Lecture 10: Related Rates

Example 1. Police are 30 feet from the side of the road. Their radar sees your car approaching at 80 feet per second when your car is 50 feet away from the radar gun. The speed limit is 65 miles per hour (which translates to 95 feet per second). Are you speeding?

First, draw a diagram of the setup (as in Fig. 1):

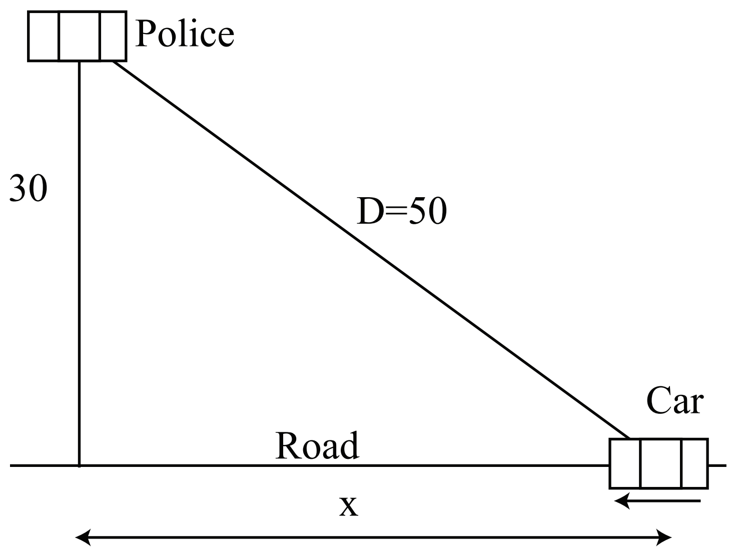

Figure 1: Illustration of example 1: triangle with the police, the car, the road, D and x labelled.

Next, give the variables names. The important thing to figure out is which variables are changing.

At $D = 50$, $x = 40$. (We know this because it's a 3-4-5 right triangle.) In addition, $\frac{dD}{dt} = D' = -80$. D' is negative because the car is moving in the $-x$ direction. Don't plug in the value for D yet! D is changing, and it depends on x.

The Pythagorean theorem says

$$30^2 + x^2 = D^2$$

Differentiate this equation with respect to time (implicit differentiation:

$$\frac{d}{dt}\left(30^2 + x^2 = D^2\right) \implies 2xx' = 2DD' \implies x' = \frac{2DD'}{2x}$$

Now, plug in the instantaneous numerical values:

$$x' = \frac{50}{40}(-80) = -100\frac{\text{feet}}{\text{s}}$$

This exceeds the speed limit of 95 feet per second; you are, in fact, speeding.

There is another, longer, way of solving this problem. Start with

$$D = \sqrt{30^2 + x^2} = (30^2 + x^2)^{1/2}$$

$$\frac{d}{dt}D = \frac{1}{2}(30^2 + x^2)^{-1/2}(2x\frac{dx}{dt})$$

Plug in the values:

$$-80 = \frac{1}{2}(30^2 + 40^2)^{-1/2}(2)(40)\frac{dx}{dt}$$

and solve to find

$$\frac{dx}{dt} = -100\frac{\text{feet}}{\text{s}}$$

(A third strategy is to differentiate $x = \sqrt{D^2 - 30^2}$). It is easiest to differentiate the equation in its simplest algebraic form $30^2 + x^2 = D^2$, our first approach.

The general strategy for these types of problems is:

1. Draw a picture. Set up variables and equations.
2. Take derivatives.
3. Plug in the given values. Don't plug the values in until *after* taking the derivatives.

Example 2. Consider a conical tank. Its radius at the top is 4 feet, and it's 10 feet high. It's being filled with water at the rate of 2 cubic feet per minute. How fast is the water level rising when it is 5 feet high?

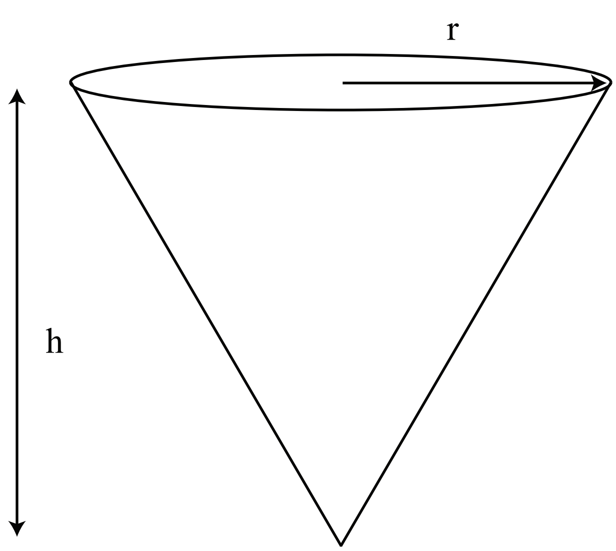

Figure 2: Illustration of example 2: inverted cone water tank.

From Fig. 2), the volume of the tank is given by

$$V = \frac{1}{3}\pi r^2 h$$

The key here is to draw the two-dimensional cross-section. We use the letters r and h to represent the variable radius and height of the water at any level. We can find the relationship between r and h from Fig. 3) using similar triangles.

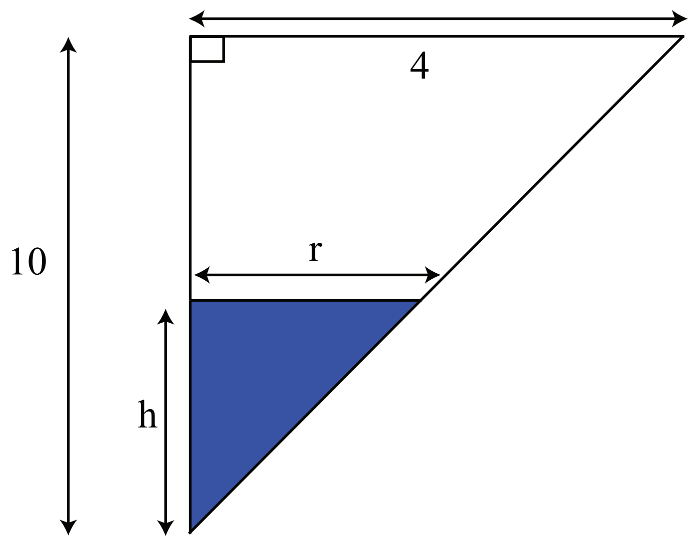

Figure 3: Relating r and h.

From Fig. 3), we see that

$$\frac{r}{h} = \frac{4}{10}$$

or, in other words,

$$r = \frac{2}{5}h$$

Plug this expression for r back into V to get

$$V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h = \frac{4}{3(25)}\pi h^3$$

$$\frac{dV}{dt} = V' = \frac{4}{25}\pi h^2 h'$$

Now, plug in the numbers ($\frac{dV}{dt} = 2$, $h = 5$):

$$2 = \left(\frac{4}{25}\right)\pi(5)^2 h'$$

$$h' = \frac{1}{2\pi}$$

Related rates also arise on Problem Set 3 (Fig. 4). There's a part II margin of error problem involving a satellite, where you're asked to find $\frac{\Delta L}{\Delta h}$.

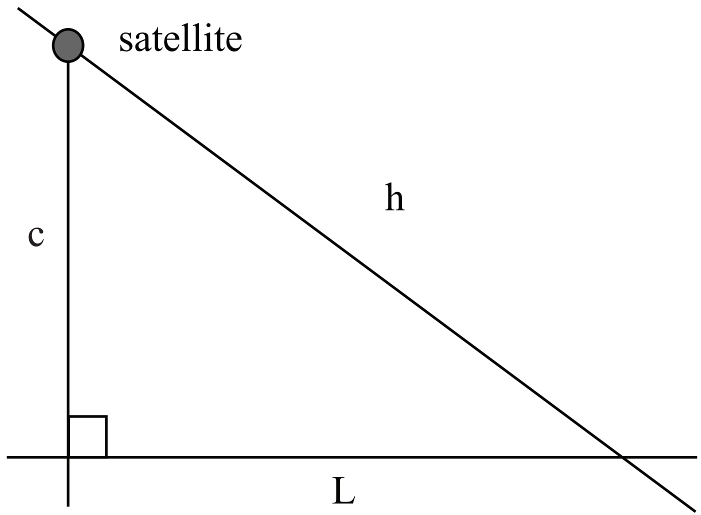

Figure 4: Illustration of the satellite problem.

$$
\begin{aligned}
L^2 + c^2 &= h^2 \\
2LL' &= 2hh' \\
\text{Hence,} \quad \frac{\Delta L}{\Delta h} \approx \frac{L'}{h'} &= \frac{h}{L}
\end{aligned}
$$

There is also a parabolic mirror problem based on similar ideas (Fig. 5).

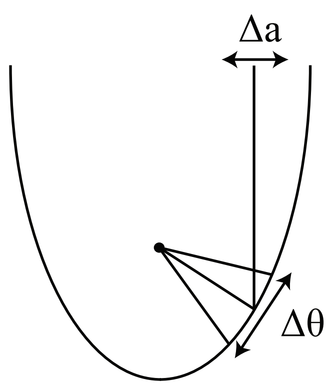

Figure 5: Illustration of the parabolic mirror problem.

Here, you want to find either $\frac{\Delta a}{\Delta \theta}$ or $\frac{\Delta \theta}{\Delta a}$. This type of sensitivity of measurement problem matters in every measurement problem, for instance predicting whether asteroids will hit Earth.