The probabilistic method in group theory

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Outline

1. The probabilistic method
2. Generation of simple groups
3. Bases for almost simple primitive groups
4. Saxl graphs

The probabilistic method

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The probabilistic method describes the **existence** and the **abundance**.

Randomly chosen elements in groups

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Question.

What if we randomly choose elements satisfying some properties?

Probabilistic methods in group theory

Let G be a finite group. Let E be an event and $\mathbb{P}_E(G)$ be the probability of randomly chosen elements satisfying E. Then

$$\begin{aligned}\text{There exist elements satisfying } E &\iff \mathbb{P}_E(G) > 0\\ &\iff 1 - \mathbb{P}_E(G) < 1.\end{aligned}$$

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Remarks:

- Sometimes it is hard to prove $\mathbb{P}_E(G) > 0$ by direct construction.
- We need to find an upper bound of $1 - \mathbb{P}_E(G)$ that is easily obtained.
- We usually have good properties if $1 - \mathbb{P}_E(G) \to 0$.

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Aim.

Find $\widehat{Q}_E(G) \geq 1 - \mathbb{P}_E(G)$ such that $\widehat{Q}_E(G) < 1$.

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Example.

- $A_n = \langle (1,2,3), (1,2,\ldots,n) \rangle$ if n is odd.
- $A_n = \langle (1,2,3), (2,3,\ldots,n) \rangle$ if n is even.

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Let G be a group and

$$\mathbb{P}(G) := \frac{|\{(x,y) \in G \times G \mid \langle x,y \rangle = G\}|}{|G|^2}$$

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where $\mathcal{M}$ is the set of maximal subgroups in G up to conjugacy.

2-generation of simple groups

Example.

Let $G = \mathrm{L}_2(13)$. Then maximal subgroups of G up to conjugacy are

Group	Class	Type
$13{:}6$	$\mathscr{C}_1$	P_1
D_{12}	$\mathscr{C}_2$	$\mathrm{GL}_1(13) \wr S_2$
D_{14}	$\mathscr{C}_3$	$\mathrm{GL}_1(13^2)$
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Thus,

$$\widehat{Q}(G) = \sum_{H \in \mathcal{M}} \frac{|H|}{|G|} = \frac{72}{1092} + \frac{12}{1092} + \frac{14}{1092} + \frac{12}{1092} = \frac{29}{273} < 1.$$

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Indeed, $\mathbb{P}(G) = 165/182$.

2-generation of simple groups

Theorem.

Every finite simple group is 2-generated.

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Problem.

Let $G = \langle x, y \rangle$.

- How abundant are such pairs (x, y)?
- What if we restrict $|x|$ and $|y|$?

Random generation of simple groups

Example.

Let $G = \mathrm{L}_2(q)$. Then maximal subgroups of G are among the following:

- P parabolic of index $q + 1$;
- $D_{q \pm 1}$;
- $\mathrm{L}_2(q_0)$ or $\mathrm{PGL}_2(q_0)$ (subfield subgroups);
- A_4, S_4, A_5.

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Note that there are at most $\log_2 \log_2 q$ subfields of $\mathbb{F}_q$. We have

$$\widehat{Q}(G) = (q+1)^{-1} + O(q^{-\frac{3}{2}} \log\log q) = O(q^{-1}).$$

Thus, $\widehat{Q}(G) \to 0$ as $q \to \infty$, and so $\mathbb{P}(G) \to 1$.

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Theorem.

Let (G_n) be any sequence of finite simple groups such that $|G_n| \to \infty$ with n. Then $\lim_{n\to\infty} \widehat{Q}(G_n) = 0$ and so $\lim_{n\to\infty} \mathbb{P}(G) = 1$.

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Theorem.

We have $\mathbb{P}(G) \geq 53/90$ for every finite simple group G, with the equality if and only if $G = A_6$.

(a, b)-generation

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Example.

- S_n is $(2, n)$-generated.
- A_n is $(3, n)$-generated if n is odd, and $(3, n-1)$-generated if n is even.
- D_{2n} is both $(2, 2)$-generated and $(2, n)$-generated.
- A $(2, 2)$-generated group is isomorphic to D_{2n}.

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Thus,

$$1 - \mathbb{P}_{a,b}(G) \leq \sum_{H \text{ maximal}} \frac{i_a(H)i_b(H)}{i_a(G)i_b(G)},$$

where $i_m(X)$ denotes the number of elements of order m in X.

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It follows that

$$\begin{aligned} 1 - \mathbb{P}_{2,3}(G) &\leq \sum_{H \text{ maximal}} \frac{i_2(H) i_3(H)}{i_2(G) i_3(G)} \\ &= \frac{338}{16562} \times 14 + \frac{14}{16562} + 0 + \frac{24}{16562} \times 14 \\ &= 45/91 < 1. \end{aligned}$$

$(2, 3)$-generation

Theorem (King, 2017).

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Conjecture.

Let G be a non-abelian finite simple group. Then one of the following cases occurs:

1. G is $(2,3)$-generated;
2. G is $(2,5)$-generated and G is one of the folloing groups:
   a. A_6, A_7, A_8;
   b. M_{11}, M_{22}, M_{23}, McL;
   c. $\mathrm{Sp}_4(2^f)$, $\mathrm{PSp}_4(3^f)$, $\mathrm{Sz}(q)$;
   d. $\mathrm{L}_2(9)$, $\mathrm{L}_3(4)$, $\mathrm{L}_4(2)$;
   e. $\mathrm{U}_3(5)$, $\mathrm{U}_4(2)$, $\mathrm{U}_4(3)$, $\mathrm{U}_5(2)$;
   f. $\mathrm{P\Omega}_8^+(2)$, $\mathrm{P\Omega}_8^+(3)$;
3. $G = \mathrm{U}_3(3)$ and G is $(2,7)$-generated.

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Bases

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- $b(G) = 1 \iff G$ has a regular orbit on Ω.

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- $G = D_{2n}$, $\Omega = \{1, \ldots, n\}$: $b(G) = 2$.
- $G = \mathrm{GL}(V)$, $\Omega = V$:
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- $G = \mathrm{GL}(V)$, $\Omega = V$:
  A subset of Ω is a base iff it contains a basis of V, so $b(G) = \dim V$.
- $G = \mathrm{PGL}(V)$, $d = \dim V > 1$, $\Omega = P(V)$: $b(G) = d + 1$
  Indeed, a base size set is $\{\langle v_1 \rangle, \dots, \langle v_d \rangle, \langle v_1 + \cdots + v_d \rangle\}$, where $v_1, \dots, v_d$ is a basis of V.

Non-standard groups

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Roughly speaking, an almost simple primitive group is called **standard** if

- $\mathrm{soc}(G) = A_n$ and G_α is primitive on $\{1, \dots, n\}$, or
- G is classical with $G_\alpha \cap \mathrm{soc}(G)$ reducible.

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Other almost simple primitive groups are called **non-standard**.

Cameron's conjecture

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Let G be a non-standard group. Then $b(G) \leq c$ for some constant c.

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For a positive integer c, let

$$\mathbb{P}(G,c) = \frac{|\{(\alpha_1,\ldots,\alpha_c) \in \Omega^c : \bigcap_{i=1}^{c} G_{\alpha_i} = 1\}|}{|\Omega|^c}$$

be the probability that a random c-tuple of points in Ω is a base for G.

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- $b(G) \leq c \iff \mathbb{P}(G,c) > 0$.
- A c-tuple is not a base if and only if it is fixed by some $x \in G$ of prime order.
- The probability of a random c-tuple is fixed by x is $\mathrm{fpr}(x)^c$, where

$$\mathrm{fpr}(x) = \frac{|C_\Omega(x)|}{|\Omega|} = \frac{|x^G \cap G_\alpha|}{|x^G|}$$

is the **fixed point ratio** of x.

Cameron's conjecture

From above, we have

$$\begin{aligned} 1 - \mathbb{P}(G, c) &\leq \sum_{x \in \mathcal{P}} \mathrm{fpr}(x)^c \\ &= \sum_{i=1}^{m} \mathrm{fpr}(x_i)^c |x_i^G| \\ &= \sum_{i=1}^{m} \left(\frac{|x_i^G \cap G_\alpha|}{|x_i^G|} \right)^c \cdot |x_i^G| =: \widehat{Q}(G, c), \end{aligned}$$

where $\mathcal{P}$ is the set of elements of prime order in G, and $\{x_1, \ldots, x_m\}$ are representatives of $\mathcal{P}$ up to G-conjugacy.

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where $\mathcal{P}$ is the set of elements of prime order in G, and $\{x_1, \dots, x_m\}$ are representatives of $\mathcal{P}$ up to G-conjugacy.

- $b(G) \leq c$ if $\widehat{Q}(G,c) < 1$.

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where $\mathcal{P}$ is the set of elements of prime order in G, and $\{x_1,\ldots,x_m\}$ are representatives of $\mathcal{P}$ up to G-conjugacy.

- $b(G) \leq c$ if $\widehat{Q}(G,c) < 1$.
- In particular, $b(G) \leq 2$ if

$$|G_\alpha|^2 \max_{1\neq x\in G_\alpha} |C_G(x)| = |G_\alpha|^2 \max_{\substack{x\in G_\alpha \\ |x| \text{ prime}}} |C_G(x)| < |G|.$$

A base-two example

Example.

Suppose $\mathrm{soc}(G) = \mathrm{L}_3^{\epsilon}(q)$ with $q = p \equiv \epsilon \pmod 3$ and $H = G_\alpha$ is of type $3^{1+2}.\mathrm{Sp}_2(3)$. Then $|H| \leq 432$ and

$$|C_G(x)| \leq \frac{|G|}{(q-1)(q^3-1)}$$

for all $x \in G$ of prime order (maximal if $\epsilon = +$ and x is unipotent with Jordan form $[J_2, J_1]$). This gives $b(G) = 2$ for all $q > 23$. When $q \leq 23$ we can also check using MAGMA that $b(G) = 2$.

Cameron's conjecture for exceptional groups

Theorem (Liebeck & Saxl, 1991).

Let G be a transitive almost simple exceptional group over $\mathbb{F}_q$. Then

$$\max_{1 \neq x \in G} \mathrm{fpr}(x) \leq \frac{4}{3q}.$$

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Let G be a transitive almost simple exceptional group over $\mathbb{F}_q$. Then

$$\max_{1\neq x\in G} \mathrm{fpr}(x) \leq \frac{4}{3q}.$$

If $\mathrm{soc}(G)$ exceptional then $|G| < q^{249}$ and so $b(G) \leq 500$ since

$$\begin{aligned}
\widehat{Q}(G,500) &= \sum_{i=1}^{m} \mathrm{fpr}(x_i)^{500}|x_i^G| \\
&\leq \left(\frac{4}{3q}\right)^{500} \sum_{i=1}^{m} |x_i^G| \\
&< \left(\frac{4}{3q}\right)^{500} |G| \\
&< \left(\frac{4}{3q}\right)^{500} q^{249} < \frac{1}{q}.
\end{aligned}$$

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Theorem (Burness, Liebeck & Shalev, 2009).

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Problem.

Determine exact base sizes for non-standard groups. In particular, classify those with $b(G) = 2$.

Saxl's base-two project

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Determine finite primitive groups G with $b(G) = 2$.

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- Affine: $G = V{:}H$. Then

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Determine pairs (V, H), where H is a finite group, V is a faithful irreducible $\mathbb{F}_p H$-module and H has a regular orbit on V.

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- Product type: no result.

Outline

Saxl graphs

Definition.

Let $G \leq \mathrm{Sym}(\Omega)$ be a base-two permutation group.
The **Saxl graph** $\Sigma(G)$: vertices Ω, $\alpha \sim \beta \iff \{\alpha, \beta\}$ is a base.

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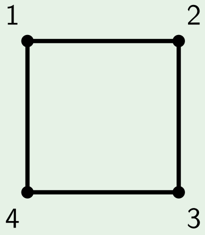
Example.

- $G = D_8$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) \cong C_4$.



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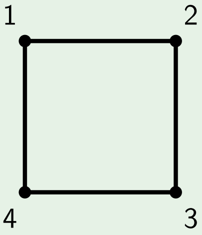
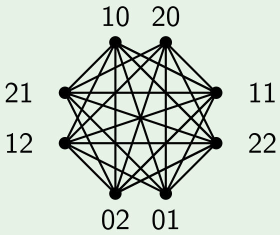


- $G = D_{10}$, $\Omega = \{1, 2, 3, 4, 5\}$: $\Sigma(G) \cong K_5$.

Some further examples

Example.

- Let $G = \mathrm{GL}_2(q)$ and $\Omega = \mathbb{F}_q^2 \setminus \{0\}$. Then $\alpha \sim \beta$ iff $\{\alpha, \beta\}$ is linearly independent. Thus, $\Sigma(G)$ is **complete multipartite** with $q+1$ parts of size $q-1$. For example, when $q = 3$ we have $\Sigma(G) \cong K_8 - 4K_2$.



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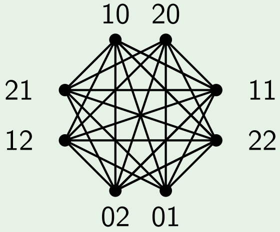

- Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$. Then α and β form a base iff they share a common 1-space. Hence, $\Sigma(G) \cong J(q+1, 2)$ is a **Johnson graph**.

First observations

Proposition.

Suppose G is transitive with $b(G) = 2$ and $\Sigma(G)$ is the Saxl graph of G.

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6. $\Sigma(G)$ has valency $r|G_\alpha|$, where r is the number of regular suborbits.
7. If $K \leq G \leq \mathrm{Sym}(\Omega)$, then $\Sigma(G)$ is a subgraph of $\Sigma(K)$.

Probabilistic methods

Let $G \leq \mathrm{Sym}(\Omega)$ be a base-two transitive permutation group with degree n. Let $\mathrm{val}(G)$ be the valency of $\Sigma(G)$. Set

$$Q(G,2) := 1 - \mathbb{P}(G,2) = \frac{|\{(\alpha,\beta) \in \Omega^2 : G_{\alpha\beta} \neq 1\}|}{n^2} = 1 - \frac{\mathrm{val}(G)}{n}.$$

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Lemma.

If $Q(G,2) < \frac{1}{t} \leq \frac{1}{2}$, then $\Sigma(G)$ has all of the following properties:

- Any t vertices in $\Sigma(G)$ has a common neighbour;
- $\Sigma(G)$ has diameter at most 2;
- $\Sigma(G)$ has clique number at least $t+1$;
- $\Sigma(G)$ is Hamiltonian.

Probabilistic methods

Recall that

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- $Q(G,2) < \frac{1}{t} \implies \Sigma(G)$ satisfies all statements in the above lemma.

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We have

$$Q(G,2) \leq \sum_{i=1}^{m} \frac{|x_i \cap H|^2}{|x_i^G|} = \widehat{Q}(G,2),$$

where $H = G_\alpha$ and $\{x_1, \ldots, x_m\}$ is the set of representatives of G-conjugacy classes of prime-ordered elements in G.

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Conjecture (Burness & Giudici, 2020).

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Example.

Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$. Then $\Sigma(G) \cong J(q+1,2)$ has valency $2(q+1)$ and thus

$$Q(G,2) = 1 - \frac{\mathrm{val}(G)}{n} = 1 - \frac{4(q-1)}{q(q+1)} \to 1 \text{ as } q \to \infty$$

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but $\Sigma(G) \cong J(q+1,2)$ still satisfies the Burness-Giudici Conjecture.

An evidence

Example.

Suppose $\mathrm{soc}(G) = \mathrm{L}_3^{\epsilon}(q)$ with $q = p \equiv \epsilon \pmod 3$ and $H = G_{\alpha}$ is of type $3^{1+2}.\mathrm{Sp}_2(3)$. Then $|H| \leq 432$ and

$$|C_G(x)| \leq \frac{|G|}{(q-1)(q^3-1)}$$

for all $x \in G$ of prime order (maximal if $\epsilon = +$ and x is unipotent with Jordan form $[J_2, J_1]$). This gives $\widehat{Q}(G,2) < 8q^{-1}$ for all $q > 23$. When $q \leq 23$ we can also check using MAGMA that the conjecture holds.

More evidences

- All primitive groups with degree $n \leq 4095$

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- Asymptotic results for many diagonal and twisted wreath type groups **(Fawcett, 2013/21)**

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Clique number: maximal size of complete subgraph.

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Theorem (Burness & H, in progress).

Let G be an almost simple primitive group with soluble stabiliser H. Suppose $G_0 \neq \mathrm{L}_2(q)$. Then one of the following holds:

- $\Sigma(G)$ has clique number at least 5.
- $G = A_5$ and $H = S_3$, the clique number of $\Sigma(G)$ is 4.

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Theorem (Burness & H, in progress).

Let $\alpha(G)$ be the independence number of $\Sigma(G)$. Then almost simple transitive groups G with $\alpha(G) = 2$ or 3 are known.

Problems

- **Connectedness.** Characterise transitive groups with connected Saxl graph. G quasiprimitive?
- **Automorphisms.**
  - When do we have $G = \mathrm{Aut}(\Sigma(G))$?
  - When is $\Sigma(G)$ Cayley?
- **Cycles.** Eulerian cycle? Hamiltonian cycle?
- **Unique regular suborbit.** Can we classify groups with $r = 1$?
- **Other invariants.** Chromatic numbers? Spectrum?

Thank you for your attention!

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