Base-two primitive permutation groups and their Saxl graphs

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- $G = \mathrm{GL}(V)$, $\Omega = V$ and Δ contains a basis of V.

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- G is primitive iff n is a prime.

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- CFSG is used. Partial results.

Saxl graphs

Definition (Burness & Giudici, 2020)

Let $G \leqslant \mathrm{Sym}(\Omega)$. Then the Saxl graph $\Sigma(G)$ is a graph with

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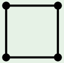
$G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) = \boldsymbol{C_4} =$ 

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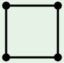
$G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) = \boldsymbol{C}_4 =$ 

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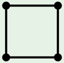
$G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) = \boldsymbol{C_4} =$ 

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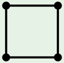
$G = D_8 = \langle (1234), (24) \rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) = \boldsymbol{C_4} =$ 

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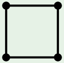
$G = D_8 = \langle(1234), (24)\rangle$, $\Omega = \{1, 2, 3, 4\}$: $\Sigma(G) = \boldsymbol{C}_4 =$ 

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- $\Sigma(G)$ is vertex-transitive;
- $\Sigma(G)$ is connected if G is primitive;
- $\Sigma(G)$ has valency $r|G_\alpha|$, where r is the number of regular G_α-orbits.

A further example

Let $G = \mathrm{PGL}_2(q)$ and Ω be the set of distinct pairs of 1-spaces in $\mathbb{F}_q^2$.

- $G_\alpha = D_{2(q-1)}$;
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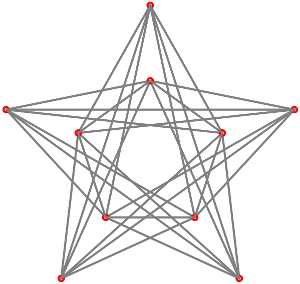
For example, when $q=4$ we have the complement of the Petersen graph.



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Theorem (Chen & Du, 2021+; Burness & H, 2021+)

$\mathrm{soc}(G) = \mathrm{L}_2(q) \implies \Sigma(G)$ has the common neighbour property.

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Theorem (Burness & H, 2021+)

$G \in \mathcal{G} \implies \Sigma(G)$ has the common neighbour property.

Probabilistic methods

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If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $Q(G) \to 1$ as $q \to \infty$. But $\Sigma(G) = J(q+1,2)$ still has the common neighbour property.

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$b(L \wr P) = 2 \iff r(L) \geqslant$ the *distinguishing number* of P.

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- If $G < L \wr P$, then when do we have $b(G) = 2$?

Thank you!