Groups, graphs and transitivity

Hong Yi Huang

University of Bristol

JAC, 21 May 2021

Outline

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Remark: one can replace "vertex" and "transitive" by almost anything. For example: edge-primitive or arc-semiregular.

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- Vertices: 2-subsets of $\{1, 2, 3, 4, 5\}$;
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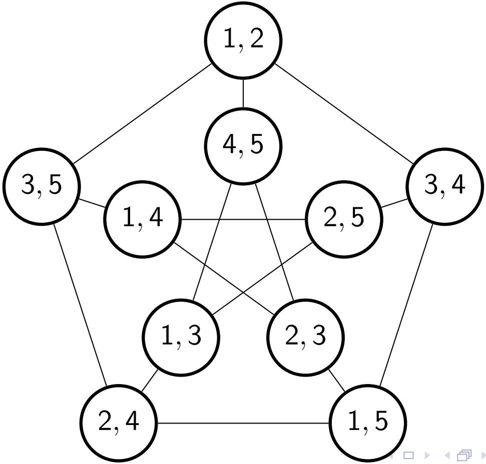

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Indeed, $1 \to s_1 \to s_1 s_2 \to \cdots \to s_1 s_2 \cdots s_n$ for $s_i \in S$ is a path.

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- $\mathrm{Cay}(G, \{x, x^{-1}\}) \cong m\mathbf{C}_n$ if $|G| = mn$ and $|x| = n$.

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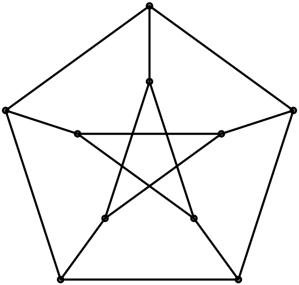

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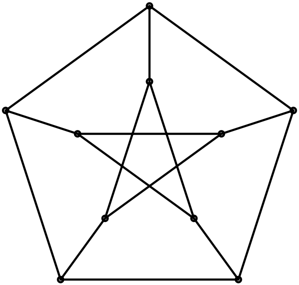

How can we use groups to construct non-Cayley vertex-transitive graphs?

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Coset graphs

Let G be a finite group, $H < G$ and $S \subseteq G \setminus H$.

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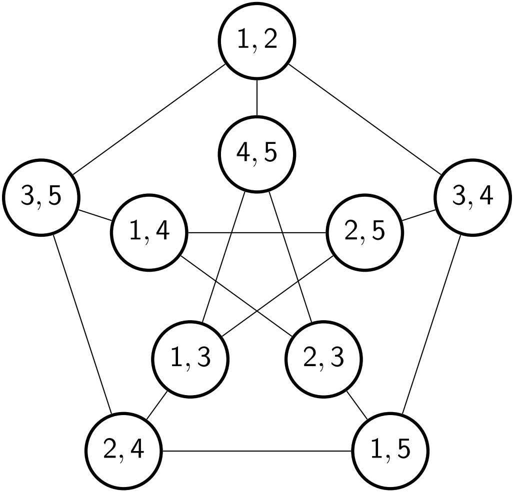
Petersen graph as a coset graph



$\Gamma = \mathrm{Cos}(S_5, H, HSH)$ with $H = \mathrm{Sym}(\{1,2\}) \times \mathrm{Sym}(\{3,4,5\})$ and

$$S = \{(13)(24), (13)(25), (14)(25)\}.$$

Arc-transitivity

Γ is called **arc-transitive** if $\mathrm{Aut}(\Gamma)$ acts transitively on arcs of Γ.

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The **orbital graph** $(x, y)^G$ of G is a graph with

- Vertices Ω;
- Edges in the orbital $(x, y)^G = \{(x^g, y^g) : g \in G\}$ (non-diagonal).

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An orbital Δ is called **self-paired** if $(x, y) \in \Delta \iff (y, x) \in \Delta$.

An orbital graph is undirected iff the orbital is self-paired.

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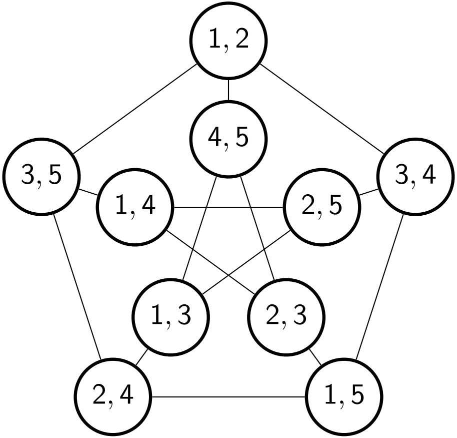

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Theorem (Meng & Huang, 1996)

- Almost all Cayley (di)graphs are connected.
- Almost all Cayley graphs are Hamiltonian.

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- $b(G) = 1 \iff G$ has a regular orbit on Ω.

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- $G = D_{2n}$, $\Omega = \{1, \dots, n\}$: $b(G) = 2$.
- $G = \mathrm{GL}(V)$, $\Omega = V$:
  A subset of Ω is a base iff it contains a basis of V, so $b(G) = \dim V$.

Saxl graphs

Definition (Burness & Giudici, 2020)

Let $G \leq \mathrm{Sym}(\Omega)$ be a permutation group with $b(G) = 2$.

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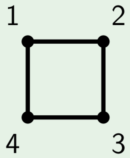
Example

- $G = D_8$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) \cong C_4$.



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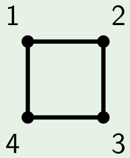

- $G = D_{10}$, $\Omega = \{1, 2, 3, 4, 5\}$: $\Sigma(G) \cong K_5$.

A further example

Let $G = \mathrm{GL}_2(q)$ and $\Omega = \mathbb{F}_q^2 \setminus \{0\}$. Then $\alpha \sim \beta$ iff $\{\alpha, \beta\}$ is linearly independent. Thus, $\Sigma(G)$ is **complete multipartite** with $q+1$ parts of size $q-1$.

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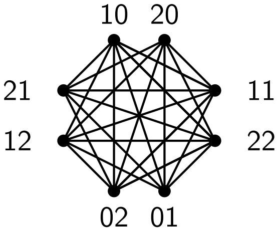
For example, when $q=3$ we have $\Sigma(G) \cong K_8 - 4K_2$.



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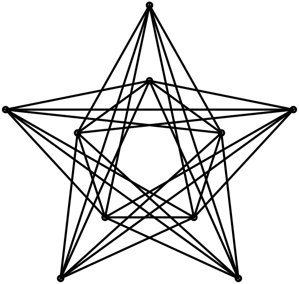
For example, when $q = 4$ we have the complement of the Petersen graph.



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5. $\Sigma(G)$ has valency $r|G_\alpha|$, where r is the number of regular suborbits.

Valencies

Theorem (Burness & Giudici, 2020)

Let G be a finite transitive base-two permutation group with degree n. Then $\Sigma(G)$ has prime valency p iff one of the following holds:

- $G = C_p \wr C_2$, $n = 2p$, $\Sigma(G) \cong K_{p,p}$.
- $G = S_3$, $n = 3$, $\Sigma(G) \cong K_3$.
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A group G is called **almost simple** if $\mathrm{soc}(G)$ is non-abelian simple.

Theorem (Chen & H, 2020)

Let G be an almost simple primitive group. Then $\Sigma(G)$ has prime-power valency p^f iff $p = 2$ and one of the following holds:

- $G = \mathrm{PGL}_2(q)$, $q = 9$ or a Fermat prime, $\Sigma(G) \cong J(q+1, 2)$.
- $G = \mathrm{M}_{10}$, $G_\alpha = 8{:}2$.

Eulerian cycles

By a famous theorem of Euler, a connected graph has an Eulerian cycle iff the degree of every vertex is even.

Theorem (Burness & Giudici, 2020; Chen & H, 2020)

Let G be an almost simple primitive group with stabiliser H. Then one of the following holds:

- $\Sigma(G)$ is Eulerian.
- $G = \mathrm{M}_{23}$ and $H = 23{:}11$.
- $\mathrm{soc}(G) = \mathrm{L}_r^{\epsilon}(q)$, r odd prime, $G \not\leq \mathrm{PGL}_r^{\epsilon}(q)$ and H is of type $\mathrm{GL}_1^{\epsilon}(q^r)$.

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Question. Is $(G, H) = (\mathrm{M}_{23}, 23{:}11)$ the only non-Eulerian example?

Unique regular suborbit

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Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$. Then $\Sigma(G) \cong J(q+1, 2)$ has valency $2(q-1)$.

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- **Burness & Giudici, 2020:** most cases when $\mathrm{soc}(G)$ alternating or sporadic.
- **Burness & H, in progress:** almost simple with soluble stabiliser.

Probabilistic methods

Let $G \leq \mathrm{Sym}(\Omega)$ be a base-two transitive permutation group with degree n. Let $\mathrm{val}(G)$ be the valency of $\Sigma(G)$. Set

$$Q(G,2) := 1 - \mathbb{P}(G,2) = \frac{|\{(\alpha,\beta) \in \Omega^2 : G_{\alpha\beta} \neq 1\}|}{n^2} = 1 - \frac{\mathrm{val}(G)}{n}.$$

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Lemma

If $Q(G,2) < \frac{1}{t} \leq \frac{1}{2}$, then $\Sigma(G)$ has all of the following properties:

- Any t vertices in $\Sigma(G)$ have a common neighbour;
- $\Sigma(G)$ has diameter at most 2;
- $\Sigma(G)$ has clique number at least $t+1$;
- $\Sigma(G)$ is Hamiltonian.

Burness-Giudici Conjecture

Recall that

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Note that if $Q(G,2) < \frac{1}{2}$, then the conjecture holds.

Some evidence

Example

Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$. Then $\Sigma(G) \cong J(q+1,2)$ has valency $2(q-1)$ and thus

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When $q \leq 23$ we can also check using MAGMA that the conjecture holds.

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- Asymptotic results for many diagonal and twisted wreath type groups **(Fawcett, 2013/21)**

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Problems

- **Connectedness.** Characterise transitive groups with connected Saxl graph. If G is quasiprimitive?
- **Automorphisms.**
  - When do we have $G = \mathrm{Aut}(\Sigma(G))$?
  - When is $\Sigma(G)$ Cayley?
- **Cycles.** Eulerian cycle? Hamiltonian cycle?
- **Unique regular suborbit.** Can we classify groups with $r = 1$?
- **Other invariants.** Chromatic numbers? Spectrum?

Thanks for your attention!