Lecture 14: Differential Equations and Separation of Variables

Ordinary Differential Equations (ODEs)

Example 1. $\dfrac{dy}{dx} = f(x)$

Solution: $y = \int f(x)dx$. We consider these types of equations as solved.

Example 2. $\left(\dfrac{d}{dx} + x\right) y = 0 \quad \left(\text{or} \quad \dfrac{dy}{dx} + xy = 0\right)$

($\left(\dfrac{d}{dx} + x\right)$ is known in quantum mechanics as the *annihilation operator*.)

Besides integration, we have only one method of solving this so far, namely, substitution. Solving for $\dfrac{dy}{dx}$ gives:

$$\frac{dy}{dx} = -xy$$

The key step is to *separate variables*.

$$\frac{dy}{y} = -xdx$$

Note that all y-dependence is on the left and all x-dependence is on the right.

Next, take the antiderivative of both sides:

$$\begin{aligned}
\int \frac{dy}{y} &= -\int xdx \\
\ln|y| &= -\frac{x^2}{2} + c \quad \text{(only need one constant } c\text{)} \\
|y| &= e^c e^{-x^2/2} \quad \text{(exponentiate)} \\
y &= ae^{-x^2/2} \quad (a = \pm e^c)
\end{aligned}$$

Despite the fact that $e^c \neq 0, a = 0$ is possible along with all $a \neq 0$, depending on the initial conditions. For instance, if $y(0) = 1$, then $y = e^{-x^2/2}$. If $y(0) = a$, then $y = ae^{-x^2/2}$ (See Fig. 1).

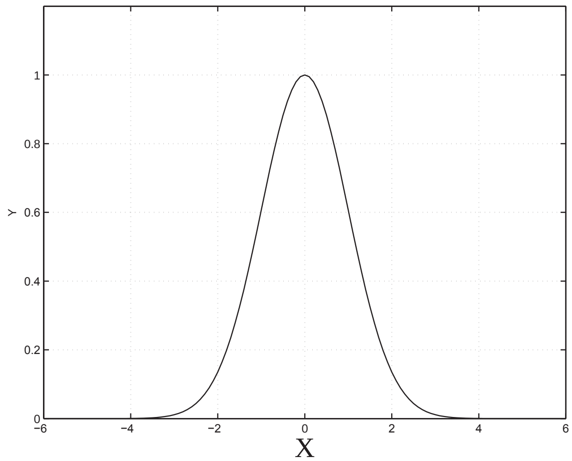

Figure 1: Graph of $y = e^{-\frac{x^2}{2}}$.

In general:

$$\begin{aligned}
\frac{dy}{dx} &= f(x)g(y) \\
\frac{dy}{g(y)} &= f(x)dx \quad \text{which we can write as} \\
h(y)dy &= f(x)dx \;\text{ where }\; h(y) = \frac{1}{g(y)}.
\end{aligned}$$

Now, we get an implicit formula for y:

$$H(y) = F(x) + c \quad (H(y) = \int h(y)dy; \quad F(x) = \int f(x)dx)$$

where $H' = h$, $F' = f$, and

$$y = H^{-1}(F(x) + c)$$

(H^{-1} is the inverse function.)

In the previous example:

$$\begin{aligned}
f(x) &= x; \quad F(x) = \frac{-x^2}{2}; \\
g(y) &= y; \quad h(y) = \frac{1}{g(y)} = \frac{1}{y}, \quad H(y) = \ln|y|
\end{aligned}$$

Example 3 (Geometric Example). $\frac{dy}{dx} = 2\left(\frac{y}{x}\right)$.
Find a graph such that the slope of the tangent line is twice the slope of the ray from $(0, 0)$ to (x, y) seen in Fig. 2.

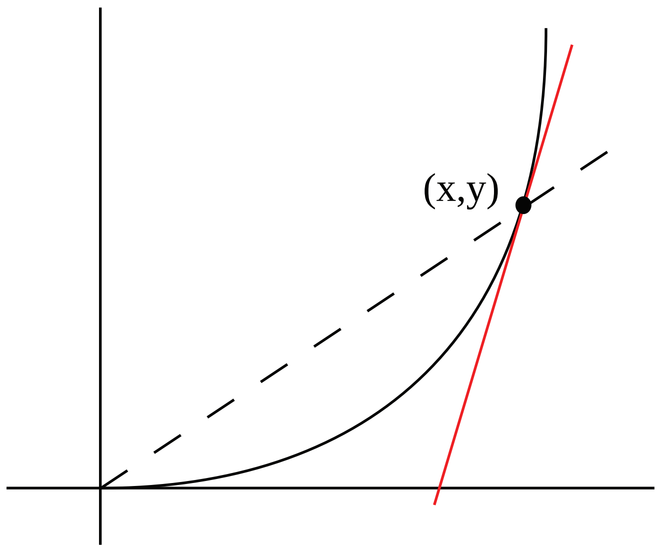

Figure 2: The slope of the tangent line (red) is twice the slope of the ray from the origin to the point (x, y).

$$\begin{aligned}
\frac{dy}{y} &= \frac{2dx}{x} \quad \text{(separate variables)} \\
\ln|y| &= 2\ln|x| + c \quad \text{(antiderivative)} \\
|y| &= e^c x^2 \quad \text{(exponentiate; remember, } e^{2\ln|x|} = x^2 \text{)}
\end{aligned}$$

Thus,

$$y = ax^2$$

Again, $a < 0$, $a > 0$ and $a = 0$ are all acceptable. Possible solutions include, for example,

$$\begin{aligned}
y &= x^2 \quad (a = 1) \\
y &= 2x^2 \quad (a = 2) \\
y &= -x^2 \quad (a = -1) \\
y &= 0x^2 = 0 \quad (a = 0) \\
y &= -2y^2 \quad (a = -2) \\
y &= 100x^2 \quad (a = 100)
\end{aligned}$$

Example 4. Find the curves that are perpendicular to the parabolas in Example 3.
We know that their slopes,

$$\frac{dy}{dx} = \frac{-1}{\text{slope of parabola}} = \frac{-x}{2y}$$

Separate variables:

$$y dy = \frac{-x}{2} dx$$

Take the antiderivative:

$$\frac{y^2}{2} = -\frac{x^2}{4} + c \qquad \Longrightarrow \qquad \frac{x^2}{4} + \frac{y^2}{2} = c$$

which is an equation for a family of ellipses. For these ellipses, the ratio of the x-semi-major axis to the y-semi-minor axis is $\sqrt{2}$ (see Fig. 3).

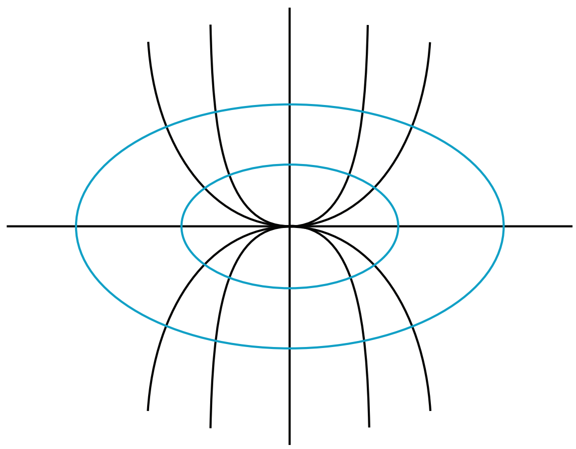

Figure 3: The ellipses are perpendicular to the parabolas.

Separation of variables leads to implicit formulas for y, but in this case you can solve for y.

$$y = \pm\sqrt{2\left(c - \frac{x^2}{4}\right)}$$

Exam Review

Exam 2 will be harder than exam 1 — be warned! Here's a list of topics that exam 2 will cover:

1. Linear and/or quadratic approximations
2. Sketches of $y = f(x)$
3. Maximum/minimum problems.
4. Related rates.
5. Antiderivatives. Separation of variables.
6. Mean value theorem.

More detailed notes on all of these topics are provided in the Exam 2 review sheet.