Base-two primitive permutation groups

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4 August 2022

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Some applications:

- Extremely primitive groups
- $3/2$-transitive groups
- Graphs defined on groups (e.g. the intersecting graph)

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- Progress where $G < L \wr P$ **(Burness & H, 2022+)**

Probabilistic methods

Consider

$$Q(G) = \frac{|\{(\alpha, \beta) \in \Omega^2 : G_\alpha \cap G_\beta \neq 1\}|}{|\Omega|^2},$$

the probability that a random pair in Ω is **not** a base for G.

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Burness & Giudici, 2020: **Saxl graph** $\Sigma(G)$:

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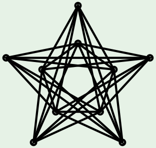
  For example, when $q = 4$ we have the complement of the Petersen.



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If $G = \mathrm{PGL}_2(q)$ and $G_\alpha = D_{2(q-1)}$, then $\Sigma(G) = J(q+1, 2)$ has the common neighbour property, though $Q(G) \to 1$ as $q \to \infty$.

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- The base-two project

Thank you!