Lecture 6: Exponential and Log, Logarithmic Differentiation, Hyperbolic Functions

Taking the derivatives of exponentials and logarithms

Background

We always assume the base, a, is greater than 1.

$$a^0 = 1; \quad a^1 = a; \quad a^2 = a \cdot a; \quad \dots$$

$$\begin{aligned} a^{x_1+x_2} &= a^{x_1}a^{x_2} \\ (a^{x_1})^{x_2} &= a^{x_1x_2} \\ a^{\frac{p}{q}} &= \sqrt[q]{a^p} \quad \text{(where } p \text{ and } q \text{ are integers)} \end{aligned}$$

To define a^r for real numbers r, fill in by continuity.

Today's main task: find $\frac{d}{dx}a^x$

We can write

$$\frac{d}{dx}a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

We can factor out the a^x:

$$\lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x} = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

Let's call

$$M(a) \equiv \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

We don't yet know what $M(a)$ is, but we can say

$$\frac{d}{dx}a^x = M(a)a^x$$

Here are two ways to describe $M(a)$:

1. Analytically $M(a) = \frac{d}{dx}a^x$ at $x = 0$.

   Indeed, $M(a) = \lim_{\Delta x \to 0} \frac{a^{0+\Delta x} - a^0}{\Delta x} = \left.\frac{d}{dx}a^x\right|_{x=0}$

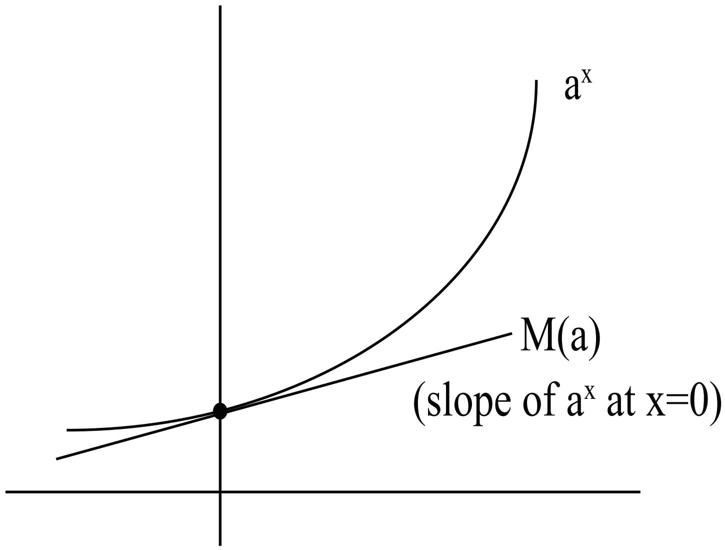

Figure 1: Geometric definition of $M(a)$

2. Geometrically, $M(a)$ is the slope of the graph $y = a^x$ at $x = 0$.

The trick to figuring out what $M(a)$ is is to beg the question and define e as the number such that $M(e) = 1$. Now can we be sure there is such a number e? First notice that as the base a increases, the graph a^x gets steeper. Next, we will estimate the slope $M(a)$ for $a = 2$ and $a = 4$ geometrically. Look at the graph of 2^x in Fig. 2. The secant line from $(0, 1)$ to $(1, 2)$ of the graph $y = 2^x$ has slope 1. Therefore, the slope of $y = 2^x$ at $x = 0$ is less: $M(2) < 1$ (see Fig. 2).

Next, look at the graph of 4^x in Fig. 3. The secant line from $(-\frac{1}{2}, \frac{1}{2})$ to $(1, 0)$ on the graph of $y = 4^x$ has slope 1. Therefore, the slope of $y = 4^x$ at $x = 0$ is greater than $M(4) > 1$ (see Fig. 3).

Somewhere in between 2 and 4 there is a base whose slope at $x = 0$ is 1.

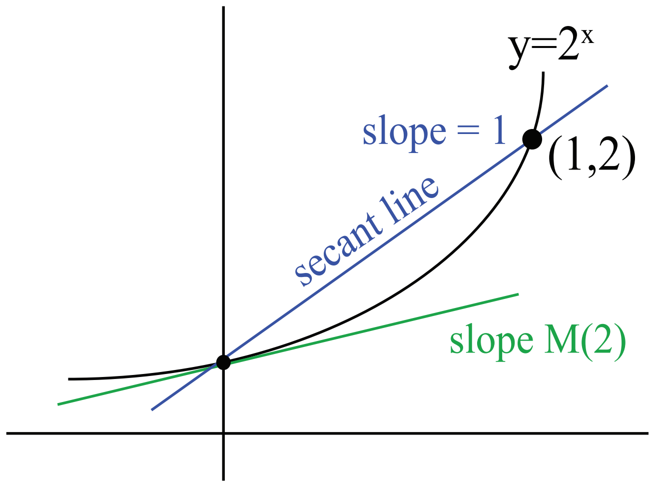

Figure 2: Slope $M(2) < 1$

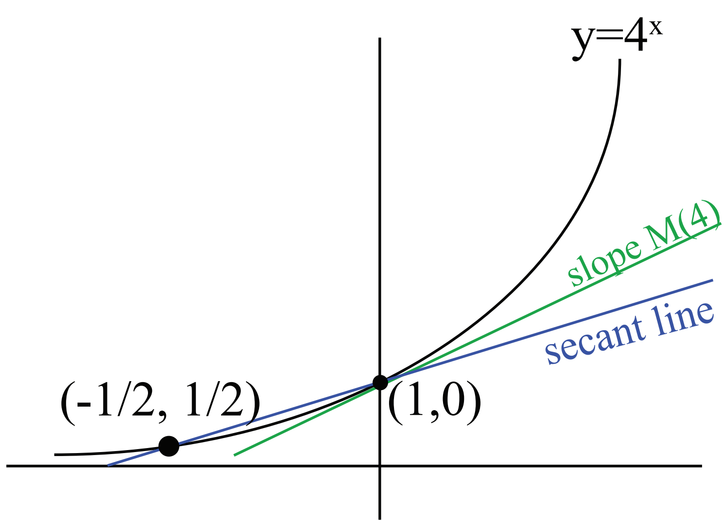

Figure 3: Slope $M(4) > 1$

Thus we can *define* e to be the unique number such that

$$M(e) = 1$$

or, to put it another way,

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

or, to put it still another way,

$$\frac{d}{dx}(e^x) = 1 \quad \text{at } x = 0$$

What is $\frac{d}{dx}(e^x)$? We just defined $M(e) = 1$, and $\frac{d}{dx}(e^x) = M(e)e^x$. So

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Natural log (inverse function of e^x)

To understand $M(a)$ better, we study the natural log function $\ln(x)$. This function is defined as follows:

$$\boxed{\text{If } y = e^x, \text{ then } \ln(y) = x}$$

(or)

$$\boxed{\text{If } w = \ln(x), \text{ then } e^x = w}$$

Note that e^x is always positive, even if x is negative.
Recall that $\ln(1) = 0$; $\ln(x) < 0$ for $0 < x < 1$; $\ln(x) > 0$ for $x > 1$. Recall also that

$$\ln(x_1 x_2) = \ln x_1 + \ln x_2$$

Let us use implicit differentiation to find $\frac{d}{dx}\ln(x)$. $w = \ln(x)$. We want to find $\frac{dw}{dx}$.

$$\begin{aligned}
e^w &= x \\
\frac{d}{dx}(e^w) &= \frac{d}{dx}(x) \\
\frac{d}{dw}(e^w)\frac{dw}{dx} &= 1 \\
e^w \frac{dw}{dx} &= 1 \\
\frac{dw}{dx} &= \frac{1}{e^w} = \frac{1}{x}
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}}$$

Finally, what about $\dfrac{d}{dx}(a^x)$?

There are two methods we can use:

Method 1: Write base e and use chain rule.

Rewrite a as $e^{\ln(a)}$. Then,

$$a^x = \left(e^{\ln(a)}\right)^x = e^{x\ln(a)}$$

That looks like it might be tricky to differentiate. Let's work up to it:

$$\frac{d}{dx}e^x = e^x$$

and by the chain rule,

$$\frac{d}{dx}e^{3x} = 3e^{3x}$$

Remember, $\ln(a)$ is just a constant number– not a variable! Therefore,

$$\frac{d}{dx}e^{(\ln a)x} = (\ln a)e^{(\ln a)x}$$

or

$$\boxed{\frac{d}{dx}(a^x) = \ln(a)\cdot a^x}$$

Recall that

$$\frac{d}{dx}(a^x) = M(a)\cdot a^x$$

So now we know the value of $M(a)$: $\quad M(a) = \ln(a)$.

Even if we insist on starting with another base, like 10, the natural logarithm appears:

$$\frac{d}{dx}10^x = (\ln 10)10^x$$

The base e may seem strange at first. But, it comes up everywhere. After a while, you'll learn to appreciate just how natural it is.

Method 2: Logarithmic Differentiation.

The idea is to find $\frac{d}{dx}f(x)$ by finding $\frac{d}{dx}\ln(f(x))$ instead. Sometimes this approach is easier. Let $u = f(x)$.

$$\frac{d}{dx}\ln(u) = \frac{d\ln(u)}{du}\frac{du}{dx} = \frac{1}{u}\left(\frac{du}{dx}\right)$$

Since $u = f$ and $\frac{du}{dx} = f'$, we can also write

$$\boxed{(\ln f)' = \frac{f'}{f} \quad \text{or} \quad f' = f(\ln f)'}$$

Apply this to $f(x) = a^x$.

$$\ln f(x) = x \ln a \implies \frac{d}{dx} \ln(f) = \frac{d}{dx} \ln(a^x) = \frac{d}{dx}(x \ln(a)) = \ln(a).$$

(Remember, $\ln(a)$ is a constant, *not* a variable.) Hence,

$$\frac{d}{dx}(\ln f) = \ln(a) \implies \frac{f'}{f} = \ln(a) \implies f' = \ln(a) f \implies \frac{d}{dx} a^x = (\ln a) a^x$$

Example 1. $\dfrac{d}{dx}(x^x) = ?$

With variable ("moving") exponents, you should use either base e or logarithmic differentiation. In this example, we will use the latter.

$$\begin{aligned} f &= x^x \\ \ln f &= x \ln x \\ (\ln f)' &= 1 \cdot (\ln x) + x \left(\frac{1}{x}\right) = \ln(x) + 1 \\ (\ln f)' &= \frac{f'}{f} \end{aligned}$$

Therefore,

$$f' = f(\ln f)' = x^x (\ln(x) + 1)$$

If you wanted to solve this using the base e approach, you would say $f = e^{x \ln x}$ and differentiate it using the chain rule. It gets you the same answer, but requires a little more writing.

Example 2. Use logs to evaluate $\displaystyle\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k$.

Because the exponent k changes, it is better to find the limit of the logarithm.

$$\lim_{k\to\infty} \ln\left[\left(1 + \frac{1}{k}\right)^k\right]$$

We know that

$$\ln\left[\left(1 + \frac{1}{k}\right)^k\right] = k \ln\left(1 + \frac{1}{k}\right)$$

This expression has two competing parts, which balance: $k \to \infty$ while $\ln\left(1 + \dfrac{1}{k}\right) \to 0$.

$$\ln\left[\left(1 + \frac{1}{k}\right)^k\right] = k \ln\left(1 + \frac{1}{k}\right) = \frac{\ln\left(1 + \frac{1}{k}\right)}{\frac{1}{k}} = \frac{\ln(1+h)}{h} \quad (\text{with } h = \frac{1}{k})$$

Next, because $\ln 1 = 0$

$$\ln\left[\left(1 + \frac{1}{k}\right)^k\right] = \frac{\ln(1+h) - \ln(1)}{h}$$

Take the limit: $h = \dfrac{1}{k} \to 0$ as $k \to \infty$, so that

$$\lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} = \frac{d}{dx}\ln(x)\Big|_{x=1} = 1$$

In all,

$$\lim_{k\to\infty} \ln\left(1+\frac{1}{k}\right)^k = 1.$$

We have just found that $a_k = \ln[\left(1+\frac{1}{k}\right)^k] \to 1$ as $k \to \infty$.

If $b_k = \left(1+\frac{1}{k}\right)^k$, then $b_k = e^{a_k} \to e^1$ as $k \to \infty$. In other words, we have evaluated the limit we wanted:

$$\lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e$$

Remark 1. We never figured out what the exact numerical value of e was. Now we can use this limit formula; $k = 10$ gives a pretty good approximation to the actual value of e.

Remark 2. Logs are used in all sciences and even in finance. Think about the stock market. If I say the market fell 50 points today, you'd need to know whether the market average before the drop was 300 points or $10,000$. In other words, you care about the percent change, or the ratio of the change to the starting value:

$$\frac{f'(t)}{f(t)} = \frac{d}{dt}\ln(f(t))$$