Lecture 15: Definite Integrals

Integrals are used to calculate cumulative totals, averages, areas.

Area under a curve: (See Figure 1.)

1. Divide region into rectangles
2. Add up area of rectangles
3. Take limit as rectangles become thin

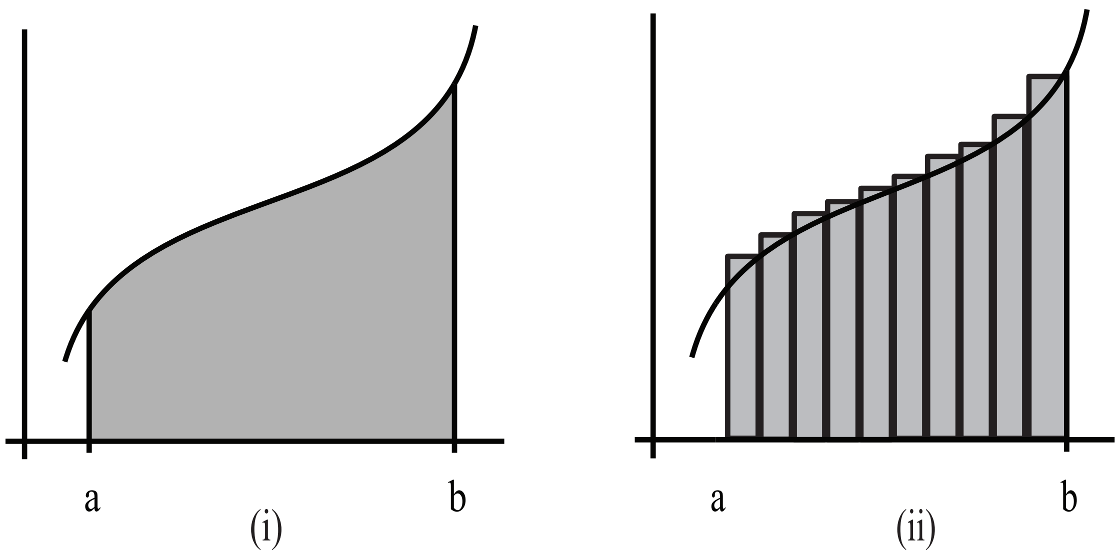

Figure 1: (i) Area under a curve; (ii) sum of areas under rectangles

Example 1. $f(x) = x^2$, $a = 0$, b arbitrary

1. Divide into n intervals
   Length $b/n =$ base of rectangle
2. Heights:
   - 1^{st}: $x = \frac{b}{n}$, height $= \left(\frac{b}{n}\right)^2$
   - 2^{nd}: $x = \frac{2b}{n}$, height $= \left(\frac{2b}{n}\right)^2$

Sum of areas of rectangles:

$$\left(\frac{b}{n}\right)\left(\frac{b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{2b}{n}\right)^2 + \left(\frac{b}{n}\right)\left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2 = \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

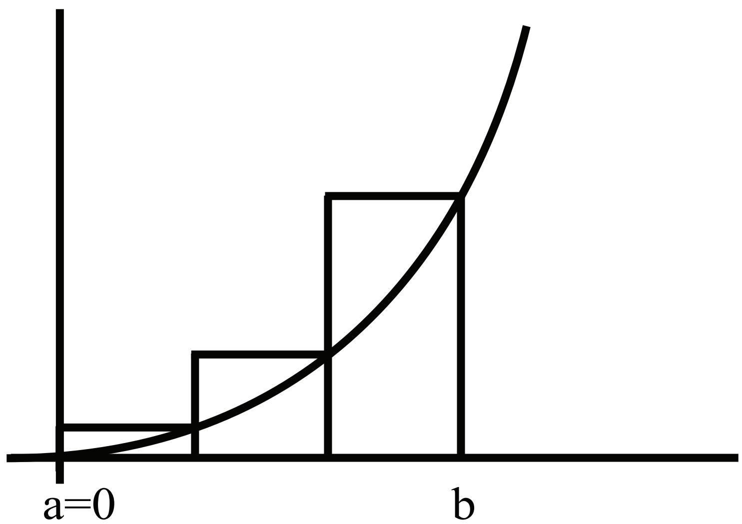

Figure 2: Area under $f(x) = x^2$ above $[0, b]$.

We will now estimate the sum using some 3-dimensional geometry.

Consider the staircase pyramid as pictured in Figure 3.

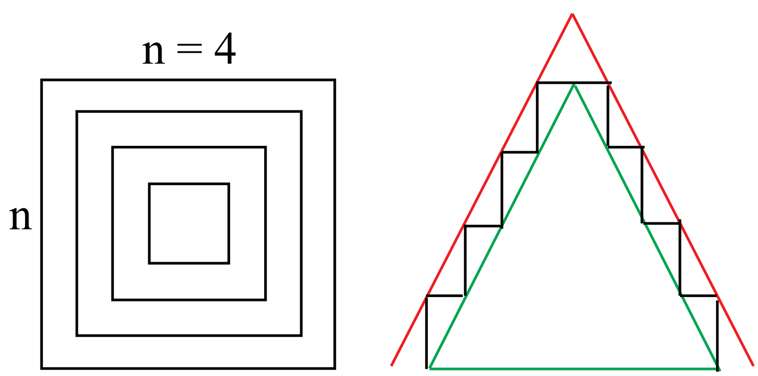

Figure 3: Staircase pyramid: left(top view) and right (side view)

1^{st} level: $n \times n$ bottom, represents volume n^2.
2^{nd} level: $(n-1) \times (n-1)$, represents volumne $(n-1)^2)$, etc.
Hence, the total volume of the staircase pyramid is $n^2 + (n-1)^2 + \cdots + 1$.

Next, the volume of the pyramid is greater than the volume of the inner prism:

$$1^2 + 2^2 + \cdots + n^2 > \frac{1}{3}(\text{base})(\text{height}) = \frac{1}{3}n^2 \cdot n = \frac{1}{3}n^3$$

and less than the volume of the outer prism:

$$1^2 + 2^2 + \cdots + n^2 < \frac{1}{3}(n+1)^2(n+1) = \frac{1}{3}(n+1)^3$$

In all,

$$\frac{1}{3} = \frac{\frac{1}{3}n^3}{n^3} < \frac{1^2 + 2^2 + \cdots + n^2}{n^3} < \frac{1}{3}\frac{(n+1)^3}{n^3}$$

Therefore,

$$\lim_{n\to\infty} \frac{b^3}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) = \frac{1}{3}b^3,$$

and the area under x^2 from 0 to b is $\dfrac{b^3}{3}$.

Example 2. $f(x) = x$; area under x above $[0, b]$. Reasoning similar to Example 1, but easier, gives a sum of areas:

$$\frac{b^2}{n^2}(1 + 2 + 3 + \cdots + n) \to \frac{1}{2}b^2 \quad (\text{as } n \to \infty)$$

This is the area of the triangle in Figure 4.

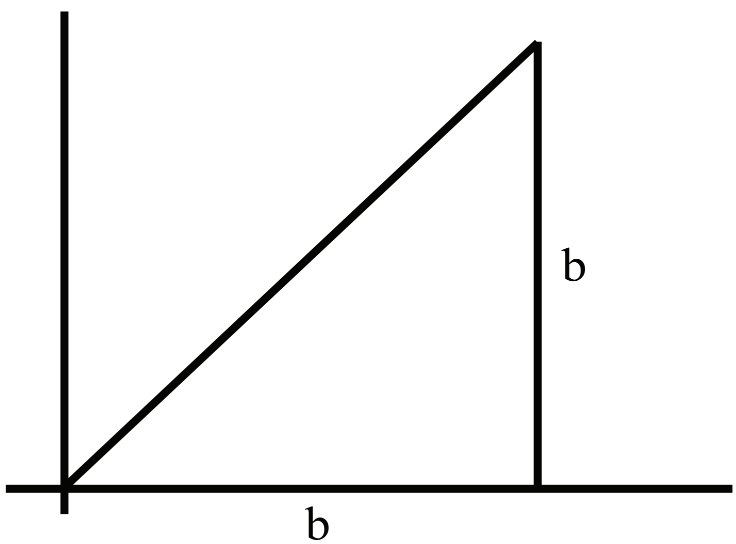

Figure 4: Area under $f(x) = x$ above $[0, b]$.

Pattern:

$$\frac{d}{db}\left(\frac{b^3}{3}\right) = b^2$$

$$\frac{d}{db}\left(\frac{b^2}{2}\right) = b$$

The area $A(b)$ under $f(x)$ should satisfy $A'(b) = f(b)$.

General Picture

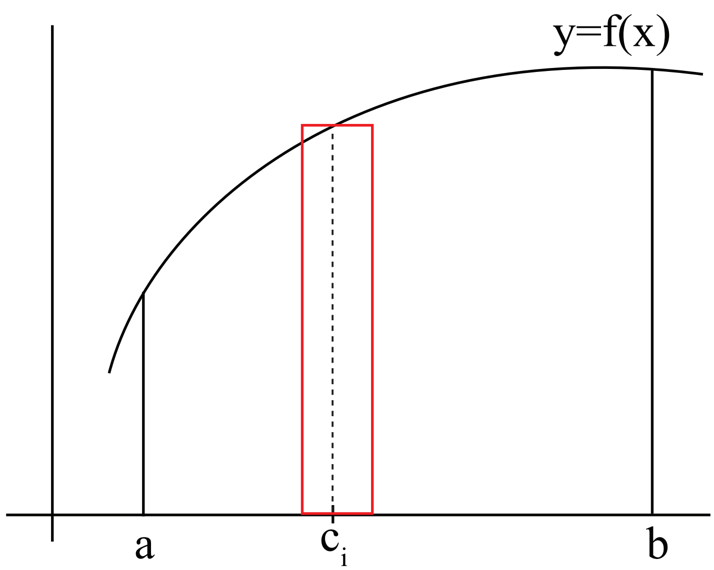

Figure 5: One rectangle from a Riemann Sum

- Divide into n equal pieces of length $= \Delta x = \dfrac{b-a}{n}$
- Pick <u>any</u> c_i in the interval; use $f(c_i)$ as the height of the rectangle
- Sum of areas: $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$

In summation notation: $\displaystyle\sum_{i=1}^{n} f(c_i)\Delta x \quad \leftarrow$ called a *Riemann sum.*

Definition:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx \ \leftarrow \ \text{called a } \textit{definite integral}$$

This definite integral represents the area under the curve $y = f(x)$ above $[a, b]$.

Example 3. (Integrals applied to quantity besides area.) Student borrows from parents.
$P =$ principal in dollars, $t =$ time in years, $r =$ interest rate (e.g., 6 % is $r = 0.06$/year).
After time t, you owe $P(1 + rt) = P + Prt$

The integral can be used to represent the total amount borrowed as follows. Consider a function $f(t)$, the “borrowing function” in dollars per year. For instance, if you borrow \$ 1000 /month, then $f(t) = 12,000$/year. Allow f to vary over time.

Say $\Delta t = 1/12$ year $= 1$ month.

$$t_i = i/12 \quad i = 1, \cdots, 12.$$

$f(t_i)$ is the borrowing rate during the i^{th} month so the amount borrowed is $f(t_i)\Delta t$. The total is

$$\sum_{i=1}^{12} f(t_i)\Delta t.$$

In the limit as $\Delta t \to 0$, we have

$$\int_0^1 f(t)dt$$

which represents the total borrowed in one year in dollars per year.

The integral can also be used to represent the total amount owed. The amount owed depends on the interest rate. You owe

$$f(t_i)(1 + r(1 - t_i))\Delta t$$

for the amount borrowed at time t_i. The total owed for borrowing at the end of the year is

$$\int_0^1 f(t)(1 + r(1 - t))dt$$