Lecture 8: Curve Sketching

Goal: To draw the graph of f using the behavior of f' and f''. We want the graph to be qualitatively correct, but not necessarily to scale.

Typical Picture: Here, y_0 is the minimum value, and x_0 is the point where that minimum occurs.

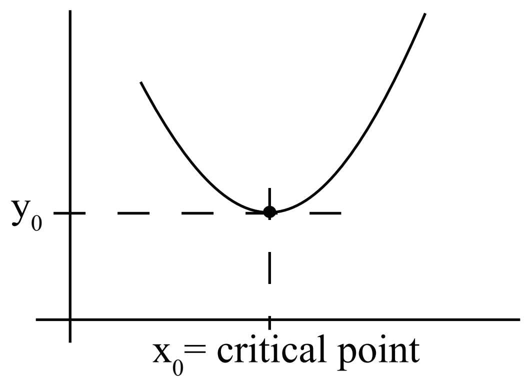

Figure 1: The critical point of a function

Notice that for $x < x_0$, $f'(x) < 0$. In other words, f is decreasing to the left of the critical point. For $x > x_0$, $f'(x) > 0$: f is increasing to the right of the critical point.

Another typical picture: Here, y_0 is the critical (maximum) value, and x_0 is the critical point. f is decreasing on the right side of the critical point, and increasing to the left of x_0.

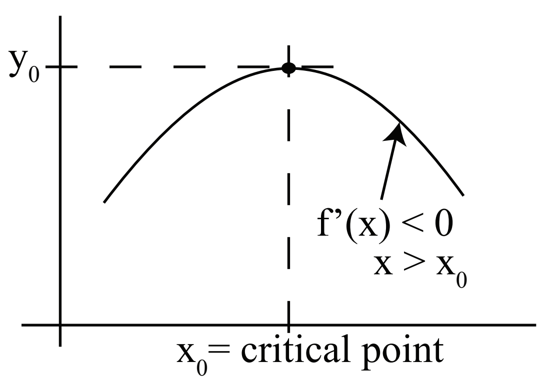

Figure 2: A concave-down graph

Rubric for curve-sketching

1. (Precalc skill) Plot the discontinuities of f — especially the infinite ones!
2. Find the critical points. These are the points at which $f'(x) = 0$ (usually where the slope changes from positive to negative, or vice versa.)
3. (a) Plot the critical points (and critical values), but only if it's relatively easy to do so.
   (b) Decide the sign of $f'(x)$ in between the critical points (if it's not already obvious).
4. (Precalc skill) Find and plot the zeros of f. These are the values of x for which $f(x) = 0$. Only do this if it's relatively easy.
5. (Precalc skill) Determine the behavior at the endpoints (or at $\pm\infty$).

Example 1. $y = 3x - x^3$

1. No discontinuities.
2. $y' = 3 - 3x^2 = 3(1 - x^2)$ so, $y' = 0$ at $x = \pm 1$.
3. (a) At $x = 1$, $y = 3 - 1 = 2$.
   (b) At $x = -1$, $y = -3 + 1 = -2$. Mark these two points on the graph.
4. Find the zeros: $y = 3x - x^3 = x(3 - x^2) = 0$ so the zeros lie at $x = 0, \pm\sqrt{3}$.
5. Behavior of the function as $x \to \pm\infty$.
   As $x \to \infty$, the x^3 term of y dominates, so $y \to -\infty$. Likewise, as $x \to -\infty$, $y \to \infty$.

Putting all of this information together gives us the graph as illustrated in Fig. 3)

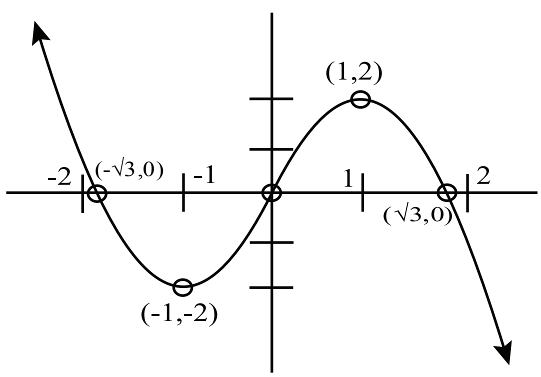

Figure 3: Sketch of the function $y = 3x - x^3$. Note the labeled zeros and critical points

Let us do step 3b (the sign of f') to double-check for consistency.

$$y' = 3 - 3x^2 = 3(1 - x^2)$$

$y' > 0$ when $|x| < 1$; $y' < 0$ when $|x| > 1$. Sure enough, y is increasing between $x = -1$ and $x = 1$, and is decreasing everywhere else.

Example 2. $y = \frac{1}{x}$.

This example illustrates why it's important to find a function's discontinuities before looking at the properties of its derivative. We calculate

$$y' = \frac{-1}{x^2} < 0$$

Warning: The derivative is never positive, so you might think that y is always decreasing, and its graph looks something like that in Fig. 4.

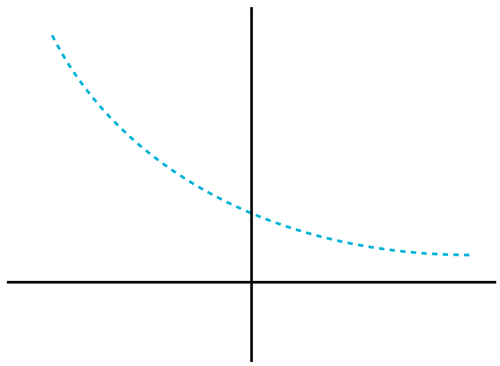

Figure 4: A monotonically decreasing function

But as you probably know, the graph of $\frac{1}{x}$ looks nothing like this! It actually looks like Fig. 5. In fact, $y = \frac{1}{x}$ is decreasing *except* at $x = 0$, where it jumps from $-\infty$ to $+\infty$. This is why we must watch out for discontinuities.

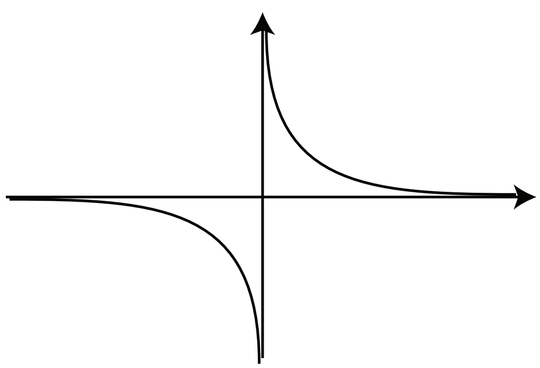

Figure 5: Graph of $y = \frac{1}{x}$.

Example 3. $y = x^3 - 3x^2 + 3x$.

$$y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$$

There is a critical point at $x = 1$. $y' > 0$ on both sides of $x = 1$, so y is increasing everywhere. In this case, the sign of y' doesn't change at the critical point, but the graph does level out (see Fig. 6.

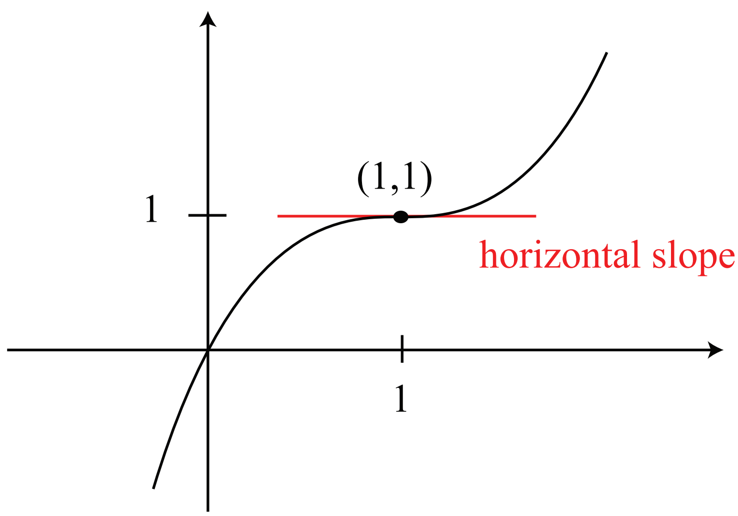

Figure 6: Graph of $y = y = x^3 - 3x^2 + 3x$

Example 4. $y = \frac{\ln x}{x}$ (Note: this function is only defined for $x > 0$)

What happens as x decreases towards zero? Let $x = 2^{-n}$. Then,

$$y = \frac{\ln 2^{-n}}{2^{-n}} = (-n \ln 2)2^n \to -\infty \,\text{as n} \to \infty$$

In other words, y decreases to $-\infty$ as x approaches zero.

Next, we want to find the critical points.

$$y' = \left(\frac{\ln x}{x}\right)' = \frac{x(\frac{1}{x}) - 1(\ln x)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$y' = 0 \implies 1 - \ln x = 0 \implies \ln x = 1 \implies x = e$$

In other words, the critical point is $x = e$ (from previous page). The critical value is

$$y(x)\mid_{x=e} = \frac{\ln e}{e} = \frac{1}{e}$$

Next, find the zeros of this function:

$$y = 0 \Leftrightarrow \ln x = 0$$

So $y = 0$ when $x = 1$.

What happens as $x \to \infty$? This time, consider $x = 2^{+n}$.

$$y = \frac{\ln 2^n}{2^n} = \frac{n \ln 2}{2^n} \approx \frac{n(0.7)}{2^n}$$

So, $y \to 0$ as $n \to \infty$. Putting all of this together gets us the graph in Fig. 7.

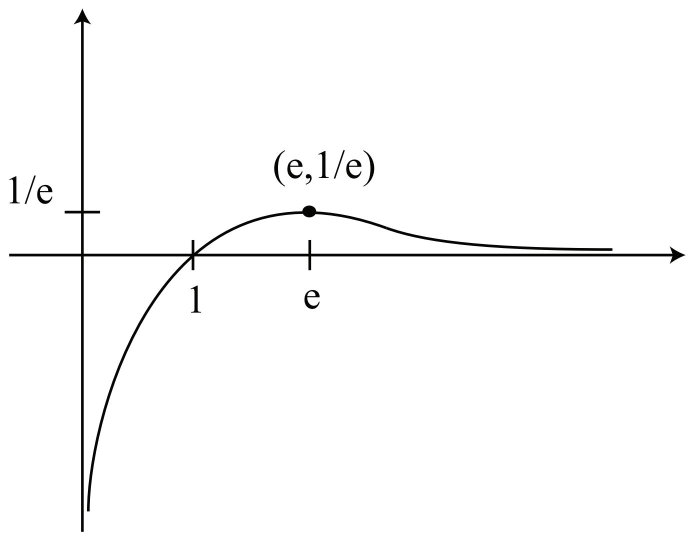

Figure 7: Graph of $y = \frac{\ln x}{x}$

Finally, let's double-check this picture against the information we get from step 3b:

$$y' = \frac{1 - \ln x}{x^2} > 0 \quad \text{for} \quad 0 < x < e$$

Sure enough, the function is increasing between 0 and the critical point.

2nd Derivative Information

When $f'' > 0$, f' is increasing. When $f'' < 0$, f' is decreasing. (See Fig. 8 and Fig. 9)

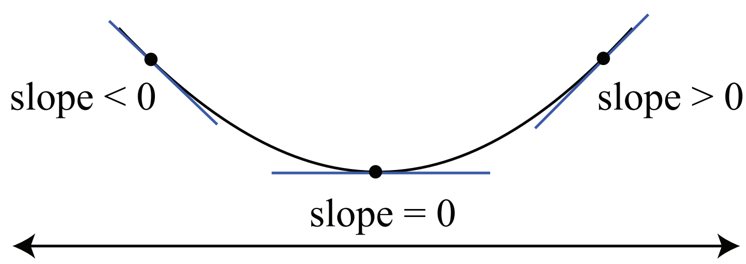

Figure 8: f is convex (concave-up). The slope increases from negative to positive as x increases.

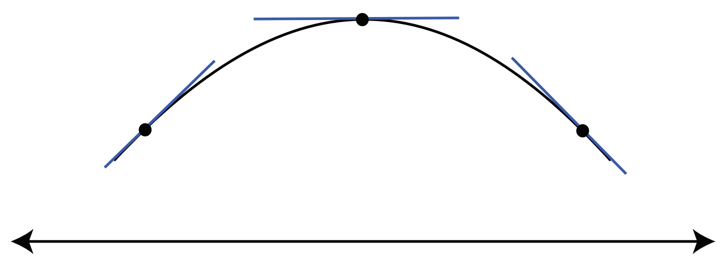

Figure 9: f is concave-down. The slope decreases from positive to negative as x increases.

Therefore, the sign of the second derivative tells us about concavity/convexity of the graph. Thus the second derivative is good for two purposes.

1. Deciding whether a critical point is a maximum or a minimum. This is known as the <u>second derivative</u> test.

$f'(x_0)$	$f''(x_0)$	Critical point is a:
0	negative	maximum
0	positive	minimum

2. Concave/convex "decoration."

The points where $f'' = 0$ are called *inflection points*. Usually, at these points the graph changes from concave up to down, or vice versa. Refer to Fig. 10 to see how this looks on Example 1.

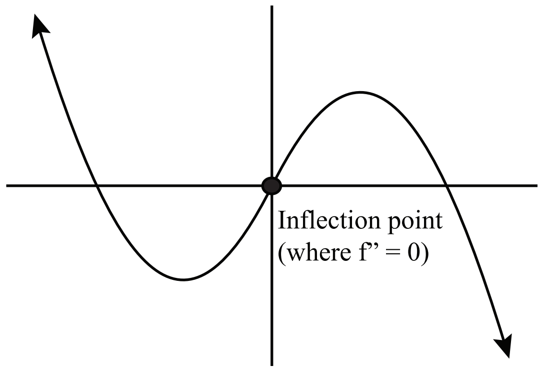

Figure 10: Inflection point: $y = 3x - x^3$, $y'' = -6x = 0$, at $x = 0$.