Lecture 19: Volumes by Disks and Shells

Disks and Shells

We will illustrate the 2 methods of finding volume through an example.

Example 1. A witch's cauldron

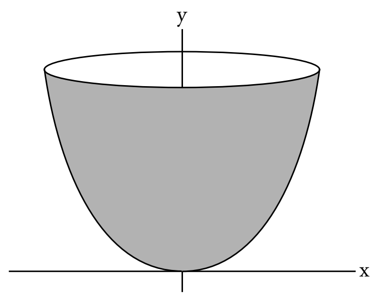

Figure 1: $y = x^2$ rotated around the y-axis.

Method 1: Disks

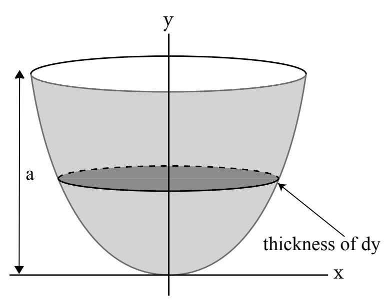

Figure 2: Volume by Disks for the Witch's Cauldron problem.

The area of the disk in Figure 2 is πx^2. The disk has thickness dy and volume $dV = \pi x^2 dy$. The volume V of the cauldron is

$$V = \int_0^a \pi x^2 \, dy \quad \text{(substitute} \quad y = x^2)$$

$$V = \int_0^a \pi y \, dy = \pi \frac{y^2}{2}\Big|_0^a = \frac{\pi a^2}{2}$$

If $a = 1$ meter, then $V = \frac{\pi}{2}a^2$ gives

$$V = \frac{\pi}{2}\ m^3 = \frac{\pi}{2}(100\text{ cm})^3 = \frac{\pi}{2}10^6\text{ cm}^3 \approx 1600\text{ liters} \quad \text{(a huge cauldron)}$$

Warning about units.

If $a = 100\,\text{cm}$, then

$$V = \frac{\pi}{2}(100)^2 = \frac{\pi}{2}10^4\text{ cm}^3 = \frac{\pi}{2}10 \sim 16\text{ liters}$$

But 100cm = 1m. Why is this answer different? The resolution of this paradox is hiding in the equation.

$$y = x^2$$

At the top, $100 = x^2 \implies x = 10\,\text{cm}$. So the second cauldron looks like Figure 3. By contrast, when

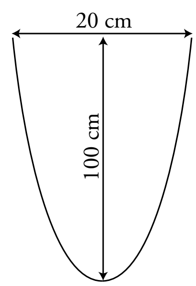

Figure 3: The skinny cauldron.

$a = 1\,\text{m}$, the top is ten times wider: $1 = x^2$ or $x = 1\,\text{m}$. Our equation, $y = x^2$, is not scale-invariant. The shape described depends on the units used.

Method 2: Shells

This really should be called the cylinder method.

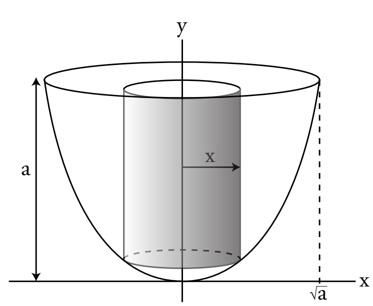

Figure 4: x = radius of cylinder. Thickness of cylinder = dx. Height of cylinder = $a - y = a - x^2$.

The thin shell/cylinder has height $a - x^2$, circumference $2\pi x$, and thickness dx.

$$
\begin{aligned}
dV &= (a - x^2)(2\pi x)dx \\
V &= \int_{x=0}^{x=\sqrt{a}} (a - x^2)(2\pi x)dx = 2\pi \int_0^{\sqrt{a}} (ax - x^3)dx \\
&= 2\pi \left(a\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^{\sqrt{a}} = 2\pi \left(\frac{a^2}{2} - \frac{a^2}{4} \right) = 2\pi \left(\frac{a^2}{4} \right) = \frac{\pi a^2}{2} \quad \text{(same as before)}
\end{aligned}
$$

Example 2. The boiling cauldron
Now, let's fill this cauldron with water, and light a fire under it to get the water to boil (at 100^oC). Let's say it's a cold day: the temperature of the air outside the cauldron is 0^oC. How much energy does it take to boil this water, i.e. to raise the water's temperature from 0^oC to 100^oC? Assume the

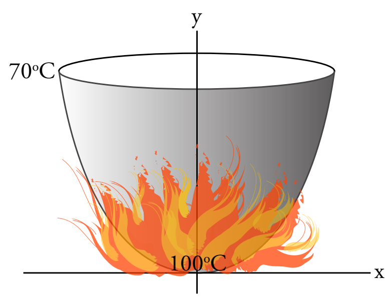

Figure 5: The boiling cauldron ($y = a = 1$ meter.)

temperature decreases linearly between the top and the bottom ($y = 0$) of the cauldron:

$$T = 100 - 30y \quad \text{(degrees Celsius)}$$

Use the method of disks, because the water's temperature is constant over each horizontal disk. The total heat required is

$$
\begin{aligned}
H &= \int_0^1 T(\pi x^2)dy \quad \text{(units are (degree)(cubic meters))} \\
&= \int_0^1 (100 - 30y)(\pi y)dy \\
&= \pi \int_0^1 (100y - 30y^2)dy = \pi(50y^2 - 10y^3)\Big|_0^1 = 40\pi\,\text{(deg.)m}^3
\end{aligned}
$$

How many calories is that?

$$\#\text{ of calories} = \frac{1\,\text{cal}}{\text{cm}^3 \cdot \text{deg}} (40\pi) \left(\frac{100\,\text{cm}}{1\,\text{m}} \right)^3 = (40\pi)(10^6)\,\text{cal} = 125 \times 10^3\,\text{kcal}$$

There are about 250 kcals in a candy bar, so there are about

$$\#\text{ of calories} = \left(\frac{1}{2}\,\text{candy bar} \right) \times 10^3 \approx 500\,\text{candy bars}$$

So, it takes about 500 candy bars' worth of energy to boil the water.

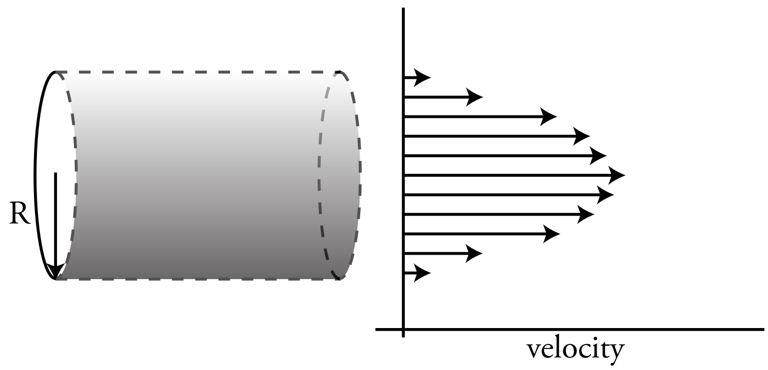

Figure 6: Flow is faster in the center of the pipe. It slows– "sticks"– at the edges (i.e. the inner surface of the pipe.)

Example 3. Pipe flow

Poiseuille was the first person to study fluid flow in pipes (arteries, capillaries). He figured out the velocity profile for fluid flowing in pipes is:

$$\begin{aligned} v &= c(R^2 - r^2) \\ v &= \text{speed} = \frac{\text{distance}}{\text{time}} \end{aligned}$$

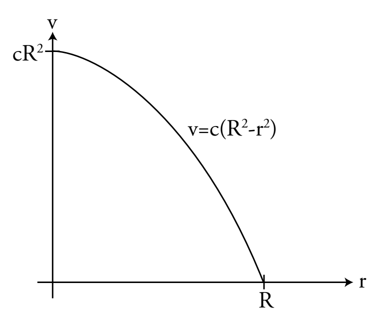

Figure 7: The velocity of fluid flow vs. distance from the center of a pipe of radius R.

The flow through the "annulus" (a.k.a ring) is (area of ring)(flow rate)

$$\text{area of ring} = 2\pi r dr \quad (\text{See Fig. } 8\text{: circumference } 2\pi r, \text{ thickness } dr)$$

v is analogous to the height of the shell.

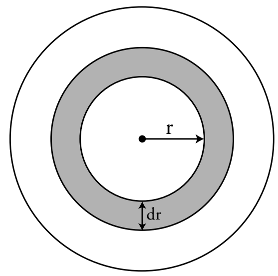

Figure 8: Cross-section of the pipe.

$$\begin{aligned}
\text{total flow through pipe} &= \int_0^R v(2\pi r dr) = c\int_0^R (R^2 - r^2)2\pi r dr \\
&= 2\pi c \int_0^R (R^2 r - r^3) dr = 2\pi c \left(\frac{R^2 r^2}{2} - \frac{r^4}{4} \right) \Big|_0^R \\
\text{flow through pipe} &= \frac{\pi}{2} c R^4
\end{aligned}$$

Notice that the flow is proportional to R^4. This means there's a big advantage to having thick pipes.

Example 4. Dart board

You aim for the center of the board, but your aim's not always perfect. Your number of hits, N, at radius r is proportional to e^{-r^2}.

$$N = ce^{-r^2}$$

This looks like:

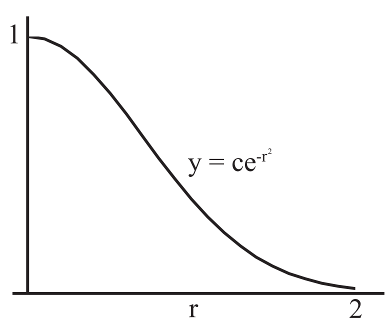

Figure 9: This graph shows how likely you are to hit the dart board at some distance r from its center.

The number of hits within a given ring with $r_1 < r < r_2$ is

$$c\int_{r_1}^{r_2} e^{-r^2}(2\pi r dr)$$

We will examine this problem more in the next lecture.