Base-two primitive permutation groups and their Saxl graphs

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- $G = \mathrm{GL}(V)$, $\Omega = V$ and Δ contains a basis of V.

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- G is primitive iff n is a prime.

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Product type: In progress

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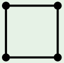
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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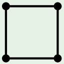
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
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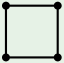
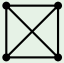
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 

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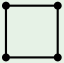
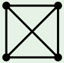
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = D_{10} = \langle (12345), (25)(34) \rangle$, $\Omega = \{1, 2, 3, 4, 5\}$:

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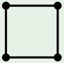
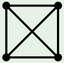
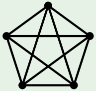
- $G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = C_4 = \langle (1234) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) =$ 
- $G = D_{10} = \langle (12345), (25)(34) \rangle$, $\Omega = \{1, 2, 3, 4, 5\}$: $\Sigma(G) =$ 

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Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$.

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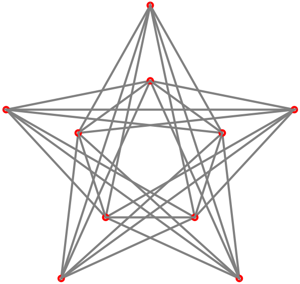
For example, when $q = 4$ we have the complement of the Petersen graph.



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Question: What is the diameter of $\Sigma(G)$ if G is primitive?

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Conjecture (Burness & Giudici, 2020)

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- **Lee & Popiel, 2021:** some affine groups

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Example: $(G, G_\alpha) = (A_5, S_3) \implies \Sigma(G) = J(5,2)$, $\omega(G) = 4$, $\alpha(G) = 2$.

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$\mathcal{S} := \{$almost simple primitive groups with soluble stabilisers$\}$

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$\mathcal{B} := \{G \in \mathcal{S} \mid b(G) = 2\}$

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Example

If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $Q(G) \to 1$ as $q \to \infty$. But $\Sigma(G) = J(q+1,2)$ still has the common neighbour property.

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Questions:

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- If $G < L \wr P$ is primitive, then when do we have $b(G) = 2$?

Thank you!