Lecture 3
Derivatives of Products, Quotients, Sine, and Cosine

Derivative Formulas

There are two kinds of derivative formulas:

1. Specific Examples: $\frac{d}{dx}x^n$ or $\frac{d}{dx}\left(\frac{1}{x}\right)$
2. General Examples: $(u+v)' = u' + v'$ and $(cu) = cu'$ (where c is a constant)

A notational convention we will use today is:

$$(u+v)(x) = u(x) + v(x); \quad uv(x) = u(x)v(x)$$

Proof of $(u+v) = u' + v'$. (General)

Start by using the definition of the derivative.

$$\begin{aligned}
(u+v)'(x) &= \lim_{\Delta x \to 0} \frac{(u+v)(x+\Delta x) - (u+v)(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u(x+\Delta x) + v(x+\Delta x) - u(x) - v(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x} \right\} \\
(u+v)'(x) &= u'(x) + v'(x)
\end{aligned}$$

Follow the same procedure to prove that $(cu)' = cu'$.

Derivatives of $\sin x$ and $\cos x$. (Specific)

Last time, we computed

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin x}{x} &= 1 \\
\frac{d}{dx}(\sin x)|_{x=0} &= \lim_{\Delta x \to 0} \frac{\sin(0+\Delta x) - \sin(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1 \\
\frac{d}{dx}(\cos x)|_{x=0} &= \lim_{\Delta x \to 0} \frac{\cos(0+\Delta x) - \cos(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = 0
\end{aligned}$$

So, we know the value of $\frac{d}{dx}\sin x$ and of $\frac{d}{dx}\cos x$ at $x = 0$. Let us find these for arbitrary x.

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

Recall:

$$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$$

So,

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{\Delta x \to 0} \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{\sin x(\cos \Delta x - 1)}{\Delta x} + \frac{\cos x \sin \Delta x}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \sin x \left(\frac{\cos \Delta x - 1}{\Delta x} \right) + \lim_{\Delta x \to 0} \cos x \left(\frac{\sin \Delta x}{\Delta x} \right)
\end{aligned}$$

Since $\dfrac{\cos \Delta x - 1}{\Delta x} \to 0$ and that $\dfrac{\sin \Delta x}{\Delta x} \to 1$, the equation above simplifies to

$$\frac{d}{dx}\sin x = \cos x$$

A similar calculation gives

$$\frac{d}{dx}\cos x = -\sin x$$

Product formula (General)

$$(uv)' = u'v + uv'$$

Proof:

$$(uv)' = \lim_{\Delta x \to 0} \frac{(uv)(x + \Delta x) - (uv)(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x+\Delta x) - u(x)v(x)}{\Delta x}$$

Now obviously,

$$u(x+\Delta x)v(x) - u(x+\Delta x)v(x) = 0$$

so adding that to the numerator won't change anything.

$$(uv)' = \lim_{\Delta x \to 0} \frac{u(x+\Delta x)v(x) - u(x)v(x) + u(x+\Delta x)v(x+\Delta x) - u(x+\Delta x)v(x)}{\Delta x}$$

We can re-arrange that expression to get

$$(uv)' = \lim_{\Delta x \to 0} \left(\frac{u(x+\Delta x) - u(x)}{\Delta x} \right) v(x) + u(x+\Delta x) \left(\frac{v(x+\Delta x) - v(x)}{\Delta x} \right)$$

Remember, the limit of a sum is the sum of the limits.

$$\left[\lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} \right] v(x) + \lim_{\Delta x \to 0} \left(u(x+\Delta x) \left[\frac{v(x+\Delta x) - v(x)}{\Delta x} \right] \right)$$

$$(uv)' = u'(x)v(x) + u(x)v'(x)$$

Note: we also used the fact that

$$\lim_{\Delta x \to 0} u(x+\Delta x) = u(x) \qquad \text{(true because } u \text{ is continuous)}$$

This proof of the product rule assumes that u and v have derivatives, which implies both functions are continuous.

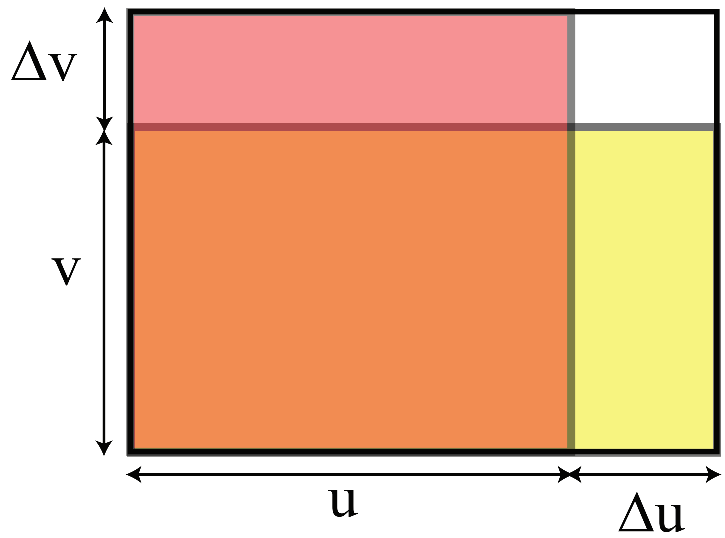

Figure 1: A graphical "proof" of the product rule

An intuitive justification:

We want to find the difference in area between the large rectangle and the smaller, inner rectangle. The inner (orange) rectangle has area uv. Define Δu, the change in u, by

$$\Delta u = u(x + \Delta x) - u(x)$$

We also abbreviate $u = u(x)$, so that $u(x + \Delta x) = u + \Delta u$, and, similarly, $v(x + \Delta x) = v + \Delta v$. Therefore the area of the largest rectangle is $(u + \Delta u)(v + \Delta v)$.

If you let v increase and keep u constant, you add the area shaded in red. If you let u increase and keep v constant, you add the area shaded in yellow. The sum of areas of the red and yellow rectangles is:

$$[u(v + \Delta v) - uv] + [v(u + \Delta u) - uv] = u\Delta v + v\Delta u$$

If Δu and Δv are small, then $(\Delta u)(\Delta v) \approx 0$, that is, the area of the white rectangle is very small. Therefore the difference in area between the largest rectangle and the orange rectangle is approximately the same as the sum of areas of the red and yellow rectangles. Thus we have:

$$[(u + \Delta u)(v + \Delta v) - uv] \approx u\Delta v + v\Delta u$$

(Divide by Δx and let $\Delta x \to 0$ to finish the argument.)

Quotient formula (General)

To calculate the derivative of u/v, we use the notations Δu and Δv above. Thus,

$$\begin{aligned}
\frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)} &= \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} \\
&= \frac{(u+\Delta u)v - u(v+\Delta v)}{(v+\Delta v)v} \quad \text{(common denominator)} \\
&= \frac{(\Delta u)v - u(\Delta v)}{(v+\Delta v)v} \quad \text{(cancel } uv - uv\text{)}
\end{aligned}$$

Hence,

$$\frac{1}{\Delta x}\left(\frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}\right) = \frac{(\frac{\Delta u}{\Delta x})v - u(\frac{\Delta v}{\Delta x})}{(v+\Delta v)v} \longrightarrow \frac{v(\frac{du}{dx}) - u(\frac{dv}{dx})}{v^2} \qquad \text{as } \Delta x \to 0$$

Therefore,

$$(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$$

.