Simple groups, fixed point ratios and applications I

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Based on the survey article by Tim Burness[1]

[1] T.C. Burness. *Simple groups, fixed point ratios and applications*. Local representation theory and simple groups, 267–322, EMS Ser. Lect. Math., Eur. Math. Soc., Zürich, 2018.

Overview

This lecture is divided into three parts:

- Introduction to simple groups and fixed point ratios;
- Generation of simple groups;
- Base sizes and Saxl graphs for almost simple groups.

Outline

CFSG and almost simple groups

Theorem (CFSG).

Let G be a non-abelian finite simple group. Then G is isomorphic to one of the following groups:

- A_n with $n \geq 5$;
- classical simple groups;
- exceptional simple groups of Lie type;
- 26 sporadic groups.

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A group G is called **almost simple** if

$$\mathrm{soc}(G) \cong \mathrm{Inn}(T) \lesssim G \lesssim \mathrm{Aut}(T)$$

for some non-abelian simple group T.

Alternating groups

If $n \geq 5$, then we have

$$\mathrm{Aut}(A_n) = \begin{cases} S_n & n \neq 6; \\ S_n.2 & n = 6. \end{cases}$$

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$$\mathrm{Aut}(A_n) = \begin{cases} S_n & n \neq 6; \\ S_n.2 & n = 6. \end{cases}$$

If $\mathrm{soc}(G) = A_n$ and H is a maximal subgroup of G, then by O'Nan-Scott, H either contains A_n or is in one of the following classes.

Class	Descriptions	Structure in S_n	Conditions
$\mathscr{A}_1$	Intransitive	$S_m \times S_k$	$m + k = n$, $m \neq k$
$\mathscr{A}_2$	Imprimitive	$S_a \wr S_b$	$n = ab$, $a, b \geq 2$
$\mathscr{A}_3$	Affine	$\mathrm{AGL}_d(p)$	$n = p^d$, p prime
$\mathscr{A}_4$	Product action	$S_a \wr S_b$	$n = a^b$, $a \geq 5$, $b \geq 2$
$\mathscr{A}_5$	Diagonal	$T^k.(\mathrm{Out}(T) \times S_k)$	$n = \lvert T \rvert^{k-1}$, $k \geq 2$, T non-abelian simple
$\mathscr{S}$	Almost simple	S	S is primitive on $[n]$

Classical groups

Let $\mathrm{soc}(G)$ be a finite simple classical group. Then G is in one of the following cases:

$$\mathrm{L}, \mathrm{U}, \mathrm{S}, \mathrm{O}, \mathrm{O}^{\pm}.$$

We also write $\mathrm{U} = \mathrm{L}^{-}$ in the literature.
The outer automorphism group of a classical group is generated by **diagonal, field and graph automorphisms**. In more detail, we have the table in the next page.

Classical groups

Case	soc(G)	Out(soc(G))	Conditions
L	$\mathrm{L}_n(p^f)$	$\mathbb{Z}_{(n,p^f-1)}{:}(\mathbb{Z}_f \times \mathbb{Z}_2)$	$n > 2$
		$\mathbb{Z}_{(2,p^f-1)} \times \mathbb{Z}_f$	$n = 2$
S	$\mathrm{PSp}_{2n}(p^f)$	$\mathbb{Z}_2.\mathbb{Z}_f$	$p \neq 2$
		$\mathbb{Z}_f.\mathbb{Z}_2$	$p = 2,\ n = 2$
		$\mathbb{Z}_f$	$p = 2,\ n > 2$
U	$\mathrm{U}_n(q)$	$\mathbb{Z}_{(n,q+1)}.\mathbb{Z}_f$	$q^2 = p^f$
O	$\Omega_{2n+1}(p^f)$	$\mathbb{Z}_{(2,p^f-1)}.\mathbb{Z}_f$	
O^+	$\mathrm{P\Omega}^+_{2n}(p^f)$	$\mathbb{Z}^2_{(2,p^f-1)}.\mathbb{Z}_f.S_3$	$n = 4$
		$\mathbb{Z}^2_{(2,p^f-1)}.\mathbb{Z}_f.\mathbb{Z}_2$	$n > 4$ even
		$\mathbb{Z}_{(4,p^{nf}-1)}.\mathbb{Z}_f.\mathbb{Z}_2$	n odd
O^-	$\mathrm{P\Omega}^-_{2n}(q)$	$\mathbb{Z}_{(4,q^n+1)}.\mathbb{Z}_f$	$q^2 = p^f$

Classical groups

Let $\mathrm{soc}(G)$ be a finite simple classical group over $\mathbb{F}_q$ with characteristic p. A famous theorem of Aschbacher implies that a maximal subgroup H of G either contains $\mathrm{soc}(G)$, or is in the following classes. We refer the reader to **Kleidman-Liebeck** for details.

Class	Descriptions
$\mathscr{C}_1$	Stabiliser of subspaces, or pairs of subspaces, of V
$\mathscr{C}_2$	Stabilisers of decompositions $V = \bigoplus_{i=1}^{t} V_i$, where $\dim V_i = a$
$\mathscr{C}_3$	Stabilisers of prime degree extension fields of $\mathbb{F}_q$
$\mathscr{C}_4$	Stabiliser of decompositions $V = V_1 \otimes V_2$
$\mathscr{C}_5$	Stabilisers of prime index subfields of $\mathbb{F}_q$
$\mathscr{C}_6$	Normalisers of symplectic-type r-groups, $r \neq p$
$\mathscr{C}_7$	Stabilisers of decompositions $V = \bigotimes_{i=1}^{t} V_i$, where $\dim V_i = a$
$\mathscr{C}_8$	Stabilisers of nondegenerate forms on V
$\mathscr{S}$	Almost simple absolutely irreducible subgroups
$\mathscr{N}$	Novelty subgroups ($\mathrm{soc}(G) = \mathrm{P}\Omega_8^+(q)$ or $\mathrm{Sp}_4(q)'$ ($p = 2$), only)

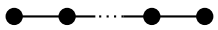
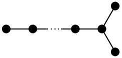
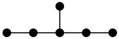
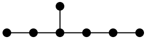
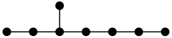
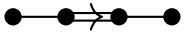
Classical groups

We also refer the reader to Table 3.5 in Kleidman-Liebeck for the types of subgroups in classes $\mathscr{C}_i$. For example, in case U we have

Class	Types	Conditions
$\mathscr{C}_1$	P_m	$1 \leq m \leq [n/2]$
	$\mathrm{GU}_m(q) \oplus \mathrm{GU}_{n-m}(q)$	$1 \leq m < n/2$
$\mathscr{C}_2$	$\mathrm{GU}_m(q) \wr S_t$	$n = mt,\ t \geq 2$
	$\mathrm{GL}_{n/2}(q^2).2$	n even
$\mathscr{C}_3$	$\mathrm{GU}_m(q^r)$	$n = mr,\ r \geq 3$ prime
$\mathscr{C}_4$	$\mathrm{GU}_{n_1}(q) \otimes \mathrm{GU}_{n_2}(q)$	$n = n_1 n_2,\ 2 \leq n_1 < \sqrt{n}$
$\mathscr{C}_5$	$\mathrm{GU}_n(q_0)$	$q = q_0^r,\ r \geq 3$ prime
	$\mathrm{O}_n(q)$	qn odd
	$\mathrm{O}_n^{\epsilon}(q)$	q odd, n even
	$\mathrm{Sp}_n(q)$	n even
$\mathscr{C}_6$	$r^{2m}.\mathrm{Sp}_{2m}(r)$	See KL
$\mathscr{C}_7$	$\mathrm{GU}_m(q) \wr S_t$	$n = m^t,\ m \geq 3,\ (q,m) \neq (2,3)$

Dynkin diagrams

The **Dynkin diagrams** have a close relation to simple groups of Lie type. Roughly speaking, a simple group of Lie type corresponds to one of the following Dynkin diagrams, where the second or third corresponding group is (if any) given by twisting points.

Groups	Dynkin diagram	Condition
$\mathrm{L}_{n+1}(q)$, $\mathrm{U}_{n+1}(q)$	A_n 	$n \geq 1$
$\Omega_{2n+1}(q)$	B_n 	$n \geq 3$
$\mathrm{PSp}_{2n}(q)$, ${}^2B_2(2^f)$	C_n 	$n \geq 2$
$\mathrm{P}\Omega^+_{2n}(q)$, $\mathrm{P}\Omega^-_{2n}(q)$, ${}^3D_4(q)$	D_n 	$n \geq 4$
$E_6(q)$, ${}^2E_6(q)$	E_6 	
$E_7(q)$	E_7 	
$E_8(q)$	E_8 	
$F_4(q)$, ${}^2F_4(2^f)$	F_4 	
$G_2(q)$, ${}^2G_2(3^f)$	G_2 	

Exceptional groups

Let $\mathrm{soc}(G)$ be a finite exceptional simple group of Lie type. Then G and $\mathrm{Out}(\mathrm{soc}(G))$ are listed as follows.

T	$\mathrm{Out}(T)$	Condition
$G_2(p^f)$	$\mathbb{Z}_f$	$p \neq 3$
	$\mathbb{Z}_f.\mathbb{Z}_2$	$p = 3$
$F_4(p^f)$	$\mathbb{Z}_f$	$p \neq 2$
	$\mathbb{Z}_f.\mathbb{Z}_2$	$p = 2$
$E_6(p^f)$	$\mathbb{Z}_{(3,p^f-1)}.\mathbb{Z}_f.\mathbb{Z}_2$	
${}^2E_6(q)$	$\mathbb{Z}_{(3,q+1)}.\mathbb{Z}_f$	$q^2 = p^f$
$E_7(p^f)$	$\mathbb{Z}_{(2,p^f-1)}.\mathbb{Z}_f$	
$E_8(p^f)$	$\mathbb{Z}_f$	
${}^3D_4(q)$	$\mathbb{Z}_f$	$q^3 = p^f$
${}^2B_2(2^{2n+1})$	$\mathbb{Z}_{2n+1}$	
${}^2G_2(3^{2n+1})$	$\mathbb{Z}_{2n+1}$	
${}^2F_4(2^{2n+1})$	$\mathbb{Z}_{2n+1}$	
${}^2F_4(2)'$	$\mathbb{Z}_2$	

Exceptional groups

The maximal subgroups of exceptional groups are similar. We "correspond" those groups to the classes of maximal subgroups of classical groups.

Description	"Correspondence"
$\overline{H}$ is maximal parabolic in $\overline{G}$	$\mathscr{C}_1$
$\overline{H}$ is non-parabolic, of maximal rank in $\overline{G}$	$\mathscr{C}_2$, $\mathscr{C}_3$
$\overline{H}$ is closed of positive dimension in $\overline{G}$	$\mathscr{C}_4$, $\mathscr{C}_7$, $\mathscr{C}_8$
Subfield subgroups	$\mathscr{C}_5$
Exotic p-local subgroups	$\mathscr{C}_6$
Almost simple (complete unless E_7 and E_8)	$\mathscr{S}$
Borovik subgroup $(A_5 \times A_6).2^2$ for $\overline{G} = E_8$	

Here we write $\overline{G}$ the associated group of Lie type defined over $\overline{\mathbb{F}_q}$. We refer the reader to a survey article by Liebeck and Seitz[2].

[2] M.W. Liebeck and G.M Seitz, *A survey of maximal subgroups of exceptional groups of Lie type*. Groups, combinatorics & geometry (Durham, 2001), 139–146, World Sci. Publ., River Edge, NJ, 2003.

Exceptional groups

Example.

Let $\mathrm{soc}(G) = \mathrm{L}_8(q)$ and $H \in \mathscr{C}_4$ of type $\mathrm{GL}_4(q) \otimes \mathrm{GL}_2(q)$. Then $\overline{G}$ is of type A_7 and $\overline{H}$ is of type A_3A_1, which is not of maximal rank.

Example.

Let $\mathrm{soc}(G) = \mathrm{L}_8(q)$ and $H \in \mathscr{C}_8$ is of type $\mathrm{Sp}_8(q)$. Then $\overline{G}$ is of type A_7 and $\overline{H}$ is of type C_4, which is not of maximal rank.

Exceptional groups

Example.

Let $\mathrm{soc}(G) = \mathrm{PSp}_8(q)$ and $H \in \mathscr{C}_2$ is of type $\mathrm{GL}_4(q).2$. The matrices of elements in H are of the form

$$\begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} \mathrm{diag}(\lambda, 1, 1, 1, \lambda^{-1}, 1, 1, 1)$$

for some $A \in \mathrm{SL}_4(q)$ and $\lambda \in \mathbb{F}_q$. Hence, $\overline{G}$ is of type C_4 and $\overline{H}$ is of type A_3T_1, which is of maximal rank.

Sporadic groups

Let $\mathrm{soc}(G)$ be a sporadic simple group. Then $|\mathrm{Out}(\mathrm{soc}(G))| \leq 2$. Specifically, the groups with $|\mathrm{Out}(\mathrm{soc}(G))| = 2$ are

$$\mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{HS}, \mathrm{J}_2, \mathrm{J}_3, \mathrm{McL}, \mathrm{Suz}, \mathrm{He}, \mathrm{HN}, \mathrm{Fi}_{22}, \mathrm{Fi}_{24}' \text{ and } \mathrm{O'N},$$

and those with trivial outer automorphism are

$$\mathrm{M}_{11}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{Co}_1, \mathrm{Co}_2, \mathrm{Co}_3, \mathrm{J}_1, \mathrm{J}_4, \mathrm{Fi}_{23}, \mathrm{Th}, \mathrm{Ly}, \mathrm{Ru}, \mathbb{B} \text{ and } \mathbb{M}.$$

[3] R.A. Wilson. *Maximal subgroups of sporadic groups*. Finite simple groups: thirty years of the atlas and beyond, 57–72, Contemp. Math., 694, Amer. Math. Soc., Providence, RI, 2017.

[4] R.A. Wilson et al., *A World-Wide-Web Atlas of finite group representations*, http://brauer.maths.qmul.ac.uk/Atlas/v3/.

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The classification of maximal subgroups of sporadic almost simple groups is complete, except for $\mathbb{M}$. See an article by Wilson[3] or the Web Atlas[4] for the classification, where some errors of the old-version Atlas are fixed.

[3] R.A. Wilson. *Maximal subgroups of sporadic groups*. Finite simple groups: thirty years of the atlas and beyond, 57–72, Contemp. Math., 694, Amer. Math. Soc., Providence, RI, 2017.

[4] R.A. Wilson et al., *A World-Wide-Web Atlas of finite group representations*, http://brauer.maths.qmul.ac.uk/Atlas/v3/.

Soluble maximal subgroups

Problem.

Classify soluble maximal subgroups of almost simple groups.

Some remarks

- Even though classification has been already done by Li and Zhang[5], it is still worth trying yourself to get familiar with maximal subgroups.
- In Li and Zhang's paper, the maximal subgroups H is presented by "`GroupName`"s. It is also worth interpreting those groups into "Type of H" considering the classification theorems.
- When dealing with some small cases, we can use MAGMA by the functions `AutomorphismGroupSimpleGroup` and `MaximalSubgroups(G:IsSolvable:=ture)`.

[5] C.H. Li and H. Zhang. *The finite primitive groups with soluble stabilizers, and the edge-primitive s-arc transitive graphs.* Proc. Lond. Math. Soc. **103** (2011), 441–472.

Outline

Fixed point ratios

Let $G \leq \mathrm{Sym}(\Omega)$ with $|\Omega|$ finite. Write

$$C_\Omega(x) = \{\alpha \in \Omega : \alpha^x = \alpha\}$$

the set of fixed points of $x \in G$.

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Definition (Fixed point ratios).

The **fixed point ratio** of $x \in G$ is

$$\mathrm{fpr}(x, \Omega) = \mathrm{fpr}(x) = \frac{|C_\Omega(x)|}{|\Omega|},$$

the proportion of points in Ω fixed by x, or the probability that a randomly chosen element of Ω is fixed by x.

S_n on 2-subsets

Let $G = S_n$ for $n \geq 5$ acting on 2-subsets of $[n]$. Set $x = (123)$. We have

$$C_\Omega(x) = \{\{a, b\} : a, b \in \{4, \dots, n\}\}$$

and so $|C_\Omega(x)| = \binom{n-3}{2}$. This implies

$$\mathrm{fpr}(x) = \frac{\binom{n-3}{2}}{\binom{n}{2}} = \frac{(n-3)(n-4)}{n(n-1)}.$$

Observations

Note that $C_\Omega(x) = \{\alpha \in \Omega : x \in G_\alpha\}$. We have

- $\mathrm{fpr}(x) = \mathrm{fpr}(y)$ for all $y \in x^G$.
- $\mathrm{fpr}(x) \leq \mathrm{fpr}(x^m)$ for all $m \in \mathbb{Z}$.

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- $\mathrm{fpr}(x) \leq \mathrm{fpr}(x^m)$ for all $m \in \mathbb{Z}$.

Moreover, if G is transitive with stabiliser H, then

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|}.$$

This provides a method to compute $\mathrm{fpr}(x)$ for an abstract action.

S_n on 2-subsets again

Let $G = S_n$ for $n \geq 5$ acting on 2-subsets of $[n]$. We have the stabiliser $H = S_{n-2} \times S_2$. Set $x = (123)$.
Since all the 3-cycles in G are conjugate, $x^G \cap H$ is the set of 3-cycles in H, which gives

$$|x^G \cap H| = |x^H| = 2\binom{n-2}{3}$$

and

$$|x^G| = 2\binom{n}{3}.$$

So the fixed point ratio is

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|} = \frac{2\binom{n-2}{3}}{2\binom{n}{3}} = \frac{(n-3)(n-4)}{n(n-1)}.$$

On computing

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Let G be transitive.

- Concrete and well known actions: directly calculate.
- General actions: very difficult!!!
- MAGMA: `CosetAction` is expensive and requires very small degrees.
- $\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|}$ is useful and most frequently used.

Problems

- Obtain upper and lower bounds on $\mathrm{fpr}(x)$.
- Compute (or bound) the minimal and maximal fixed point ratios $\min\{\mathrm{fpr}(x) : x \in G\}$ and $\max\{\mathrm{fpr}(x) : 1 \neq x \in G\}$.
- Given $S \subseteq G \setminus \{1\}$, compute (or bound) $\min\{\mathrm{fpr}(x) : x \in S\}$ and $\max\{\mathrm{fpr}(x) : x \in S\}$.

Minimal degrees and fixities

- **Minimal degree**:

$$\mu(G) = \min_{1 \neq x \in G} (n - |C_\Omega(x)|) = n \left(1 - \max_{1 \neq x \in G} \mathrm{fpr}(x)\right),$$

  which is the smallest number of points moved by any non-identity element in G.

- **Fixity**:

$$f(G) = n \left(\max_{1 \neq x \in G} \mathrm{fpr}(x)\right) = n - \mu(G),$$

  the largest number of fixed points of a non-identity element in G.

- **Involution fixity**:

$$\mathrm{ifix}(G) = n \left(\max_{|x|=2} \mathrm{fpr}(x)\right).$$

Base sizes

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- Let $Q(G, c)$ be the probability that a randomly chosen c-tuple $\{\alpha_1, \ldots, \alpha_c\}$ in Ω is NOT a base. That is,

$$Q(G, c) = \frac{|\{(\alpha_1, \ldots, \alpha_c) \in \Omega^c : G_{\alpha_1 \cdots \alpha_c} \neq 1\}|}{|\Omega|^c}.$$

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- If $\{\alpha_1, \ldots, \alpha_c\}$ is not a base, then there exists an element x (of prime order) such that

$$x \in \bigcap_{i=1}^{c} G_{\alpha_i} \neq 1,$$

which gives $\{\alpha_1, \ldots, \alpha_c\} \subseteq C_\Omega(x)$ and $c \leq |C_\Omega(x)|$.

Base sizes

- Let $\mathcal{P}$ be the set of elements of prime order in G. Then

$$Q(G,c) \leq \sum_{x\in\mathcal{P}} \left(\frac{|C_\Omega(x)|}{|\Omega|}\right)^c = \sum_{x\in\mathcal{P}} \mathrm{fpr}(x)^c =: \hat{Q}(G,c).$$

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- Note that $\mathrm{fpr}(x) = \mathrm{fpr}(x^g)$ for all $g \in G$. This implies

$$Q(G,c) \leq \hat{Q}(G,c) = \sum_{i=1}^{k} |x_i^G| \cdot \left(\frac{|x_i^G \cap H|}{|x_i^G|}\right)^c,$$

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where $x_1, \dots, x_k$ are representatives of $\mathcal{P}$ up to conjugacy in G.

- We have $b(G) \leq c$ if $\hat{Q}(G,c) < 1$.
- In particular, $b(G) \leq 2$ if

$$|H|^2 \max_{1 \neq x \in H} |C_G(x)| = |H|^2 \max_{\substack{x \in H \\ |x| \text{ prime}}} |C_G(x)| < |G|.$$

Base sizes

Example.

Suppose $\mathrm{soc}(G) = \mathrm{L}_3^{\epsilon}(q)$ with $q = p \equiv \epsilon \pmod 3$ and H is of type $3^{1+2}.\mathrm{Sp}_2(3)$. Then $|H| \leq 432$ and

$$|C_G(x)| \geq \frac{|G|}{(q-1)(q^3-1)}$$

for all $x \in G$ of prime order (maximal if $\epsilon = +$ and x is unipotent with Jordan form $[J_2, J_1]$). This gives $b(G) = 2$ for all $q > 23$. When $q \leq 23$ we can also check using MAGMA that $b(G) = 2$.

Next time

In the next talk, we will introduce the generation problem of simple groups and some related results, which includes

- The 2-generation property of simple groups;
- Random generations of simple groups;
- Spreads and uniform spreads;
- Generating graphs;

and how fixed point ratios are applied to these problems.

Thank you for your attention!