Lecture 17: Second Fundamental Theorem

Recall: First Fundamental Theorem of Calculus (FTC 1)

If f is continuous and $F' = f$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$

We can also write that as
$$\int_a^b f(x)dx = \int f(x)dx\Big|_{x=a}^{x=b}$$
Do all continuous functions have antiderivatives? Yes. However...
What about a function like this?
$$\int e^{-x^2}dx = ??$$
Yes, this antiderivative exists. No, it's not a function we've met before: it's a new function.

The new function is defined as an integral:
$$F(x) = \int_0^x e^{-t^2}dt$$
It will have the property that $F'(x) = e^{-x^2}$.

Other new functions include antiderivatives of $e^{-x^2}, x^{1/2}e^{-x^2}, \dfrac{\sin x}{x}, \sin(x^2), \cos(x^2), \ldots$

Second Fundamental Theorem of Calculus (FTC 2)

If $F(x) = \int_a^x f(t)dt$ and f is continuous, then
$$F'(x) = f(x)$$

Geometric Proof of FTC 2: Use the area interpretation: $F(x)$ equals the area under the curve between a and x.

$$\begin{aligned}
\Delta F &= F(x+\Delta x) - F(x) \\
\Delta F &\approx \text{(base)(height)} \approx (\Delta x)f(x) \qquad \text{(See Figure 1.)} \\
\frac{\Delta F}{\Delta x} &\approx f(x) \\
\text{Hence } \lim_{\Delta x \to 0}\frac{\Delta F}{\Delta x} &= f(x)
\end{aligned}$$

But, by the definition of the derivative:
$$\lim_{\Delta x \to 0}\frac{\Delta F}{\Delta x} = F'(x)$$

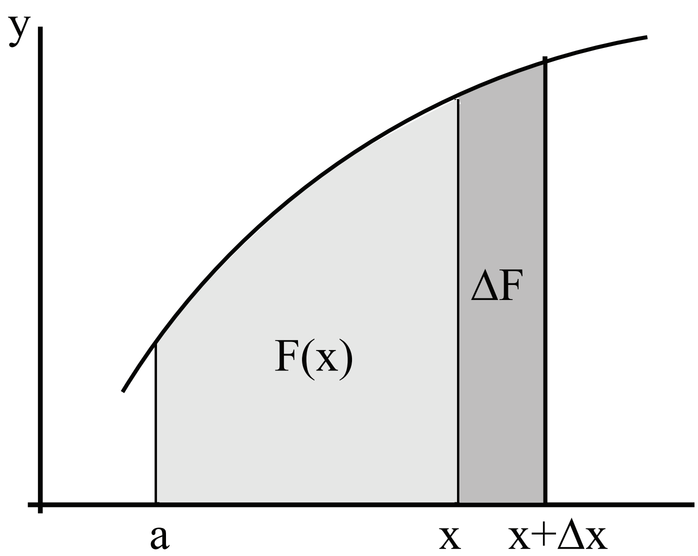

Figure 1: Geometric Proof of FTC 2.

Therefore,

$$F'(x) = f(x)$$

Another way to prove FTC 2 is as follows:

$$\begin{aligned}\frac{\Delta F}{\Delta x} &= \frac{1}{\Delta x}\left[\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt\right] \\ &= \frac{1}{\Delta x}\int_x^{x+\Delta x} f(t)dt \quad \text{(which is the “average value” of } f \text{ on the interval } x \le t \le x+\Delta x.\text{)}\end{aligned}$$

As the length Δx of the interval tends to 0, this average tends to $f(x)$.

Proof of FTC 1 (using FTC 2)

Start with $F' = f$ (we assume that f is continuous). Next, define $G(x) = \displaystyle\int_a^x f(t)dt$. By FTC2, $G'(x) = f(x)$. Therefore, $(F - G)' = F' - G' = f - f = 0$. Thus, $F - G =$ constant. (Recall we used the Mean Value Theorem to show this).

Hence, $F(x) = G(x) + c$. Finally since $G(a) = 0$,

$$\int_a^b f(t)dt = G(b) = G(b) - G(a) = [F(b) - c] - [F(a) - c] = F(b) - F(a)$$

which is FTC 1.

Remark. In the preceding proof G was a definite integral and F could be any antiderivative. Let us illustrate with the example $f(x) = \sin x$. Taking $a = 0$ in the proof of FTC 1,

$$G(x) = \int_0^x \cos t\, dt = \sin t\Big|_0^x = \sin x \quad \text{and } G(0) = 0.$$

If, for example, $F(x) = \sin x + 21$. Then $F'(x) = \cos x$ and

$$\int_a^b \sin x\,dx = F(b) - F(a) = (\sin b + 21) - (\sin a + 21) = \sin b - \sin a$$

Every function of the form $F(x) = G(x) + c$ works in FTC 1.

Examples of "new" functions

The *error function*, which is often used in statistics and probability, is defined as

$$\begin{aligned} \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt \\ \text{and } \lim_{x\to\infty} \operatorname{erf}(x) &= 1 \quad \text{(See Figure 2)} \end{aligned}$$

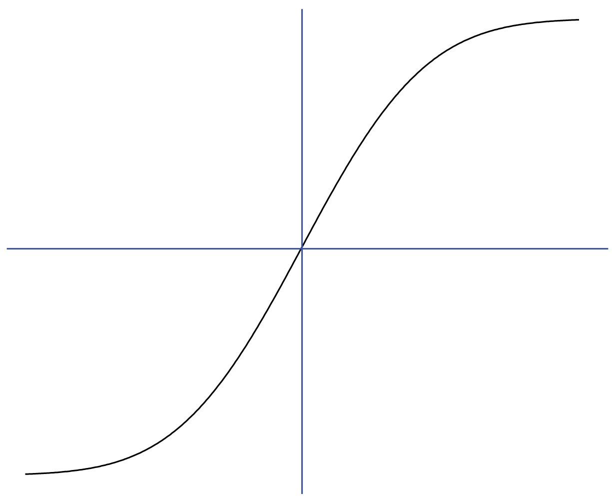

Figure 2: Graph of the error function.

Another "new" function of this type, called the *logarithmic integral*, is defined as

$$\operatorname{Li}(x) = \int_2^x \frac{dt}{\ln t}$$

This function gives the approximate number of prime numbers less than x. A common encryption technique involves encoding sensitive information like your bank account number so that it can be sent over an insecure communication channel. The message can only be decoded using a secret prime number. To know how safe the secret is, a cryptographer needs to know roughly how many 200-digit primes there are. You can find out by estimating the following integral:

$$\int_{10^{200}}^{10^{201}} \frac{dt}{\ln t}$$

We know that

$$\ln 10^{200} = 200\ln(10) \approx 200(2.3) = 460 \quad \text{and} \quad \ln 10^{201} = 201\ln(10) \approx 462$$

We will approximate to one significant figure: $\ln t \approx 500$ for $200 \leq t \leq 10^{201}$.

With all of that in mind, the number of 200-digit primes is roughly [1]

$$\int_{10^{200}}^{10^{201}} \frac{dt}{\ln t} \approx \int_{10^{200}}^{10^{201}} \frac{dt}{500} = \frac{1}{500}\left(10^{201} - 10^{200}\right) \approx \frac{9 \cdot 10^{200}}{500} \approx 10^{198}$$

There are LOTS of 200-digit primes. The odds of some hacker finding the 200-digit prime required to break into your bank account number are very very slim.

Another set of "new" functions are the Fresnel functions, which arise in optics:

$$\begin{aligned} C(x) &= \int_0^x \cos(t^2)dt \\ S(x) &= \int_0^x \sin(t^2)dt \end{aligned}$$

Bessel functions often arise in problems with circular symmetry:

$$J_0(x) = \frac{1}{2\pi}\int_0^\pi \cos(x \sin\theta)d\theta$$

On the homework, you are asked to find $C'(x)$. That's easy!

$$C'(x) = \cos(x^2)$$

We will use FTC 2 to discuss the function $L(x) = \displaystyle\int_1^x \frac{dt}{t}$ from first principles next lecture.

[1] The middle equality in this approximation is a very basic and useful fact

$$\int_a^b c\,dx = c(b-a)$$

Think of this as finding the area of a rectangle with base $(b-a)$ and height c. In the computation above, $a = 10^{200}, b = 10^{201}, c = \frac{1}{500}$