Transitive subgroups of primitive groups

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Group factorisations

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The expression $G = HK$ is called a **factorisation** of G.

Factorisations of almost simple groups (thanks Cheryl!)

Theorem. The factorisations of almost simple groups are classified.

Contributed earlier by Giudici, Liebeck, Praeger, Saxl...

Recent advances:

- H or K has at least two insoluble composition factors:

  Li & Xia, 2019, JAlg

- Both H and K are insoluble (for classical groups):

  Li, Wang & Xia, 2024, arXiv (102 pages)

- At least one of H and K is soluble:

  Li & Xia, 2022, AMS Memoirs

  Burness & Li, 2021, Adv

  Feng, Li, Li, Wang, Xia & Zou, 2024, arXiv (final step)

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- Almost simple (AS) ✓
- Diagonal type (HS & SD)
- Product type (HC, CD & PA)
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Problem. Classify the **regular** subgroups and the **soluble** transitive subgroups of primitive groups of **diagonal type**, up to conjugacy.

Holomorph simple groups (HS)

Let T be a non-abelian finite simple group and let

$$G = \mathrm{Hol}(T) = T{:}\mathrm{Aut}(T) = T^2.\mathrm{Out}(T)$$

be the **holomorph** of T. So $G \leqslant \mathrm{Sym}(T)$ is primitive.

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$T = A_{q+1}$, $B \cong (A_{q-2} \times \mathrm{PSL}_2(q)).2$ associated to $S_{q+1} = S_{q-2}\,\mathrm{PGL}_2(q)$.

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Conversely, if $T = HK$, then there exists a transitive subgroup of G isomorphic to $H \times K$.

The key step

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If B is regular, then there exists $N \trianglelefteqslant H$ and $M \trianglelefteqslant K$ such that

$$H/N \cong K/M \text{ and } |H : N| = |HK : T||H \cap K|. \qquad (\star\star)$$

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It is NOT easy to determine the factorisations satisfying $(\star)$ and $(\star\star)$.

Suppose $T = A_n$ and $L \in \{A_n, S_n\}$. If $L = HK$ then H is transitive on m-subsets and $A_{n-m} \trianglelefteqslant K \leqslant S_{n-m} \times S_m$, with $1 \leqslant m \leqslant 5$.

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Example

Assume $(\star)$ and $(\star\star)$, with $m = 1$. Then H is regular on $[n]$, and

- the Sylow 2-subgroups of H are not cyclic $\implies (L, K) = (A_n, A_{n-1})$.
- otherwise $(L, K) = (S_n, S_{n-1})$.

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Example

Assume $(\star)$ and $(\star\star)$, with $m = 3$. Then (basically) $n = q + 1$ and

- q even, $L = A_n$, $H = \mathrm{PSL}_2(q)$, $K = A_{n-3}$.
- $q \equiv 3 \pmod 4$, $L = A_n$, $H = \mathrm{PSL}_2(q)$, $K = S_{n-3}$.
- $\sqrt{q} \equiv 3 \pmod 4$, $L = A_n$, $H = \mathrm{PSL}_2(q).2_3$, $K \in \{A_{n-3}, S_{n-3}\}$.
- $L = S_n$, $H = \mathrm{PGL}_2(q)$, $K = S_{n-3}$.
- $q \equiv 1 \pmod 4$, $L = S_n$, $H = \mathrm{PGL}_2(q)$, $K = A_{n-3} \times S_2$.
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Main results on holomorph simple groups

Theorem (H & Wang, 2025+)

For every finite simple group T, the **soluble transitive** subgroups of $\mathrm{Hol}(T)$ are determined, up to conjugacy.

Theorem (H & Wang, 2025+)

For every finite simple group T, the **regular** subgroups of $\mathrm{Hol}(T)$ are determined, up to conjugacy.

Application: Hopf-Galois structure and skew braces

For a finite group G, TFAE:

- B is isomorphic to a regular subgroup of $\mathrm{Hol}(G)$;
- There exists a **Hopf-Galois structure** of type B on any Galois G-extension.
- $G \cong (X, +)$ and $B \cong (X, \circ)$ for some **skew brace** $(X, +, \circ)$.

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Theorem (H & Wang, 2025+)

The types of Hopf-Galois structures are determined on any Galois extension whose Galois group is finite simple.

Theorem (H & Wang, 2025+)

The skew braces with finite simple additive groups are classified.

General diagonal type groups

Let $k \geqslant 2$, T be a non-abelian simple group, and let

$$X := \{(t, \ldots, t) : t \in T\} \leqslant T^k.$$

Then $T^k \leqslant \mathrm{Sym}(\Omega)$ with $\Omega = T^k/X$ of size $|T|^{k-1}$.

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A group $G \leqslant \mathrm{Sym}(\Omega)$ is of **diagonal type** if

$$T^k \trianglelefteqslant G \leqslant N_{\mathrm{Sym}(\Omega)}(T^k) \cong T^k.(\mathrm{Out}(T) \times S_k).$$

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$$P_G \text{ is primitive (SD), or } k = 2 \text{ and } P_G = A_2 = 1 \text{ (HS).}$$

Example. $\mathrm{Hol}(T) = T^2.\,\mathrm{Out}(T)$.

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Example. $B = T^{k-2} \times H \times K$, with $H \times K$ a regular subgroup of $\mathrm{Hol}(T)$.

The main result

Theorem (H & Wang, 2025+). The regular and the soluble transitive subgroups of diagonal type groups are determined, up to conjugacy.

Danke schön!