Lecture 4
Chain Rule, and Higher Derivatives

Chain Rule

We've got general procedures for differentiating expressions with addition, subtraction, and multiplication. What about composition?

Example 1. $y = f(x) = \sin x, x = g(t) = t^2$.
So, $y = f(g(t)) = \sin(t^2)$. To find $\dfrac{dy}{dt}$, write

$t_0 = t_0$	$t = t_0 + \Delta t$
$x_0 = g(t_0)$	$x = x_0 + \Delta x$
$y_0 = f(x_0)$	$y = y_0 + \Delta y$

$$\frac{\Delta y}{\Delta t} = \frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta t}$$

As $\Delta t \to 0$, $\Delta x \to 0$ too, because of continuity. So we get:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \leftarrow \textbf{The Chain Rule!}$$

In the example, $\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dx} = \cos x$.

$$\begin{aligned} \text{So,} \quad \frac{d}{dt}\left(\sin(t^2)\right) &= \left(\frac{dy}{dx}\right)\left(\frac{dx}{dt}\right) \\ &= (\cos x)(2t) \\ &= (2t)\left(\cos(t^2)\right) \end{aligned}$$

Another notation for the chain rule

$$\frac{d}{dt}f(g(t)) = f'(g(t))g'(t) \qquad \left(\text{or} \quad \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)\right)$$

Example 1. (continued) Composition of functions $f(x) = \sin x$ and $g(x) = x^2$

$$\begin{array}{rcccl} (f \circ g)(x) & = & f(g(x)) & = & \sin(x^2) \\ (g \circ f)(x) & = & g(f(x)) & = & \sin^2(x) \\ \text{Note:} \quad f \circ g & \neq & g \circ f. & & \textit{Not Commutative!} \end{array}$$

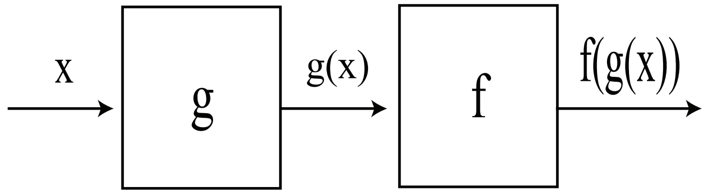

Figure 1: Composition of functions: $f \circ g(x) = f(g(x))$

Example 2. $\dfrac{d}{dx}\cos\left(\dfrac{1}{x}\right) = ?$

Let $u = \dfrac{1}{x}$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
\frac{dy}{du} &= -\sin(u); \qquad \frac{du}{dx} = -\frac{1}{x^2} \\
\frac{dy}{dx} &= \frac{\sin(u)}{x^2} = (-\sin u)\left(\frac{-1}{x^2}\right) = \frac{\sin\left(\frac{1}{x}\right)}{x^2}
\end{aligned}$$

Example 3. $\dfrac{d}{dx}\left(x^{-n}\right) = ?$

There are two ways to proceed. $x^{-n} = \left(\dfrac{1}{x}\right)^n$, or $x^{-n} = \dfrac{1}{x^n}$

1. $\dfrac{d}{dx}\left(x^{-n}\right) = \dfrac{d}{dx}\left(\dfrac{1}{x}\right)^n = n\left(\dfrac{1}{x}\right)^{n-1}\left(\dfrac{-1}{x^2}\right) = -nx^{-(n-1)}x^{-2} = -nx^{-n-1}$
2. $\dfrac{d}{dx}\left(x^{-n}\right) = \dfrac{d}{dx}\left(\dfrac{1}{x^n}\right) = nx^{n-1}\left(\dfrac{-1}{x^{2n}}\right) = -nx^{-n-1}$ (Think of x^n as u)

Higher Derivatives

Higher derivatives are derivatives of derivatives. For instance, if $g = f'$, then $h = g'$ is the second derivative of f. We write $h = (f')' = f''$.

Notations

$f'(x)$	Df	$\frac{df}{dx}$
$f''(x)$	$D^2 f$	$\frac{d^2 f}{dx^2}$
$f'''(x)$	$D^3 f$	$\frac{d^3 f}{dx^3}$
$f^{(n)}(x)$	$D^n f$	$\frac{d^n f}{dx^n}$

Higher derivatives are pretty straightforward —- just keep taking the derivative!

Example. $D^n x^n = ?$
Start small and look for a pattern.

$$\begin{aligned}
Dx &= 1 \\
D^2x^2 &= D(2x) = 2 \quad (= 1 \cdot 2) \\
D^3x^3 &= D^2(3x^2) = D(6x) = 6 \quad (= 1 \cdot 2 \cdot 3) \\
D^4x^4 &= D^3(4x^3) = D^2(12x^2) = D(24x) = 24 \quad (= 1 \cdot 2 \cdot 3 \cdot 4) \\
D^n x^n &= n! \leftarrow \text{we guess, based on the pattern we're seeing here.}
\end{aligned}$$

The notation $n!$ is called "n factorial" and defined by $n! = n(n-1)\cdots 2 \cdot 1$

Proof by Induction: We've already checked the base case ($n = 1$).

Induction step: Suppose we know $D^n x^n = n!$ (n^{th} case). Show it holds for the $(n+1)^{st}$ case.

$$\begin{aligned}
D^{n+1}x^{n+1} &= D^n\left(Dx^{n+1}\right) = D^n\left((n+1)x^n\right) = (n+1)D^n x^n = (n+1)(n!) \\
D^{n+1}x^{n+1} &= (n+1)!
\end{aligned}$$

Proved!