Base-two primitive permutation groups

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- $G = S_n$, $\Omega = \{1, \dots, n\}$ and $\Delta = \{1, \dots, n-1\}$.
- $G = \mathrm{GL}(V)$, $\Omega = V \setminus \{0\}$ and Δ contains a basis of V.

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- $G = \mathrm{GL}(V)$, $\Omega = V \setminus \{0\}$: $b(G) = \dim(V)$.

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Observation: If Δ is a base and $x, y \in G$, then

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Some applications:

- Minimal dimensions
- Generation of finite groups
- Graphs defined on groups

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- G is primitive iff n is a prime.

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- Progress where G_α is soluble and $G < L \wr P$ **(Burness & H, 2022+)**

Probabilistic methods

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group of degree n. Then

$$Q(G) = \frac{|\{(\alpha,\beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{n^2}$$

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Saxl graphs

Definition (Burness & Giudici, 2020)

Let $G \leqslant \mathrm{Sym}(\Omega)$. Then the Saxl graph $\Sigma(G)$ is a graph with

- vertex set Ω;
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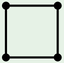
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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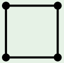
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
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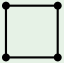
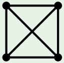
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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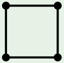
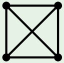
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = D_{10} = \langle (12345), (25)(34) \rangle$, $\Omega = \{1, 2, 3, 4, 5\}$:

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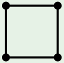
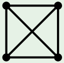
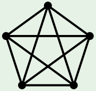
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1,2,3,4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1,2,3,4\}$: $\Sigma(G) =$ 
- $G = D_{10} = \langle (12345), (25)(34) \rangle$, $\Omega = \{1,2,3,4,5\}$: $\Sigma(G) =$ 

Another example

Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$.

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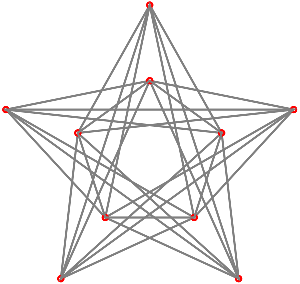
For example, when $q = 4$ we have the complement of the Petersen graph.



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Remark. $r(G) = 1 \iff \Sigma(G)$ is an orbital graph of G.

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- **Lee & Popiel, 2021+:** some affine groups

Probabilistic methods

Recall that $\Sigma(G)$ has valency $v(G) = r(G)|G_\alpha|$ and

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If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $Q(G) \to 1$ as $q \to \infty$. But $\Sigma(G) = J(q+1, 2)$ still has the common neighbour property.

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Thank you!