Lecture 9: Max/Min Problems

Example 1. $y = \dfrac{\ln x}{x}$ (same function as in last lecture)

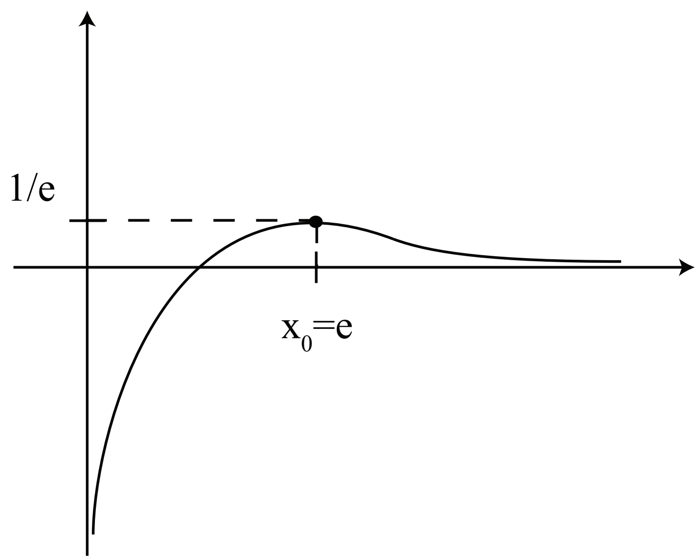

Figure 1: Graph of $y = \dfrac{\ln x}{x}$.

- What is the maximum value? Answer: $y = \dfrac{1}{e}$.
- Where (or at what point) is the maximum achieved? Answer: $x = e$. (See Fig. 1).)

Beware: Some people will ask "What is the maximum?". The answer is *not* e. You will get so used to finding the critical point $x = e$, the main calculus step, that you will forget to find the maximum value $y = \dfrac{1}{e}$. Both the critical point $x = e$ and critical value $y = \dfrac{1}{e}$ are important. Together, they form the point of the graph $(e, \dfrac{1}{e})$ where it turns around.

Example 2. Find the max and the min of the function in Fig. 2

Answer: If you've already graphed the function, it's obvious where the maximum and minimum values are. The point is to find the maximum and minimum without sketching the whole graph.

Idea: Look for the max and min among the critical points and endpoints.You can see from Fig. 2 that we only need to compare the heights or y-values corresponding to endpoints and critical points. (Watch out for discontinuities!)

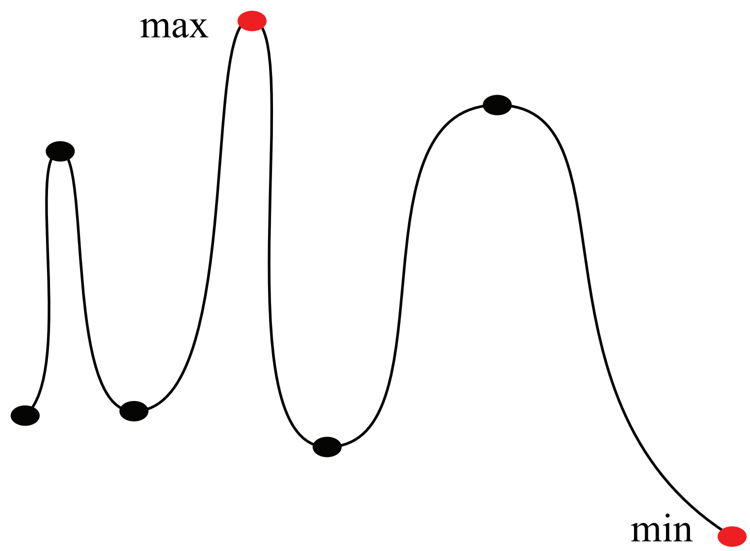

Figure 2: Search for max and min among critical points and endpoints

Example 3. Find the open-topped can with the least surface area enclosing a fixed volume, V.

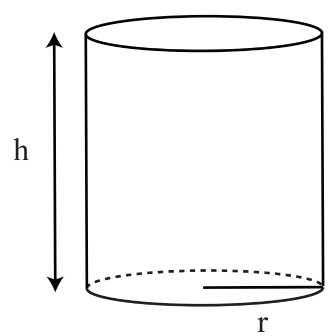

Figure 3: Open-topped can.

1. Draw the picture.
2. Figure out what variables to use. (In this case, r, h, V and surface area, S.)
3. Figure out what the constraints are in the problem, and express them using a formula. In this example, the constraint is

   $$V = \pi r^2 h = \text{constant}$$

   We're also looking for the surface area. So we need the formula for that, too:

   $$S = \pi r^2 + (2\pi r)h$$

   Now, in symbols, the problem is to minimize S with V constant.

4. Use the constraint equation to express everything in terms of r (and the constant V).

$$h = \frac{V}{2\pi r}; \quad S = \pi r^2 + (2\pi r)\left(\frac{V}{\pi r^2}\right)$$

5. Find the critical points (solve $dS/dr = 0$), as well as the endpoints. S will achieve its max and min at one of these places.

$$\frac{dS}{dr} = 2\pi r - \frac{2V}{r^2} = 0 \implies \pi r^3 - V = 0 \implies r^3 = \frac{V}{\pi} \implies r = \left(\frac{V}{\pi}\right)^{1/3}$$

We're not done yet. We've still got to evaluate S at the endpoints: $r = 0$ and "$r = \infty$".

$$S = \pi r^2 + \frac{2V}{r}, \quad 0 \le r < \infty$$

As $r \to 0$, the second term, $\frac{2}{r}$, goes to infinity, so $S \to \infty$. As $r \to \infty$, the first term πr^2 goes to infinity, so $S \to \infty$. Since $S = +\infty$ at each end, the minimum is achieved at the critical point $r = (V/\pi)^{1/3}$, not at either endpoint.

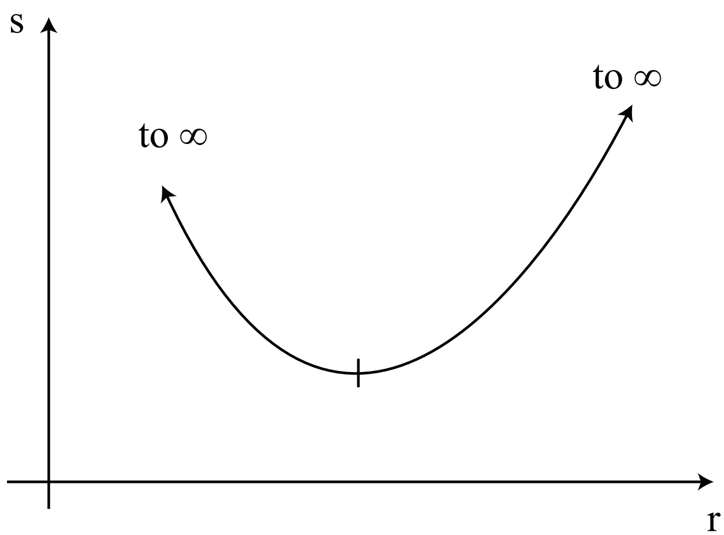

Figure 4: Graph of S

We're <u>still</u> not done. We want to find the minimum value of the surface area, S, and the values of h.

$$r = \left(\frac{V}{\pi}\right)^{1/3}; \quad h = \frac{V}{\pi r^2} = \frac{V}{\pi\left(\frac{V}{\pi}\right)^{2/3}} = \frac{V}{\pi}\left(\frac{V}{\pi}\right)^{-2/3} = \left(\frac{V}{\pi}\right)^{1/3}$$

$$S = \pi r^2 + 2\frac{V}{r} = \pi\left(\frac{V}{\pi}\right)^{2/3} + 2V\left(\frac{V}{\pi}\right)^{1/3} = 3\pi^{-1/3}V^{2/3}$$

Finally, another, often better, way of answering that question is to find the proportions of the can. In other words, what is $\frac{h}{r}$? Answer: $\frac{h}{r} = \frac{(V/\pi)^{1/3}}{(V/\pi)^{1/3}} = 1$.

Example 4. Consider a wire of length 1, cut into two pieces. Bend each piece into a square. We want to figure out where to cut the wire in order to enclose as much area in the two squares as possible.

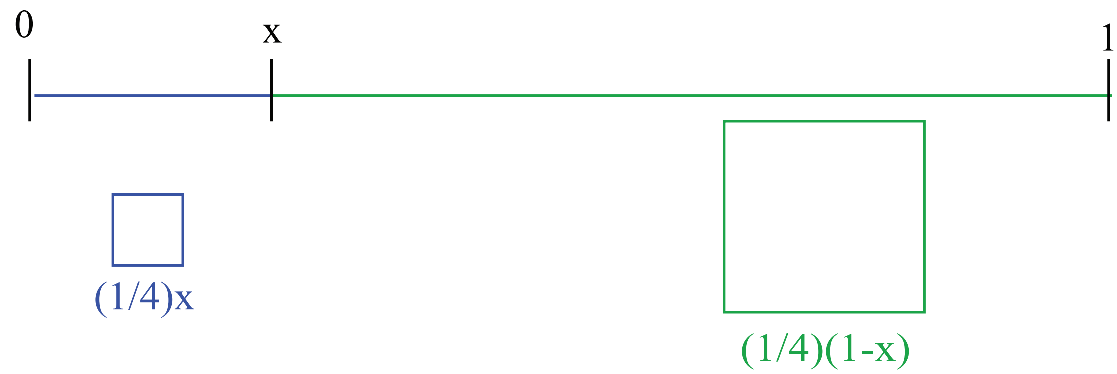

Figure 5: Illustration for Example 5.

The first square will have sides of length $\frac{x}{4}$. Its area will be $\frac{x^2}{16}$. The second square will have sides of length $\frac{1-x}{4}$. Its area will be $\left(\frac{1-x}{4}\right)^2$. The total area is then

$$\begin{aligned} A &= \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2 \\ A' &= \frac{2x}{16} + \frac{2(1-x)}{16}(-1) = \frac{x}{8} - \frac{1}{8} + \frac{x}{8} = 0 \implies 2x - 1 = 0 \implies x = \frac{1}{2} \end{aligned}$$

So, one extreme value of the area is

$$A = \left(\frac{\frac{1}{2}}{4}\right)^2 + \left(\frac{\frac{1}{2}}{4}\right)^2 = \frac{1}{32}$$

We're not done yet, though. We still need to check the endpoints! At $x = 0$,

$$A = 0^2 + \left(\frac{1-0}{4}\right)^2 = \frac{1}{16}$$

At $x = 1$,

$$A = \left(\frac{1}{4}\right)^2 + 0^2 = \frac{1}{16}$$

By checking the endpoints in Fig. 6, we see that the *minimum* area was achieved at $x = \frac{1}{2}$. The maximum area is not achieved in $0 < x < 1$, but it is achieved at $x = 0$ or 1. The maximum corresponds to using the whole length of wire for one square.

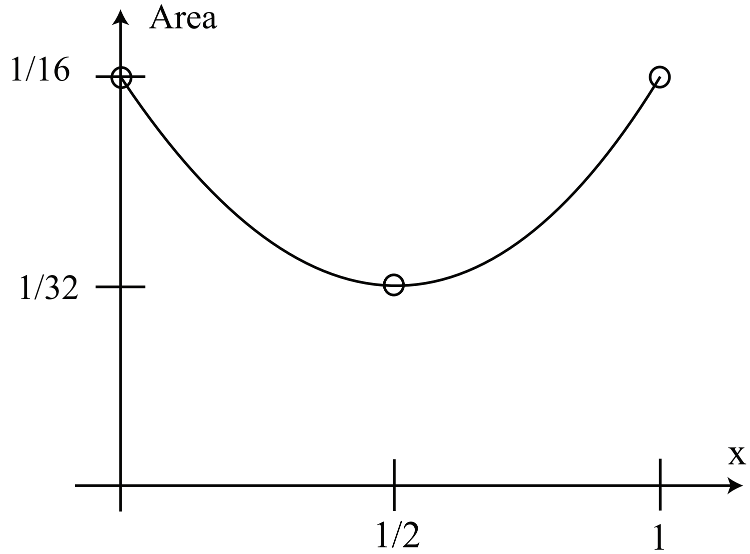

Figure 6: Graph of the area function.

Moral: Don't forget endpoints. If you only look at critical points you may find the worst answer, rather than the best one.