Correct discretization of process noise

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When discretizing a continuous state space model as shown in Eq. (1), extra care must be shown when discretizing the white noise term $\mathbf{Ew}$. The discrete counterpart of Eq. (1) is shown in Eq. (2).

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} + \mathbf{Ew} \\ E\{\mathbf{w}\mathbf{w}^\top\} &= \mathbf{Q}\delta(0)\end{aligned} \tag{1}$$

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A_d}\mathbf{x}[k] + \mathbf{B_d}\mathbf{u}[k] + \mathbf{w_d}[k] \\ E\{\mathbf{w_d}\mathbf{w_d}^\top\} &= \mathbf{Q_d}\end{aligned} \tag{2}$$

The $\mathbf{A_d}$ and $\mathbf{B_d}$ term can as previously be calculated using exact discretization. In MATLAB this can simply be done as follows where Δt is the discretization time-step.

$$[\mathbf{A_d} \quad \mathbf{B_d}] = \mathrm{c2d}(\mathbf{A}, \mathbf{B}, \Delta t) \tag{3}$$

It is not possible to discretize $\mathbf{E}$ using exact discretization. This is caused by exact discretization assuming the input constant over the time-interval Δt. This holds for the input, $\mathbf{u}$, since $\mathbf{u}$ is driven by a controller with the same time-step. But for the process noise $\mathbf{w}$, this does not hold. Instead van Loans method has to be utilized. It is important to note that after the discretization the $\mathbf{E}$ term is no longer present, as shown in equation 2. The information from $\mathbf{E}$ is instead contained in the new $\mathbf{Q_d}$, which might be of a different dimension than $\mathbf{Q}$. $\mathbf{w_d}$ will have the same size as the rows or columns in the square matrix $\mathbf{Q_d}$. See the example below for further details.

1 Van Loans method

1. Generate a matrix M as follows. In MATLAB remember to use a zero matrix of correct dimension. Δt is the discretization time-step.

$$\mathbf{M} = \left[\begin{array}{c:c} -\mathbf{A} & \mathbf{EQE}^\top \\ \hdashline \mathbf{0} & \mathbf{A}^\top \end{array}\right]\Delta t \tag{4}$$

$$\tag{5}$$

2. Calculate $\mathbf{N} = e^{\mathbf{M}}$. This can in MATLAB simply be done as follows

$$\mathbf{N} = \mathrm{expm}(\mathbf{M}) \tag{6}$$

$\mathbf{N}$ will have the following form

$$\mathbf{N} = \left[\begin{array}{c:c} \cdots & \mathbf{A_d}^{-1}\mathbf{Q_d} \\ \hdashline 0 & \mathbf{A_d}^{\top} \end{array}\right] \tag{7}$$

3. Extract $\mathbf{A_d}$, which will be equal to $\mathbf{A_d}$ from exact discretization.

$$\mathbf{A_d} = \text{“transpose of lower-right partition of } \mathbf{N}\text{”} \tag{8}$$

4. Extract $\mathbf{Q_d}$

$$\mathbf{A_d}^{-1}\mathbf{Q_d} = \text{“upper-right partition of } \mathbf{N}\text{”} \tag{9}$$
$$\mathbf{Q_d} = \mathbf{A_d}\text{“upper-right partition of } \mathbf{N}\text{”} \tag{10}$$

[Example on next page]

2 Example

The following system with a scalar input u that is affected by a scalar Gaussian white noise process is to be discretized with a time-step $\Delta t = 0.1$.

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \tag{11}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{E}w \tag{12}$$

$$E\{w^2\} = q = 4$$

In discrete form this system can be written as

$$\mathbf{x}[k+1] = \mathbf{A_d}\mathbf{x}[k] + \mathbf{B_d}u[k] + \mathbf{w_d}[k] \tag{13}$$

$$E\{\mathbf{w_d}\mathbf{w_d}^\top\} = \mathbf{Q_d}$$

We have to calculate $\mathbf{A_d}$, $\mathbf{B_d}$, and $\mathbf{Q_d}$. $\mathbf{A_d}$ and $\mathbf{B_d}$ can be calculated using exact discretization in MATLAB

$$[\mathbf{A_d} \quad \mathbf{B_d}] = \text{c2d}(\mathbf{A}, \mathbf{B}, 0.1) \tag{14}$$

$$\mathbf{A_d} = \begin{bmatrix} 0.9 & 0.09 \\ 0 & 0.9 \end{bmatrix} \tag{15}$$

$$\mathbf{B_d} = \begin{bmatrix} 0.005 \\ 0.09 \end{bmatrix} \tag{16}$$

The $\mathbf{Q_d}$ is calculated using van Loans method.

1. Generate the M matrix

$$\mathbf{M} = \left[\begin{array}{c:c} -\mathbf{A} & \mathbf{E}\mathbf{Q}\mathbf{E}^\top \\ \hdashline \mathbf{0} & \mathbf{A}^\top \end{array}\right] \Delta t \tag{17}$$

$$\mathbf{M} = \left[\begin{array}{cc:cc} 0.1 & -0.1 & 0.004 & 0.036 \\ 0 & 0.1 & 0.036 & 0.32 \\ \hdashline 0 & 0 & -0.1 & 0 \\ 0 & 0 & 0.1 & -0.1 \end{array}\right] \tag{18}$$

2. Calculate $\mathbf{N} = e^{\mathbf{M}}$.

$$\mathbf{N} = \text{expm}(\mathbf{M}) \tag{19}$$

$$\mathbf{N} = \left[\begin{array}{c:c} \cdots & \mathbf{A_d}^{-1}\mathbf{Q_d} \\ \hdashline 0 & \mathbf{A_d}^\top \end{array}\right] \tag{20}$$

$$\mathbf{N} = \left[\begin{array}{cc:cc} 1.1 & -0.11 & 0.0033 & 0.019 \\ 0 & 1.1 & 0.052 & 0.32 \\ \hdashline 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0.09 & 0.9 \end{array}\right] \tag{21}$$

3. Extract $\mathbf{A_d}$, which is equal to $\mathbf{A_d}$ from exact discretization.

$$\mathbf{A_d} = \mathbf{N}(3:4, 3:4)^\top \tag{22}$$
$$\mathbf{A_d} = \begin{bmatrix} 0.9 & 0.09 \\ 0 & 0.9 \end{bmatrix} \tag{23}$$
$$\tag{24}$$

4. Extract $\mathbf{Q_d}$

$$\mathbf{Q_d} = \mathbf{A_d}\mathbf{N}(1:2, 3:4) \tag{25}$$
$$\mathbf{Q_d} = \begin{bmatrix} 0.0077 & 0.0468 \\ 0.0468 & 0.2937 \end{bmatrix} \tag{26}$$

Note that $\mathbf{Q_d}$ is noe a 2x2 matrix. This means that the new discrete time disturbance $\mathbf{w_d[k]}$ is a 2x1 vector instead of a scalar.