Permutation groups of rank three

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Group Theory in Florence IV

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joint with C.H. Li and Y.Z. Zhu

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Example. $G = \mathrm{GL}_n(3)$ and $\Omega = \mathbb{F}_3^n \setminus \{0\}$.

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Note. Every normal subgroup of $\mathrm{GL}_n(3)$ is transitive or semiregular (such a group is said to be **semiprimitive**).

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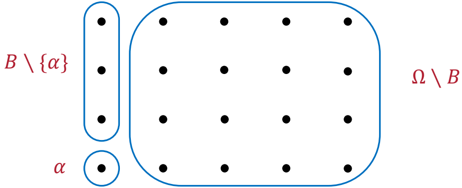

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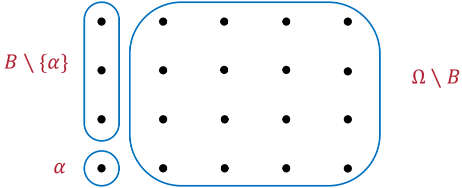

- The induced groups $G^{\mathcal{B}}$ (on $\mathcal{B}$) and G_B^B (on B) are 2-transitive.

Imprimitive rank three groups

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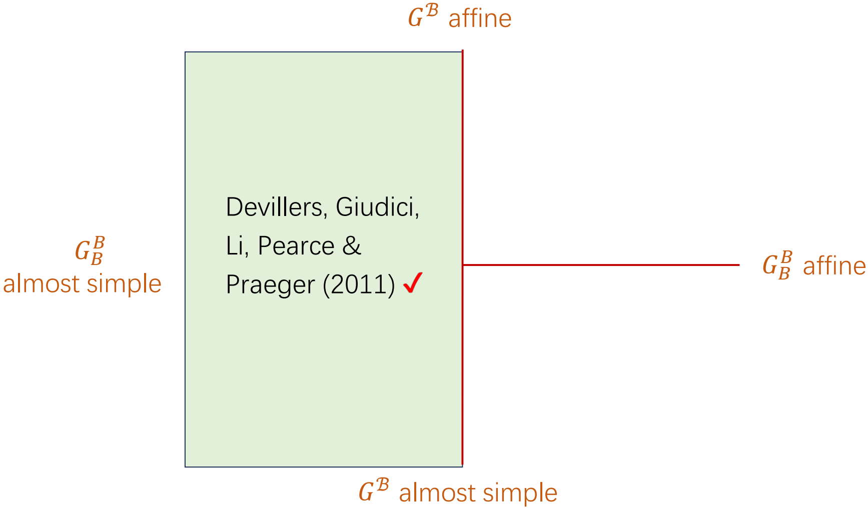

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- $K_{(B)}$ is transitive on B';
- $K_{(B)} \neq 1$ is intransitive on B', and G has an elementary abelian self-centralising normal subgroup,

where $K = G_{(\mathcal{B})}$ and $B, B' \in \mathcal{B}$.

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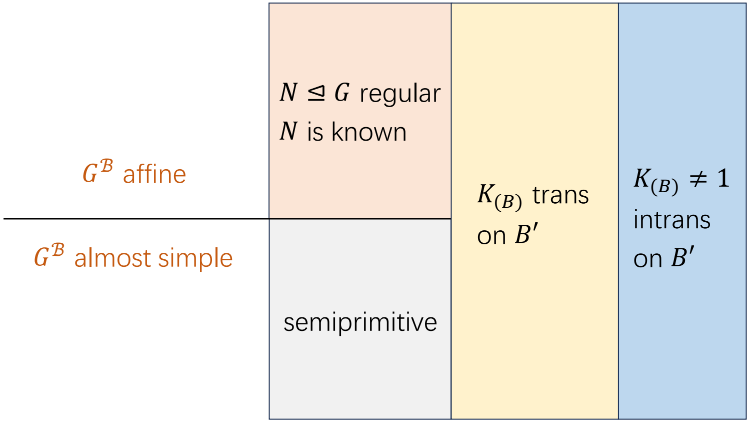
Imprimitive rank three groups (G_B^B affine)



Semiprimitive linear groups

Let $X = \mathrm{GL}_n(q) \leqslant \mathrm{Sym}(\Gamma)$ with $\Gamma = \mathbb{F}_q^n \setminus \{0\}$.

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If $|Z(X) : C| = 2$ (so $G = 2.\mathrm{PGL}_n(q)$), then $\mathrm{rank}(G) = 3$ with

- $|B| = 2$ and $G_B^B = S_2$;
- $\mathcal{B} = \{1\text{-spaces}\}$ and $G^{\mathcal{B}} = \mathrm{PGL}_n(q)$.

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- $(G, G_\alpha) = (3.S_6, S_5)$ or $(2.\mathrm{M}_{12}, \mathrm{M}_{11})$.
- Ω is the set of C-orbits on $\mathbb{F}_q^n \setminus \{0\}$ for $C \leqslant Z(\mathrm{GL}_d(q))$, and

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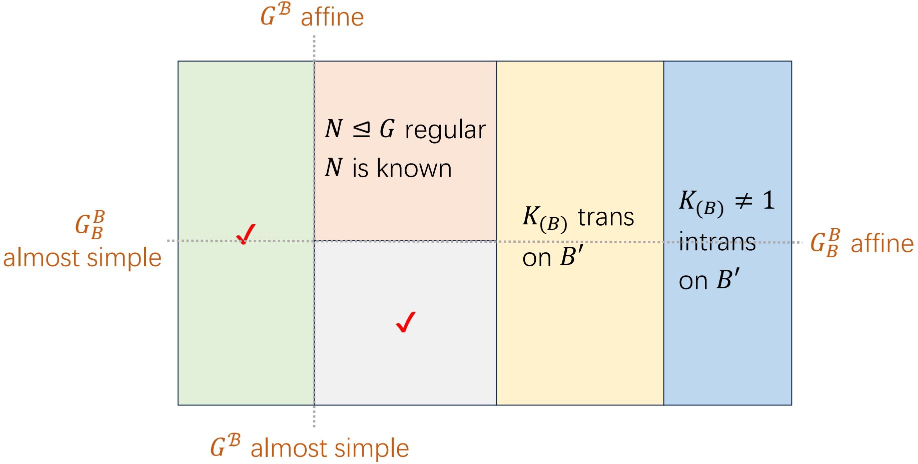
Li, Yi & Zhu (last Sunday): such G is determined.

Summary



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$N \trianglelefteq G$ regular
N is known
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$K_{(B)}$ trans on B'
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