Base-two primitive permutation groups

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23rd Postgraduate Group Theory Conference

8 July 2022

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- $G = \mathrm{GL}(V)$, $\Omega = V \setminus \{0\}$ and Δ contains a basis of V.

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Observation. If Δ is a base and $x, y \in G$, then

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Question. How small can a base be?

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- G is primitive iff n is a prime.

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- Progress where G_α is soluble and $G < L \wr P$ **(Burness & H, 2022+)**

Saxl graphs

Definition (Burness & Giudici, 2020)

Let $G \leqslant \mathrm{Sym}(\Omega)$. Then the Saxl graph $\Sigma(G)$ is a graph with

- vertex set Ω;
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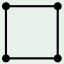
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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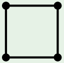
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
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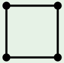
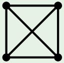
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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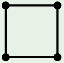
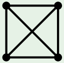
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = D_{10} = \langle (12345), (25)(34) \rangle$, $\Omega = \{1, 2, 3, 4, 5\}$:

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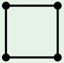
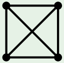
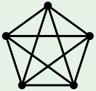
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = D_{10} = \langle (12345), (25)(34) \rangle$, $\Omega = \{1, 2, 3, 4, 5\}$: $\Sigma(G) =$ 

Another example

Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$.

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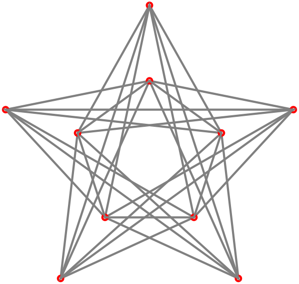
For example, when $q = 4$ we have the complement of the Petersen graph.



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Remark. $r(G) = 1 \iff \Sigma(G)$ is an orbital graph.

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In particular, it asserts that $\Sigma(G)$ has diameter at most 2.

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- **Burness & H, 2021+:** almost simple groups with G_α soluble ✓

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- All primitive groups of degree up to 4095 ✓
- "Most" almost simple groups with alternating or sporadic socles
- **Chen & Du, 2020+; Burness & H, 2021+:** $\mathrm{soc}(G) = \mathrm{L}_2(q)$ ✓
- **Burness & H, 2021+:** almost simple groups with G_α soluble ✓
- **Lee & Popiel, 2021+:** some affine groups

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Example

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $Q(G) \to 1$ as $q \to \infty$. But $\Sigma(G) = J(q+1, 2)$ still has the common neighbour property.

A strong conjecture

Let $\Sigma(\alpha)$ be the set of neighbours of α in $\Sigma(G)$.

Conjecture (Burness & H, 2022+)

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- Generalised Saxl graphs?

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Thank you!