# On Valency Problems of Saxl Graphs

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# Outline

# Bases

Let $G \leq \mathrm{Sym}(\Omega)$ with $|\Omega| < \infty$.

- Base: $\Delta \subset \Omega$ such that the point-wise stabiliser $G_{(\Delta)} = 1$.
- Base size: minimal cardinality of bases, denoted by $b(G)$.
- Base size set: a base $\Delta$ such that $|\Delta| = b(G)$.

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With the natural actions,

- $b(S_n) = n - 1$;
- $b(A_n) = n - 2$;
- $b(\mathrm{GL}_n(q)) = n$, and a base size set is exactly a basis of $\mathbb{F}_q^n$ over $\mathbb{F}_q$.

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Suppose $G$ is transitive.

- $G$ is regular $\iff b(G) = 1$.
- If $G$ is Frobenius then $b(G) = 2$.
- If $G$ is sharply $k$-transitive then $b(G) = k$.

# Primitive Groups

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Li-Zhang 2011: Classified all primitive groups with soluble stabilisers.

# Almost Simple Groups

**Theorem (Classification of Finite Simple Groups).**

Let $G$ be a non-abelian finite simple group. Then $G$ is isomorphic to one of the following:

- an alternating group $A_n$ with $n \geq 5$;
- a group of Lie type;
- one of 26 sporadic groups.

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**Theorem (Schreier Conjecture).**

The outer automorphism group of a finite simple group is soluble.

A group $G$ is called almost simple if

$$\mathrm{soc}(G) = T \cong \mathrm{Inn}(T) \lesssim G \lesssim \mathrm{Aut}(T)$$

for some non-abelian simple group $T$.

# Bases for Primitive Groups

Let $G \leq \mathrm{Sym}(\Omega)$ be an almost simple primitive group.

- Cameron-Kantor 1993: Conjectured $b(G) \leq c$ if $G$ is non-standard.
- Liebeck-Shalev 1999: $c$ exists.
- Burness-Liebeck-Shalev 2009: $c = 7$ is optimal.
- Burness 2018: Determined groups with $b(G) = 6$.

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Let $G \leq \mathrm{Sym}(\Omega)$ be primitive with soluble stabiliser.

- Seress 1996: $b(G) \leq 4$ if $G$ is also soluble.
- Burness 2020+: $b(G) \leq 5$.

# Saxl Graphs

Saxl first proposed determining all primitive groups $G$ with $b(G) = 2$.
Burness-Giudici 2020: Saxl graph $\Sigma(G)$:

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We have

- $b(G) \geq 3 \implies \Sigma(G)$ empty;
- $b(G) = 1$ and $G$ transitive $\implies \Sigma(G)$ complete.

# First Observations

**Proposition.**

Suppose $G$ is transitive with $b(G) = 2$ and $\Sigma(G)$ is the Saxl graph of $G$.

1. $\Sigma(G)$ is $G$-vertex-transitive.
2. $\Sigma(G)$ is connected if $G$ is primitive.
3. $\Sigma(G)$ is complete if and only if $G$ is Frobenius.
4. $\Sigma(G)$ is $G$-arc-transitive if $G$ is 2-transitive.
5. $\Sigma(G)$ is $G$-arc-semiregular.

Indeed, $\Sigma(G)$ is the union of all regular orbital graphs of $G$.

# Burness-Giudici Conjecture

**Conjecture (Burness-Giudici 2020).**

Let $G$ be primitive and $b(G) = 2$. Then any two vertices in $\Sigma(G)$ has a common neighbour.

Note that if $\mathrm{val}(\Sigma(G)) > \frac{1}{2}|\Omega|$ then the conjecture is verified. This gives a motivation to study the valency problems.

# $\mathrm{val}(\Sigma(G)) = r|H|$

**Proposition.**

Suppose $G$ is transitive with $b(G) = 2$ and $\Sigma(G)$ is the Saxl graph of $G$. Then $\Sigma(G)$ has valency $r|H|$, where $H$ is the point stabiliser and $r$ is the number of regular suborbits of $G$.

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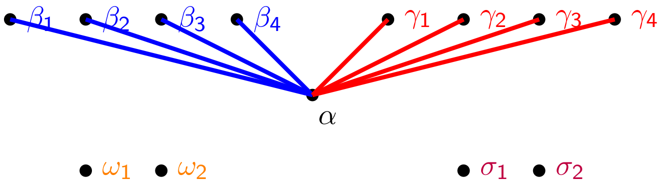

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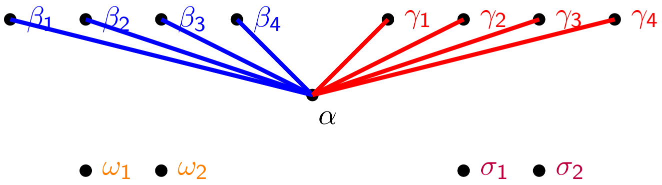


By $\mathrm{val}(G, H)$ we mean the valency of the Saxl graph of $G$ with stabiliser $H$. In particular, $|H|$ divides $\mathrm{val}(G, H)$.

# Outline

# Arc Stabilisers

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive and $H = G_\alpha$ be the point stabiliser. Set

$$\delta(A) := \{g \in G \mid H \cap H^g = A\} = \{g \in G \mid G_{(\alpha,\alpha^g)} = A\}.$$

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**Lemma.**

We have

1. $G = \biguplus_{A \leq H} \delta(A)$.
2. $\mathrm{val}(G, H) = \frac{|\delta(1)|}{|H|}$.
3. For any $A \leq H$ and $h \in H$, $\delta(A^h) = \delta(A)h$ and so $|\delta(A^h)| = |\delta(A)|$.

It is difficult to determine $|\delta(A)|$ directly.

# The Strategy to Calculate $|\delta(A)|$

Consider the poset $P = (\{A \mid A \leq H\}, \leq)$. Define

$$\Delta(A) = \{g \in G \mid H \cap H^g \geq A\}$$

and

$$c(A, B) = \begin{cases} 1 & \text{if } A \leq B; \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that

$$|\Delta(A)| = \sum_{B \geq A} |\delta(B)| = \sum_{B \in P} c(A, B)|\delta(B)|.$$

# Reduction

The size of $P$ is generally very large. We "reduce" the size of $P$ by the following methods.

$$\begin{array}{l} P \\ \downarrow \text{conjugacy in } H \\ \mathcal{S} \\ \downarrow \text{possible arc stabilisers} \\ \mathcal{I} \end{array}$$

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It follows that

$$|\Delta(A)| = \sum_{B \in \mathcal{I}} \eta(A, B)|\delta(B)|,$$

where $\eta(A, B) = |\{B^h \mid B^h \geq A\}|$.

# Matrix Form

Note that $|\Delta(A)| = \sum_{B\in\mathcal{I}} \eta(A,B)|\delta(B)|$ is a system of linear equations. Write this in the matrix form we have

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$$|\Delta(A)| = \sum_{B\in\mathcal{S}\cap A^G} \frac{|H||N_G(B)|}{|N_H(B)|}.$$

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We only need to find $\mathcal{I}$ and normalisers. This is generally very difficult!

# An Example

Let $G = \mathrm{PSL}_2(17)$ and $H = \langle x\rangle{:}\langle y\rangle \cong D_{16}$. Then $H \cap H^g \cong 1, \mathbb{Z}_2, \mathbb{Z}_2^2$ or $H$. Indeed,

$$\mathcal{I} = \{1, \langle y\rangle, \langle xy\rangle, \langle x^4, y\rangle, \langle x^4, xy\rangle, H\}$$

and

$$\delta = M^{-1}\Delta = \begin{bmatrix} 1 & 4 & 4 & 2 & 2 & 1 \\ & 1 & 0 & 1 & 0 & 1 \\ & & 1 & 0 & 1 & 1 \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2448 \\ 144 \\ 144 \\ 48 \\ 48 \\ 16 \end{bmatrix},$$

which implies $|\delta(1)| = 1536$ and so $\mathrm{val}(G, H) = 96$.

# Outline

# Prime Valency

**Proposition (Burness-Giudici 2020).**

Suppose $G$ is transitive with $b(G) = 2$ and $\Sigma(G)$ is the Saxl graph of $G$. Then $\Sigma(G)$ has prime valency $p$ if and only if $G$ is one of the following:

1. $G = \mathbb{Z}_p \wr \mathbb{Z}_2$ and $\Sigma(G) \cong K_{p,p}$.
2. $G = S_3$, $p = 2$ and $\Sigma(G) \cong K_3$.
3. $G = \mathrm{AGL}_1(2^f)$, where $p = 2^f - 1$ is a Mersenne prime and $\Sigma(G) \cong K_{p+1}$.

# Prime-power Valency

**Theorem (Chen-H. 2020+).**

Suppose $G$ is almost simple primitive with $b(G) = 2$ stabiliser $H$. Then the Saxl graph $\Sigma(G)$ has prime-power valency if and only if $(G, H)$ is one of the following:

1. $(G, H) = (\mathrm{M}_{10}, 8{:}2)$ and $\mathrm{val}(G, H) = 32$.
2. $(G, H) = (\mathrm{PGL}_2(q), D_{2(q-1)})$, where $q \geq 17$ is a Fermat prime or $q = 9$, $\Sigma(G)$ is isomorphic to the Johnson graph $J(q+1, 2)$ and $\mathrm{val}(G, H) = 2(q-1)$.

# Frobenius Group

Recall that a group $H$ is called Frobenius if there exists a non-trivial proper subgroup $L < H$ such that $L \cap L^h = 1$ for any $h \in H \setminus L$.

- Frobenius complement: $L$.
- Frobenius kernel: the subgroup $K$ consisting the identity element and those elements that are not in any conjugate of $L$.

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- $H = K{:}L$.
- If $K$ is cyclic, then so does $L$.

# Frobenius Groups with Cyclic Kernel

**Theorem (Chen-H. 2020+).**

Suppose $G$ is a finite primitive permutation group with stabiliser $H$, where $H = K{:}L$ is Frobenius with cyclic kernel $K$. Write $L = \langle y \rangle$. Then

$$\mathrm{val}(G,H) = |G:H| + |K| - 1 + \frac{|K|}{|L|} \sum_{1 \neq d \mid |L|} \mu(d) |N_G(\langle y^{\frac{|L|}{d}} \rangle)|,$$

where $\mu$ is the Möbius function.

# Alternating and Symmetric Groups

This can be applied to various problems. For example

**Corollary.**

Let $G = S_p$ and $H = \mathrm{AGL}_1(p) \cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ with $p \geq 5$ a prime. Then

$$\mathrm{val}(G,H) = (p-2)! + p - 1 + p \sum_{1 \neq d \mid (p-1)} \mu(d)\phi(d) d^{\frac{p-1}{d}-1} \left(\frac{p-1}{d} - 1\right)!.$$

**Corollary.**

Let $G = A_p$ and $H = \mathrm{AGL}_1(p) \cap A_p \cong \mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$ with $p \geq 5$ a prime and $p \neq 7, 11, 17, 23$. Then

$$\mathrm{val}(G,H) = (p-2)! + p - 1 + p \sum_{1 \neq d \mid \frac{p-1}{2}} \mu(d)\phi(d) d^{\frac{p-1}{d}-1} \left(\frac{p-1}{d} - 1\right)!.$$

# Alternating and Symmetric Groups

**Theorem (Chen-H. 2020+).**

Let $G$ be an almost simple primitive group with socle $A_n$ and soluble stabiliser $H$. If $b(G) = 2$, then $(G, H, \mathrm{val}(G, H))$ is listed in the following.

| $G$ | $H$ | $\mathrm{val}(G, H)$ |
|---|---|---|
| $A_5$ | $S_3$ | 6 |
| $\mathrm{M}_{10}$ | $\mathrm{AGL}_1(5)$ | 20 |
| $\mathrm{M}_{10}$ | 8:2 | 32 |
| $\mathrm{PGL}_2(9)$ | $D_{16}$ | 16 |
| $A_9$ | $\mathrm{ASL}_2(3)$ | 432 |
| $A_p$ | $\mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$ | See above |
| $S_p$ | $\mathrm{AGL}_1(p)$ | See above |

# Odd Valency

**Proposition (Burness-Giudici 2020).**

Let $G$ be an almost simple primitive group with stabiliser $H$ and $b(G) = 2$. If $\mathrm{val}(G, H)$ is odd then one of the following holds:

1. $(G, H) = (\mathrm{M}_{23}, 23{:}11)$.
2. $(G, H) = (A_p, \mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2})$, where $p \equiv 3 \pmod 4$ is a prime and $(p-1)/2$ is composite.
3. $\mathrm{soc}(G) = \mathrm{L}_r^{\epsilon}(q)$ and $H \cap \mathrm{soc}(G) = \mathbb{Z}_a{:}\mathbb{Z}_r$, where $r$ is an odd prime, $a = \frac{q^r - \epsilon}{(q-\epsilon)(r, q-\epsilon)}$ and $G \neq \mathrm{soc}(G)$.

# Odd Valency

Case (2) can be easily shown impossible by above. Moreover, we analysis the case when $G = \mathrm{PGL}_r^{\epsilon}(q)$. These lead the following.

**Theorem (Chen-H. 2020+).**

Let $G$ be an almost simple primitive group with stabiliser $H$ and $b(G) = 2$. Then $\mathrm{val}(G, H)$ is odd only if one of the following holds:

1. $G = \mathrm{M}_{23}$ and $H = 23{:}11$.
2. $G = \mathrm{L}_r^{\epsilon}(q).O \leq \mathrm{P\Gamma L}_r^{\epsilon}(q)$ with $r$ prime and $O \leq \mathrm{Out}(\mathrm{L}_r^{\epsilon}(q))$, but $G \not\leq \mathrm{PGL}_r^{\epsilon}(q)$, with $H = \mathbb{Z}_a{:}\mathbb{Z}_r.O$, where $a = \frac{q^r-\epsilon}{(q-\epsilon)(r,q-\epsilon)}$.

# Outline

# Conjectures

To calculate the valency we need to determine all possible arc stabilisers $H \cap H^g$ for $g \in G$. This leads the following conjecture, which may be of independent interest.

**Conjecture.**

Let $G$ be a finite primitive permutation group with stabiliser $H$. Then for any $g \notin H$, either $H \cap H^g = 1$ or $H \cap H^g$ is not normal in $H$.

The conjecture is verified when:

- $|\Omega| \leq 4095$;
- $H \cap H^g$ has odd order.

# Conjectures

The only known genuine example of almost simple primitive group with odd valency is $\mathrm{M}_{23}$ with stabiliser 23:11. Is there any more?

**Conjecture.**

Let $G$ be an almost simple primitive group with stabiliser $H$. Then $\mathrm{val}(G, H)$ is odd if and only if $G = \mathrm{M}_{23}$ and $H = 23{:}11$.

# Other Problems on Saxl Graphs

Connectivity:

- How to characterise the connectivity of Saxl graphs of transitive permutation groups?
- The Burness-Giudici Conjecture.
- When does $\mathrm{val}(G, H) = |H|$?

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Cycles:

- Euler cycle? The conjecture above.
- Hamiltonian cycle?

Thank you for your attention!