Problem Chosen	2023 ShuWei Cup Summary Sheet	Team Control Number
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Summary

Laundry can be seen everywhere on the street, which brings great convenience to people's lives. Through research and experiment, we can find the most economical and environmentally friendly washing solutions to help laundries and families achieve their goal of cleaning clothes, while also contributing to the protection of the environment and conservation of resources. It is a very meaningful research topic and mainly involves the theory of multi-objective optimization. In this paper, some models are constructed and the accurate solutions are obtained by Cplex.

For problem 1, the amount of dirt in clothes and the amount of water available are given, and the solubility of dirt in water a_k decreases as the number of rounds increases. **The model**: To find the optimal solution of the number of washings and the amount of water used each time, we construct a model to minimize the final dirt residue after k rounds of washing and minimize the amount of water used. The constraints are related to the amount of water available and the minimum amount of dirt remaining on clothing. **The solutions**: The optimal amount of water used in each washing is the maximum amount of water used to achieve the best effect of dirt dissolution (H), H is positively correlated with the weight of clothes in each washing. The optimal number of washings is $n_{opt1} = \lfloor W/H \rfloor$, W is the amount of available water. **The discussion on the effect of variables on the objective:** (1) The experimental results indicate that the larger the solubility a_k, the better the washing effect; when a_k is constant, the washing effect is more in line with common sense. (2) The initial amount of dirt D has no impact on the number of washings, however, the larger the D, the larger the amount of water used, and the worse the final washing effect when W is given. (3) The larger the amount of water available, the more the number of washings, and the better the washing effect.

For problem 2, it removes the limit on the amount of water available but adds the limit that the ratio of the remaining dirt residue to the initial dirt amount is less than 0.001, while specifying the same washing time for each round. **The model**: To find the most time-efficient cleaning plan, based on the model of problem 1, we remove one of the objectives "*minimize the water used in the washing process*", and add a constraint "*the final dirt residue should be no more than one-thousandth of the initial dirt amount*". **The most time-efficient cleaning plan** is to ensure the solubility $a_k = 1 (k = 1, 2, ...)$ during each washing by adding sufficient amount of detergent and water. In this way, the predefined washing effect can be achieved after 4 rounds of washing, and the washing time is $4 \times R$ (R is the time taken by each washing). **The discussion on the effect of variables on the solution:** (1) The experimental results show that the larger the a_k, the fewer number of washings are required and the less washing time will be spent. (2) Since there is no limit on the amount of available water and H can be flexibly adjusted according to the initial amount of dirt, the initial amount of dirt has no impact on the number of washings and the total washing time.

For problem 3, the types and amount of dirt, the price of water per ton, as well as the solubility and price of various detergents are given. We follow the time-efficient cleaning plan in problem 2 to ensure the solubility $a_k = 1$ during each round of washing. The goal of problem 3 is to give both cost savings and a good cleaning program. **The optimal cleaning plan is defined as:** less cost of detergent, less cost of water, fewer number of washings, various dirt on the clothes should be dissolved, and the final dirty water remaining on the clothes should be no more than ε of the initial dirt amount. **The solutions**: The parameters setting: the weight of each piece of clothing

$m = 0.2$ kilogram, the water absorption rate $c = 80\%$, the optimal amount of water $H = 1.5$ liter and the minimum amount of water $L = 0.5$ liter. Three different values of ε are considered, 0.01,0.001,0.0001, and the results are shown in **Table 3**. It's shown that only the 2nd detergent is always used, and the used amount is 1646.667 gram. Through analysis, it's found that the 2nd detergent has the highest cost performance. Besides, the experimental results when considering the maximum capacity of machine are further discussed in **Table 6**.

For problem 4, we are requested to further consider the restriction on mixing clothing of different materials for washing, indicating that the clothing need to be divided into piles when washing. **The model**: The restriction can be regarded as the adjacency matrix of an undirected graph. Therefore, the types of clothing that can be classified into the same pile are essentially a complete subgraph of this undirected graph or the single node. Also, the weight limit of the single round of the washing machine should be taken into consideration while dividing clothes into piles. **The solving method**: To make the result more accurate and reasonable, based on the model of problem 3, we use the neighborhood search algorithm to determine the dividing plan of clothing. and then solve it precisely by Cplex. **The solutions**: The parameters setting: $H = 7.5$, $L = 2.5$, the weight of each material $m = [0.5, 0.15, 0.3, 0.4, 0.6, 0.4, 0.5, 0.45]$, $c_i = 0.8, i = 1,...,8$, and the washing effect threshold is $\varepsilon = 0.001$. Four different values of the maximum capacity of machine are considered. The results are shown in **Table 5**. It's found that the optimal cleaning plan needs to divide the eight types of materials into 4 piles, and there are various combinations. The optimal cost of washing is 754.6165. Besides, the comparison with **Table 7** verifies the rationality of the adopted solving method.

Keywords: Multi-objective optimization; Neighborhood search; Graph theory; Laundry cleaning; Cplex

Content

1. Introduction

1.1 Problem background

Washing clothes is an important activity in people's daily life, and so as small-scale household operations to professional operations in hotels and specialized institutions. This paper mainly focuses on how to clean clothes and keep them clean at a lower cost or with the least amount of water in small and medium-scale laundries.

In the washing process, detergents play a key role. Detergents are usually composed of surfactants, auxiliaries and washing auxiliaries. Surfactants are the main components in detergents, which have the effect of reducing surface tension, penetrating fibers and dirt. Auxiliaries such as alkaline substances can improve the cleaning capacity of detergents, while washing auxiliaries such as bleach and softener can enhance the washing effect and improve the feel of clothes.

The operation process of the washing machine includes three main steps: adding water, rinsing and dewatering. In the process of adding water, the clothing comes into contact with the detergent and water, and the dirt begins to be dissolved and loosened. During the rinsing process, the rotation and vibration of the washing machine bring the clothing into full contact with the water, separating the dirt from the clothing and washing it away. Finally, during the dehydration process, the detergent and dirt are thoroughly rinsed off and the clothing is cleaned.

In order to improve the washing effect and save water resources, researchers continue to explore new washing technologies and methods. For example, using more efficient detergents, optimizing washing procedures, choosing the right washing mode, etc. In addition, some new washing machines also have intelligent control functions, which can automatically adjust the washing time and water level according to the type of clothing and the level of dirt, so as to use water more efficiently.

In general, with the increasing concern about the shortage of water resources, saving water resources and improving washing efficiency have become important goals in the laundry process. By understanding the washing principle and the operation process of the washing machine, small and medium-scale laundries can better choose the appropriate washing methods and technologies to achieve water saving and high efficiency in the laundry process.

1.2 Clarification and restatement

Given the background information and constraints identified in the problem statement we need to address the following questions:

- Problem 1: The amount of dirt, the amount of water available and the solubility of the dirt are given, it is requested to find the optimal solution regarding the number of washings and the amount of water used each time and discuss the effect of the above parameters.

- Problem 2: Assuming each washing takes the same time and there is no limit on available water, with other conditions similar to problem 1. Provide the most time-efficient cleaning plan that enables final dirt residue to be no more than one-thousandth of its initial amount, and analyze the impact of the solubility and the initial amount of dirt on the optimal solution.
- Problem 3: According to the type and quantity of dirt on clothing, along with the solubility of ten detergents on the dirt and its price, as well as the water cost per ton are shown, give both cost savings and a good cleaning program.
- Problem 4: Considering the differences in material, some clothes cannot be mixed for washing, as shown in Table 4. Under the same conditions as problem 2, provide a cost-effective and efficient cleaning plan.

2. Problem Analysis and Solutions

2.1 Analysis of problem 1

The conditions given in problem 1 include: Given the amount of dirt in clothes and the amount of water available, the solubility of dirt in water at the k th washing is a_k, where $a_1 = 0.8, a_k = 0.5a_{k-1}, k = 2,3,...,L$. The final requirement of problem 1 is: under the above conditions, find the best way to clean clothes, and find the optimal solution of the number of washings and the amount of water used in each washing; then, discuss the effect of a_k, initial amount of dirt, and the amount of water available.

Therefore, it can be seen that the objectives to be considered in the construction of the model is to minimize the final dirt residue after *k* rounds of washing and minimize the amount of water used in the washing process. The constraints that need to be considered are also related to the amount of water used and the minimum amount of dirt remaining in clothing. It is worth noting that detergent residue on the clothing will do harm to the skin and clothing. As a result, **the dirt that can be removed includes not only the original dirt on the clothing, but also the detergents remaining in it.**

In addition, “with other conditions similar to problem 1” is mentioned in Problem 2, and “Under the same conditions as problem 2” is mentioned in problem 4. Therefore, during the model construction of problem 1, it is also necessary to consider the consistency of the required models in other problems to be solved, and to give some assumptions required.

2.2 Analysis of problem 2

Compared with problem 1, problem 2 removes the limit on the amount of water available but adds the limit that the ratio of the remaining dirt residue to the initial dirt amount is less than 0.001, while specifying the same washing time for each round.

Therefore, for the model of problem 2, based on the model of problem 1, we remove one of the objectives “minimize the amount of water used in the washing process”, and add a constraint “the final dirt residue should be no more than one-thousandth of the initial dirt amount”.

Since the optimal amount of water used in each washing can be figured out in problem 1 and $a_k, k = 1, 2, ...$ can be calculated through recursive relation given in problem 1, we can finally figure out the optimal number of washings n_{opt2}. Since the washing time of each round $R_k = R$ is the same, the duration of the most time-efficient washing plan is $n_{opt2} \times R$.

2.3 Analysis of problem 3

In problem 3, the types and amount of dirt, the price of water per ton, as well as the solubility and price of various detergents are given. We are requested to give both cost savings and a good cleaning program. Therefore, the optimal cleaning plan is defined as: less cost of detergent, less cost of water, the final dirt residue should be no more than ε of the initial dirt amount, and fewer number of washings. Among them, the number of washings also reflects the cost of washing, such as time and electricity. According to this, the model can be established.

To simplify the model solving process, we should set the values of parameters to combine the goals. Besides, the conclusions obtained from problem 1 and problem 2 can facilitate the establishment and solution of problem 3.

2.4 Analysis of problem 4

In problem 4, the materials of clothing are given. We are requested to further consider the restrictions on mixing clothing of different materials for washing, indicating that the clothing need to be divided into piles first before washing. The restrictions can be regarded as the adjacency matrix of an undirected graph. Therefore, the types of clothing that can be classified into the same pile are essentially a complete subgraph of this undirected graph or the single node.

In this problem, we need to do a broad investigation to give different attributes of materials such as the weight, the water absorption characteristics. In reality, it is impossible for a laundry to clean all the clothing at once, indicating that the capability of machines needs to be considered. The weight of each divided pile should be less than the maximum weight of clothes can be washed in a single round. After dividing the clothing, the optimal cleaning plan can be obtained by the method and model used in problem 3.

Therefore, the core of problem 4 is how to give an optimum piling plan to get a cost-effective and time-efficient clothing dividing and cleaning solution. Besides, the conditions in problem 2 should be considered.

2.5 Overview of our solutions

The full flow chart of our solutions is shown as Fig. 1.

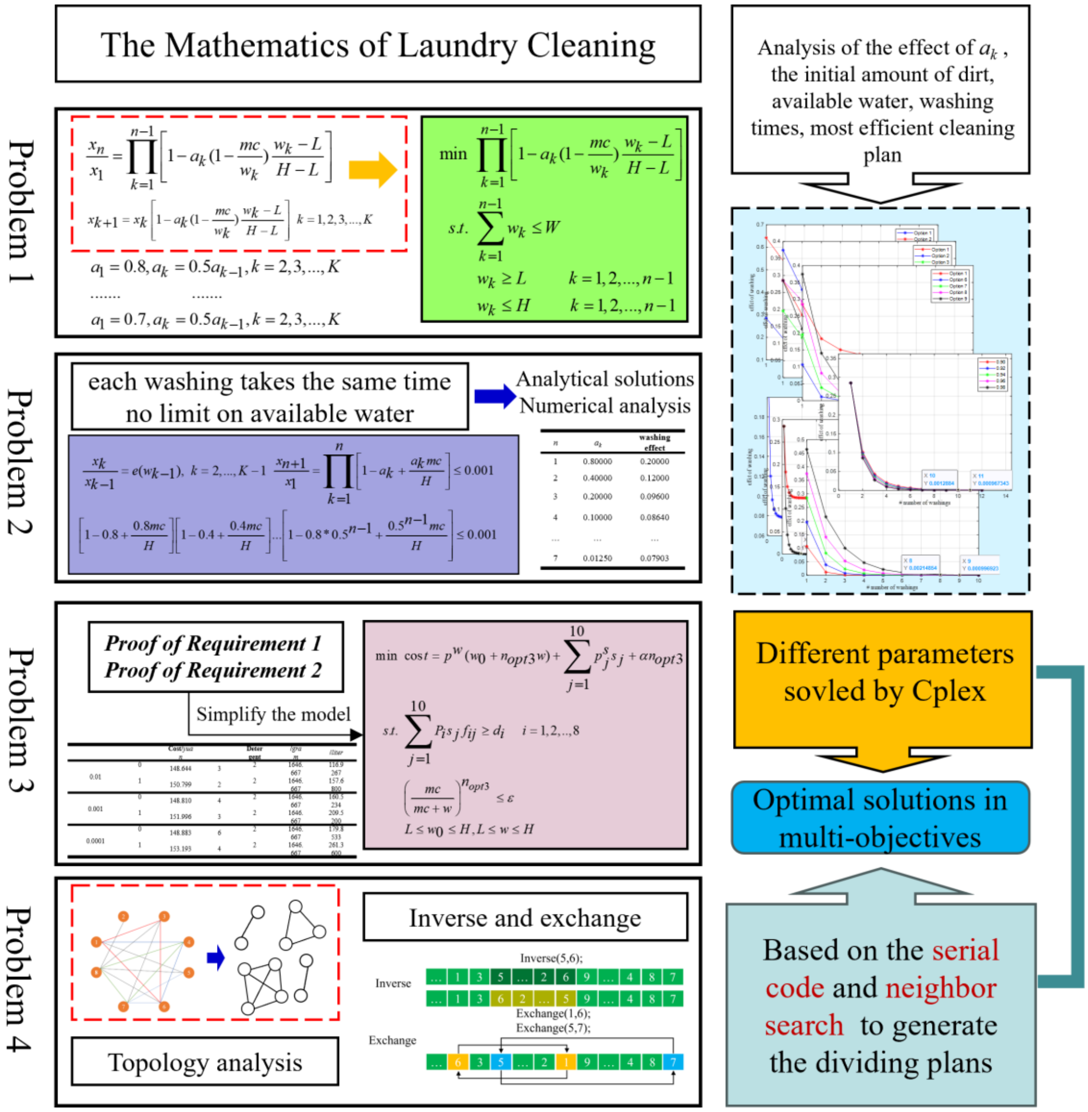

Fig. 1 The full flow chart of our solutions

3. Symbol and Assumptions

3.1 Symbol description

The symbols used in this paper are summarized in Table 5 in appendix.

3.2 Fundamental assumptions

To simplify the given problems and modify it more appropriate for simulating real life conditions, we make the following basic hypotheses:

1. The process of each round of washing is: *adding water - rinsing - dehydration.*
2. Detergent is usually only added in the first round of washing, and then rinsed with water in the following rounds, no more detergent is added.
3. The temperature and pressure of each round of washing are the same and will not affect the solubility of the dirt.

4. Washing machine has certain requirements on the water injection volume, when the water injection volume is small, it cannot complete the washing task; At the same time, optimal amount of water used to achieve the best dirt dissolution results without causing water waste is H, and H is proportional to the initial weight of clothing m, i.e., $H = \lambda m$. Thus, it is assumed that the water injection volume cannot be lower than L and cannot be greater than H.
5. The time available for each round of washing is sufficient. That is, in each washing, water can be fully mixed with the clothing, so that the dirt on the clothing can be fully dissolved.
6. The amount of water left after each round of dehydration is a fixed value that is related to the water absorption characteristic of the clothing material and the total amount of clothing currently washed.

4. Model Establishment and Solution of Problem 1

4.1 Model establishment

4.1.1 The established model

Assuming that there are K rounds of washing, and the water used per round be $w_k, k = 1, 2, ..., K$. At the beginning, the dirt on the clothes is x_1, including the amount of original dirt and the detergent added in the first round, after the k th round of dehydration, the amount of dirt remaining on the clothes is x_{k+1}.

After the $(k-1)$ th round of rinsing and before the $(k-1)$ th dehydration, the amount of dirt x_k consists of two parts:

$$x_k = p_k + q_k \tag{1}$$

where p_k represents the dirt that has been dissolved into the water, and q_k represents the dirt that has not been dissolved into the water.

p_k is related to the amount of water added in the $(k-1)$ th round. Its value conforms to the following law: when w_k is the smallest, $w_k = L, p_k = 0$; when w_k is maximum, $w_k = H, p_k = a_k \times x_k$, where a_k is the solubility of the dirt. Therefore, a simple linear relationship is used to describe the relationship between dissolution and water injection volume.

$$p_k = a_k x_k \frac{w_k - L}{H - L} \tag{2}$$

After $(k-1)$ rounds of rinsing, the remaining dirt on the clothes is $q_k = x_k - p_k$.

Let m be the total weight of the clothes, and c be the water absorption characteristics of the clothes. The amount of water retained by the clothes after each dehydration is mc, which is a fixed value according to fundamental assumptions, and the amount of dirt contained in this part is $(p_k / w_k) \times mc$. Therefore, it can be inferred that after $(k-1)$ rounds of washing, the total amount of dirt remaining on the clothes is:

$$x_{k+1} = x_k - p_k + \frac{p_k}{w_k} mc \tag{3}$$

Inserting (2) into (3), the following equation can be sorted out:

$$x_{k+1} = x_k \left[1 - a_k (1 - \frac{mc}{w_k}) \frac{w_k - L}{H - L} \right] \quad k = 1, 2, 3, ..., K \tag{4}$$

The amount of dirt after each round of washing can be calculated by Eq. (4), and x_{k+1}/x_1 can reflect the washing effect of the k th round. Therefore, the index of washing effect can be defined as Eq. (5):

$$\frac{x_n}{x_1} = \prod_{k=1}^{n-1} \left[1 - a_k (1 - \frac{mc}{w_k}) \frac{w_k - L}{H - L} \right] \tag{5}$$

Eq. (5) denotes the proportion of remaining dirt compared to the initial amount of dirt after $(n-1)$ rounds of washing.

Further, the total amount of water used W_{used} can be calculated as:

$$W_{used} = \sum_{k=1}^{n-1} w_k \tag{6}$$

Based on the given amount of water and dirt, the optimization model can be established as follows:

$$\begin{aligned} \min \ & \prod_{k=1}^{n-1} \left[1 - a_k (1 - \frac{mc}{w_k}) \frac{w_k - L}{H - L} \right] \\ s.t. \ & \sum_{k=1}^{n-1} w_k \le W \\ & w_k \ge L \qquad k = 1, 2, ..., n-1 \\ & w_k \le H \qquad k = 1, 2, ..., n-1 \end{aligned} \tag{7}$$

where W is the amount of water available.

In Eq. (7), the optimization objective is the final washing effect. The smaller the proportion of the residual dirt relative to the initial amount of dirt, the better the washing effect. w_k is the amount of water used in the k th round, and the total amount of water used needs to meet the constraint of less than or equal to the given amount of water. At the same time, the water added at each round of washing needs to be greater than or equal to the minimum amount of water L and less than or equal to the maximum amount of water H.

4.1.2 The property of the model

In order to facilitate the analysis of the objective function, let

$$v_k = \frac{w_k - L}{H - L} \tag{8}$$

$$w_k = v_k(H-L)+L \tag{9}$$

v_k is the normalized amount of water used at the k th round of washing. Due to $L \le w_k \le H$, the value of v_k is between [0,1].

Then, the following objective Eq. (10) is equal to Eq. (7):

$$\min \prod_{k=1}^{n-1}\left[1-a_k v_k + \frac{a_k v_k}{Av_k + B}\right] \tag{10}$$

$$A = \frac{H-L}{mc} = B(\frac{H}{L}-1),\ B = \frac{L}{mc} \tag{11}$$

a_k is the solubility rate, its value can be calculated through the recurrence relationship of $a_1 = 0.8,\ a_k = 0.5a_{k-1}$. Thus, it is a constant rather than a variable.

According to Eq. (5), the washing effect of a single round can be defined as:

$$e(t) = 1 - at + \frac{at}{At+B},\ 0 \le t \le 1\left(t = \frac{w_k - L}{H-L}\right) \tag{12}$$

Taking the derivative of Eq. (12) with respect to t:

$$e'(t) = a\left[\frac{B}{(At+B)^2} - 1\right] < 0\ ,\ 0 \le t \le 1 \tag{13}$$

Obviously, $e'(t) < 0$ is constant. Thus, $e(t)$ monotonically decreases between [0,1].

4.2 Model solution

4.2.1 The optimal amount of water used

According to Eq. (13), it can be preliminarily drawn when the maximum amount of water H used at each round, the washing effect will be optimal. **This conclusion is further proofed in the following.**

Assuming that the solubility at the k th round of washing is a_k, then, the solubility at the $(k+1)$ th round of washing is $a_{k+1} = 0.5a_k$. In response to the purpose of proof, we should verify that: when the amount of water used becomes from t_1 to $t_2 = t_1 + \Delta t$, the change in washing effect of the k th washing is greater than that of the $(k+1)$ th washing.

According to Eq. (12), for the k th washing: (1) When the amount of water used is t_1, the washing effect is $e(t_1) = 1 - a_k t_1 + a_k t_1/(At_1 + B)$; (2) When the amount of water used becomes $t_1 + \Delta t$, the washing effect is $e(t_2) = 1 - a_k(t_1 + \Delta t) + a_k(t_1+\Delta t)/(A(t_1+\Delta t)+B)$. Then, the change of the washing effect is $\Delta e_k = e(t_1) - e(t_2) = a_k \times \Delta t + a_k t_1/(At_1+B) - a_k(t_1+\Delta t)/(A(t_1+\Delta t)+B)$.

Similarly, for the $(k+1)$ th washing, the change of the washing effect after the amount of water used increasing is $\Delta e_{k+1} = 0.5 \times a_k \times \Delta t + 0.5 \times a_k t_1 / (At_1 + B) - 0.5 \times a_k (t_1 + \Delta t) / (A(t_1 + \Delta t) + B)$.

Then, comparing Δe_k with Δe_{k+1}:

$$\Delta e_k - \Delta e_{k+1} = 0.5 * a_k * \Delta t + \frac{0.5 * a_k t_1}{At_1 + B} - \frac{0.5 * a_k (t_1 + \Delta t)}{A(t_1 + \Delta t) + B} = \Delta e_{k+1} \tag{14}$$

Due to $e(t)$ monotonically decreases and $t_1 < t_2$. Thus, $\Delta e_{k+1} > 0$, and $\Delta e_k - \Delta e_{k+1} > 0$. Therefore, when the amount of water used is the same, the bigger the solubility, the better the washing effect. The above conclusion is confirmed, and it is in line with common sense in daily life.

The above conclusion can also be explained more intuitively with a specific example. Assuming that there is only a piece of clothing, and the values of various parameters are assumed to be: $H = 1.5$ liter; $L = 0.5$ liter; $m = 0.2$ kilogram; according to work [1], $c = 80\%$. Therefore, $A = 6.25$, $B = 3.125$. If the amount of water available is $W = 5 \times H$, and there are two options for washing:

Option 1: The maximum amount of water H is used in each washing, then the number of washings is 5.

Option 2: The amount of water used in each washing is $0.5 \times H$, then the number of washings is 10.

According to Eq. (12), the relationship between the washing effect and washing times under the above two options is shown in Fig. 2.

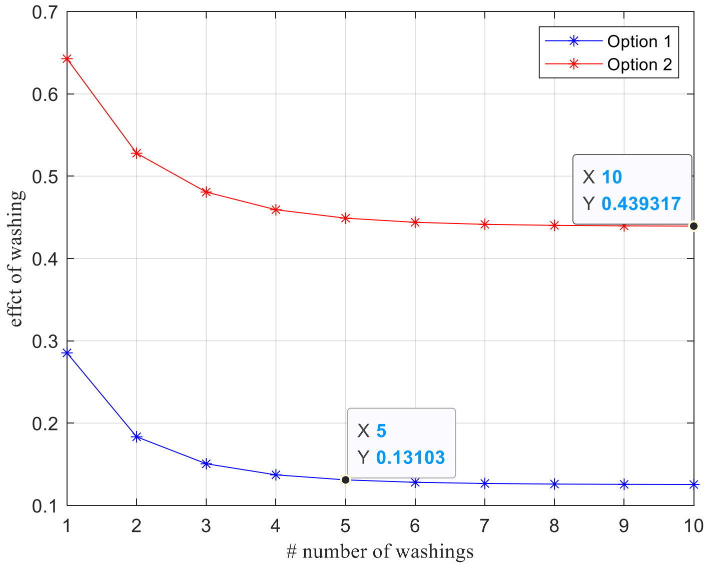

Fig. 2 The relationship between washing effect and washing times under different amount of water

It can be seen from Fig. 2 that when Option 1 is adopted, the final remaining proportion of dirt is 0.1310, while when Option 2 is adopted, the final remaining proportion of dirt is 0.4393. Obviously, Option 1 performs better.

In a word, the optimal amount of water used at each washing is the maximum amount of water used to achieve the best effect of dirt dissolution, i.e., H.

4.2.2 The optimal number of washings

The amount of water available is W, and the optimal amount of water used at each washing is H. Therefore, the optimal number of washings is $n_{opt1} = \lfloor W/H \rfloor$, which means rounding down the value of W/H.

4.3 Discussion on the effect of variables on the objective

4.3.1 The solubility of the dirt in water at the k th wash a_k

The influence of a_k is analyzed from two different perspectives. On the one hand, a_k changes with the number of washings. On the other hand, a_k can also be a constant value. In this section, the amount of water used in each washing is H.

Firstly, when a_k changes with the number of washings, suppose there are several different rules of change:

Option 1: $a_1 = 0.8, a_k = 0.5a_{k-1}, k = 2,3,...,K$. This is the condition given in problem 1.

Option 2: $a_1 = 0.7, a_k = 0.5a_{k-1}, k = 2,3,...,K$. That is, a_1 decreases.

Option 3: $a_1 = 0.9, a_k = 0.5a_{k-1}, k = 2,3,...,K$. That is, a_1 increases.

Option 4: $a_1 = 0.8, a_k = 0.3a_{k-1}, k = 2,3,...,K$. That is, a_k decreases faster.

Option 5: $a_1 = 0.8, a_k = 0.7a_{k-1}, k = 2,3,...,K$. That is, a_k decreases slower.

When the parameters are assumed to be the same as those in Section 4.2.1, the relationship between washing effect and washing times under the above five options are shown in Fig. 3.

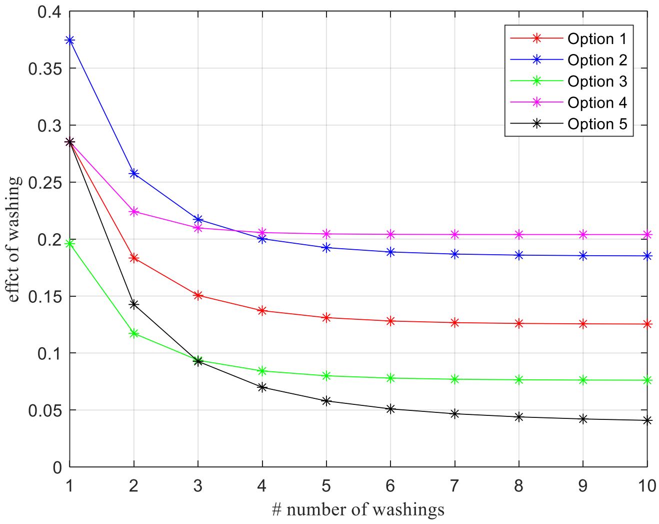

Fig. 3 The relationship under different solubility

The following conclusions can be drawn from Fig. 3:

(1) Comparing Option 2 with Option 1: When a_1 decreases, the washing effect becomes worse.

(2) Comparing Option 3 with Option 1: When a_1 increases, the washing effect

becomes better.

(3) Comparing Option 4 with Option 1: When a_k decreases faster, the washing effect becomes worse.

(4) Comparing Option 5 with Option 1: When a_k decreases slower, the washing effect becomes better.

These conclusions are consistent with theory and practice: The larger the solubility of the dirt in water, the better the washing effect.

Besides, $a_k, k=1,2,...,K$ can be a constant value. In this case, the following four options are considered:

Option 6: $a_k = 1, k=1,2,...,K$; **Option 7:** $a_k = 0.9, k=1,2,...,K$;

Option 8: $a_k = 0.8, k=1,2,...,K$; **Option 9:** $a_k = 0.7, k=1,2,...,K$.

When the parameters keep the same, the relationship between washing effect and washing times under the above four options are shown in Fig. 4.

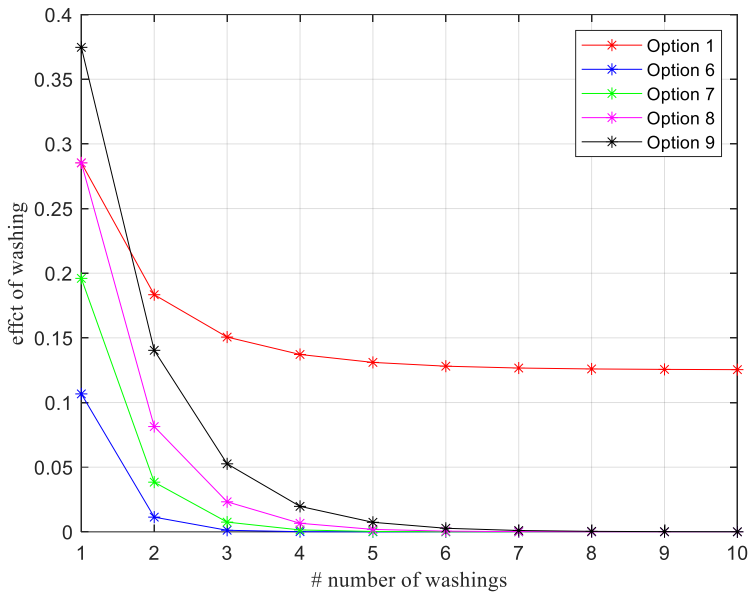

Fig. 4 The relationship under different constant solubility

Obviously, the situation shown in Fig. 4 is similar to that in Fig. 3, that is, the larger the solubility, the better the washing effect. Besides, when a_k is a constant value, the tested 4 values can almost completely eliminate the dirt within 10 times of washing; While when a_k decrease with the number of washings, after 10 rounds of washing, there is still a lot of dirt left in the clothing, **which is not in line with common sense in daily life.**

In summary, the effect of a_k on washing effect is summarized as follows: (1) The larger the solubility a_k, the better the washing effect; (2) When the value of a_k is constant at each washing, the washing effect more consistent with common sense in daily life can be achieved.

4.3.2 The initial amount of dirt

The initial amount of dirt has an impact on the amount of water used and the final washing effect.

For the amount of water used: The optimal amount of water used at each round of washing H increases with the increase of the weight of clothes (including the initial amount of dirt) in this paper. Therefore, the larger the initial amount of dirt, the larger the amount of water used.

For the final washing effect: The washing effect at each round of washing (Eq. (12)) is proportional to the amount of water used. Therefore, when the amount of water available is given, the larger the initial amount of dirt, the worse the final washing effect.

Besides, for the number of washings: Since H is adjusted according to the weight of clothes, the number of washings $n_{opt1} = \lfloor W/H \rfloor$ is not affected by the initial amount of dirt D.

4.3.3 The amount of water available

The amount of water available W affects the total number of washings and also affects the final washing effect. The specific relationship is: The larger the amount of water available W, the greater number of washings can be achieved. Besides, when other conditions remaining the same, a larger amount of water available W can achieve better washing effect. The analysis is as follows:

(1) The optimal number of washings is $n_{opt1} = \lfloor W/H \rfloor$. Obviously, it is critically affected by W and is directly proportional.

(2) The situations shown in Fig. 2, Fig. 3 and Fig. 4 indicate that the washing effect becomes better with the increase number of washings, which means an increase in the amount of water used. Therefore, the washing effect improves with an increase in the amount of water available.

5. Model Establishment and Solution of Problem 2

5.1 Model establishment

In problem 2, the amount of water used is no longer limited, and the washing time of each round $R_k, k = 1,2,3,...,K$ has the same value R. The goal is to figure out the most time-efficient cleaning plan. Since the amount of water available are sufficient, and it has been proved in problem 1 that the larger amount of water in the previous rounds can get the best washing effect. The optimal amount of water used in each round is $w_k = H$, $k = 1,2,3,...,K$.

According to Eq. (4) and Eq. (12), the function of the washing effect of the $(k-1)$ th round is:

$$\frac{x_k}{x_{k-1}} = e(w_{k-1}),\ \ k = 2,...,K-1 \tag{15}$$

It can be calculated that after n rounds of washing, the washing effect is the percentage of the final remaining dirt to the initial amount of dirt:

$$\frac{x_{n+1}}{x_1} = \prod_{k=1}^{n}\left[1 - a_k + \frac{a_k mc}{H}\right] \tag{16}$$

According to problem 2, the final washing effect should be less than 0.001, i.e.,

$$\frac{x_{n+1}}{x_1}=\prod_{k=1}^{n}\left[1-a_k+\frac{a_k mc}{H}\right]\le 0.001 \tag{17}$$

When k is determined, the value of $a_1,a_2,\ldots,a_k$ can be determined according to the relationship "$a_1=0.8,\ a_k=0.5a_{k-1}$". Eq. (17) is actually a one-variate inequality:

$$\left[1-0.8+\frac{0.8mc}{H}\right]\left[1-0.4+\frac{0.4mc}{H}\right]\ldots\left[1-0.8\times 0.5^{n-1}+\frac{0.5^{n-1}mc}{H}\right]\le 0.001 \tag{18}$$

where n is the number of washings.

When m,c,H is determined, we can finally figure out n_{opt2} that satisfies Eq. (18). Since the washing time of each round $R_k=R$ is the same, the duration of the most time-efficient cleaning plan is $n_{opt2}\times R$.

5.2 Model solution

5.2.1 The given solubility a_k in problem 1

In Eq. (17), $a_k mc/H$ is derived from the idea that after each round of washing, there will be sewage left in the clothes due to their water absorption characteristics. In this section, we firstly assume that this part does not exist, that is, washing is carried out in a completely ideal situation, and there is no residual sewage in the clothes during each round of cleaning. Then, Eq. (18) is changed as:

$$[1-0.8][1-0.4]\ldots\left[1-0.8\times 0.5^{n-1}\right]\le 0.001 \tag{19}$$

n is a positive integer. When $n=100$, the relationship between the washing effect and washing times is shown in Fig. 5.

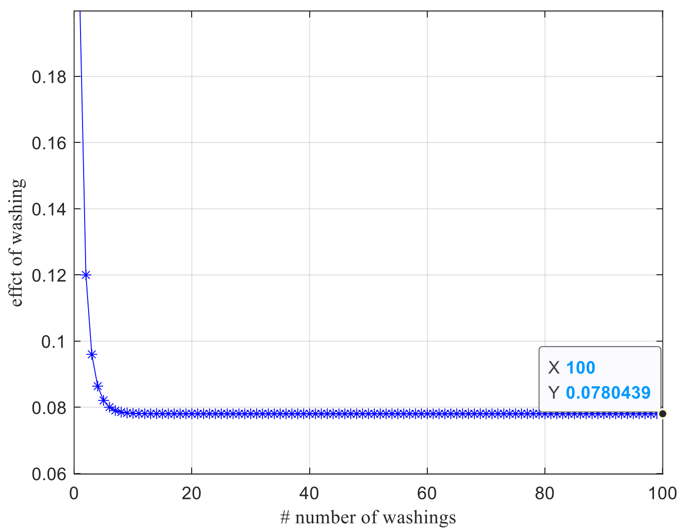

Fig. 5 The solution under no absorption of water on clothes at all

It can be concluded that after 100 washings, the proportion of residual dirt in clothes to the initial amount of dirt is 0.07804, and the objective has not been satisfied.

The specific washing effect of the first 19 washings is shown in Table 1. It can be seen that from the 7th washing, the change of washing effect is very small; and from the 17th washing, the washing effect almost converges to 0.07804 and no longer changes, which is contrary to the actual practice.

Table 1 The washing effect of the first 19 washings

n	a_k	**washing effect**	n	a_k	**washing effect**	n	a_k	**washing effect**
1	0.80000	0.20000	8	0.00625	0.07853	15	0.00005	0.07805
2	0.40000	0.12000	9	0.00313	0.07829	16	0.00002	0.07805
3	0.20000	0.09600	10	0.00156	0.07817	17	0.00001	0.07804
4	0.10000	0.08640	11	0.00078	0.07810	18	0.00001	0.07804
5	0.05000	0.08208	12	0.00039	0.07807	19	0.00000	0.07804
6	0.02500	0.08003	13	0.00020	0.07806			
7	0.01250	0.07903	14	0.00010	0.07805			

Therefore, the result n_{opt2} that satisfies Eq. (19) cannot be obtained. And Eq. (19) is more ideal than Eq. (18), thus, we cannot find the result satisfying Eq. (18) as well.

After analysis, the main cause of this phenomenon is that the solubility a_k decreases exponentially with the increase of washing times, which is too fast. This leads to gradually insignificant washing effect after several washings. The discussion in Section 4.3.1 shows that when a_k decreases slower (Fig. 3) or takes a constant value (Fig. 4), the washing effect can converge quickly. Therefore, the performance of these two cases is further considered below.

5.2.2 The solubility a_k decreases slower

a_k decreases slower means that in the relationship "$a_1 = 0.8$, $a_k = 0.5a_{k-1}$", 0.5 becomes larger. Assuming that it becomes 0.9, i.e., $a_k = 0.9a_{k-1}$. Following the setting of parameters in Section 4.2.1: $m = 0.2, c = 80\%, H = 1.5$. The results are presented in Fig. 6.

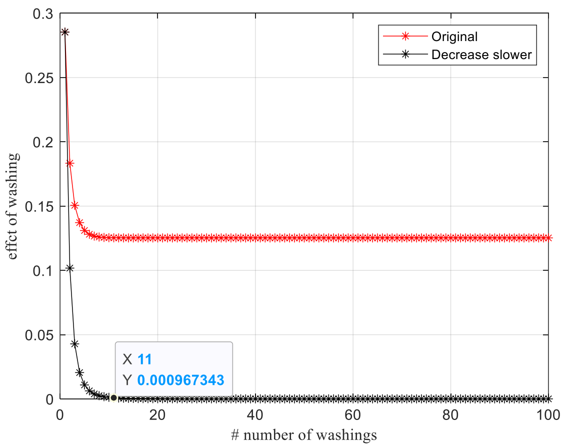

Fig. 6 The solution decreases slower

It can be judged that $n_{opt2} = 11$. That is, under this parameter setting, the predefined washing effect can be achieved after 11 washings.

5.2.3 The solubility a_k is constant

For soluble dirt, if it can be dissolved by detergent, there is an optimal concentration of detergent to water. So that after sufficient soaking and stirring, the dirt can be completely dissolved in water. Considering the actual working process of laundry, it is not a reasonable behavior to perform multiple rounds of cleaning and add detergent at each round. **A more realistic approach is** to add sufficient detergent in the first round of washing, so that all dirt can detach from the clothes after soaking and stirring in the first round; then, after continuous rinsing with water, the amount of detergent and partially dissolved dirt in the clothes are reduced.

Based on the analysis, we believe that after adding sufficient detergent in the first round, all the dirt will be dissolved, i.e., $a_1 = 1, a_k = a_{k-1}$. Then, only the impact of dirt contained in the water remaining on the clothing needs to be considered.

Taking $a_1 = 1, a_k = a_{k-1}$ into Eq. (17):

$$\left(\frac{mc}{H}\right)^n \leq 0.001 \tag{20}$$

Then, n_{opt2} can be calculated:

$$n_{opt2} = \left\lceil \frac{-3\log(10)}{\log(\frac{mc}{H})} \right\rceil \tag{21}$$

where $\lceil x \rceil$ denotes rounding up the value of x.

Similarly, following the setting of parameters in Section 4.2.1, it can be calculated that $n_{opt2} = 4$.

In summary, (1) when the solubility a_k given in problem 1 is adopted, there is no result satisfying Eq. (18); (2) when a_k decrease slower, $a_k = 0.9a_{k-1}$, under the set parameters, the predefined washing effect can be achieved after 11 washings; (3) when a_k is fixed to be 1, under the set parameters, the predefined washing effect can be achieved by 4 washings, which is more in line with common sense in daily life.

Therefore, the most time-efficient cleaning plan is to ensure the solubility $a_k = 1$ during each round of washing by adding sufficient amount of detergent and water. In this way, the predefined washing effect can be achieved after 4 rounds of washing, and the washing time is $4 \times R$.

5.3 Discussion on the effect of variables on the solution

5.3.1 The solubility of the dirt in water at the k th wash a_k

When a_k decreases slower or is constant, n_{opt2} can be calculated. The value of a_k has an impact on the number of washings, which is discussed separately.

Case 1: a_k decreases slower

The form of relationship is $a_k = b \times a_{k-1}$, five cases are considered in this section, where the value of b is 0.90, 0.92, 0.94, 0.96 and 0.98 respectively. The results are shown in Fig. 7.

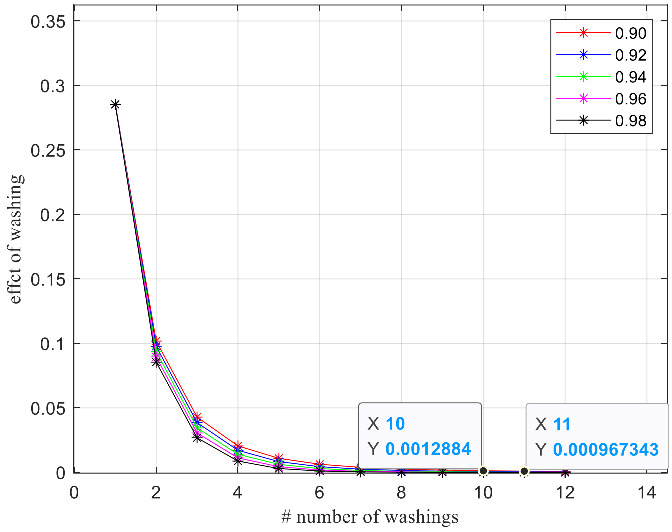

Fig. 7 The solution decreases with different speeds

It can be found that all of them can achieve predefined effect within 11 times. When $b = 0.98$, only 7 rounds of washing are needed. Besides, there is a law that the slower the decrease rate of a_k, the fewer washing times are needed to achieve the pre-defined washing effect, resulting in lower washing time.

Case 2: a_k is constant

When a_k is a constant value, five cases (a_k = 1, 0.9, 0.8, 0.7, 0.6) are considered. The results are shown in Fig. 8. It can be seen that all of them can achieve predefined washing effect after 9 rounds of washing. And when $a_k = 1$, only 4 rounds are needed, which is the fastest.

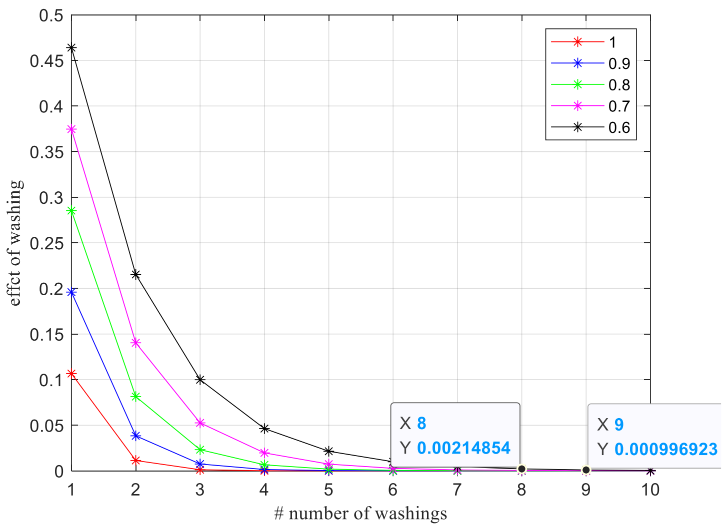

Fig. 8 The solution with different constant values

In summary, the larger the a_k, the fewer rounds of washing are required and the less washing time will be spent.

5.3.2 The initial amount of dirt

The impact of the initial amount of dirt on the optimal solution is analyzed as follows:

Since there is no limit on the amount of available water and the detergent and water used in each washing are flexibly adjusted according to the weight of clothes (including the initial amount of dirt), the initial amount of dirt has no effect on the number of washings and the total washing time. That is, it will not affect the optimal solution obtained for this problem.

6. Model Establishment and Solution of Problem 3

6.1 Model establishment

We conclude from problem 2 that the most time-efficient cleaning plan is to ensure the solubility $a_k = 1$ during each round of washing by adding sufficient amount of detergent and water. Thus, it is assumed that the amount of detergent added in the first round can dissolve all dirt with fully soaking and stirring.

Eight types of contaminates and ten types of detergents are given, and the following variables are defined: i represents the types of the original dirt; j represents the types of the detergents; d_i represents the amount of the i th original dirt, and the total amount of the original dirt is $D = \sum_{i=1}^{8} d_i$; s_j represents the amount of the j th detergent; the proportion of a detergent consumed by the i th dirt is $P_i = d_i / D$.; the amount of the detergent consumed by the dirt is directly proportional to the amount of the dirt itself, so the amount of the j th detergent consumed by the i th dirt is $P_i s_j$; f_{ij} represents the solubility of the j th detergent to the i th dirt, i.e., the amount of dirt that can be dissolved by a unit of detergent, and the amount of the i th dirt dissolved by the j th detergent is $P_i s_j f_{ij}$. The initial amount of dirt including the amount of ten detergents is $x_1 = \sum_{i=1}^{8} d_i + \sum_{j=1}^{10} s_j$.

To ensure that all the dirt is dissolved in the first round, i.e., the amount of detergent added is sufficient to dissolve all the dirt. The following constraints need to be satisfied:

$$\sum_{j=1}^{10} P_i s_j f_{ij} \ge d_i \tag{22}$$

Under the assumed conditions of the above analysis, when all the dirt in the clothing is dissolved, the total amount of dissolved dirt will not change, and the continued addition of water will only affects the concentration of dirt. It is assumed that the minimum amount of water satisfying the condition is $w_{\min}$, and the initial amount of water added is w_0 , $w_0 \ge w_{\min}$. The concentration of dirt is expressed by Eq. (23):

$$Q_0 = \frac{x_1}{w_0} \tag{23}$$

The amount of dirt left in the clothes after the first round of washing is:

$$\Delta x_1 = Q_0 mc \tag{24}$$

where m is the total weight of the clothes, and c is the water absorption characteristics of the clothes. The amount of water retained by the clothes after each dehydration is mc, which is a fixed value according to fundamental assumptions.

After k rounds of rinsing, the concentration of dirt in the remaining water is:

$$Q_k = \frac{mc}{mc + w_k} Q_{k-1} \tag{25}$$

Therefore, the effect of washing can be described as:

$$\frac{Q_k}{Q_0} = \left(\frac{mc}{mc + w_k} \right)^k \tag{26}$$

The desired washing effect threshold is ε:

$$\frac{Q_k}{Q_0} \le \varepsilon \tag{27}$$

Suppose the optimal number of washings is $k = n_{opt3}$, the goal of the problem is: Under a certain washing effect, the washing cost is as small as possible. The washing cost mainly includes the cost of detergent and water, as well as the cost of electricity. Besides, the rounds of washing determine the time spent. Therefore, the rounds of washing should be as small as possible. The model can be expressed as:

$$\begin{aligned} &\min \ \cos t = p^w (w_0 + \sum_{k=1}^{n_{opt3}} w_k) + \sum_{j=1}^{10} p_j^s s_j + \alpha n_{opt3} \\ &s.t. \ \sum_{j=1}^{10} P_i s_j f_{ij} \ge d_i \quad i = 1, 2, .., 8 \\ &\qquad L \le w_k \le H, \quad k = 0, 1, 2, ... \end{aligned} \tag{28}$$

where α is the coefficient of the rounds of washing; p_j^s is the price of the j th detergent and p^w is the price of water per ton.

Through the above analysis, it can be found that the washing effect reaches the optimal value when the water used in each round meets the following requirements:

Requirement 1: In addition to the first round of washing, when the amount of water is certain, the optimal washing effect can be achieved when the same amount of water is added in each subsequent round.

Requirement 2: When the total amount of water used is fixed, the more the rounds of washing, the less the remaining dirt.

***Proof of Requirement* 1:**

(1) Suppose that after n rounds of rinsing, the concentration of the dirt is Q_n, and the amount of water added in each round is not exactly the same.

(2) Suppose that after n rounds of rinsing, the concentration of dirt is $Q_n{}'$, and the amount of water added in each round is the same: $w = \sum_{k=1}^{n} w \big/ n$. Through the reverse proving, it can be found that $Q_n{}' < Q_n$ is valid.

If $Q_n{}' > Q_n$, we can get Eq. (29) and Eq. (30) based on Eq. (25):

$$\left(\frac{mc}{mc+w}\right)^n Q_0 > \frac{(mc)^n}{(mc+w_1)(mc+w_2)...(mc+w_n)} Q_0 \tag{29}$$

$$(mc+w)^n < (mc+w_1)(mc+w_2)...(mc+w_n) \tag{30}$$

Since $w = \sum_{k=1}^{n} w \big/ n$, let $y_1 = mc + w_1, y_2 = mc + w_2, ..., y_n = mc + w_n$:

$$\frac{y_1 + y_2 + ... + y_n}{n} < \sqrt[n]{y_1 y_2 ... y_n} \tag{31}$$

This result contradicts with the important inequality $(y_1 + y_2 + ... + y_n)/n \geq \sqrt[n]{y_1 y_2 ... y_n}$. Therefore, adding the same amount of water to each rinse works best.

***Proof of Requirement* 2:**

When the amount of water used W is fixed, the washing effect after $(n+m)$ rounds is better than that after n rounds:

$$Q_n = \left(\frac{mc}{mc + \dfrac{W}{n}}\right)^n Q_0 \tag{32}$$

Regarding n as a continuous variable, taking the derivative of Q_n:

$$Q_n{}' = Q_0 n \left(\frac{mc}{mc + \dfrac{W}{n}}\right)^{n-1} \frac{\dfrac{W}{n} mc}{\left(mc + \dfrac{W}{n}\right)^2} \left(-\frac{W}{n^2}\right) < 0 \tag{33}$$

Obviously, Eq. (32) monotonically decreases. Thus, the washing effect after $(n+m)$ rounds is better than that after n rounds.

Besides, the minimum amount of water cannot be lower than the minimum amount of water L and cannot be higher than the maximum amount of water H. Based on the two requirements proved, we can obtain the following expressions:

$$w = \frac{mc}{\sqrt[n]{\dfrac{Q_n}{Q_0}}} - mc \quad (L \leq w \leq H) \tag{34}$$

$$\frac{Q_n}{Q_0} \le \varepsilon \tag{35}$$

Then, the model can be simplified as:

$$\begin{aligned}
&\min\ \ cost = p^w(w_0 + n_{opt3}w) + \sum_{j=1}^{10} p_j^s s_j + \alpha n_{opt3} \\
&s.t.\ \sum_{j=1}^{10} P_i s_j f_{ij} \ge d_i \quad i = 1,2,..,8 \\
&\left(\frac{mc}{mc+w}\right)^{n_{opt3}} \le \varepsilon \\
&L \le w_0 \le H, L \le w \le H
\end{aligned} \tag{36}$$

6.2 Model solution

In the constructed model Eq. (36), there are some constant variables m, c, H, L. To solve the model, the setting of parameters in Section 4.2.1 is still followed in this section, $m = 0.2, c = 80\%, H = 1.5, L = 0.5$. Besides, the weight of 36 pieces of clothing is assumed to be the same, i.e., the total weight is $m_{total} = 36 \times 0.2 = 7.2$. w_0 is set to be the maximum amount of water for each piece of clothing, then, $w_0 = 36 \times 1.5 = 54$.

6.2.1 Regardless of the maximum capacity of machine

According to 2.3, **the optimal cleaning plan is defined as:** less cost of detergent, less cost of water, the final dirt residue should be no more than ε of the initial dirt amount, and fewer number of washings. Among them, the number of washings also reflects the cost of washing, such as time and electricity, and it is determined by α in Eq. (36).

Since there is no explicit requirement for ε in problem 3, three different values of ε are considered, $0.01, 0.001, 0.0001$. Besides, to observe the impact of the number of washings, two values of α are tested: $\alpha = 0$ indicates that there is no limit on the number of washings; $\alpha = 1$ indicates that the number of washings is expected to be as small as possible. Accordingly, 6 different models are solved, and the results are listed in Table 2.

Table 2 The solutions of problem 3 under different values of ε and α

ε	α	**Cost**/*yuan*	k	**Detergent**	s /*gram*	w /*liter*
0.01	0	148.644	3	2	1646.667	116.9267
	1	150.799	2	2	1646.667	157.6800
0.001	0	148.810	4	2	1646.667	160.5234
	1	151.996	3	2	1646.667	209.5200
0.0001	0	148.883	6	2	1646.667	179.8533
	1	153.193	4	2	1646.667	261.3600

The following conclusions can be drawn from Table 2:

(1) Smaller ε means higher requirement for washing effect and higher cost of washing. Obviously, the calculated results are in line with this idea.

(2) Under each value of ε, the number of washings under $\alpha = 0$ is larger than that under $\alpha = 1$, which is in line with the objective. However, the former corresponds to a lower total cost and a lower amount of water. The main reason is that larger number of washings can bring higher cost of other factors, such as time, electricity, wear and tear of machine, etc.

(3) In each result, the amount of detergent s is identical, and the 2^{nd} detergent is always used. The reason is that the cost performance of the 2^{nd} detergent is the highest. The cost performance of detergent is defined as the amount of dirt that can be eliminated by one yuan of detergent, in *gram/yuan*. According to the definition and the data given by problem 3, the cost performance of each detergent can be calculated, as shown in Table 3. It can be found that the 2^{nd} detergent has the highest cost performance for eliminating the 4 types of dirt, and its total cost performance is also the highest. Therefore. The solution for problem 3 is reasonable.

Table 3 The cost performance of each detergent

Detergent	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**total**
1	2.1600	3.0000	3.1200	2.7600	3.2000	2.6400	2.2000	2.9200	22.0000
2	**8.5556**	7.4444	**6.5556**	6.4444	**7.8889**	5.3333	**5.0000**	7.3333	**54.5556**
3	6.3636	5.8182	5.6364	6.4545	5.7273	**7.0909**	4.5455	7.2727	48.9091
4	2.6842	2.8421	3.4737	4.3158	3.6842	3.1579	2.4211	2.8947	25.4737
5	4.8750	**9.0000**	5.3750	**7.1250**	5.7500	6.6250	**5.0000**	7.5000	51.2500
6	4.5000	6.0000	5.3000	4.8000	5.5000	4.5000	4.9000	**7.7000**	43.2000
7	5.7500	4.5833	5.1667	4.6667	3.9167	3.1667	3.6667	5.3333	36.2500
8	4.7273	4.0000	5.7273	6.4545	5.0909	6.1818	3.2727	6.0000	41.4545
9	4.4444	3.6111	3.1111	2.7222	4.0556	3.0556	2.6111	3.3889	27.0000
10	2.1364	3.6818	3.5000	2.4091	2.9091	2.6818	2.4091	1.9091	21.6364

6.2.2 Considering the maximum capacity of machine

In the above solutions, there is no limit on the maximum capacity of the machine. In fact, in real life, the weight of clothes that a washing machine can clean at once is usually limited. Therefore, this section assumes it as Max (unit: kilogram) to further solve and discuss this situation. Four different values of Max are considered, the experimental results and corresponding analysis are presented in Table 6 in appendix.

7. Model Establishment and Solution of Problem 4

7.1 Model establishment

Based on problem 3, problem 4 further considers the mixing problem between different materials of clothes. If a material of clothes are regarded as a node, " √ " marks in Table 4 is regarded as value "1", and "×" is regarded as value "0". Then an undirected graph adjacency matrix can be obtained. The relationship between each material can be described by an undirected graph $G=(V,E)$, which is shown in Fig. 9. If the two materials can be mixed for washing, there is a line between the corresponding nodes.

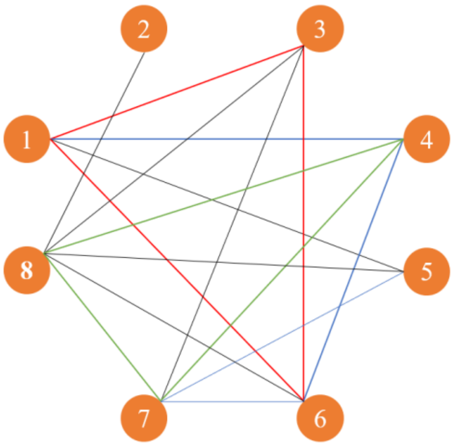

Fig. 9 The undirected graph of the eight materials

The materials that can be mixed for washing form a complete subgraph $G'=(V',E')$. If $V' \in V, E' \in E$, and any two points in G' are connected by edges, then G' is a complete subgraph of G.

In problem 3, we assume that the amount of detergent added in the first round can dissolve all dirt with fully soaking and stirring, and the water absorption characteristics of the clothes c is only determined by the material of the clothing. Based on this, problem 4 adds the assumption that the weight of different materials of clothing is different, and the weight of the clothing composed by the same material is the same. Therefore, the model of problem 4 is obtained by modifying the model of problem 3.

Following problem 2, it is assumed that the washing time of each round is the same, and the amount of water available is sufficient, problem 4 requires an economical and efficient washing plan and the final washing effect requires that the residual dirt is less than one thousandth of the initial dirt.

Suppose that Max is the maximum amount of washing clothes in a single round, m_i^r is the weight of garment i in pile r, c_i^r is the water absorption characteristics of garment i in pile r, $|m^r|$ is the number of clothing in pile r, $types(m^r)$ is the type of the clothing in pile r, G_{sub} is the set of complete subgraphs derived from the mixed washing relation of the materials, V is the node set abstracted from the clothing material types. Based on these parameters, the model is established as Eq. (37).

The first constraint means that the amount of detergent added should be able to clear all the dirt. The second constraint means that each pile of clothes to be cleaned should not exceed the maximum capacity that the washing machine can hold. The third constraint is on the washing effect. The fourth constraint is on the mixing relationship between two materials.

$$
\begin{aligned}
&\min \ \cos t=\sum_{r=1}^{|m^r|}(p^w(w_0^r+n_{opt3}^r w^r)+\sum_{j=1}^{10}p_j^s s_j^r+\alpha n_{opt3}^r) \\
&s.t.\ \sum_{j=1}^{10}P_i^r s_j^r f_{ij}\ge d_i^r \qquad i=1,2,..,8 \ \ r=0,1,... \\
&\quad \sum_{i=1}^{|m^r|}m_i^r \le Max \qquad r=0,1,... \\
&\quad \left(\frac{\sum_{i=1}^{|m^r|}m_i^r c_i^r}{\sum_{i=1}^{|m^r|}m_i^r c_i^r+w^r}\right)^{n_{opt3}} \le \varepsilon \qquad r=0,1,2,... \\
&\quad types(m^r)\in(G_{sub}\cup V) \qquad r=0,1,... \\
&\quad H\ge {w_0}^r \ge L, H\ge w^r \ge L
\end{aligned} \tag{37}
$$

7.2 Model solution

7.2.1 Solving method

In order to solve the model Eq. (37), it is necessary to construct an efficient and feasible method to divide the clothing into piles first. Considering the actual situation of small and medium-sized laundry equipment, the feasible dividing rules are as follows:

(1) Each round of cleaning gives priority to the clothes that are difficult to be mixed with other clothes, adding clothes until they meet the maximum capacity of the washing machine. According to graph theory, it means that the nodes with the least degree have priority to be cleaned.

(2) If the weight of the current round of clothing has not reached the maximum weight and this type of material has been all washed up, then try to add the largest degree of clothing that does not conflict with it for mixed washing.

Considering the characteristics of this problem, we use the neighborhood search algorithm and Cplex algorithm to solve problem 4.

Step 1: The neighborhood search algorithm is used to randomly generate a sorting list of 8 materials, such as [1,8,2,3,4,5,6,7]. The material sorting list is generated by inverse operator and exchange operator, in which inverse operator can arrange a certain sequence in reverse order. The exchange operator can exchange the values of two points in the sequence. The schematic diagram of the operator is shown in Fig. 10.

Step 2: Based on the material sorting list, considering the weight limit of the single round washing machine, the clothes of different materials are divided into piles (the clothes that are washed in the same round are regarded as a pile). Read the material sorting list from left to right, if the total weight of the clothes from a certain material exceeds the weight limit of a single round of washing machine, the remaining clothes are placed in the next pile; If the total weight of a material does not reach the weight limit of a single round washing machine, the material sorting list is read from right to left, and the clothes that are not stacked and do not conflict with the material preselected are also placed in the pile until the weight limit of the washing machine is reached.

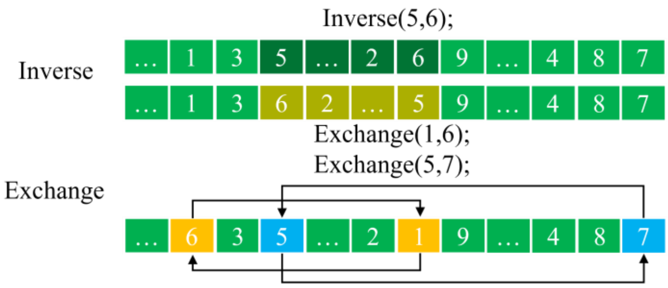

Fig. 10 The schematic diagram of the operator

Step 3: Based on the results of pile separation in Step 2, combined with the objectives and constraints of problem 4, the optimal washing rounds and the optima consumption of detergent and water in each round are obtained through Cplex.

Repeat steps 1 to 3, the material dividing list is different at each time, and finally the optimal solution can be obtained.

7.2.2 Results and analysis

In the constructed model Eq. (37), there are some constant variables. To solve the model, they are set as follows: $H = 7.5$, $L = 2.5$, the weight of each material $m =$ [0.5, 0.15, 0.3, 0.4, 0.6, 0.4, 0.5, 0.45], $c_i = 0.8, i = 1,...,8$. The coefficient $\alpha = 5$, and the washing effect threshold is $\varepsilon = 0.001$. Besides, four different values of Max are tested.

When solving the model, we first attempt to find feasible solutions based on the degree of each node in Fig. 9, and the results are shown in Table 7 in appendix. Then, the method based on neighborhood search is further adopted to expand the range of searching feasible solutions. The results solved by Cplex are listed in Table 4, and the number of iterations is always 100.

The following conclusions can be drawn from Table 5:

(1) The value of Max influences the number of divided piles. When $Max = 10$, the optimal plan is to divide into 5 piles; when Max takes other three larger values, only 4 piles are needed, and the total cost decreases. This phenomenon is in line with the practice.

(2) When Max takes 15,20,25, the total cost is 754.6165, and the amount of water used is 1530.66. But the optimal washing plan for achieving the goal varies, and the amount of detergent consumed also varies. This is because 8 types of materials have multiple feasible combination schemes, and the model and solution method constructed in this paper can provide the optimal solution that meets the objective constraints under various value of Max.

(3) It can be found that compared with Table 5, the results in Table 7 are simpler. Only when $Max = 25$, can the optimal cost similar to Table 5 be obtained, indicating that the solving method corresponding to Table 5 is more reasonable.

Table 4 The solutions of problem 4 under different values of Max

Max	r	**Material**	**Cost/$yuan$**	k	**Detergent**	s/$gram$	w/$liter$
10	1	3,7	152.473	3	2; 5	51.472; 1647.094	282.27
	2	2,8	209.295	3	2	2146.667	288.09
	3	4,6,7	126.284	3	2; 5	41.585; 1330.717	285.18
	4	1,4	146.706	3	2	1451.111	291
	5	5	94.4616	3	2	871.111	279.36
	6	1	51.9317	3	2	406.667	87.3
	Total		**781.1513**			**7946.424**	**1513.2**
15	1	2,8	209.295	3	2	2146.667	288.09
	2	4,7	191.814	3	2	1946.667	424.86
	3	1,5	150.026	3	2	1482.222	427.77
	4	1,3,6	203.482	3	2	2077.778	389.94
	Total		**754.6165**			**7653.334**	**1530.66**
20	1	4,6,8	263.162	3	2	2733.333	568.905
	2	5,7	169.702	3	2	1697.778	500.52
	3	1,3	194.404	3	2	1977.778	369.57
	4	2	127.348	3	2	1244.444	91.665
	Total		**754.6165**			**7653.333**	**1530.66**
25	1	1,6,4	263.989	3	2	2742.222	576.18
	2	5,7,8	251.648	3	2	2600	696.945
	3	3	111.63	3	2	1066.667	165.87
	4	2	127.348	3	2	1244.444	91.665
	Total		**754.6165**			**7653.333**	**1530.66**

When $Max = 15$, the convergence curve during the solving process is shown in Fig. 11. Obviously, the solution adopted in this paper can fully converge to the optimal value after 100 iterations, indicating that the solution is reasonable. The convergence curve in other cases is similar and will not be repeatedly described here.

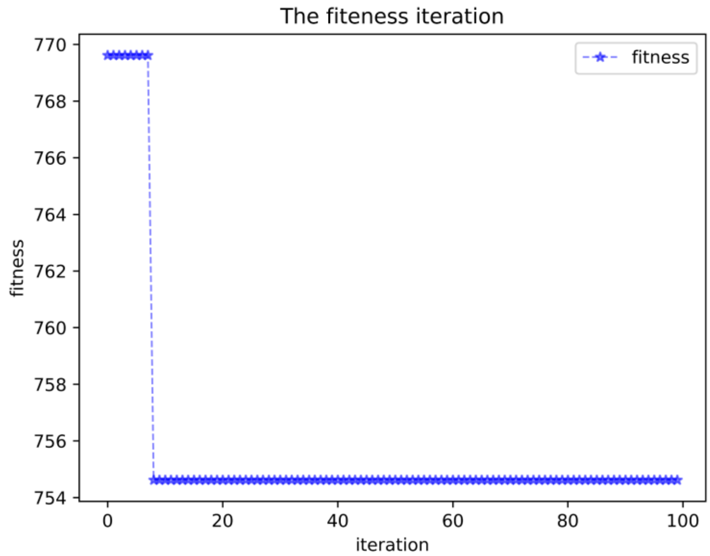

Fig. 11 The convergence curve under $Max = 15$

8. Model Evaluation

According to the model established in the above paper, its advantages mainly include the following points:

1. Multiple influencing factors are considered: the model takes into account many factors such as washing time, washing liquid concentration, initial amount of stains and available water, so as to evaluate the washing effect more comprehensively.

2. The model has theoretical basis: the model is based on the observation and experimental data of the actual washing process, which makes the results more theoretical basis and easier to be accepted and recognized.

3. Suitable for different situations: the model can be applied to different washing conditions and washing items, so as to provide effective guidance for washing in different situations.

However, there are some drawbacks to the model:

1. Simplified some factors: in the process of establishing the model, some complex factors are simplified, which may lead to the accuracy of the model is affected to a certain extent.

2. Not applicable to all detergents: This model is based on specific detergents and stain types, so it may not be applicable to all detergents and stain types.

3. Need further verification: Although the model has a certain theoretical basis, it still needs further practical verification to ensure its accuracy and reliability.

Overall, the model established in this paper has certain advantages and value, but it still needs to be further refined and validated to apply to a wider range of situations and washing items.

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Appendix

The symbol description

Table 5 Notations

Symbol	Notation	Unit
H	the optimal amount of water used to achieve the best effect of dirt dissolution without causing waste	liter
L	the minimum amount of water required for washing	liter
W	the amount of water available	liter
$w_k, k=1,2,...,K$	the amount of water used in the k th wash	liter
$a_k, k=1,2,...,K$	the solubility of the dirt in water at the k th wash	
x_1	the initial amount of dirt on the clothes including detergents	gram
$x_k, k=1,2,...,K$	the amount of residual dirt after $(k-1)$ th round of washing, including the residual detergents	gram
p_k	the amount of dirt that has been dissolved into the water	gram
q_k	the amount of dirt that has not been dissolved into the water	gram
m	the total weight of the clothes to be washed, composed by the weight of clothes and the initial amount of dirt	kilogram
c	the water absorption characteristics of the clothes	%
$R_k, k=1,2,3,...,K$	the time cost at the k th washing	
$n_{opti}, i=1,2,3,4$	the optimal number of washings in problem i	
d_i	the amount of dirt i	gram
s_j	the amount of detergent j	gram
f_{ij}	the solubility of detergent j to dirt i, i.e., the amount of dirt that can be dissolved by a unit of detergent	gram
p_j^s	the unit price of detergent j	yuan/gram
p^w	the unit price of water, 3.8 yuan per ton	yuan/ton
D	the initial amount of dirt without detergents	gram
P_i	the ratio of the amount of dirt i to the initial amount of dirt	
Q_0	the concentration of dirt after the first round of washing and before dehydration	
Q_k	the concentration of dirt after the k th round of rinsing	
w	the optimal amount of water used during rinsing	liter
ε	the threshold of washing effect	
Max	the maximum weight of clothes can be washed in a single round	kilogram
m_i^r	the weight of garment i in pile r	gram
c_i^r	the water absorption characteristics of garment i in pile r	
$\|m^r\|$	the number of clothes in pile r	
$types(m^r)$	the type of the clothes in pile r	
G_{sub}	the set of complete subgraphs derived from the mixed washing relation of the materials	
V	the node set abstracted from the clothing material types	

Some solutions in Section 6.2.2 for problem 3

Table 6 The solutions of problem 3 under different values of Max

Max	ε	r	**Cost**/*yuan*	k	**Detergent**	s /*gram*	w /*liter*
3	0.01	1	63.633	2	2	682.222	61.32-
		2	62.8497	2	2	673.333	65.7
		3	28.3165	2	2	291.111	30.66
	0.001	1	64.7096	3	2	682.222	81.48
		2	63.9317	3	2	673.333	87.3
		3	29.3548	3	2	291.111	40.74
	0.0001	1	65.7862	4	2	682.222	101.64
		2	65.0138	4	2	673.333	108.9
		3	30.3931	4	2	291.111	50.82
5	0.01	1	105.599	2	2	1146.667	105.12
		2	47.1997	2	2	500	52.56
	0.001	1	106.731	3	2	1146.667	139.68
		2	48.2654	3	2	500	69.84
	0.0001	1	107.862	4	2	1146.667	174.24
		2	49.3311	4	2	500	87.12
8	0.01	1	150.799	2	2	1646.667	157.68
	0.001	1	151.996	3	2	1646.667	209.52
	0.0001	1	153.193	4	2	1646.667	261.36
10	0.01	1	150.799	2	2	1646.667	157.68
	0.001	1	151.996	3	2	1646.667	209.52
	0.0001	1	153.193	4	2	1646.667	261.36

It can be concluded from Table 6 that:

(1) When $Max = 3$ and $Max = 5$, the clothes need to be divided into multiple heaps. When $Max = 8$ and $Max = 10$, all the clothes can be cleaned together, i.e., there is only a heap, this situation is the same with Section 6.2.1 and the experimental results are the same as Table 3.

(2) Under different values of Max, when ε takes the same value, the calculated number of washings, the type and amount of detergent used, and the amount of water used are all the same.

(3) However, the smaller the value of Max, the larger the number of divided heaps, and the more the total cost. This is in line with practice, because larger number of heaps will require larger number of washings, leading to higher cost.

Some solutions in Section 7.2.2 for problem 4

Table 7 The solutions of problem 4 under different values of *Max*

Max	*r*	**Material**	**Cost/***yuan*	*k*	**Detergent**	*s* /*gram*	*w* /*liter*
15	1	2,8	209.2950	3	2	2146.667	288.09
	2	1,6,4	174.2820	3	2	1753.333	389.94
	3	3,7	183.4040	3	2	1855.556	369.57
	4	4	104.7080	3	2	988.889	186.24
	5	5	97.9279	3	2	908.889	296.82
	Total		**769.6169**			**7653.334**	**1530.66**
20	1	2,8	209.2950	3	2	2146.667	288.09
	2	1,6,4	230.1240	3	2	2368.889	506.34
	3	3,7	183.4040	3	2	1855.556	369.57
	4	4	48.8654	3	2; 5	12.679; 405.736	69.84
	5	5	97.9279	3	2	908.889	296.82
	Total		**769.6163**			**7698.416**	**1530.66**
25	1	2,8	209.2950	3	2	2146.667	288.09
	2	1,6,4	263.9890	3	2	2742.222	576.18
	3	3,7	183.4040	3	2	1855.556	369.57
	4	5	97.9279	3	2	908.889	296.82
	Total		**754.6159**			**7653.334**	**1530.66**