On Valency Problems of Saxl Graphs

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Joint work with Jiyong Chen

Outline

Bases

Let $G \leq \mathrm{Sym}(\Omega)$ with $|\Omega| < \infty$.

- Base: $\Delta \subset \Omega$ such that the point-wise stabiliser $G_{(\Delta)} = 1$.
- Base size: minimal cardinality of bases, denoted by $b(G)$.
- Base size set: a base Δ such that $|\Delta| = b(G)$.

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With the natural actions,

- $b(S_n) = n - 1$;
- $b(A_n) = n - 2$;
- $b(\mathrm{GL}_n(q)) = n$, and a base size set is exactly a basis of $\mathbb{F}_q^n$ over $\mathbb{F}_q$.

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Suppose G is transitive.

- G is regular $\iff b(G) = 1$.
- If G is Frobenius then $b(G) = 2$.
- If G is sharply k-transitive then $b(G) = k$.

Primitive Groups

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Li-Zhang 2011: Classified all primitive groups with soluble stabilisers.

Almost Simple Groups

Theorem (Classification of Finite Simple Groups).

Let G be a non-abelian finite simple group. Then G is isomorphic to one of the following:

- an alternating group A_n with $n \geq 5$;
- a group of Lie type;
- one of 26 sporadic groups.

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A group G is called almost simple if

$$\mathrm{soc}(G) = T \cong \mathrm{Inn}(T) \lesssim G \lesssim \mathrm{Aut}(T)$$

for some non-abelian simple group T.

Bases for Primitive Groups

Let $G \leq \mathrm{Sym}(\Omega)$ be an almost simple primitive group.

- Cameron-Kantor 1993: Conjectured $b(G) \leq c$ if G is non-standard.
- Liebeck-Shalev 1999: c exists.
- Burness-Liebeck-Shalev 2009: $c = 7$ is optimal (M_{24}).
- Burness 2018: Determined groups with $b(G) = 6$.

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- Burness 2018: Determined groups with $b(G) = 6$.

Let $G \leq \mathrm{Sym}(\Omega)$ be primitive with soluble stabiliser.

- Seress 1996: $b(G) \leq 4$ if G is also soluble.
- Burness 2020+: $b(G) \leq 5$.

Saxl Graphs

Saxl first proposed determining all primitive groups G with $b(G) = 2$.
Burness-Giudici 2020: Saxl graph $\Sigma(G)$:

- Vertex set Ω;
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We have

- $b(G) \geq 3 \implies \Sigma(G)$ empty;
- $b(G) = 1$ and G transitive $\implies \Sigma(G)$ complete.

First Observations

Proposition.

Suppose G is transitive with $b(G) = 2$ and $\Sigma(G)$ is the Saxl graph of G.

1. $\Sigma(G)$ is G-vertex-transitive.
2. $\Sigma(G)$ is connected if G is primitive.
3. $\Sigma(G)$ is complete if and only if G is Frobenius.
4. $\Sigma(G)$ is G-arc-transitive if G is 2-transitive.
5. $\Sigma(G)$ is G-arc-semiregular.

Indeed, $\Sigma(G)$ is the union of all regular orbital graphs of G.

Burness-Giudici Conjecture

Conjecture (Burness-Giudici 2020).

Let G be primitive and $b(G) = 2$. Then any two vertices in $\Sigma(G)$ has a common neighbour.

Note that if $\mathrm{val}(\Sigma(G)) > \frac{1}{2}|\Omega|$ then the conjecture is verified. This gives a motivation to study the valency problems.

$\mathrm{val}(\Sigma(G)) = r|H|$

Proposition.

Suppose G is transitive with $b(G) = 2$ and $\Sigma(G)$ is the Saxl graph of G. Then $\Sigma(G)$ has valency $r|H|$, where H is the point stabiliser and r is the number of regular suborbits of G. In particular, $\Sigma(G)$ is G-arc-transitive if and only if $r = 1$.

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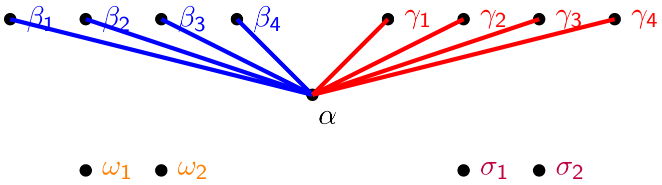

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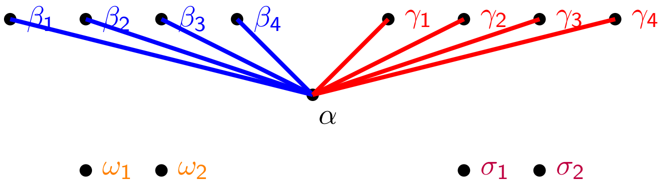

By $\mathrm{val}(G, H)$ we mean the valency of the Saxl graph of G with stabiliser H. In particular, $|H|$ divides $\mathrm{val}(G, H)$.

An Example

Let $G = A_5$ and $H = \langle (123), (23)(45) \rangle \cong S_3$. Then

$$\begin{aligned} |\Omega| = 10 \text{ and } b(G) = 2 &\implies \mathrm{val}(G, H) = 6 \text{ and } r = 1 \\ &\implies \Sigma(G) \text{ is } G\text{-arc-transitive} \\ &\implies \overline{\Sigma(G)} \text{ is Petersen.} \end{aligned}$$

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Indeed, $H \cap H^g = 1$ if and only if $g \in HS$, where

$$S = \{(345), (354), (12345), (12354), (13452), (235)\}.$$

Hence, $\Sigma(G)$ is isomorphic to the coset graph $\mathrm{Cos}(G, H, HSH)$.

Outline

The Strategy

Let $G \leq \mathrm{Sym}(\Omega)$ be transitive with stabiliser $H = G_\alpha$.

- $\mathcal{I}$: possible arc stabilisers $H \cap H^g$ up to conjugacy in H.
- $\delta(A) := \{g \in G \mid H \cap H^g = A\}$ for $A \in \mathcal{I}$.
- $\Delta(A) := \{g \in G \mid H \cap H^g \geq A\}$ for $A \in \mathcal{I}$.

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Lemma.

We have

$$|\Delta(A)| = \sum_{B \in \mathcal{I}} \eta(A, B)|\delta(B)|,$$

where $\eta(A, B) = |\{B^h \mid B^h \geq A\}|$.

Matrix Form

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$$|\Delta(A)| = \sum_{B\in\mathcal{S}\cap A^G} \frac{|H||N_G(B)|}{|N_H(B)|} = |H||N_G(A)| \sum_{B\in\mathcal{S}\cap A^G} \frac{1}{|N_H(B)|},$$

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  where $\mathcal{S}$ is the set of subgroups of H up to conjugacy.

We only need to find $\mathcal{I}$ and normalisers. This is generally very difficult!

An Example

Let $G = \mathrm{PSL}_2(17)$ and $H = \langle x \rangle{:}\langle y \rangle \cong D_{16}$. Then $H \cap H^g \cong 1, \mathbb{Z}_2, \mathbb{Z}_2^2$ or H. Indeed,

$$\mathcal{I} = \{1, \langle y \rangle, \langle xy \rangle, \langle x^4, y \rangle, \langle x^4, xy \rangle, H\}.$$

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Note that all involutions in G are conjugate. It follows that

- $|\Delta(1)| = |G| = 2448$;
- $|\Delta(\langle y\rangle)| = |H||N_G(\langle y\rangle)|(\frac{1}{|N_H(\langle x^4\rangle)|} + \frac{1}{|N_H(\langle y\rangle)|} + \frac{1}{|N_H(\langle xy\rangle)|}) = 144$;
- $|\Delta(\langle xy\rangle)| = |H||N_G(\langle xy\rangle)|(\frac{1}{|N_H(\langle x^4\rangle)|} + \frac{1}{|N_H(\langle y\rangle)|} + \frac{1}{|N_H(\langle xy\rangle)}|) = 144$;
- $|\Delta(\langle x^4, y\rangle)| = \frac{|H||N_G(\langle x^4,y\rangle)|}{|N_H(\langle x^4,y\rangle)|} = \frac{|H||S_4|}{|D_8|} = 48$;
- $|\Delta(\langle x^4, xy\rangle)| = \frac{|H||N_G(\langle x^4,xy\rangle)|}{|N_H(\langle x^4,xy\rangle)|} = \frac{|H||S_4|}{|D_8|} = 48$;
- $|\Delta(H)| = |H| = 16$.

An Example

Finally,

$$\delta = M^{-1}\Delta = \begin{bmatrix} 1 & 4 & 4 & 2 & 2 & 1 \\ & 1 & 0 & 1 & 0 & 1 \\ & & 1 & 0 & 1 & 1 \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2448 \\ 144 \\ 144 \\ 48 \\ 48 \\ 16 \end{bmatrix},$$

which implies $|\delta(1)| = 1536$ and so $\mathrm{val}(G, H) = 96$.

Prime Valency

Proposition (Burness-Giudici 2020).

Suppose G is transitive with $b(G) = 2$ and $\Sigma(G)$ is the Saxl graph of G. Then $\Sigma(G)$ has prime valency p if and only if G is one of the following:

1. $G = \mathbb{Z}_p \wr \mathbb{Z}_2$ and $\Sigma(G) \cong K_{p,p}$.
2. $G = S_3$, $p = 2$ and $\Sigma(G) \cong K_3$.
3. $G = \mathrm{AGL}_1(2^f)$, where $p = 2^f - 1$ is a Mersenne prime and $\Sigma(G) \cong K_{p+1}$.

Prime-power Valency

Theorem (Chen-H. 2020+).

Suppose G is almost simple primitive with $b(G) = 2$ stabiliser H. Then the Saxl graph $\Sigma(G)$ has prime-power valency if and only if (G, H) is one of the following:

1. $(G, H) = (\mathrm{M}_{10}, 8{:}2)$ and $\mathrm{val}(G, H) = 32$.
2. $(G, H) = (\mathrm{PGL}_2(q), D_{2(q-1)})$, where $q \geq 17$ is a Fermat prime or $q = 9$, $\Sigma(G)$ is isomorphic to the Johnson graph $J(q + 1, 2)$ and $\mathrm{val}(G, H) = 2(q - 1)$.

Frobenius Group

Recall that a group H is called Frobenius if there exists a non-trivial proper subgroup $L < H$ such that $L \cap L^h = 1$ for any $h \in H \setminus L$.

- Frobenius complement: L.
- Frobenius kernel: the subgroup K consisting the identity element and those elements that are not in any conjugate of L.

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- Frobenius kernel: the subgroup K consisting the identity element and those elements that are not in any conjugate of L.
- $H = K{:}L$.
- If K is cyclic, then so does L.

Frobenius Groups with Cyclic Kernel

Theorem (Chen-H. 2020+).

Suppose G is a finite primitive permutation group with stabiliser H, where $H = K{:}L$ is Frobenius with cyclic kernel K. Write $L = \langle y \rangle$. Then

$$\mathrm{val}(G,H) = |G : H| + |K| - 1 + \frac{|K|}{|L|} \sum_{1 \neq d \mid |L|} \mu(d) |N_G(\langle y^{\frac{|L|}{d}} \rangle)|,$$

where μ is the Möbius function.

Alternating and Symmetric Groups

This can be applied to various problems. For example

Corollary.

Let $G = S_p$ and $H = \mathrm{AGL}_1(p) \cong \mathbb{Z}_p{:}\mathbb{Z}_{p-1}$ with $p \geq 5$ a prime. Then

$$\mathrm{val}(G,H) = (p-2)! + p - 1 + p \sum_{1 \neq d \mid (p-1)} \mu(d)\phi(d) d^{\frac{p-1}{d}-1} \left(\frac{p-1}{d} - 1\right)!.$$

Corollary.

Let $G = A_p$ and $H = \mathrm{AGL}_1(p) \cap A_p \cong \mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$ with $p \geq 5$ a prime and $p \neq 7, 11, 17, 23$. Then

$$\mathrm{val}(G,H) = (p-2)! + p - 1 + p \sum_{1 \neq d \mid \frac{p-1}{2}} \mu(d)\phi(d) d^{\frac{p-1}{d}-1} \left(\frac{p-1}{d} - 1\right)!.$$

Alternating and Symmetric Groups

Theorem (Chen-H. 2020+).

Let G be an almost simple primitive group with socle A_n and soluble stabiliser H. If $b(G) = 2$, then $(G, H, \mathrm{val}(G, H))$ is listed in the following.

G	H	$\mathrm{val}(G, H)$
A_5	S_3	6
M_{10}	$\mathrm{AGL}_1(5)$	20
M_{10}	8:2	32
$\mathrm{PGL}_2(9)$	D_{16}	16
A_9	$\mathrm{ASL}_2(3)$	432
A_p	$\mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2}$	See above
S_p	$\mathrm{AGL}_1(p)$	See above

Odd Valency

Proposition (Burness-Giudici 2020).

Let G be an almost simple primitive group with stabiliser H and $b(G) = 2$. If $\mathrm{val}(G, H)$ is odd then one of the following holds:

1. $(G, H) = (\mathrm{M}_{23}, 23{:}11)$.
2. $(G, H) = (A_p, \mathbb{Z}_p{:}\mathbb{Z}_{(p-1)/2})$, where $p \equiv 3 \pmod 4$ is a prime and $(p-1)/2$ is composite.
3. $\mathrm{soc}(G) = \mathrm{L}_r^{\epsilon}(q)$ and $H \cap \mathrm{soc}(G) = \mathbb{Z}_a{:}\mathbb{Z}_r$, where r is an odd prime, $a = \frac{q^r-\epsilon}{(q-\epsilon)(r,q-\epsilon)}$ and $G \neq \mathrm{soc}(G)$.

Odd Valency

Case (2) can be easily shown impossible by above. Moreover, we analysis the case when $G = \mathrm{PGL}_r^{\epsilon}(q)$. These lead the following.

Theorem (Chen-H. 2020+).

Let G be an almost simple primitive group with stabiliser H and $b(G) = 2$. Then $\mathrm{val}(G, H)$ is odd only if one of the following holds:

1. $G = \mathrm{M}_{23}$ and $H = 23{:}11$.
2. $G = \mathrm{L}_r^{\epsilon}(q).O \leq \mathrm{P\Gamma L}_r^{\epsilon}(q)$ with r prime and $O \leq \mathrm{Out}(\mathrm{L}_r^{\epsilon}(q))$, but $G \not\leq \mathrm{PGL}_r^{\epsilon}(q)$, with $H = \mathbb{Z}_a{:}\mathbb{Z}_r.O$, where $a = \frac{q^r-\epsilon}{(q-\epsilon)(r,q-\epsilon)}$.

Outline

1. Preliminaries
2. Our Results
3. Problems

Conjectures

To calculate the valency we need to determine all possible arc stabilisers $G_{(\alpha,\alpha^g)}$ for $g \in G$. This leads the following conjecture, which may be of independent interest.

Conjecture.

Let G be a finite primitive permutation group with stabiliser G_α. Then for any $g \notin G_\alpha$, either $G_{(\alpha,\alpha^g)} = 1$ or $G_{(\alpha,\alpha^g)}$ is not normal in G_α.

The conjecture is verified when:

- $G_{(\alpha,\alpha^g)} < G_{\{\alpha,\alpha^g\}}$;
- $|\Omega| \leq 4095$;
- $G_{(\alpha,\alpha^g)}$ has odd order.

Conjectures

The only known genuine example of almost simple primitive group with odd valency is M_{23} with stabiliser 23:11. Is there any more?

Conjecture.

Let G be an almost simple primitive group with stabiliser H. Then $\mathrm{val}(G, H)$ is odd if and only if $G = \mathrm{M}_{23}$ and $H = 23{:}11$.

Connectivity

- How to characterise the connectivity of Saxl graphs of transitive permutation groups? We know that

$$G \text{ primitive } \Longrightarrow \Sigma(G) \text{ connected.}$$

The converse? Simple quasi-primitive groups?

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Connectivity

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$$G \text{ primitive} \implies \Sigma(G) \text{ connected.}$$

  The converse? Simple quasi-primitive groups?
- The Burness-Giudici Conjecture.
- When does $\mathrm{val}(G, H) = |H|$? That is, there is exactly one regular suborbit, especially when G is primitive.

Example.

When $(G, H) = (\mathrm{PGL}_2(q), D_{2(q-1)})$ for $q \geq 5$ we have $\mathrm{val}(G, H) = |H|$.

Automorphisms

- We have $G \leq \mathrm{Aut}(\Sigma(G))$. When we have $G = \mathrm{Aut}(\Sigma(G))$?

Example.

- When $(G, H) = (\mathrm{Sp}_{2m}(2), S_{2m+2})$ with $m \geq 6$ even, $G = \mathrm{Aut}(\Sigma(G))$.
- When $(G, H) = (\mathrm{PGL}_2(q), D_{2(q-1)})$ with $q \geq 13$ odd, we have $\Sigma(G) \cong J(q+1, 2)$ and so $G < \mathrm{Aut}(\Sigma(G)) \cong S_{q+1}$.

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- To what extent does $\Sigma(G)$ determine G up to permutation isomorphism?
- When is $\Sigma(G)$ Cayley? That is, when $\mathrm{Aut}(\Sigma(G))$ has a regular subgroup?

Example.

- When $(G, H) = (\mathrm{M}_{10}, 8{:}2)$, $\Sigma(G)$ is not Cayley.
- When $(G, H) = (S_7, \mathrm{AGL}_1(7))$, $\Sigma(G)$ is Cayley.

Cycles

- Euler cycle? The conjecture above on odd valency.

Cycles

- Euler cycle? The conjecture above on odd valency.
- Hamiltonian cycle? Note that $\Sigma(G)$ is G-vertex-transitive.

Lemma.

All the known examples of vertex-transitive non-Hamiltonian graphs of order at least 3 are cubic, and hence not Saxl graphs of transitive groups.

Other Problems

- When is a vertex-transitive graph the Saxl graph of a transitive group?

Example.

- Most vertex-transitive graphs with prime valency are not.
- The Johnson graph $J(q+1,2)$ for any prime-power $q \geq 5$ is isomorphic to the Saxl graph of $\mathrm{PGL}_2(q)$ with stabiliser $D_{2(q-1)}$.

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- General primitive groups with prime-power valency?

Example.

- $\mathrm{val}(\mathrm{PSU}_3(2), Q_8) = 8$.
- $\mathrm{val}(\mathrm{M}_{10} \wr C_2, (8{:}2) \wr C_2) = 512 = 2^9$, while $\mathrm{val}(\mathrm{M}_{10} \wr C_4, (8{:}2) \wr C_4) = 786432 = 2^{18} \cdot 3$.

Other Problems

- When is a vertex-transitive graph the Saxl graph of a transitive group?

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- Other invariants of graphs:
  - chromatic number;
  - total domination number;
  - independence number.

Thank you for your attention!