Lecture 12: Mean Value Theorem and Inequalities

Mean-Value Theorem

The Mean-Value Theorem (MVT) is the underpinning of calculus. It says:

> If f is differentiable on $a < x < b$, and continuous on $a \le x \le b$, then
> $$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{(for some } c,\ a < c < b)$$

Here, $\frac{f(b) - f(a)}{b - a}$ is the slope of a secant line, while $f'(c)$ is the slope of a tangent line.

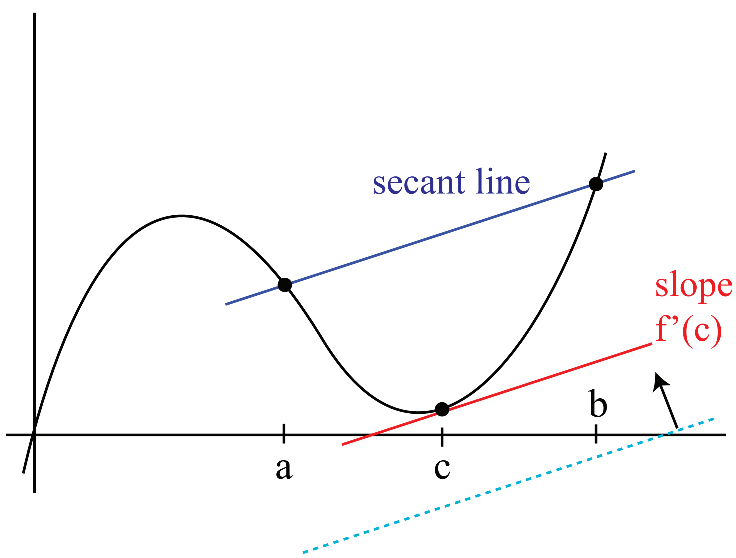

Figure 1: Illustration of the Mean Value Theorem.

Geometric Proof: Take (dotted) lines parallel to the secant line, as in Fig. 1 and shift them up from below the graph until one of them first touches the graph. Alternatively, one may have to start with a dotted line above the graph and move it down until it touches.

If the function isn't differentiable, this approach goes wrong. For instance, it breaks down for the function $f(x) = |x|$. The dotted line always touches the graph first at $x = 0$, no matter what its slope is, and $f'(0)$ is undefined (see Fig. 2).

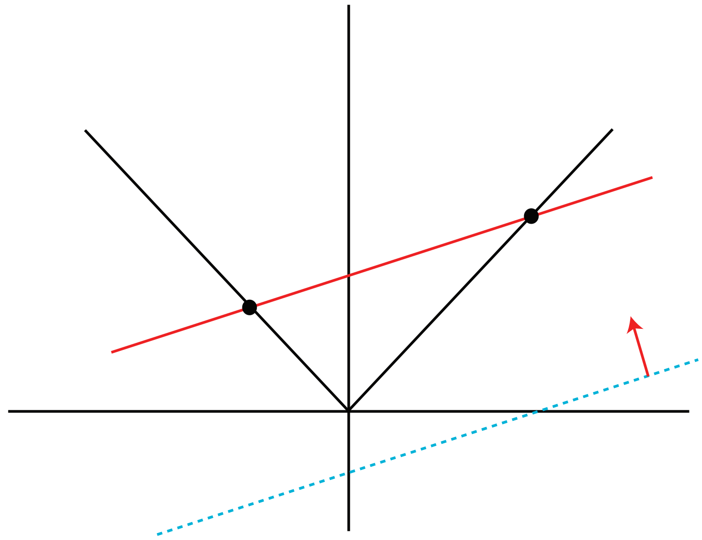

Figure 2: Graph of $y = |x|$, with secant line. (MVT goes wrong.)

Interpretation of the Mean Value Theorem

You travel from Boston to Chicago (which we'll assume is a 1,000 mile trip) in exactly 3 hours. At some time in between the two cities, you must have been going at exactly $\frac{1000}{3}$ mph.

$f(t) =$ position, measured as the distance from Boston.

$$f(3) = 1000, \quad f(0) = 0, \quad a = 0, \text{ and b} = 3.$$

$$\frac{1000}{3} = \frac{f(b) - f(a)}{3} = f'(c)$$

where $f'(c)$ is your speed at some time, c.

Versions of the Mean Value Theorem

There is a second way of writing the MVT:

$$\begin{aligned} f(b) - f(a) &= f'(c)(b-a) \\ f(b) &= f(a) + f'(c)(b-a) \quad (\text{for some } c, a < c < b) \end{aligned}$$

There is also a third way of writing the MVT: change the name of b to x.

$$\boxed{f(x) = f(a) + f'(c)(x-a) \quad \text{for some } c, a < c < x}$$

The theorem does not say what c is. It depends on f, a, and x.

This version of the MVT should be compared with linear approximation (see Fig. 3).

$$f(x) \approx f(a) + f'(a)(x-a) \quad x \text{ near } a$$

The tangent line in the linear approximation has a definite slope $f'(a)$. by contrast formula is an exact formula. It conceals its lack of specificity in the slope $f'(c)$, which could be the slope of f at any point between a and x.

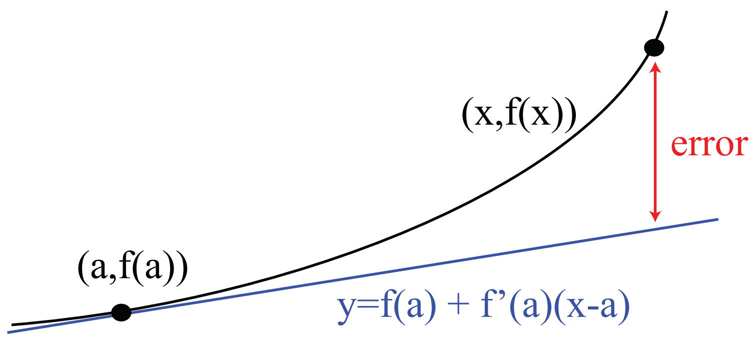

Figure 3: MVT vs. Linear Approximation.

Uses of the Mean Value Theorem.

Key conclusions: (The conclusions from the MVT are theoretical)

1. If $f'(x) > 0$, then f is increasing.
2. If $f'(x) < 0$, then f is decreasing.
3. If $f'(x) = 0$ all x, then f is constant.

Definition of increasing/decreasing:
Increasing means $a < b \Rightarrow f(a) < f(b)$. Decreasing means $a < b \implies f(a) < f(b)$.

Proofs:
Proof of 1:

$$\begin{aligned} a &< b \\ f(b) &= f(a) + f'(c)(b-a) \end{aligned}$$

Because $f'(c)$ and $(b-a)$ are both positive,

$$f(b) = f(a) + f'(c)(b-a) > f(a)$$

(The proof of 2 is omitted because it is similar to the proof of 1)

Proof of 3:

$$f(b) = f(a) + f'(c)(b-a) = f(a) + 0(b-a) = f(a)$$

Conclusions 1,2, and 3 seem obvious, but let me persuade you that they are not. Think back to the definition of the derivative. It involves infinitesimals. It's not a sure thing that these infinitesimals have anything to do with the non-infinitesimal behavior of the function.

Inequalities

The fundamental property $f' > 0 \implies f$ is increasing can be used to deduce many other inequalities.

Example. e^x

1. $e^x > 0$
2. $e^x > 1$ for $x > 0$
3. $e^x > 1 + x$

Proofs. We will take property 1 ($e^x > 0$) for granted. Proofs of the other two properties follow:

<u>Proof of 2</u>: Define $f_1(x) = e^x - 1$. Then, $f_1(0) = e^0 - 1 = 0$, and $f_1'(x) = e^x > 0$. (This last assertion is from step 1). Hence, $f_1(x)$ is increasing, so $f(x) > f(0)$ for $x > 0$. That is:

$$e^x > 1 \text{ for } x > 0$$

.

<u>Proof of 3:</u> Let $f_2(x) = e^x - (1 + x)$.

$$f_2'(x) = e^x - 1 = f_1(x) > 0 \quad (\text{if } x > 0).$$

Hence, $f_2(x) > 0$ for $x > 0$. In other words,

$$e^x > 1 + x$$

Similarly, $e^x > 1 + x + \frac{x^2}{2}$ (proved using $f_3(x) = e^x - (1 + x + \frac{x^2}{2})$). One can keep on going: $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$ for $x > 0$. Eventually, it turns out that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \quad (\text{an infinite sum})$$

We will be discussing this when we get to Taylor series near the end of the course.