of subsets of $\mathbb{R}$. G_λ is open $\forall \lambda \in \Lambda \Rightarrow \bigcup_{\lambda\in\Lambda} G_\lambda$ is open

(b) The $\boxed{\text{intersection}}$ of a $\boxed{\text{finite}}$ collection of open subsets of $\mathbb{R}$ is open

Equivalently, $G_1, G_2, \dots, G_n$ are open subsets of $\mathbb{R} \Rightarrow \bigcap_{k=1}^{n} G_k$ is open.

For (b), Use mathematical induction.

11.1.5 Closed Set Properties.

(a) The $\boxed{\text{intersection}}$ of an arbitrary collection of closed subsets of $\mathbb{R}$ is closed in $\mathbb{R}$.

(b) The $\boxed{\text{union}}$ of any $\boxed{\text{finite}}$ collection of closed subsets of $\mathbb{R}$ is closed.

11.1.7 Characterization of Closed Sets.

$F :\subseteq \mathbb{R}$ is closed

$\Leftrightarrow \forall$ convergent $(x_n) \subseteq F$, $\lim_{n\to\infty} x_n \in F$.

Proof:

"$\Rightarrow$"

$X := (x_n) \subseteq F$ is convergent and $x := \lim_{n\to\infty} x_n$.

Suppose $x \notin F \Rightarrow x \in C(F)$ which is open.

$\therefore \exists \varepsilon > 0.\ V_\varepsilon(x) \subseteq C(F)$

$\therefore V_\varepsilon(x) \cap F = \emptyset$

But some $x_n \in V_\varepsilon(x)$ since $x := \lim_{n\to\infty} x_n$.

This contradicts to $x_n \in F$.

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"$\Leftarrow$"

Suppose F is not closed $\Rightarrow$ $C(F)$ is not open.

$\exists y_0 \in C(F)$ s.t. $\forall \varepsilon > 0$, $V_\varepsilon(y_0) \nsubseteq C(F)$.

$\therefore$ $V_\varepsilon(y_0) \setminus C(F) \neq \emptyset$.

$\forall n \in \mathbb{N}$, take $y_n \in V_{\frac{1}{n}}(y_0) \setminus C(F)$

$\therefore$ $y_n \notin C(F) \Rightarrow y_n \in F$.

By squeeze theorem, $y_0 = \lim_{n\to\infty} y_n$, which

by the assumption, $\in F$. Contradicts.

11.1.8 Characterization of closed sets.

$F: \subseteq \mathbb{R}$ is closed

$\Leftrightarrow$ F contains all of its cluster points.

Proof:

"$\Rightarrow$"

Suppose x is a cluster point of F but $x \notin F$.

$\therefore$ $x \in C(F)$. which is open.

$\therefore$ $\exists \varepsilon > 0$, $V_\varepsilon(x) \subseteq C(F) \Rightarrow V_\varepsilon(x) \cap F = \emptyset$

$\Rightarrow V_\varepsilon(x) \setminus \{x\} \cap F = \emptyset$

Contradicts with the fact that x is a cluster point.

"$\Leftarrow$"

Given $y \in C(F)$, y is not a cluster point of F

$\therefore$ $\exists \varepsilon > 0$, $V_\varepsilon(y) \setminus \{y\} \cap F = \emptyset$

Since $y \notin F$, $V_\varepsilon(y) \cap F = \emptyset$

$\therefore$ $V_\varepsilon(y) \subseteq C(F)$.

$\therefore$ $C(F)$ is a neighbourhood of all $y \in C(F)$

$\therefore$ $C(F)$ is open

$\therefore$ F is closed.

11.3.1 Continuity in terms of open sets.

$f: A \to \mathbb{R}$ is continuous at c.

$\Leftrightarrow$ $\forall$ neighbourhood U of $f(c)$, $\exists$ neighbourhood V of c s.t. $x \in V \cap A \Rightarrow f(x) \in U$.

$\Leftrightarrow$ $\forall$ neighbourhood U of $f(c)$, $\exists$ neighbourhood V of C. $f(V \cap A) \subseteq U$

Remark: The difference from 5.1.2 is that set neighborhoods are used here.

Proof:

Use 5.1.2 directly

11.3.2 Global Continuity Theorem.

$f: A \to \mathbb{R}$

(a) f is continuous on A.

$\Leftrightarrow$

(b) $\forall$ open set $G \subseteq \mathbb{R}$, $\exists$ open set $H \subseteq \mathbb{R}$ s.t.

$H \boxed{\cap} A = f^{-1}(G)$

Remark. $f^{-1}(G) := \{x \in A : f(x) \in G\}$

Proof:

(b) $\Rightarrow$ (a).

Let $c \in A$, G be an open neighbourhood of $f(c)$.

$\therefore$ $\exists$ open $H \subseteq \mathbb{R}$, s.t. $c \in H \cap A = f^{-1}(G)$

$\therefore$ H is also a neighbourhood of $c \Rightarrow f(H \cap A) \subseteq G$

$\therefore$ By 11.3.1, f is continuous on A.

(a) $\Rightarrow$ (b).

Strategy: ① $H := \bigcup V(c)$, $H \cap A \subseteq f^{-1}(G)$

② $f^{-1}(G) \subseteq H \cap A$

$\therefore H \cap A = f^{-1}(G)$

$\forall c \in f^{-1}(G) \Rightarrow f(c) \in G$.

By 11.3.1, since G is an open neighbourhood of $f(c)$, $\exists V(c)$ as an open neighbourhood of c.

Let $H := \bigcup_{c \in f^{-1}(G)} V(c)$, which is open.

$\therefore H \cap A \subseteq f^{-1}(G)$.
$\forall c \in f^{-1}(G)$, $c \in V(c) \cap A \Rightarrow c \in H \cap A$
$\therefore f^{-1}(G) \subseteq H \cap A$
$\therefore H \cap A = f^{-1}(G)$.

11.3.3 Corollary of Global Continuity Theorem

A function $f: \mathbb{R} \to \mathbb{R}$ is continuous
$\Leftrightarrow$ $f^{-1}(G)$ is open in $\mathbb{R}$ whenever G is open in $\mathbb{R}$.

11.4.1 Metric Space → aka "distance function"

A metric on a set S is a function
$d: S \times S \to \mathbb{R}$ s.t.
① $d(x,y) \geq 0$, $\forall x, y \in S$ (positivity)
② $d(x,y) = 0 \Leftrightarrow x = y$ (definiteness)
③ $d(x,y) = d(y,x)$, $\forall x, y \in S$ (symmetry)
④ $d(x,y) \leq d(x,z) + d(z,y)$ $\forall x, y, z \in S$
(triangle inequality)

A metric space (S,d) is a set S and a metric d on S.

Proof of 2D space's Triangle Inequality:
$[d(x,y)]^2 \leq [d(x,z) + d(z,y)]^2$

Euclidean Space $\mathbb{R}^n$.

Let $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n} = \{\vec{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\ \forall 1 \leq i \leq n\}$

$$d_2(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$$

$(\mathbb{R}^n, d)$ is a metric space. $\mathbb{R}^n$ is called n-dimensional Euclidean space and d is called Euclidean distance.

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More Metric Spaces.

① (S, d) is a metric space and $X \subseteq S \Rightarrow$
(X, d) is a metric space.

② $d_1(\vec{p}, \vec{q}) := |p_1 - q_1| + |p_2 - q_2|$
$d_2(\vec{p}, \vec{q}) := \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$
$d_\infty(\vec{p}, \vec{q}) := \max\{|p_1 - q_1|, |p_2 - q_2|\}$

③ Let $C([0,1])$ be set of all continuous functions on $[0,1]$.
$d_1(f, g) := \int_0^1 |f(x) - g(x)|\, dx$
$d_2(f, g) := \sqrt{\int_0^1 |f(x) - g(x)|^2 dx}$
$d_\infty(f, g) := \sup\{|f(x) - g(x)| : x \in [0,1]\}$.

11.4.3 Neighbourhood in Metric Space

$\forall \varepsilon > 0$, ε-neighbourhood of a point x_0 in $S :=$
$\{x \in S, d(x_0, x) < \varepsilon\}$
A set U is a neighbourhood of x_0 if $V_\varepsilon(x_0) \subseteq U$ for some $\varepsilon > 0$.

11.4.4 Convergence in Metric Space.

$(x_n) :=$ a sequence in (S, d)
$x_n \to x$
$\Leftrightarrow \forall \varepsilon > 0, \exists K \in \mathbb{N}$, s.t $x_n \in V_\varepsilon(x)$ for all $n \geq K$.

11.4.6 Cauchy Sequence in Metric Space

(x_n) in S is Cauchy
$\Leftrightarrow \forall \varepsilon > 0, \exists H \in \mathbb{N}$, s.t $d(x_n, x_m) < \varepsilon, \forall m, n \geq H$.

11.4.9 Open Sets in Metric Space

A subset G in S is open in S
$\Leftrightarrow G$ is a neighbourhood for $\forall x \in G$

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11.4.10 Continuity in Metric Space

$(S_1, d_1), (S_2, d_2) :=$ metric spaces, $f: S_1 \to S_2$

f is continuous

$\Leftrightarrow \forall \varepsilon$-neighbourhood $V_\varepsilon(f(c))$ of $f(c)$ in S_2, $\exists\ \delta$-neighbourhood $V_\delta(c)$ of c in S_1, s.t., $x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$

$\Leftrightarrow \forall \varepsilon$-neighbourhood $V_\varepsilon(f(c))$, $\exists\ \delta$-neighbourhood $V_\delta(c)$

$f(V_\delta(c)) \subseteq V_\varepsilon(f(c))$

Open/Closed Set Properties still remain in metric spaces.

11.4.11 Global Continuity Theorem in Metric Spaces.

$(S_1, d_1), (S_2, d_2) :=$ metric spaces, $f: S_1 \to S_2$

f is continuous

$\Leftrightarrow \forall G$, which is an open set in S_2, $f^{-1}(G)$ is also an open set in S_1.

Fact: ① $C(A \times B) = (\bar{A} \times \mathbb{R}) \cup (\mathbb{R} \times \bar{B})$

② In $(C([0,1]), d_1)$, the set $E := \{g(x) \in C([0,1]) : \int_0^1 |g(x)|\,dx \geq 1\}$ is closed.

Let $f_0(x) = 0$

$C([0,1]) \setminus E = V_1(f_0) = \{g \in C([0,1]) : \int_0^1 |g(x)|\,dx < 1\}$ is open.

11.2.1 Open Cover

An open cover of a subset A of a metric space S is a collection $\mathcal{G} := \{G_\lambda : \lambda \in \Lambda\}$ of open subsets of S whose union contains A, i.e., $A \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$

If $\mathcal{G}'$ is a subcollection of sets from $\mathcal{G}$ s.t. $\mathcal{G}'$ is an open cover of A, then $\mathcal{G}'$ is ~~#~~ a subcover of $\mathcal{G}$

If $|\mathcal{G}'| \in \mathbb{N}$, then $\mathcal{G}'$ is a finite subcover of $\mathcal{G}$

11.2.2 Compact Set

A subset K of a metric space S is said to be compact if $\forall$ open cover of K has a finite subcover.

If K has finitely many points, K is compact.

Boundedness in Metric Space.

$(S, d) :=$ metric space,

$A :\subseteq S$ is bounded

$\Leftrightarrow \exists M > 0, \exists x \in S$ not necessarily in A. s.t. $A \subseteq V_M(x)$

11.2.5 "Closed & Bounded $\Rightarrow$ Compact" Hiene-Borel Theorem

$(S, d) :=$ metric space

$K :\subseteq S$ is compact

$\Leftrightarrow$ K is closed and bounded.

3.4.8' "Revised Bolzano-Weierstrass Theorem"

A bounded sequence in a metric space (S, d) has a convergent subsequence.

Proof: for every entry of $(x_n) \subseteq$

The first coordinates form a convergent sequence.

The second ...

The m-th ...

$d(x_n^1, x_0^1) \to 0$

$d(x_n^2, x_0^2) \to 0$

...

$d(x_n^m, x_0^m) \to 0$.

$\therefore x_n \to x_0$

11.2.6' "Compact sets & Convergent Subsequences" Similar to 11.1.7

$K :\subseteq S$ is compact.

$\Leftrightarrow \forall (x_n) \subseteq K, \exists x_{n_k} \to x \in K$.

Proof: Use 11.1.7 and contradiction

Fact: K is compact in $\mathbb{R}$.

$\Rightarrow \sup K, \inf K \in K$.

11.4.13 Preservation of Compactness

(S, d) is a compact metric space and $f: S \to \mathbb{R}$ is continuous

$\Rightarrow f(S)$ is compact in $\mathbb{R}$.

Extreme value theorem (Use 11.4.13 and Fact above).

$K :\subseteq S$ is compact and $f: K \to \mathbb{R}$ is continuous.

$\Rightarrow \exists x^*, x_* \in K$ s.t.

$f(x^*) = \sup f(K)$, $f(x_*) = \inf f(K)$.

Connected Sets.

$U :\subseteq S$ is disconnected

$\Leftrightarrow \exists$ open cover $\{A, B\}$ s.t.

$A \cap B \cap U = \emptyset$ but $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$

Otherwise, U is connected.

All interval I are connected

$U = [0,1] \cup [3,4]$ is disconnected.

$\mathbb{N}$ is disconnected

"Connected Sets $\Leftrightarrow$ Intervals"

$E :\subseteq \mathbb{R}$ is connected

$\Leftrightarrow$ E is an interval (open or closed or neither).

$\Leftrightarrow x, y \in E \Rightarrow [x, y] \subseteq E$

Proof:

"$\Rightarrow$" Let E not be an interval

Let $E \subseteq (-\infty, z) \cup (z, \infty) \Rightarrow$ Contradiction.

"$\Leftarrow$"

Suppose E is disconnected.

$\therefore \exists \{A, B\}$, let $A' = A \cap E$ and $B' = B \cap E$.

Since $E \subseteq A \cup B$, $E = (A \cup B) \cap E \Rightarrow E = A' \cup B'$

$\therefore A', B' \neq \emptyset$ and $A' \cap B' = \emptyset$

Let $x \in A'$ and $y \in B'$ and $x < y$

Let $z = \sup(A' \cap [x, y])$, $z \in A'$ or $z \in B'$.

If $z \in A'$, $z \neq y$ since $y \in B'$. $\therefore (z, y)$ is not empty,

$\therefore \exists \varepsilon > 0$, $V_\varepsilon(z) \subseteq A'$ since A' is open.

$\therefore (z, z + \min\{y - z, \varepsilon\}) \subseteq A$ and

$(z, z + \min\{y - z, \varepsilon\}) \subseteq (z, y)$.

$\therefore z$ is not $\sup(A' \cap [x, y])$. Contradition.

Similarly, if $z \in B'$, $(z - \min\{z - x, \varepsilon\}, z) \subseteq B'$

$\therefore (z - \min\{z - x, \varepsilon\}, z) \cap A = \emptyset$, z is not sup.

"Revised Intermediate Value Theorem"

$f : S \Rightarrow \mathbb{R}$ is continuous

E is connected in $S \Rightarrow f(E)$ is connected in $\mathbb{R}$.

"Corollary of Revised Intermediate Value Theorem"

$f : S \to \mathbb{R}$ is continuous.

K is compact connected set

$\Rightarrow f(K) = [\inf f(K), \sup f(K)]$, a closed bounded I.