Lecture 13: Differentials and Antiderivatives

Differentials

New notation:

$$\boxed{dy = f'(x)dx} \quad (y = f(x))$$

Both dy and $f'(x)dx$ are called *differentials*. You can think of

$$\frac{dy}{dx} = f'(x)$$

as a quotient of differentials. One way this is used is for linear approximations.

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

Example 1. Approximate $65^{1/3}$

Method 1 (review of linear approximation method)

$$\begin{aligned} f(x) &= x^{1/3} \\ f'(x) &= \frac{1}{3}x^{-2/3} \\ f(x) &\approx f(a) + f'(a)(x-a) \\ x^{1/3} &\approx a^{1/3} + \frac{1}{3}a^{-2/3}(x-a) \end{aligned}$$

A good base point is $a = 64$, because $64^{1/3} = 4$.

Let $x = 65$.

$$65^{1/3} = 64^{1/3} + \frac{1}{3}64^{-2/3}(65-64) = 4 + \frac{1}{3}\left(\frac{1}{16}\right)(1) = 4 + \frac{1}{48} \approx 4.02$$

Similarly,

$$(64.1)^{1/3} \approx 4 + \frac{1}{480}$$

Method 2 (review)

$$65^{1/3} = (64+1)^{1/3} = [64(1+\frac{1}{64})]^{1/3} = 64^{1/3}[1+\frac{1}{64}]^{1/3} = 4\left[1+\frac{1}{64}\right]^{1/3}$$

Next, use the approximation $(1+x)^r \approx 1 + rx$ with $r = \dfrac{1}{3}$ and $x = \dfrac{1}{64}$.

$$65^{1/3} \approx 4(1 + \frac{1}{3}(\frac{1}{64})) = 4 + \frac{1}{48}$$

This is the same result that we got from Method 1.

Method 3 (with differential notation)

$$\begin{aligned} y &= x^{1/3}|_{x=64} = 4 \\ dy &= \frac{1}{3}x^{-2/3}dx|_{x=64} = \frac{1}{3}\left(\frac{1}{16}\right)dx = \frac{1}{48}dx \end{aligned}$$

We want $dx = 1$, since $(x + dx) = 65$. $dy = \dfrac{1}{48}$ when $dx = 1$.

$$(65)^{1/3} = 4 + \frac{1}{48}$$

What underlies all three of these methods is

$$\begin{aligned} y &= x^{1/3} \\ \frac{dy}{dx} &= \frac{1}{3}x^{-2/3}|_{x=64} \end{aligned}$$

Anti-derivatives

$F(x) = \displaystyle\int f(x)dx$ means that F is the antiderivative of f.

Other ways of saying this are:

$$F'(x) = f(x) \quad \text{or,} \quad dF = f(x)dx$$

Examples:

1. $\displaystyle\int \sin x dx = -\cos x + c$ where c is any constant.
2. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
3. $\displaystyle\int \frac{dx}{x} = \ln|x| + c$ (This takes care of the exceptional case $n = -1$ in 2.)
4. $\displaystyle\int \sec^2 x dx = \tan x + c$
5. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$ (where $\sin^{-1} x$ denotes "inverse sin" or arcsin, and not $\dfrac{1}{\sin x}$)
6. $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$

Proof of Property 2: The absolute value $|x|$ gives the correct answer for both positive and negative x. We will double check this now for the case $x < 0$:

$$\begin{aligned} \ln|x| &= \ln(-x) \\ \frac{d}{dx}\ln(-x) &= \left(\frac{d}{du}\ln(u)\right)\frac{du}{dx} \quad \text{where } u = -x. \\ \frac{d}{dx}\ln(-x) &= \frac{1}{u}(-1) = \frac{1}{-x}(-1) = \frac{1}{x} \end{aligned}$$

Uniqueness of the antiderivative up to an additive constant.

If $F'(x) = f(x)$, and $G'(x) = f(x)$, then $G(x) = F(x) + c$ for some constant factor c.

Proof:

$$(G - F)' = f - f = 0$$

Recall that we proved as a corollary of the Mean Value Theorem that if a function has a derivative zero then it is constant. Hence $G(x) - F(x) = c$ (for some constant c). That is, $G(x) = F(x) + c$.

Method of substitution.

Example 1. $\int x^3(x^4 + 2)^5 dx$

Substitution:

$$u = x^4 + 2, \quad du = 4x^3 dx, \quad (x^4 + 2)^5 = u^5, \quad x^3 dx = \frac{1}{4} du$$

Hence,

$$\int x^3(x^4 + 2)^5 dx = \frac{1}{4}\int u^5 du = \frac{u^6}{4(6)} = \frac{u^6}{24} + c = \frac{1}{24}(x^4 + 2)^6 + c$$

Example 2. $\int \frac{x}{\sqrt{1 + x^2}} dx$

Another way to find an anti-derivative is "advanced guessing." First write

$$\int \frac{x}{\sqrt{1 + x^2}} dx = \int x(1 + x^2)^{-1/2} dx$$

Guess: $(1 + x^2)^{1/2}$. Check this.

$$\frac{d}{dx}(1 + x^2)^{1/2} = \frac{1}{2}(1 + x^2)^{-1/2}(2x) = x(1 + x^2)^{-1/2}$$

Therefore,

$$\int x(1 + x^2)^{-1/2} dx = (1 + x^2)^{1/2} + c$$

Example 3. $\int e^{6x} dx$

Guess: e^{6x}. Check this:

$$\frac{d}{dx} e^{6x} = 6e^{6x}$$

Therefore,

$$\int e^{6x} dx = \frac{1}{6} e^{6x} + c$$

Example 4. $\int xe^{-x^2}dx$

Guess: e^{-x^2} Again, take the derivative to check:

$$\frac{d}{dx}e^{-x^2} = (-2x)(e^{-x^2})$$

Therefore,

$$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2} + c$$

Example 5. $\int \sin x \cos x dx = \frac{1}{2}\sin^2 x + c$

Another, equally acceptable answer is

$$\int \sin x \cos x dx = -\frac{1}{2}\cos^2 x + c$$

This seems like a contradiction, so let's check our answers:

$$\frac{d}{dx}\sin^2 x = (2\sin x)(\cos x)$$

and

$$\frac{d}{dx}\cos^2 x = (2\cos x)(-\sin x)$$

So both of these are correct. Here's how we resolve this apparent paradox: the difference between the two answers is a constant.

$$\frac{1}{2}\sin^2 x - (-\frac{1}{2}\cos^2 x) = \frac{1}{2}(\sin^2 x + \cos^2 x) = \frac{1}{2}$$

So,

$$\frac{1}{2}\sin^2 x - \frac{1}{2} = \frac{1}{2}(\sin^2 x - 1) = \frac{1}{2}(-\cos^2 x) = -\frac{1}{2}\cos^2 x$$

The two answers are, in fact, equivalent. The constant c is shifted by $\frac{1}{2}$ from one answer to the other.

Example 6. $\int \frac{dx}{x\ln x}$ (We will assume $x > 0$.)

Let $u = \ln x$. This means $du = \frac{1}{x}dx$. Substitute these into the integral to get

$$\int \frac{dx}{x\ln x} = \int \frac{1}{u}du = \ln u + c = \ln(\ln(x)) + c$$