Bases & Saul graphs over the past 4 years.

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Throughout, let $G \leq \mathrm{Sym}(\Omega)$ and assume $|\Omega| < \infty$.

§1 Bases

Base $\Delta \subseteq \Omega$ s.t. $\bigcap_{\alpha \in \Delta} G_\alpha = 1$.

Base size $b(G)$ min size of a base for G.

Note $b(G) = \min \{ k \mid G$ has a regular orbit on $\Omega^k \}$.

Let $\mathrm{reg}(G)$ be the number of regular G-orbits on $\Omega^{b(G)}$.

Examples

- $G = S_n$, $|\Omega| = n$, $\Delta = \{1, \dots, n-1\}$, $b(G) = n-1$, $\mathrm{reg}(G) = 1$
- $G = GL(V)$, $\Omega = V \setminus \{0\}$:

  Δ contains a basis of V. $b(G) = \dim V$, $\mathrm{reg}(G) = 1$

- $G = D_{2n}$. $|\Omega| = n$, $b(G) = 2$, $\mathrm{reg}(G) = \lceil \frac{n}{2} \rceil - 1$.
- T non-abelian simple, $\Omega = T$, $G = T : \mathrm{Aut}(T) = \mathrm{Hol}(T)$

$G_1 = \mathrm{Aut}(T)$; $G_1 \cap G_x = C_{\mathrm{Aut}(T)}(x) \neq 1 \Rightarrow b(G) \geq 3$.

Fact $\exists x, y \in T$ s.t. $\langle x, y \rangle = T \Rightarrow b(G) = 3$.

§2 Saxl graphs

Generalised Saxl graph $\Sigma(G)$ (BG, 2020) (FHLR, 2021+)

vertex set Ω;

$\alpha \sim \beta \iff \{\alpha, \beta\}$ is ~~a base~~ a subset of a base of size $b(G)$.

Examples

- $G = S_n$, $|\Omega| = n$: $\Sigma(G)$ is complete
- G is 2-transitive : $\Sigma(G)$ is complete
- $G = GL(V)$, $\Omega = V \setminus \{0\}$: $\Sigma(G)$ is complete multipartite.
- $G = D_{2n}$, $|\Omega| = n$: $\Sigma(G) \cong \begin{cases} K_n, & n \text{ odd} \\ K_n - \frac{n}{2} K_2, & n \text{ even.} \end{cases}$
- $G = \mathrm{Hol}(T)$, $\Omega = T$: $\Sigma(G)$ is complete.

Guralnick & Kantor, 2000 : $\forall\, 1 \neq x \in T$, $\exists y \in T$ s.t. $\langle x, y \rangle = T$

- $G = PGL_2(q)$, $\Omega = \{$2-subsets of $\{$1-spaces of $\mathbb{F}_q^2\}\}$

Then $G_\alpha \cong D_{2(q-1)}$, and $G_\alpha \cap G_\beta = 1 \iff |\alpha \cap \beta| = 1$.

So $\Sigma(G) \cong J(q+1, 2)$ is a <u>Johnson graph</u>.

Basic properties

- $G \leq \mathrm{Aut}(\Sigma(G))$.
- G is transitive $\Rightarrow$ $\Sigma(G)$ is G-vertex-transitive.
- G is primitive $\Rightarrow$ $\Sigma(G)$ is connected

  $(G_\alpha \underset{\max}{<} G)$

- $\mathrm{reg}(G) = 1 \underset{b(G)=2}{\overset{\Longrightarrow}{\Longleftarrow}} \Sigma(G)$ is G-arc-transitive

- $$\left(\frac{\mathrm{val}(\Sigma(G))}{|\Omega|}\right)^{b(G)-1} \underset{b(G)=2}{\overset{\geqslant}{=\!=}} \frac{\mathrm{reg}(G)\,|G|}{|\Omega|^{b(G)}}$$

Probability

Let $$Q(G,k) = \frac{|\{(\alpha_1,\dots,\alpha_k)\in\Omega^k : G_{\alpha_1}\cap\dots\cap G_{\alpha_k} \neq 1\}|}{|\Omega|^k}$$

Note
- $Q(G,k) < 1 \iff b(G) \leq k$
- $Q(G,b(G)) = 1 - \dfrac{\mathrm{reg}(G)\,|G|}{|\Omega|^{b(G)}}$

Lemma If $t \in \mathbb{N}$, $t \geqslant 2$ and $Q(G,b(G)) < \frac{1}{t}$, then
- Any t vertices in $\Sigma(G)$ have a common neighbour.
- $\Sigma(G)$ is connected with diameter $\geqslant 2$.
- $\Sigma(G)$ has clique number $\geqslant t+1$
- $\Sigma(G)$ is Hamiltonian.

§3 Problems & results.

Let $\mathcal{S} = \{G \mid G$ almost simple primitive, G_α soluble$\}$
(Li & Zhang, 2011)

$\mathcal{D} = \{G \mid G$ diagonal type primitive, top group $P\}$
(e.g. $P = A_2 \Rightarrow G \leq \mathrm{Hol}(T)$)

$\mathcal{L} = \{G \mid G$ primitive, $\mathrm{soc}(G) = \mathrm{PSL}_2(q)\}$.

1. $b(G)$ for primitive groups

O'Nan-Scott. Finite primitive groups are divided into 5 types.

- Almost simple, $T \trianglelefteq G \leq \mathrm{Aut}(T)$. $T = \mathrm{soc}(G)$ simple.

  Precise $b(G)$ when
  - $T = A_n$ or sporadic ✓
  - $G \in \mathcal{L}$ ✓ (B, 2007/21 ; FHLR, 2024+).
  - $G \in \mathcal{S}$ ✓ (B, 2021)
- $G \in \mathcal{D}$: precise $b(G)$ ✓ (Fawcett, 2013 ; H, 2024)

2. Common neighbour

<u>Conjecture</u> (BG, 2020 ; FHLR, 2024+)

G primitive $\Rightarrow$ any two vertices in $\Sigma(G)$ have a common neighbour.

<u>Recall</u> $Q(G, b(G)) < \frac{1}{2}$ is sufficient.

<u>Evidence</u>

- $G \in \mathcal{L}$ & $b(G) = 2$ (BH, 2022).
- $G = PSL_2(q)$ (FHLR, 2024+).
- G almost simple sporadic & $b(G) \geq 3$ (FHLR, 2024+).
- $G \in \mathcal{D}$, $P \notin \{A_k, S_k\}$ (H, 2024+).

3. Arc-transitivity.

<u>Problem</u> Classify the primitive groups G with $\mathrm{reg}(G) = 1$.

- $G \in \mathcal{S}$ ✓ (BH, 2022/23)

  e.g. $(G, G_\alpha) = (PGL_2(q), D_{2(q-1)})$.
- $G \in \mathcal{L}$ & $b(G) = 2$, or $G = PSL_2(q)$ ✓ (FHLR, 2024+)
- $G \in \mathcal{D}$ ✓ (H, 2024 ; FHLR, 2024+).

4. Completeness

Note If $b(G) = 2$, then $\Sigma(G)$ is complete $\Leftrightarrow$ G is Frobenius.

Problem Classify the primitive groups G s.t. $\Sigma(G)$ is complete.

e.g. 2-transitive groups ; $\mathrm{Hol}(T)$.

FHLR, 2024+ :
- $G \in \mathcal{L}$ ✓
- Partial results for $G \in \mathcal{D}$.

5. Other invariants

$\mathrm{val}(\Sigma(G))$: Chen & H, 2022

clique & independence numbers : BH, 2022.

6. Rank 3 groups

Conjecture? Theorem? (H & Zhu, ?)

G imprimitive & $\mathrm{rank}(G) = 3$

$\Rightarrow$ $\Sigma(G)$ is complete or complete multipartite.

Corollary $\Sigma(G)$ disconnected $\Rightarrow$ $\mathrm{rank}(G) \geq 4$.