MA2104 Notes.

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Chapter I.

2.1 Projection and Component.

$$\text{proj}_{\underset{\sim}{b}} \underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\|\underset{\sim}{b}\|^2} \cdot \underset{\sim}{b}.$$

$$\text{comp}_{\underset{\sim}{b}} \underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\|\underset{\sim}{b}\|}.$$

$\text{proj}_{\underset{\sim}{b}} \underset{\sim}{a}$ is the projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$.

2.3 Dot Product

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\| \, \|\underset{\sim}{b}\| \cos\theta.$$

In MA2104, parallelograms and parallelpipeds are boundariless.

$$\underset{\sim}{a} \times \underset{\sim}{b} = -\underset{\sim}{b} \times \underset{\sim}{a}.$$

$$\underset{\sim}{a} \times (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$$

3.3 Cross Product.

$$\|\underset{\sim}{a} \times \underset{\sim}{b}\| = \|\underset{\sim}{a}\| \, \|\underset{\sim}{b}\| \sin\theta.$$

3.5 $V_{parallelpiped} = |\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c})| = |\underset{\sim}{b} \cdot (\underset{\sim}{a} \times \underset{\sim}{c})| = |\underset{\sim}{c} \cdot (\underset{\sim}{a} \times \underset{\sim}{b})|$

Chap II.

Vector Equation.

$$\underset{\sim}{r}(t) = \underset{\sim}{a} + t\underset{\sim}{u}.$$

$\underset{\sim}{r}(t)$ is a unit speed parameterization if $\|\underset{\sim}{u}\| = 1$.

the normal vector.

$$(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0.$$

any vector in a plane

a vector in the plane.

$$\underset{\sim}{r}(t, s) = \underset{\sim}{a} + t\underset{\sim}{v} + s\underset{\sim}{w}$$

$$\underset{\sim}{v} \times \underset{\sim}{w} = \lambda \underset{\sim}{n}, \quad \lambda \in \mathbb{R}.$$

Angle between 2 Planes:

$$\cos \angle (P_1, P_2) = \frac{|\underset{\sim}{n_1} \cdot \underset{\sim}{n_2}|}{\|\underset{\sim}{n_1}\| \, \|\underset{\sim}{n_2}\|}$$

normal vectors.

Intersection of 2 Planes

$\underset{\sim}{n_1} \times \underset{\sim}{n_2}$ is a direction vector of the line of intersection.

Continuously Differentiable:

If the derivative itself is continuous.

Tangent Vector:

$$\underset{\sim}{R}'(a) = \begin{pmatrix} f'(a) \\ g'(a) \\ h'(a) \end{pmatrix}.$$

Arc Length:

$$\int_a^b \|\underset{\sim}{R}'(t)\| \, dt = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \, dt.$$

Surface: $\{(x, y, f(x,y)) : (x,y) \in D\}$.

Level Curves (sets).

$k \in \mathbb{R}$ (can be negative).

$\{(x,y) \in \mathbb{R}^2 : f(x,y) = k\}$.

$\{(x,y,z) \in \mathbb{R}^3 : f(x,y,z) = k\}$.

Chap III

Limits.

$$\lim_{(x,y) \to (a,b)} f(x,y) = L.$$

$$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon.$$

Showing a limit does not exist:

① Let $x = ky$ for different k.

② Let $x = k$ for some constant k.

③ Let $x = ky^p$.

Must ensure $x = a \Rightarrow kb$ or k or $kb^p = a$.

Showing a limit does not exist.

- Transformation by limit operations. • show that $\lim \frac{f(a+h,b+k)-f(a,b)-f_x h-f_y k}{\sqrt{h^2+k^2}}=0$.
- Squeeze Theorem.

Limit Operations: The prereq is that the resulting limits both exist!

Continuity:

Iff: $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

$h(x,y) = (g \circ f)(x,y) = g(f(x,y))$ is continuous at (a,b) iff $f(x,y)$ is continuous at (a,b) and $g(z)$ is continuous at $f(a,b)$.

Partial Derivative.

$$\frac{\partial f}{\partial x} = f_x(x,y) := \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_{xy} = (f_x)_y = \frac{\partial z^2}{\partial x \partial y}.$$

first then.

Clairaut's Theorem.

If f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)\ \forall (a,b)\in D$.

Interior Point.

P is an interior (point) $\Leftrightarrow$.

$\exists\, \varepsilon>0,\ B_\varepsilon(P) := \{Q\in\mathbb{R}^n : d(P,Q)<\varepsilon\} \subseteq D$.

The interior of D is the set of IPs.

The boundary of D = D − {IP}.

Squeeze Theorem V2.

$|f(x,y)-L| \le g(x,y)\ \forall (x,y)$ near (a,b).

and $\lim_{(x,y)\to(a,b)} g(x,y) = 0$

then $\lim f(x,y) = L$.

Differentiability:

① $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

OR ② $\lim_{h \to 0} \frac{f(a+h) - f(a) - ch}{|h|} = 0.$

In multivariable cases,

$$\lim_{(h,k) \to (0,0)} \frac{f(a+h, b+k) - f(a,b) - \ell(h,k)}{\sqrt{h^2+k^2}} = 0.$$

The map $(x,y) \mapsto Df_{(a,b)}(x-a, y-b) + f(a,b)$

If differentiable, then

$$Df_{(a,b)}(h,k) = f_x(a,b)h + f_y(a,b)k.$$

If $g(a,b) \neq 0$.

$$D\left(\frac{f}{g}\right)_{(a,b)} = \frac{1}{(g(a,b))^2}\left(g(a,b)Df_{(a,b)} - f(a,b)Dg_{(a,b)}\right).$$

Differentiability Theorem ①.

If f_x and f_y are defined on D and are continuous at (a,b), then f is differentiable at (a,b).

(The converse might not be true).

Linear Approximations.

$$f(a+h, b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k.$$

For any function $f, f_x, f_y, \ldots,$ that 'potentially' contains multiple variables, use the multivariate version of chain rule.

- Suppose $f(x,y) = f(x(t), y(t))$
  then $f'(x,y) = \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$

① Notes on Differentiability Theorem:

$\Delta f(a,b) = f_x(a,b)\Delta x + f_y(a,b)\Delta y + (\Delta x)^2 + (\Delta y)^2$ is true

$\Rightarrow$ differentiable.

$\Delta f(a,b) = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

with $\varepsilon_1, \varepsilon_2 \to 0, 0)$

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General Chain Rule.

f is differentiable with n variables x_i and each x_i is differentiable with m variables t_j.

$$\therefore \frac{\partial f}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j}.$$

Directional Derivatives.

$\hat{u}$ is a unit vector. $\underset{\sim}{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

$$Df_{(a,b)}(\underset{\sim}{u}) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

It is possible that f is not differentiable at (a,b) even if direction derivative exists for all directions.

If f is diff-able at (a,b), then we have.

$$Df_{(a,b)}(\underset{\sim}{u}) = f_x(a,b)u_1 + f_y(a,b)u_2.$$

Gradient Vector

$$\nabla f(a,b) = \begin{pmatrix} f_x(a,b) \\ f_y(a,b) \end{pmatrix}$$

$$\nabla f(a,b)(\underset{\sim}{u}) = \nabla f(a,b) \cdot \underset{\sim}{u}.$$

The graph of $Df_{(a,b,c)}$ translated to pass through $(a,b,c, f(a,b,c))$ is the tangent space of f at (a,b,c).

$\nabla f(a,b) \neq \underset{\sim}{0} \Rightarrow \nabla f(a,b)$ is orthogonal to the level curve of f that contains (a,b).

$\nabla f(a,b) \neq \underset{\sim}{0} \Rightarrow \nabla f(a,b)$ is the direction of max rate of change, $-\nabla f(a,b)$ is the direction of min rate of change.

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Implicit Differentiation.

Suppose $F(x,y,z)=k$ defines z as a differentiable function of x and y near (a,b,c), If $F_z(a,b,c)\neq 0$, then

$$\frac{\partial z}{\partial x}(a,b,c) = -\frac{F_x(a,b,c)}{F_z(a,b,c)}.$$

Calculating tangent planes.

Suppose S is a surface and has a point (a,b,c) where z can be expressed as a function of x,y, then we can find a tangent plane of S at (a,b,c).

Tangent vectors can be

$$\begin{pmatrix} 1 \\ 0 \\ -\frac{F_x(a,b,c)}{F_z(a,b,c)} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ -\frac{F_y(a,b,c)}{F_z(a,b,c)} \end{pmatrix}.$$

$\therefore$ A normal vector of the tangent plane is

$$\begin{pmatrix} \frac{\partial z}{\partial x}(a,b,c) \\ \frac{\partial z}{\partial y}(a,b,c) \\ -1 \end{pmatrix}$$

The equation of the tangent plane is

$$\frac{\partial z}{\partial x}(a,b,c)x + \frac{\partial z}{\partial y}(a,b,c)y - z = \frac{\partial z}{\partial x}(a,b,c)a + \frac{\partial z}{\partial y}(a,b,c)b - c.$$

Extrema

$f: D \to \mathbb{R}$, $D \subset \mathbb{R}^2$.

a disk centred at (a,b).

(a,b) is $\begin{cases} \text{local max if } f(x,y) \le f(a,b) \ \forall (x,y) \in B \cap D \\ \text{global max if } f(x,y) \le f(a,b) \ \forall (x,y) \in D. \end{cases}$

Critical Points: $\Leftrightarrow f_x(a,b) = 0 \wedge f_y(a,b) = 0$.

If f is diff-able, then a local extremum must be a critical point.

Saddle Point

$\Leftrightarrow$ A critical point of f.

AND. every disk D centred at (a,b) contains (x_1, y_1) s.t. $f(x_1, y_1) < f(a,b)$ and (x_2, y_2) s.t. $f(x_2, y_2) > f(a,b)$.

A set $D \subseteq \mathbb{R}^2$ is bounded if there is some $r > 0$ s.t. $D \subset B_r((0,0))$.

Extreme Value

$f: D \to \mathbb{R}$ is continuous on a closed and bounded set $D \subseteq \mathbb{R}^2$, then f has.

- A global max, and.
- A global min.

Lagrange Multipliers.

If the constraint is given by $g(x,y) = k$ which g is a diff-able function, then a critical point subject to the constraint C is given by.

$$\underbrace{\nabla f(a,b)}_{\text{can be zero}} = \lambda \underbrace{\nabla g(a,b)}_{\text{cannot}}, \underbrace{\lambda}_{\text{can be zero}} \in \mathbb{R}$$

Finding Extrema.

Without Constraint	With Constraint.
• Find critical points. • Find boundary points.	• All you do for extrema without constraints as well. • Find all (a,b) and λ s.t. $\nabla f(a,b) = \lambda \nabla g(a,b)$ and $g(a,b) = k$. • Find boundary points of the constraint

check corners as well!!!

Cauchy Inequality.

$$|xy| \le \frac{1}{2}(x^2 + y^2).$$

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Chap V.

Riemann Sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \Rightarrow \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

[sample point

Theorem "Integral Existence"

- For any # vars, if f is continuous on a closed and bounded domain D, then $\iint_D f(\cdots)\,dA$ exists.
  - Close: not open, contains boundary, e.g. $\{(x,y): x \geq 0\}$.
  - Bounded: is in a unit disk/sphere/...

Fubini's Theorem: → Applies to any dimension.

$$\iint_D f(x,y)\,dA = \underbrace{\int_a^b\int_c^d f(x,y)\,dx\,dy = \int_c^d\int_a^b f(x,y)\,dy\,dx}$$

Proof: Use Riemann Sum and pick min: K_{ij}, max: K_{ij} as sample points.

Type I Region.

- $D_I = \{(x,y): a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$ where g_1 and g_2 are continuous on $[a,b]$.
- Type II is defined similarly.
- If f is continuous on D_I, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

- Similarly for Type II D_{II}.

Polar Coordinates

- $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2+y^2$, $\theta = \tan^{-1}(\frac{y}{x})$ subject to quadrant!
- Polar Rectangle: $R = \{(r,\theta): a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$.
- $\Delta A = \Delta x \Delta y = \Delta r\,(r\Delta\theta)$.
- If f is continuous on R, then $\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\,\underline{r}\,dr\,d\theta$

Type I/II Polar Regions

$D_I = \{(r,\theta) : a \le r \le b,\ g_1(r) \le \theta \le g_2(r),\ 0 \le g_2(r) - g_1(r) \le 2\pi\}$

$D_{II} = \{(r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$.

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Area:

$A(D) = \iint_D 1\, dA$

Density / Average:

Density function $f(x,y)$,

$\text{density} = \dfrac{\boxed{\iint_D f(x,y)\,dA}}{\iint_D 1\, dA}$. → "Aggregate from density"

Chap VI

Type I/II/III Solids.

$E_I = \{(x,y,z) : \underline{(x,y) \in D},\ u_1(x,y) \le z \le u_2(x,y)\}$.

$E_{II} = \{\ \cdots\ (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$.

$E_{III} = \{\ \cdots\ \cdots\ \cdots\ \}$.

Cylindrical Coordinates:

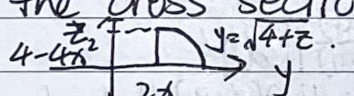
(r, θ, z) based on xy-plane.

(r, θ, x) ⋯ yz ⋯

(r, θ, y) ⋯ xz ⋯

Tricks when changing order of integral.

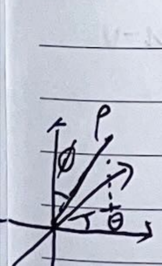
① Determine the outmost range first, say x, then draw the cross section parallel to $x=0$ in y-z. E.g.



② Be careful about $\min(f(x), g(y))$ if z is the inner. Figure out when $f(x) > g(y)$ and vice versa.

③ Avoid letting x depend on y, z depend on y, for example

Spherical Coordinates



- $\rho \ge 0$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$
- $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$
- $\rho = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$ subject to Q.

E.g:

- $\rho = c$: a sphere
- $\theta = c$: a vertical half-plane
- $\phi = c$: a half-cone

Spherical Wedge.

$E = \{(\rho, \theta, \phi) : a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$.

$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\boxed{\rho^2\sin\phi}\, d\rho\, d\theta\, d\phi$

Chap VII

Plane Transformations.

$S \leftrightarrow R$ is a <u>differentiable</u> map and has a differentiable inverse

↳ does not have boundaries → defined on an open D.

To show the image of S after $S \to T$.

- Look at the boundaries of S one by one.
- $T(u,v) = T(f(u,v), g(u,v))$ Let $x = f(u,v)$, $y = g(u,v)$.
  Find a function $h(x,y) = 0$ along the boundary.
- Resemble the mapped boundaries to get the image.

To show a transformation is planar.

- Show that $T(u,v) = (f(u,v), g(u,v))$ is differentiable,
  that is, ∇f, ∇g exist and are continuous
- Show that $T^{-1}(x,y) = (u(x,y), v(x,y))$ is differentiable,
  that is, ∇u, ∇v exist and are continuous.
- Don't need to deal with $\frac{\partial T}{\partial(u,v)}$ or $\frac{\partial T^{-1}}{\partial(x,y)}$

2D Jacobian

- Intuition: consider how a <u>small rectangle</u> (R with ΔA) changes to a parallelogram (S with $\Delta A'$) with sides $\underset{\sim}{a}$ and $\underset{\sim}{b}$
- $\Delta A \approx \|\underset{\sim}{a} \times \underset{\sim}{b}\| = \|\Delta u T_u(u,v) \times \Delta v T_v(u,v)\|$
  $= \|T_u(u,v) \times T_v(u,v)\| \Delta u \Delta v. = \left\|\frac{\partial(x,y)}{\partial(u,v)}\right\| \Delta A'$
- Define $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u.$

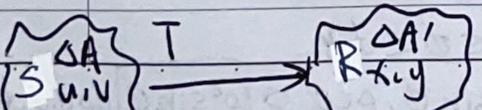
$$\therefore \iint_R f(x,y)\,dA = \iint_S f \circ T(u,v) \left\|\frac{\partial(x,y)}{\partial(u,v)}\right\| dA'$$
$$= \iint_S f(x(u,v), y(u,v)) \left\|\frac{\partial(x,y)}{\partial(u,v)}\right\| du\,dv.$$

Note: On occasions we can only express u and v in terms of x and y, then use $\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}$.
$\therefore$ We are basically using T^{-1} to send x-y back to u-v and do the computation there.

※ Trick. $\int_a^b \int_c^d f(u) g(v)\,du\,dv = \int_a^b g(v)\,dv \cdot \int_c^d f(u)\,du$.

3D Jacobian.

- Defined similarly.
- $\iiint_R f(x,y,z)\,dV = \iiint_S f \circ T(u,v,w) \left\|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right\| dV'$
  $= \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left\| \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \right\| du\,dv\,dw.$



$\Delta A = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta A'$

Chap VIII.

Line Integrals of Functions.

- Let $\vec{C}$ be $R(t) = (x(t), y(t))$. then
  $\int_{\vec{C}} f(x,y)\, ds = \int_a^b f(x(t), y(t)) \|R'(t)\|\, dt$. ☆
- The integral does not depend on the choice of R.
- If f is continuous along $\vec{C}$ and $\vec{C}$ is bounded and closed, then the integral always exists.
- $ds = \|R'(t)\|\, dt$.

Vector Fields.

A vector field on D is a map $\vec{F}$ that assigns to each point $P \in D$ a vector $\vec{F}(P)$ with initial point P.

Oriented curve.: $\vec{C} = (C, o)$.

Line Integrals of Vector Fields.

- $\int_{\vec{C}} \vec{F}\, d\vec{r} = \int_{\vec{C}} \vec{F} \cdot \underline{\underline{\vec{T}}}\, ds = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot R'(t)\, dt$.
- $\int_{-\vec{C}} \vec{F}\, d\vec{r} = -\int_{\vec{C}} \vec{F}\, d\vec{r}$

  $\vec{T}$ unit tangent vector, sometimes can use it directly, no need to find an R, esp. when spherical coord. is ok.
- $\int_{\vec{C}} \vec{F} \cdot d\vec{r} = \int_C X\, dx + Y\, dy + Z\, dz$
- $d\vec{r} = \vec{T}\, ds$

Gradient Theorem

- If $\vec{C} = (C, o)$ is a smooth (continuous first-order derivative.) oriented curve parameterized by $R(t)$, $a \le t \le b$. and ∇f is continuous along C, then
  $\int_{\vec{C}} \nabla f \cdot d\vec{r} = f(R(b)) - f(R(a))$

Conservative Fields

- $\vec{F}$ is conservative if $F = \nabla f$ for some differentiable potential function f. $\int_{\vec{C}} \vec{F} \cdot d\vec{r} = 0$ if C is a loop.

Test of Conservativity.

- To show $\vec{F}$ is conservative: $\boxed{1}$ find f. $\boxed{2}$ $\frac{\partial X}{\partial y} = \frac{\partial Y}{\partial x}$ [$\frac{\partial Z}{\partial x} = \frac{\partial X}{\partial z}$, $\frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z}$] (for 3-vars) iff (the domain on which f is defined is simply connected and X, Y, Z have continuous first-order partial derivatives).
- To show f is not, find $\oint_{\vec{C}} \vec{F}\, d\vec{r} \neq 0$ for a loop C.

Simply Connected $\Leftrightarrow$ every loop on D can be contracted to a point

1. Let the space be $\mathbb{R}^n$, D is not simply connected if there is a whole on D which a line can go thru.
2. A hollow D can be simply connected!

Simple Loops $\Leftrightarrow$ no self-intersections.

Green's Theorem:

* Let $\vec{C} = (C, o)$ be piesewise differentiable, simple loop in $\mathbb{R}^2$ that is oriented couter-clockwise, and let D be bounded by $\vec{C}$.

* Let $F = \binom{X}{Y}$ have continuous partial derivatives on a open region containing D. Then

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$$\int_{\vec{C}} \vec{F}\, d\vec{r} = \int_{\vec{C}} X\,dx + Y\,dy = \iint_D \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\, dA$$

notice the order!

Application of Green's Theorem with the same conditions *.

- Let $\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = 1$ to get the area of D.
  - Some choices: $\begin{cases} X=0 \\ Y=x \end{cases}$, $\begin{cases} X=-y \\ Y=0 \end{cases}$, $\begin{cases} X=-\frac{1}{2}y \\ Y=\frac{1}{2}x \end{cases}$.
- $A = \int_{\vec{C}} x\,dy = \int_{\vec{C}} -y\,dx = \frac{1}{2}\left(\int_{\vec{C}} x\,dy - y\,dx\right)$

2D Divergence

$$\operatorname{div} \vec{F} = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$ the order!

2D Flux with Conditions *.

- $\tilde{n}(x,y)$: the outward-pointing unit normal vector to $\vec{C}$.

$$\underbrace{\int_{\vec{C}} \vec{F}\cdot\tilde{n}\, ds}_{\text{outward flux of } \vec{F} \text{ across } \vec{C}} = \iint_D \operatorname{div}\vec{F}\, dA$$

- $\tilde{n} = \dfrac{\binom{y'(t)}{-x'(t)}}{\|R'(t)\|}$ s.t. $\tilde{n}\cdot T = 0$.
- Prove by letting $G = \binom{-Y}{X}$

Chap IX

Surface Parameterization.

- Intuitively, a surface is a composition of 2 mappings, the first sending two vars to a flat plane, the second sending the flat plane to a surface.
- $dS = \|R_u \times R_v\|\, dA$. If the surface is parallel to the flat plane, $\|R_u \times R_v\| = \|(0,0,1)\| = 1 \Rightarrow dS = dA$.

Surface Integrals of Functions:

- $\iint_S f(x,y,z)\, dS = \iint_D f(x(u,v), y(u,v), z(u,v))\, \|R_u \times R_v\|\, dA$ ☆
- If $z = g(x,y)$, then $= \iint_D f(x,y,g(x,y))\left(\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\right) dA$ ☆
- $\operatorname{Area}(S) = \iint_S 1\, dS = \iint_D \|R_u \times R_v\|\, dA$

Surface Orientation.

A (differentiable) surface on $\mathbb{R}^3$ is orientable if it is possible to define, for every $(x,y,z)\in S$, $\exists\, \hat{n}(x,y,z)$ is normal to S with initial point (x,y,z), s.t. $\hat{n}$ varies continuously with (x,y,z).

※ Exception: Möbius strip.
An orientation is upward if its $\underline{k}$ component is positive. vice versa.
There is also so-called inward/outward orientation.

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Flux thru a Membrane

- The flux is $comp_{\hat{n}} \vec{F} \cdot \Delta S = \vec{F} \cdot \hat{n} \Delta S$
- The flux of $\vec{F}$ across $\vec{S}$ is

  ☆ $\iint_{\vec{S}} \vec{F} \cdot d\vec{S} = \iint_{\vec{S}} \vec{F} \cdot \hat{n}\, dS.$

- If $R_u \times R_v$ is non-zero at every point and is in the same direction as $\hat{n}$ then

  ☆ $\iint_{\vec{S}} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (R_u \times R_v)\, dA.$

- Common parameterization: $(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$

  $\underline{R}_\theta \times \underline{R}_\phi = -r^2(\cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi).$

  $\underline{R}_r \times \underline{R}_\theta = -r(\cos\theta\cos\phi\sin\phi, \sin\theta\cos\phi\sin\phi, -\sin^2\phi)$

  $R_r \times R_\phi = r(-\sin\theta, \cos\theta, 0)$

- If $R(x,y) = (x, y, g(x,y))$, $\vec{F} = (X, Y, Z)$

  ☆ $\iint_{\vec{S}} \vec{F}\, d\vec{S} = \iint_D (-X\frac{\partial g}{\partial x} - Y\frac{\partial g}{\partial y} + Z)\, dA$ (for upward).

  Only use this when D can be described as rectangles or Type I/II regions. Be careful with Jacobian.

※ Normal Vector VS Tangent Vector.

If we have a function $f(x,y)$, then $\nabla f_{(a,b)} \begin{pmatrix} f_x \\ f_y \end{pmatrix}$ is normal to k-level-set. where $k = f(a,b)$.

If we have a parameterization, then $R'(t)$ is tangent to the line. $R_u \times R_v$ is normal to the surface.

$(x,y) \mapsto (x-a)R_u(a,b) + (y-b)R_v(a,b) + R(a,b)$.
is the tangent plane at (a,b).

Divergence and Gauss' Theorem

- $\text{div}\, \vec{F}(x,y,z) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}$
- Gauss' Theorem: Let E be a solid region where the boundary surface $\vec{S}$ of E is piesewise smooth and equipped with outward orientation. $\vec{F}$ has continuous first-order partial derivatives. We have

  ☆ $\iint_{\vec{S}} \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\, \vec{F}\, dV$

Surface Orientation

Right-hand Rule: thumb: $\hat{n}$, other fingers: $\vec{C}$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ X & Y & Z \end{vmatrix}$$

Stoke's Theorem.

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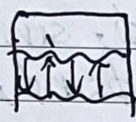
- S: piesewise differentiable.
  C: simple, closed, piesewise-smooth loop.
  $\vec{S}$: $(S, \hat{n})$. $\vec{C}$: $(C$, induced orientation$)$
  $\vec{F}$: continuous first-order partial derivatives.
  ☆ $\iint_{\vec{S}} \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{\vec{C}} \vec{F}\, d\vec{r}$

※ Tricks about Stoke's

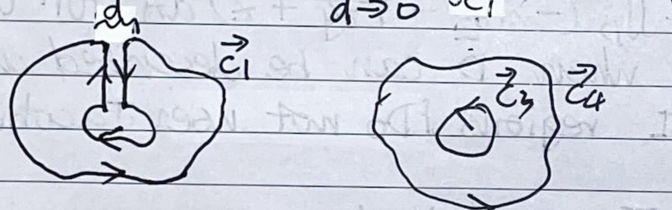
① If $\vec{S}_1$ and $\vec{S}_1$ share the same $\vec{C}$ and orientation,

$$\iint_{\vec{S}_1} \operatorname{curl} \vec{F}\, d\vec{S}_1 = \iint_{\vec{S}_2} \operatorname{curl} \vec{F}\, d\vec{S}_2$$

② 

S_1 and $S_2 \cup \{$side surface$\}$ share the same $\vec{C}$ and orientation.

③ Similarly, though Green's theorem requires simple loops, we can let $\lim_{d \to 0} \int_{\vec{C}_1} \vec{F}\, d\vec{r} = \int_{\vec{C}_2} \vec{F}\, d\vec{r}$.



$\vec{C}_2 = \vec{C}_3 \cup \vec{C}_4$

④ $\operatorname{curl} \vec{F}(x_0, y_0, z_0) \cdot \hat{n}(x_0, y_0, z_0) = \lim_{a \to 0} \frac{1}{\pi a^2} \int_{\vec{C}_a} \vec{F} \cdot d\vec{r}$

θ in Spherical Coordinates.

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x > 0 \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x < 0 \text{ and } y \geq 0 \\ -\pi + \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x < 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0. \end{cases}$$

For any field G, $\operatorname{div}(\operatorname{curl} G) = 0$.

Test of saddle points: $D := f_{xx} \cdot f_{yy} - [f_{xy}]^2$

$D > 0$, $f_{xx} > 0$ min / $D > 0$, $f_{xx} < 0$ max
$D < 0$ saddle / $D = 0$ inconclusive.

Area of Arc: $r\theta$. Area of Segment: $\frac{1}{2}(\theta - \sin\theta) r^2$

Area of Sector: $\frac{\theta r^2}{2}$ D

Volume of Spherical Cap. $\frac{1}{3}\pi h^2 (3p - h) = \frac{1}{6}\pi h (3r^2 + h^2)$.