Lecture 16: First Fundamental Theorem of Calculus

Fundamental Theorem of Calculus (FTC 1)

If $f(x)$ is continuous and $F'(x) = f(x)$, then
$$\int_a^b f(x)dx = F(b) - F(a)$$

Notation: $F(x)\Big|_a^b = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$

Example 1. $F(x) = \dfrac{x^3}{3}$, $F'(x) = x^2$; $\displaystyle\int_a^b x^2 dx = \frac{x^3}{3}\Big|_a^b = \frac{b^3}{3} - \frac{a^3}{3}$

Example 2. Area under one hump of $\sin x$ (See Figure 1.)

$$\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2$$

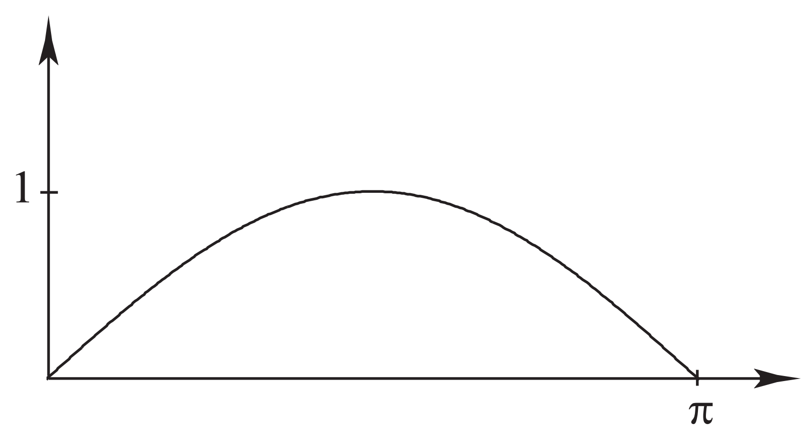

Figure 1: Graph of $f(x) = \sin x$ for $0 \le x \le \pi$.

Example 3. $\displaystyle\int_0^1 x^5 dx = \frac{x^6}{6}\Big|_0^1 = \frac{1}{6} - 0 = \frac{1}{6}$

Intuitive Interpretation of FTC:

$x(t)$ is a position; $\quad v(t) = x'(t) = \dfrac{dx}{dt}$ is the speed or rate of change of x.

$$\int_a^b v(t)dt = x(b) - x(a) \qquad \text{(FTC 1)}$$

R.H.S. is how far $x(t)$ went from time $t = a$ to time $t = b$ (difference between two odometer readings). L.H.S. represents speedometer readings.

$$\sum_{i=1}^{n} v(t_i)\Delta t \quad \text{approximates the sum of distances traveled over times } \Delta t$$

The approximation above is accurate if $v(t)$ is close to $v(t_i)$ on the i^{th} interval. The interpretation of $x(t)$ as an odometer reading is no longer valid if v changes sign. Imagine a round trip so that $x(b) - x(a) = 0$. Then the positive and negative velocities $v(t)$ cancel each other, whereas an odometer would measure the total distance not the net distance traveled.

Example 4. $\displaystyle\int_0^{2\pi} \sin x\, dx = -\cos x\Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0) = 0.$

The integral represents the sum of areas under the curve, above the x-axis minus the areas below the x-axis. (See Figure 2.)

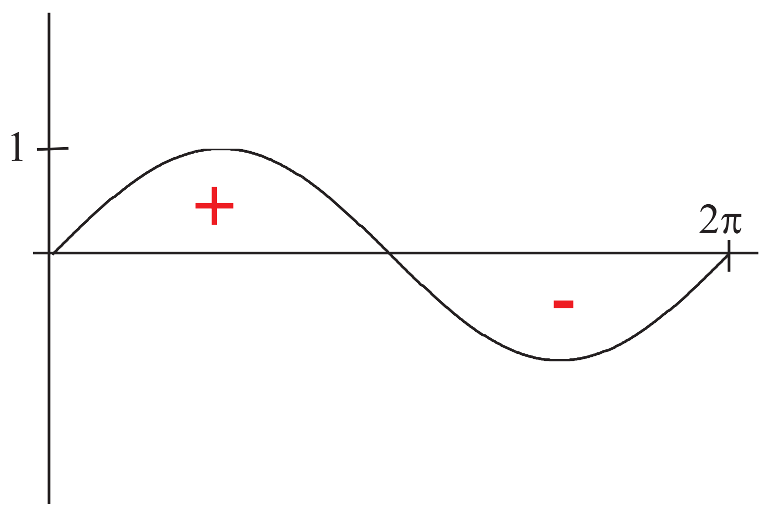

Figure 2: Graph of $f(x) = \sin x$ for $0 \leq x \leq 2\pi$.

Integrals have an important additive property (See Figure 3.)

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

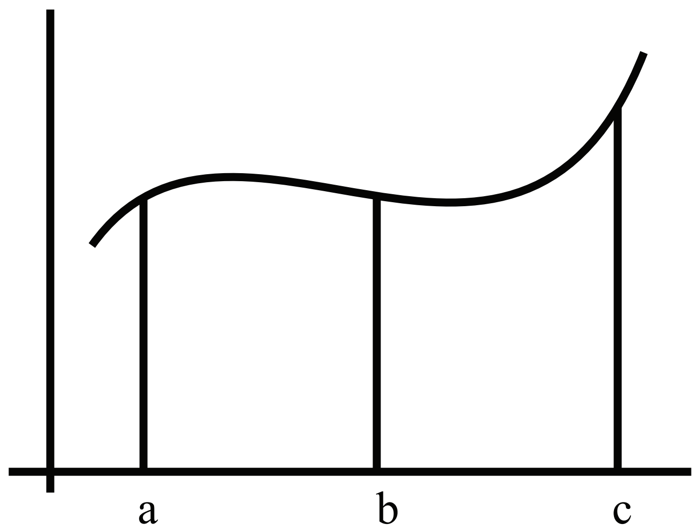

Figure 3: Illustration of the additive property of integrals

New Definition:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

This definition is used so that the fundamental theorem is valid no matter if $a < b$ or $b < a$. It also makes it so that the additive property works for a, b, c in any order, not just the one pictured in Figure 3.

Estimation:

If $f(x) \leq g(x)$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$ (only if $a < b$)

Example 5. Estimation of e^x
Since $1 \leq e^x$ for $x \geq 0$,

$$\int_0^1 1dx \leq \int_0^1 e^x dx$$

$$\int_0^1 e^x dx = e^x \Big|_0^1 = e^1 - e^0 = e - 1$$

Thus $1 \leq e - 1$, or $e \geq 2$.

Example 6. We showed earlier that $1 + x \leq e^x$. It follows that

$$\int_0^1 (1+x)dx \leq \int_0^1 e^x dx = e - 1$$

$$\int_0^1 (1+x)dx = \left(x + \frac{x^2}{2}\right)\Big|_0^1 = \frac{3}{2}$$

Hence, $\frac{3}{2} \leq e - 1$,or, $e \geq \frac{5}{2}$.

Change of Variable:

If $f(x) = g(u(x))$, then we write $du = u'(x)dx$ and

$$\int g(u)du = \int g(u(x))u'(x)dx = \int f(x)u'(x)dx \qquad \text{(indefinite integrals)}$$

For definite integrals:

$$\int_{x_1}^{x_2} f(x)u'(x)dx = \int_{u_1}^{u_2} g(u)du \quad \text{where } u_1 = u(x_1),\ u_2 = u(x_2)$$

Example 7. $\int_1^2 \left(x^3+2\right)^4 x^2 dx$

$$\text{Let } u = x^3 + 2. \quad \text{Then } du = 3x^2 dx \implies x^2 dx = \frac{du}{3};$$

$$x_1 = 1, x_2 = 2 \implies u_1 = 1^3 + 2 = 3,\ u_2 = 2^3 + 2 = 10,\ \text{and}$$

$$\int_1^2 \left(x^3+2\right)^4 x^2 dx = \int_3^{10} u^4 \frac{du}{3} = \frac{u^5}{15}\Big|_3^{10} = \frac{10^5 - 3^5}{15}$$