
Lecture 5

Implicit Differentiation and Inverses

Implicit Differentiation

Example 1. $\frac{d}{dx}(x^a) = ax^{a-1}$.

We proved this by an explicit computation for $a = 0, 1, 2, \dots$. From this, we also got the formula for $a = -1, -2, \dots$. Let us try to extend this formula to cover rational numbers, as well:

$$a = \frac{m}{n}; \quad y = x^{\frac{m}{n}} \quad \text{where } m \text{ and } n \text{ are integers.}$$

We want to compute $\frac{dy}{dx}$. We can say $y^n = x^m$ so $ny^{n-1} \frac{dy}{dx} = mx^{m-1}$. Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{m}{n} \frac{x^{m-1}}{y^{n-1}}$$

We know that $y = x^{(\frac{m}{n})}$ is a function of x .

$$\begin{aligned} \frac{dy}{dx} &= \frac{m}{n} \left(\frac{x^{m-1}}{y^{n-1}} \right) \\ &= \frac{m}{n} \left(\frac{x^{m-1}}{(x^{m/n})^{n-1}} \right) \\ &= \frac{m}{n} \frac{x^{m-1}}{x^{m(n-1)/n}} \\ &= \frac{m}{n} x^{(m-1) - \frac{m(n-1)}{n}} \\ &= \frac{m}{n} x^{\frac{n(m-1) - m(n-1)}{n}} \\ &= \frac{m}{n} x^{\frac{nm - n - nm + m}{n}} \\ &= \frac{m}{n} x^{\frac{m}{n} - \frac{n}{n}} \\ \text{So, } \frac{dy}{dx} &= \frac{m}{n} x^{\frac{m}{n} - 1} \end{aligned}$$

This is the same answer as we were hoping to get!

Example 2. Equation of a circle with a radius of 1: $x^2 + y^2 = 1$ which we can write as $y^2 = 1 - x^2$. So $y = \pm\sqrt{1 - x^2}$. Let us look at the positive case:

$$\begin{aligned} y &= +\sqrt{1 - x^2} = (1 - x^2)^{\frac{1}{2}} \\ \frac{dy}{dx} &= \left(\frac{1}{2} \right) (1 - x^2)^{-\frac{1}{2}} (-2x) = \frac{-x}{\sqrt{1 - x^2}} = \frac{-x}{y} \end{aligned}$$

Now, let's do the same thing, using *implicit* differentiation.

$$\begin{aligned}x^2 + y^2 &= 1 \\ \frac{d}{dx}(x^2 + y^2) &= \frac{d}{dx}(1) = 0 \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) &= 0\end{aligned}$$

Applying chain rule in the second term,

$$\begin{aligned}2x + 2y \frac{dy}{dx} &= 0 \\ 2y \frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= \frac{-x}{y}\end{aligned}$$

Same answer!

Example 3. $y^3 + xy^2 + 1 = 0$. In this case, it's not easy to solve for y as a function of x . Instead, we use implicit differentiation to find $\frac{dy}{dx}$.

$$3y^2 \frac{dy}{dx} + y^2 + 2xy \frac{dy}{dx} = 0$$

We can now solve for $\frac{dy}{dx}$ in terms of y and x .

$$\begin{aligned}\frac{dy}{dx}(3y^2 + 2xy) &= -y^2 \\ \frac{dy}{dx} &= \frac{-y^2}{3y^2 + 2xy}\end{aligned}$$

Inverse Functions

If $y = f(x)$ and $g(y) = x$, we call g the *inverse function* of f , f^{-1} :

$$x = g(y) = f^{-1}(y)$$

Now, let us use implicit differentiation to find the derivative of the inverse function.

$$\begin{aligned}y &= f(x) \\ f^{-1}(y) &= x \\ \frac{d}{dx}(f^{-1}(y)) &= \frac{d}{dx}(x) = 1\end{aligned}$$

By the chain rule:

$$\begin{aligned}\frac{d}{dy}(f^{-1}(y)) \frac{dy}{dx} &= 1 \\ \text{and} \\ \frac{d}{dy}(f^{-1}(y)) &= \frac{1}{\frac{dy}{dx}}\end{aligned}$$

So, implicit differentiation makes it possible to find the derivative of the inverse function.

Example. $y = \arctan(x)$

$$\begin{aligned}\tan y &= x \\ \frac{d}{dx} [\tan(y)] &= \frac{dx}{dx} = 1 \\ \frac{d}{dy} [\tan(y)] \frac{dy}{dx} &= 1 \\ \left(\frac{1}{\cos^2(y)} \right) \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \cos^2(y) = \cos^2(\arctan(x))\end{aligned}$$

This form is messy. Let us use some geometry to simplify it.

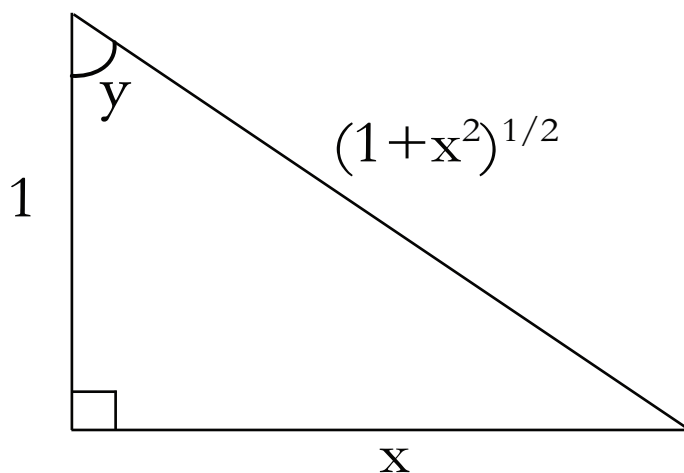


Figure 1: Triangle with angles and lengths corresponding to those in the example illustrating differentiation using the inverse function \arctan

In this triangle, $\tan(y) = x$ so

$$\arctan(x) = y$$

The Pythagorean theorem tells us the length of the hypotenuse:

$$h = \sqrt{1+x^2}$$

From this, we can find

$$\cos(y) = \frac{1}{\sqrt{1+x^2}}$$

From this, we get

$$\cos^2(y) = \left(\frac{1}{\sqrt{1+x^2}} \right)^2 = \frac{1}{1+x^2}$$

So,

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

In other words,

$$\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

Graphing an Inverse Function.

Suppose $y = f(x)$ and $g(y) = f^{-1}(y) = x$. To graph g and f together we need to write g as a function of the variable x . If $g(x) = y$, then $x = f(y)$, and what we have done is to trade the variables x and y . This is illustrated in Fig. 2

$f^{-1}(f(x)) = x$	$f^{-1} \circ f(x) = x$
$f(f^{-1}(x)) = x$	$f \circ f^{-1}(x) = x$

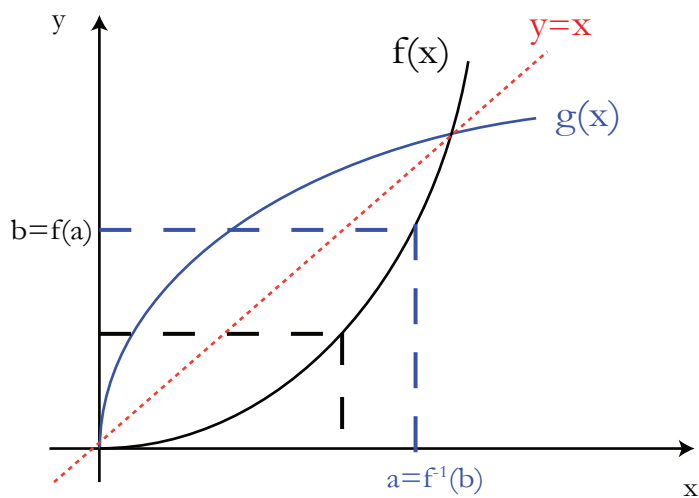


Figure 2: You can think about f^{-1} as the graph of f reflected about the line $y = x$