Base-two primitive groups and their Saxl graphs

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Groups and Graphs

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- $G = \mathrm{GL}(V)$, $\Omega = V \setminus \{0\}$ and Δ contains a basis of V.

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Observation: If Δ is a base and $x, y \in G$, then

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Some applications:

- Minimal dimensions
- Generation of finite groups
- Graphs defined on groups

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- G is primitive iff n is a prime.

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- Progress where $G < L \wr P$ **(Burness & H, 2022+)**

Probabilistic methods

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group of degree n. Then

$$Q(G) = \frac{|\{(\alpha, \beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{n^2}$$

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Definition (Burness & Giudici, 2020)

Let $G \leqslant \mathrm{Sym}(\Omega)$. Then the **Saxl graph** $\Sigma(G)$ is a graph with

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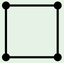
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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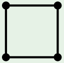
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
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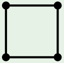
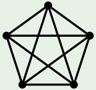
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = D_{10} = \langle (12345), (25)(34) \rangle$, $\Omega = \{1, 2, 3, 4, 5\}$: $\Sigma(G) =$ 

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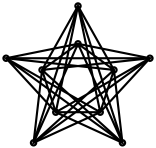
  For example, when $q = 4$ we have the complement of the Petersen.



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Diameter

Conjecture (Burness & Giudici, 2020)

G primitive $\implies$ any two vertices in $\Sigma(G)$ have a common neighbour.

Example

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $\Sigma(G) = J(q+1, 2)$ has the common neighbour property.

Evidence:

- All primitive groups of degree up to 4095 ✓
- "Most" almost simple groups with alternating or sporadic socles
- **Chen & Du, 2020+; Burness & H, 2021+:** $\mathrm{soc}(G) = \mathrm{L}_2(q)$ ✓
- **Burness & H, 2021+:** almost simple groups with G_α soluble ✓
- **Lee & Popiel, 2021+:** some affine groups

Probabilistic methods

Let $v(G)$ be the valency of $\Sigma(G)$ and recall that

$$Q(G) = 1 - \frac{v(G)}{|\Omega|}$$

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Example

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $Q(G) \to 1$ as $q \to \infty$. But $\Sigma(G) = J(q+1, 2)$ still has the common neighbour property.

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Evidence:

- All primitive groups of degree up to 4095 ✓
- $G = \mathrm{PSL}_2(q)$ and G_α of type $\mathrm{GL}_1(q) \wr S_2$

Future work

Saxl graphs:

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- Valencies (**Chen & H, 2022:** some results)
- Other invariants of Saxl graphs (e.g. the independence number)

Other problems on bases:

- Irredundant bases
- Bounds for $b(G)$ in a general setting

Thank you!