Price Competition and Assortment Display in Online Marketplace

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Abstract

Online platforms have been expanding the seller base to widen their product assortment to match the individual preferences of consumers. Nevertheless, the increasing number of sellers leads to intensified competition and results in sellers setting lower prices for the products. Thus, it is unclear whether the current practice employed by most platforms, i.e., displaying all the sellers to the entire customer base, maximizes platform revenue. Motivated by the unique setting of Airbnb, we consider a game theoretical setup in which each seller on the platform provides a single-unit product and competes with one another on price. We investigate sellers' optimal pricing decisions and the platform's optimal assortment display policy, which is characterized by the partitioning of products and traffic assigned to each partition. We find that the platform should display the entire assortment to all the customers when demand is sufficiently high. Moreover, we propose a tabularization algorithm and a mixed-integer programming formulation to effectively solve for the sellers' and the platform's optimal decisions. Additionally, we introduce two fairness definitions for the display policy, namely (α, δ)-fairness and envy level, to gauge how the requirements on the closeness of the attractiveness of each partition and the traffic allocated to each partition affect total platform revenue. Using data from Airbnb, we present a case study to illustrate how our model framework can be applied in practice. Finally, we extend the case in which each seller supplies a distinct product with inventory size of one by considering scenarios in which each product has more than one unit.

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1 Introduction

It has become a norm for online platforms to constantly widen their seller base. For example, Airbnb now offers 5.6 million listings worldwide[1]; Amazon has 6.3 million total sellers worldwide, 1.5 million of whom are active[2]. Such rapid expansion of sellers is partly due to the network nature of platforms, yet it also reflects the platforms' effort to satisfy the heterogeneous tastes of their customers with a constellation of products. Unlike brick-and-mortar retailers whose product assortment is limited by shelf space, online platforms can increase the number of sellers and the size of the product assortment almost indefinitely, thus enabling platforms that are already large to further expand their product variety.

Although a large seller base increases the likelihood the consumer will find the preferred variety, it may also jeopardize platform revenue because sellers would set lower prices for their products in response to the resulting heightened competition. Thus, it is unclear whether the current practice employed by most platforms, i.e., displaying all the products to the entire customer base, actually maximizes platform revenue. In this paper, we focus on a unique setting in which each seller provides a distinct product with one-unit inventory per time period (e.g., Airbnb, eBay), we contribute to this growing discussion in the literature on assortment optimization by answering the following questions: (i) how does competition affect the seller's pricing decision when each seller supplies one unit of a distinct product, (ii) when would showing different subsets of products to different subsets of customers generate higher platform revenue than that generated by the current practice and (iii) how to design a fair assortment display policy and what is the associated cost of implementing such a policy.

To answer the above questions, we construct a game-theoretical model in which the platform decides whether and how each seller should be grouped into different partitions and how much traffic each partition should receive to maximize revenue. Furthermore, each seller supplies a distinct product with one-unit inventory and sets the price of the product in response to other sellers' prices within the same partition. Under this setup, we theoretically demonstrate that when a platform faces sufficient demand, it is optimal to display the entire product assortment. When the platform demand is moderate, however, the derivation of the equilibrium price requires solving a system of nonlinear equations, which can be computationally infeasible because the number of display policies increases rapidly with the number of sellers. To derive the equilibrium price for each seller and ultimately prescribe the display policy, we propose a novel approach that is able to calculate the equilibrium price under flexible display constraints. Specifically, we first discretize and tabularize the left- and right-hand sides of the first-order conditions (FOCs) of seller's pricing decision. Incorporating the outputs from the tabularization procedure, we rewrite the platform's problem as a mixed integer programming (MIP) model. The MIP framework is able to effectively solve for the optimal display policy and can easily be adjusted to include new constraints that reflect restrictions on the display policy.

[1]https://news.airbnb.com/about-us/
[2]https://www.marketplacepulse.com/amazon/number-of-sellers

Table 1: Summary of Optimal Display Policies Under Different Settings and Demand Levels

	Demand/Supply Ratio		
	Very High	Moderate	Very Low
Unit Inventory	**Display Everything**	**MIP**†	**Multiple Partitions**
Limited Inventory	Display Everything	MIP	Multiple Partitions
Infinite Inventory	Display Subset	Display Subset	Display Single Item

† MIP indicates that the specific display policy is inconclusive and needs to be advised by the solution of the MIP problem. The optimal display strategies under the remaining market conditions are known, yet the specific partitions of sellers and customers still require solving the MIP problem.

Leveraging our game-theoretical model and the MIP framework, we summarize the optimal display policy under different market conditions in the first row of Table 1. Similar to Heese and Martínez-de Albéniz (2018), we note that it is sometimes economically sensible to display only a subset of products to the entire customer base. While such policy might be practical in offline retail settings, in which the limited shelf space naturally prevents the store from displaying all the products, this display policy essentially denies a proportion of the listing owners the ability to join the platform and could spur concerns over an unfair business environment on online marketplaces. To evaluate how fair a given partition is, we first introduce (α, δ)-fairness, defined as the closeness in the attractiveness of each partition and the traffic allocated to each partition. We integrate such fairness constraints into the MIP framework to provide the optimal partition policy that maximizes total revenue given a combination of α and δ. Additionally, we define the 'envy level' as a seller's revenue difference when moving to other partitions as an alternative fairness measure. We showcase how the upper bound on a seller's envy level is determined by the (α, δ) combination.

Additionally, we present a case study that draws transaction data from Airbnb to showcase how our framework can be applied in practice. Specifically, we estimate the quality of each individual listing by minimizing the squared discrepancy between each listing's rental probability and the realized demand outcome. We then conduct counterfactual analysis under different demand scenarios to recommend the optimal display policy for the platform. Furthermore, we calculate the revenue loss when the platform shifts from the $(0, 0)$-fair display, i.e., the unrestricted case, to $(0, 1)$-fair, in which each partition receives equal demand. We find that the platform incurs less than a 20% loss in revenue when the demand level is low or moderate. Nevertheless, the revenue gap ceases to exist when the demand level is high, as the optimal policy under the unrestricted case is to display all the sellers to all customers, which is also $(0, 1)$-fair.

Finally, we extend the case in which each seller supplies a distinct product with inventory of size one by considering two scenarios where the product each seller provides has (i) limited inventory or (ii) infinite inventory. We formulate the objective function for each seller and demonstrate that when facing sufficient demand, it is still optimal to display all the products when the seller possesses multiple units of the product, but the platform should display only the highest-quality item when each seller has infinite units of each product and the number of sellers is large. The insights for optimal display policies under different settings are summarized in Table 1.

Contribution Our paper makes four contributions to the literature. First, we provide a theoretical justification for the 'display everything' practice employed by large platforms such as Airbnb. Second, we propose a tabularization procedure and translate the optimal product partition and traffic allocation problem of the platform, which is combinatorial by nature and computationally burdensome, into a tractable MIP formulation. The MIP formulation allows us to derive the optimal display policy for the platform under arbitrary market conditions characterized by the demand level and size of the platform. Third, we define two notions of fairness, and through numerical analysis and a case study using transaction data from Airbnb, we demonstrate the cost of implementing a fair policy. Finally, we derive the optimal display policy when each seller has more than one unit of the product when the platform is sufficiently large. We also showcase that our MIP is able to cope with such non-unit scenarios, which allows our insights to be applied to other platforms such as Amazon and Alibaba where each product has more than one unit.

The remainder of the paper is organized as follows. We review the related literature in Section 2. In Section 3, we present our model framework. In Section 4, we develop an MIP model to characterize the price equilibrium and prescribe the optimal display policy. Then, in Section 5, we fit our model to an Airbnb dataset and conduct counterfactual analysis to derive the optimal display policy under different market conditions. Furthermore, we extend our single-unit inventory model to the case in which each seller holds multiple units of inventory in Section 6. Finally, we conclude the paper in Section 7.

2 Literature Review

Our work is related to two streams of literature, namely, assortment optimization and fair resource allocation.

The field of assortment optimization has been active in recent decades, including both dynamic assortment planning and static assortment display. Dynamic assortment optimization focuses on optimal product planning over time under various customer behaviors; see Ferreira and Goh (2021), Bernstein et al. (2015), Caro et al. (2014), Bernstein and Martínez-de Albéniz (2017), **?**), and Wagner and Martínez-de Albéniz (2020).

Static assortment display only focuses on the optimal assortment policy for a single period, and Kök et al. (2008) provide a comprehensive review of the literature in this field. Talluri and Van Ryzin (2004) show that under the MNL demand model, the optimal assortment set is that with the highest profit margins. However, this structural property no longer holds in our setting, as the limited inventory size could lead to unsatisfied customers, which a larger assortment could alleviate. Under the static setup, the interplay between assortment and pricing has also been studied extensively. Anderson et al. (1992) show that it is optimal for the retailer to use the same markup for all products, although this result no longer holds when price sensitivities differ across products (e.g., Wang 2012 or Gallego and Wang 2014). Additionally, Besbes and Sauré (2016) consider the game-theoretic setup where two retailers compete on assortment and pricing and show that competition would lead to larger assortment breadth. By contrast, we model a similar process as a two-stage game in which the platform decides the assortment strategy. Perhaps the work that relates the most to ours is Heese and Martínez-de Albéniz (2018), where they also model the assortment planning process as a two-stage game, which is between a retailer and its upstream manufacturers, and study the optimal decision for the assortment breadth. However, Heese and Martínez-de Albéniz (2018) mainly analyze fixed subset display policies instead of partition policies, as in our paper; moreover, our subgames also differ substantially since our listing owners face different submarkets and compete for revenue, while in their paper, the upstream manufacturers compete for position in the assortment.

Our paper discusses the optimal display policy under a number of fairness constraints and is thus related to previous literature that investigates the tradeoff between fairness and efficiency. Most of the previous work that investigates the optimal decisions under a number of fairness constraints adopts the notion of α-fairness proposed in Mo and Walrand (2000). Specifically, Mo and Walrand (2000) provide a generalized definition of fairness using a parametric function, for which higher values of α indicate a fairer allocation, and the parameters $\alpha = 0, 1$, and ∞ correspond to the utilitarian, proportional and max-min fairness, respectively. Using the definition of α-fairness, Bertsimas et al. (2011, 2012) study the efficiency loss incurred to achieve a certain fairness level in offline resource allocation problems; McCoy and Lee (2014) quantify how equity and efficiency interact for humanitarian and health delivery supply chains; and Bateni et al. (2018) investigate an online resource allocation problem on a marketplace and provide empirical validation of the proposed allocation algorithm. Our work also uses the notion of α-fairness, but our notion of α-fairness departs from the traditional definition and requires instead the relative ratio of incoming traffic and the total competitiveness between any two partitions to remain within a certain range in a way similar to Cohen et al. (2019).

3 Model Framework

3.1 Price Competition under Full Display

We first consider a simple setup where a platform (e.g., Airbnb) hosts N sellers (e.g., listing owners) and displays everyone by default. Each seller i supplies a single-unit product with quality a_i on the platform and observes the qualities of all the other competitors $\boldsymbol{a_{-i}}$. We denote the expected number of customer arrivals by M, which we assume is common knowledge to the platform and the sellers. Observing $(a_i, \boldsymbol{a_{-i}})$ and M, every seller i simultaneously decides the price for its product to maximize revenue. The pricing game is characterized as follows. We assume that M customers arrive simultaneously and observe the full product assortment. The utility that customer m obtains from product i is $u_{im} = a_i - \beta p_i + \epsilon_{im}$, where a_i is the quality of the product, β is the coefficient for price p_i and ϵ_{im} follows a $Gumbel(0, 1)$ distribution that captures customers' heterogeneous tastes. In this way, customers select products according to the multinomial logit (MNL) choice model. Notably, as each product has only unit inventory, if a product is requested by multiple customers, the seller will accept the offer from only one buyer, and the remaining customers will be rejected and unable to select another product. In this setting, seller i's payoff function can be written as:

$$\max_{p_i} \Pi_i(M, \boldsymbol{p_{-i}}) = p_i \left(1 - \left(1 - \frac{\exp{(a_i - \beta p_i)}}{1 + \sum_{j=1}^{N} \exp{(a_j - \beta p_j)}}\right)^M\right). \tag{1}$$

Equation (1) states that seller i's expected revenue is the product of the price and probability that product i is selected by at least one customer, which equals one minus the probability that all M customers choose options other than i. The utility of the outside option is normalized to one as in the classic MNL setup.

For brevity, we first denote the attractiveness of listing i as $v_i = \exp{(a_i - \beta p_i)}$ and write the probability that a customer chooses product i as $q_i = v_i/(1+\sum_{j=1}^{N} v_j)$. In this way, the price equilibrium, $p_i^*, i = 1, ..., N$, can be characterized by the FOC of Equation (1) as:

$$(1 + M\beta p_i^* q_i^*)(1 - q_i^*)^M = 1, \quad \forall i = 1, ..., N. \tag{2}$$

We next describe the properties of the equilibrium prices. To start, the pricing game always permits a unique equilibrium, which we formally state in the following Proposition 1:

Proposition 1. *Facing M customers, there always exists a unique pure-strategy Nash equilibrium for the seller's pricing game.*

This property guarantees that, regardless of the number of listings and customers, there always exists one and only one equilibrium. In addition to existence and uniqueness, the equilibrium price has the following characteristics:

Proposition 2. *The equilibrium price p_i^* is increasing in a_i, i.e., products with higher qualities will charge higher prices. Furthermore, $\lim_{M\to\infty} \partial p_i^*/\partial a_i = 1/\beta,\ \forall i = 1, ..., N$.*

Proposition 2 has two implications. First, sellers with higher product quality charge higher prices in equilibrium. Second, in the limiting case in which the platform faces considerable demand, the price change is linear in quality. As a result, all the listings share the same attractiveness $a_i - \beta p_i^*$. While Propositions 1 and 2 characterize the properties of the equilibrium price, they do not reveal what equilibrium price each seller should set. We showcase how the exact value of the equilibrium prices, which is the solution to a system of nonlinear FOCs, can be computed in Section 4.

3.2 Price Competition under Partitioned Display

Different from the common practice in which the platform displays all the sellers, we now consider a case in which the platform has the freedom to choose which product to display, and to which customers. Specifically, we consider a two-stage game with the platform as the leader and the sellers the follower. In the first stage, the platform announces a display policy $(\mathcal{S}_k, M_k)$ to the sellers. The display policy has two components.

First, it describes how the platform partitions products from the full assortment into different subsets $\mathcal{S}_k$. Second, it indicates the number of customers M_k assigned to each product partition $\mathcal{S}_k$[3]. Importantly, in the rest of the paper, we refer to a display policy as the division of the assortment $\mathcal{S}$ and demand M that satisfies the following two conditions:

1. Each seller is assigned to one and only one partition, i.e., $\cup_k \mathcal{S}_k = \mathcal{S}$ and $\mathcal{S}_i \cap \mathcal{S}_j = \varnothing$.

2. Every customer can observe one and only one partition, i.e., $\sum_k M_k = M$.

In the second stage of the game, after observing the display policy and $(a_i, \boldsymbol{a_{-i}})$, the qualities of all other competitors with whom seller i displays together in the same partition, every seller i maximizes its expected utility by simultaneously setting p_i. Finally, customers assigned to each partition arrive and make purchase decisions after observing the attractiveness of each product in that partition.

We now elaborate seller i's optimization problem facing the partitioned display. In the second stage, each seller i in each assortment $\mathcal{S}_k$ observes the quality of the competing listings within the same partition. After observing the quality for each seller $(a_i, \boldsymbol{a_{-i}}), i \in \mathcal{S}_k$, total arrival M_k traffic of the subset, and a belief on competitor prices p_{-i}, seller i faces the following optimization problem:

$$\max_{p_i} \Pi_i^s(\mathcal{S}_k, M_k, p_{-i}) = p_i \left(1 - \left(1 - \frac{\exp{(a_i - \beta p_i)}}{1 + \sum_{\{j \in \mathcal{S}_k\}} \exp{(a_j - \beta p_j)}}\right)^{M_k}\right). \tag{3}$$

Similar to the scenario of displaying everything, we can write the FOC for Equation (3) as:

$$(1 + M_k \beta p_i^* q_i^*)(1 - q_i^*)^{M_k} = 1, \quad \forall i \in \mathcal{S}_k. \tag{4}$$

Notably, both Propositions 1 and 2 hold valid for Equation (4), as each seller's equilibrium price is determined only by the competing sellers in the same partition, not by the equilibrium prices in other partitions. Foreseeing the equilibrium prices set by each seller, the platform decides the optimal partition to maximize the total revenue, which, in cases such as Airbnb or Uber, is a fraction of the total revenue from each individual seller. Thus, omitting the commission rate, which is a constant, the platform's optimization

[3]In reality, as customer arrivals can be perceived as uniform within a short time window (e.g., every 10 seconds), the traffic allocation can by achieved by common modulo operations.

problem can be formulated as

$$
\begin{aligned}
&\max_{\mathcal{S}_k, M_k} \sum_k \sum_{i \in \mathcal{S}_k} \Pi_i^*(\mathcal{S}_k, M_k) \\
&s.t. \cup_k \mathcal{S}_k = \mathcal{S}, \quad \mathcal{S}_i \cap \mathcal{S}_j = \varnothing \\
&\quad \sum_k M_k = M \\
&\quad \Pi_i^*(\mathcal{S}_k, M_k) = p_i^*(1-(1-q_i^*)^{M_k}) \\
&\quad (1 + M_k \beta p_i^* q_i^*)(1-q_i^*)^{M_k} = 1, \quad \forall i \in \mathcal{S}_k.
\end{aligned}
\tag{5}
$$

The first two constraints in Equation (5) manifest our assumptions that each product exclusively belongs to one subset and that each customer observes only one such subset of products. The last two constraints capture the incentive compatibility (IC) of the seller's pricing decision and imply that each seller sets the optimal price given a set of display policies $(\mathcal{S}_k, M_k)$. Intuitively, partitioning the entire assortment into several distinct product sets dampens the competition in the whole market, thus inducing sellers to charge higher prices. However, because the demand allocated to each partition is also lower than that under the full display policy, the probability that each product is selected by a customer can be lower. Thus, the overall impact of product partitioning on equilibrium pricing and the resulting platform revenue is unclear. In what follows, we first provide a theoretical analysis of the optimal display strategy when the total demand M is sufficiently large, which is often the case for large online marketplaces such as Airbnb. First, we are able to characterize the following limiting behaviors:

Lemma 1. *For a fixed partition display $\mathcal{S}_k$ and any product $i \in \mathcal{S}_k$, define $p_i^*(\mathcal{S}_k, M_k)$ and $q_i^*(\mathcal{S}_k, M_k)$ to be the equilibrium price and selection probability of product i under partition $\mathcal{S}_k$ and demand M_k. Then, in the limiting case in which $M_k \to \infty$, we have*

1. $\lim_{M_k\to\infty} p_i^*(\mathcal{S}_k, M_k) = +\infty, \lim_{M_k\to\infty} q_i^*(\mathcal{S}_k, M_k) = 0$
2. $\lim_{M_k\to\infty} \frac{M_k q_i^*(\mathcal{S}_k, M_k)}{\ln(-\ln q_i^*(\mathcal{S}_k, M_k))} = 1$.

Lemma 1 describes the order of the equilibrium purchasing probability q_i^* in the limiting case, which we use to calculate the revenue a product generates under different partitions as stated in Lemma 2.

Lemma 2. *Consider two subsets, $\mathcal{S}_1$ and $\mathcal{S}_2$, and a product i that is in both subsets, i.e., $i \in \mathcal{S}_1 \cap \mathcal{S}_2$. If the number of customers assigned to each subset maintains a fixed ratio $\gamma > 0$, then $\lim_{M\to\infty} \Pi_i^*(\mathcal{S}_1, M) - \Pi_i^*(\mathcal{S}_2, M/\gamma) = \ln \gamma$.*

Note that $\ln \gamma$ is strictly positive as long as $\gamma > 1$. Thus, Lemma 2 implies that in the limiting case, as long as the number of customers assigned to partition i remains a fixed proportion of the total number of customers, presenting product i to the entire customer base generates higher revenue than that to a fraction of the incoming traffic. This result holds regardless of how many other products are also included in subsets

S_1 and S_2. Nevertheless, it is important to note that Lemma 2 does not imply that the platform should not partition over listings and customers. This is because Lemma 2 does not shed light on the optimal size the platform should set for each subset S_k, and the number of customers M_k rationed to each subset of products may not always remain at a constant ratio of the total incoming traffic. To complement Lemma 2, we then provide the following result about the monotonicity of seller revenue on M.

Lemma 3. *For a fixed assortment display $\mathcal{S}$, revenue $\Pi_i(\mathcal{S}, M)$ is monotonically increasing in M.*

Combining Lemmas 2 and 3, we now provide the sufficient condition to make it optimal for the platform to display everything as a whole when demand becomes high through Theorem 1.

Theorem 1. *Suppose that there are N products $\mathcal{S} = \{1, 2, \ldots, N\}$. For any $\gamma > 1$, there exists a threshold $M(\gamma)$, such that when $M > M(\gamma)$, for any display policy $\{\mathcal{S}_k, M_k\}_{k=1}^K$ that satisfies $M_k < M/\gamma$, we have*

$$\sum_{i=1}^{N} \Pi_i^*(\mathcal{S}, M) > \sum_{k=1}^{K} \sum_{i \in \mathcal{S}_k} \Pi_i^*(\mathcal{S}_k, M_k).$$

Proof for Theorem 1. As the number of subsets $\mathcal{S}_1 \subset \mathcal{S}$ is finite, according to Lemma 2, when $M > M(\gamma)$, where $M(\gamma)$ is the threshold that depends on γ, we have

$$\Pi_i^*(\mathcal{S}, M) - \Pi_i^*(\mathcal{S}_1, M/\gamma) > 0, \quad \forall \mathcal{S}_1 \subset \mathcal{S}, i \in \mathcal{S}_1.$$

Lemma 3 further states that $\Pi_i(\mathcal{S}_k, M)$ is monotonically increasing in M, and we can conclude that

$$\begin{aligned}\sum_{i=1}^{N} \Pi_i^*(\mathcal{S}, M) &\geq \sum_{k=1}^{K} \sum_{i \in \mathcal{S}_k} \Pi_i^*(\mathcal{S}_k, M/\gamma) \\ &\geq \sum_{k=1}^{K} \sum_{i \in \mathcal{S}_k} \Pi_i^*(\mathcal{S}_k, M_k).\end{aligned}$$

It is helpful to discuss the implication behind Theorem 1. For a platform facing sufficiently high demand, the platform enjoys the highest total revenue when the platform exhibits all products concurrently to all customers, i.e., there is no need to group products into different partitions and display each partition to a subset of customers. Importantly, for large platforms such as Airbnb and eBay on which most sellers offer a single-unit product, Theorem 1 provides a theoretical justification for their current display practice of displaying the entire assortment to customers. To verify Theorem 1 and to gain insight into the optimal display policy under an arbitrary M, we perform a numerical analysis and present the results in Figure 1. [4] Specifically, Figure 1 compares the optimal total revenue under different partition scenarios. We select an instance with $N = 100$ sellers and let each partition receive the same level of demand M/K. $K = 1$ corresponds to the default policy that displays all sellers to all customers. We normalize the revenue of the

[4]The detailed derivation of equilibrium prices and optimal display policies will be introduced in Section 4.

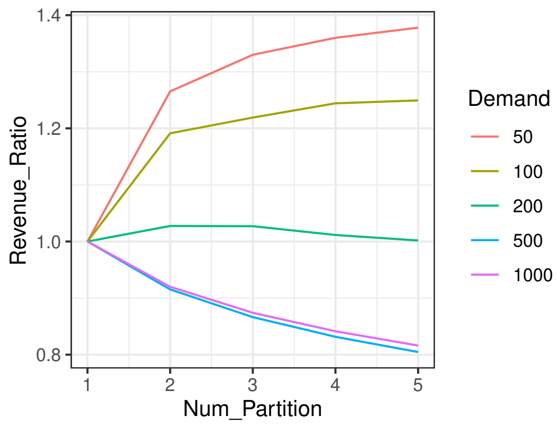

Figure 1: Revenue under Partition Display with Equal Demand

default policy to one and compare the revenue under different display policies under various demand levels. We observe that when the demand level is low ($M = 50$ and $M = 100$), it is more profitable to divide the sellers into more partitions. When demand level is high ($M = 500$ or $M = 1000$), it is optimal to maintain the display-everything policy, which is consistent with the conclusion of Theorem 1.

4 Characterizing the Price Equilibrium

In this section, we derive the platform's optimal partition strategy and sellers' optimal pricing decisions under such partitioning. Given an arbitrary display policy, the derivation of the equilibrium prices requires jointly solving a system of FOCs specified in Equations (2). These FOCs are highly nonlinear and cannot be characterized by the Lambert-W function, which Heese and Martínez-de Albéniz (2018) use to analyze the infinite inventory scenario. It is thus impossible to derive a closed-form solution for the equilibrium price p_i^*.

There are several viable approaches to solve for the equilibrium prices. One option is to compute through simulations. Since we have demonstrated the existence and uniqueness of the equilibrium, we can initialize a set of feasible prices and iteratively update each seller's price as the best responses to the competitor's prices until they converge to an equilibrium where no seller has the incentive to deviate. However, this approach will no longer be applicable when the platform attempts to find the optimal display policy. This is because the possible number of display policies grows exponentially with the number of sellers N, and enumerating all the policies becomes computationally burdensome, if not infeasible.

In our work, we propose a novel approach that transforms Equation (5) into an MIP problem to determine the optimal display policy and the induced optimal price. The idea is to define and discretize an intermediate $\mathcal{Z}$ in Equation (2) such that each FOC is decoupled from the system of FOCs given $\mathcal{Z}$. Then, we precalculate and tabularize the required inputs into tables and let the MIP solve for the display policy and the corresponding equilibrium prices that maximize the platform's total revenue. This approach enhances the computational tractability by discretizing the nonlinear FOCs into a set of precomputed values. Moreover, the MIP framework provides a flexible infrastructure for the platform to impose different constraints on display policies. Granted, solving a large-scale MIP can still be computationally intensive, but we demonstrate in Section 4.2 that our approach can be solved within a reasonable running time.

To better introduce the tabularization process, we first define the total attractiveness of all the sellers in partition k as $\mathcal{Z}^k = \sum_{i \in \mathcal{S}_k} \exp(a_i - \beta p_i)$. Furthermore, as each partition's equilibrium can be solved independently, we focus on solving the equilibrium for a single partition k and omit the superscript for ease of notation in the remainder of the section. In fact, there is a unique value of $\mathcal{Z}$ that can allow for a set of prices, $\{p_i\}$, to simultaneously satisfy Equation (4) and the definition of $\mathcal{Z}$. Moreover, under such $\mathcal{Z}$, the resulting set of prices represents the equilibrium price. Thus, we construct a feasible range $\mathcal{Z} \in [\underline{\mathcal{Z}}, \bar{\mathcal{Z}}]$, over which we discretize $\mathcal{Z}$ into L levels to form a ladder. For each seller $i \in N$ and each fixed value $\mathcal{Z}_j, j \in L$ in the ladder, we can compute the best response $p_{i,j}$ by applying common built-in root finding package to the FOCs. Although the FOCs are nonlinear, finding each root under a fixed $\mathcal{Z}_j$ reduces to a one-dimensional search problem that can be efficiently solved. By doing so, we can tabulate the best responses into a $N \times L$ table with each entry (i, j) corresponding to the equilibrium price for the i-th seller under the j-th value in the ladder of $\mathcal{Z}$. Similarly, we also track seller i's attractiveness $E_{i,j} = \exp(a_i - \beta p^*_{i,j})$ and the induced revenue $\Pi_{i,j}$ in separate tables. These three tables $p^*_{i,j}, E_{i,j}$ and $\Pi_{i,j}$ $(i = 1, ..., N, j = 1, ..., L)$ are connected in the sense that the price in the (i, j)-th entry in the $E_{i,j}$ and $\Pi_{i,j}$ tables is identical to the value of the (i, j)-th entry in the table for $p^*_{i,j}$. We summarize the tabulation procedure in Table 1. This process allows us to formulate the problem as a mixed integer linear programming (MILP) that we introduce in the next subsection.

Algorithm 1: Tabularization of Formulation Input

input : $\{a_i,\ i = 1, ..., N\}, \{\mathcal{Z}_j,\ j = 1, ..., L\}, M$

for $j \in \{1, ..., L\}$ **do** `// Iterate over` $\mathcal{Z}$

  for $i \in \{1, ..., N\}$ **do** `// Iterate over sellers`

    Solve FOC

$$p^*_{i,j} = \arg_p \ \{(M\beta pq + 1)(1 - q)^M = 1\},$$

    to get the best response price $p^*_{i,j}$

    Tabulate $E_{i,j} = \exp(a_i - \beta p^*_{i,j})$

    Tabulate $\Pi_{i,j} = p^*_{i,j}\big(1 - (1 - \exp(a_i - \beta p^*_{i,j})/(1 + \mathcal{Z}_j))^M\big)$

  end for

end for

output: $\{p^*_{i,j}, E_{i,j}, \Pi_{i,j},\ i = 1, ..., N, j = 1, ..., L\}$

4.1 MIP Formulation and Fixed Subset Display

With the pretabulated inputs, we now construct the MILP model to solve the price equilibrium. We start with the simplest case, in which the platform displays all the products to all the customers as currently employed by platforms such as Airbnb and ebay. Given the qualities $\{a_i,\ i = 1, ..., N\}$, demand M, the total attractiveness $\{\mathcal{Z}_j,\ j = 1, ..., L\}$ and tabulated inputs $\{p^*_{i,j}, E_{i,j}, \Pi_{i,j},\ i = 1, ..., N, j = 1, ..., L\}$, the goal is to find a column j^* from table $E_{i,j}$, such that $\sum_i E_{i,j^*} = \mathcal{Z}_{j^*}$. The existence and uniqueness of the equilibrium are guaranteed by Proposition 1. For simplicity, we will use this exact form of constraints with equality in the remainder of the paper. However, in practice, it is unlikely that there exists a unique $\mathcal{Z}_{j^*}$ value that makes the equilibrium coincide with one point on the ladder. To ensure the MIP is feasible, we could replace the abovementioned equality constraint with two inequalities $\sum_i E_{i,k} \geq 0.5(\mathcal{Z}_k + \mathcal{Z}_{k-1})$ and $\sum_i E_{i,k} \leq 0.5(\mathcal{Z}_k + \mathcal{Z}_{k+1})$. Intuitively, the solution will approach the true optimum as the discretization of the $\mathcal{Z}$ ladder becomes finer.

In the example above, we compute the equilibrium prices under the default display policy. The platform does not need to make any decisions because the policy is fixed. We now present the case in which the platform chooses to displays a fixed subset of the sellers. This scenario is a special case of our partition display policy allocating all the traffic M to a single partition and is in line with the classic setting in the assortment optimization literature (e.g., Talluri and Van Ryzin 2004). Formally, we denote by z_j the binary decision variable on whether the jth column is selected and by $x_{i,j}$ the binary decision variable indicating whether seller i is assigned to the group with z value z_j. The formulation is as follows.

$$\begin{aligned}
\max_{x,z} \quad & \sum x_{i,j}\ \Pi_{i,j} \\
s.t. \quad & \sum_i x_{i,j} E_{i,j} = z_j \mathcal{Z}_j, \qquad\qquad \forall j = 1, ..., L \\
& \sum_j z_j = 1, \\
& x, z\ \text{ binary}
\end{aligned} \tag{6}$$

This formulation selects one column $z_j = 1$ in the table and sellers $x_{i,j} = 1$ within the column. The objective maximizes the total revenue of all the sellers, which aligns with the platform's incentive. The first constraint in Equation (6) ensures that the definition of $\mathcal{Z}$ is satisfied. In particular, $z_j = 0$ when column j is not selected. Thus, we must have $x_{i,j} = 0$ for all i, because the inputs $E_{i,j}$ are nonzero. When column j is selected, the model will select a subset of rows to satisfy the definition of $\mathcal{Z}$. Furthermore, the second constraint ensures that only one column will be selected, as only one fixed assortment will be displayed to the customers. Notably, similar to the full display case, the first constraint cannot be met exactly due to the discrete nature of $\mathcal{Z}$. To guarantee the feasibility of the MIP, in practice, we could relax the constraint by adding buffers of the following form: $\sum_i x_{i,j} E_{i,j} \geq 0.5 z_j(\mathcal{Z}_j + \mathcal{Z}_{j-1})$ and $\sum_i x_{i,j} E_{i,j} \leq 0.5 z_j(\mathcal{Z}_j + \mathcal{Z}_{j+1})$.

Exploiting Equation (6), we now conduct numerical analysis to explore the price equilibrium under the default display policy. We simulate a market with $N = 100$ sellers whose product quality is normally distributed and follows $a \sim \mathcal{N}(3, 1)$. We set the price coefficient to be $\beta = 1$ and vary the demand level from 50 to 1,000. Panel (a) in Figure 2 plots the relationship between the equilibrium price and product quality under different demand scenarios. Specifically, Panel (a) is consistent with Proposition 2 in that prices are monotonically increasing in quality, and when facing high demand, the relationship between price and quality becomes linear.

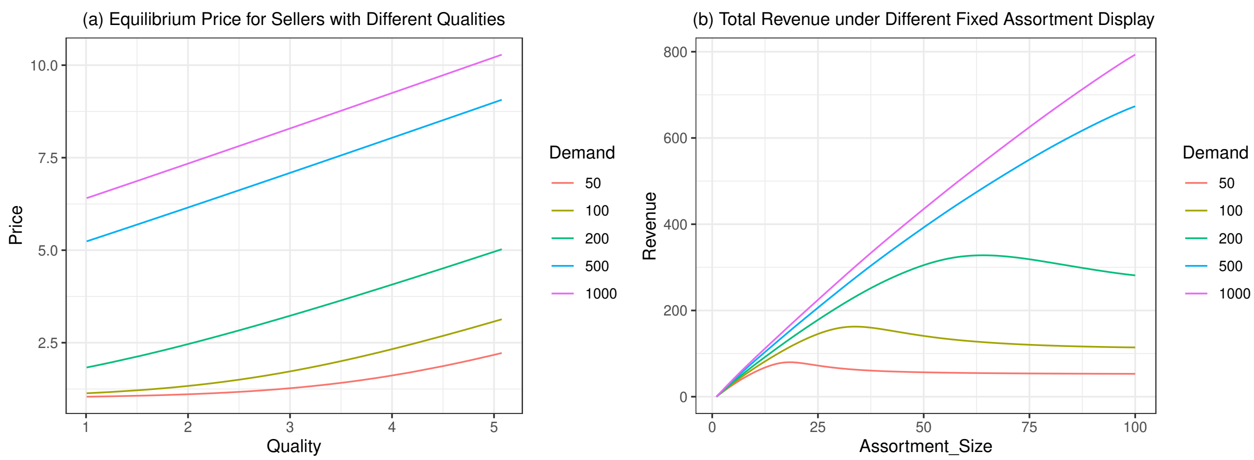

Figure 2: Price and Revenue under Fixed Assortment Display

We then conduct numerical analysis to study the total revenue under the optimal display policy given different assortment sizes. The results are presented in Panel (b) in Figure 2. Specifically, the x and y axes represent the cardinality of the displayed subset and the total revenue, respectively. In this way, the platform showcases the entire assortment to the customers at $x = 100$ and displays a subset over the remaining region. Interestingly, when demand is low, displaying a subset of products actually induces higher total revenue than displaying the whole assortment. As demand increases, the cardinality of the optimal assortment also grows. Consistent with the results from Theorem 1, it is optimal for the platform to display all N products when demand becomes sufficiently high. Notably, our numerical results provide important guidelines for platforms to decide their expansion strategy. Newly launched platforms usually provide subsidies to incentivize service providers to join. However, our results suggest that attracting a large number of service providers at an early stage can be suboptimal due to insufficient demand. Nevertheless, for large platforms that face heavy traffic on a daily basis, increasing the base of service providers can actually boost total revenue, as a reduction in the lost demand caused by (i) customers choosing the product and (ii) customers choosing the outside option outweigh the dampened product price caused by the more intense competition.

4.2 Partitioned Display

In Section 4.1, we proposed an MIP formulation to characterize the equilibrium price and the optimal display policy under fixed assortment. We now move to a more general case in which the platform has the flexibility to partition the products into arbitrary subsets and allocate the incoming traffic to each subset. To derive the optimal partition $(\mathcal{S}_k, M_k)$ that maximizes total revenue and characterize the equilibrium price, we now describe the tabularization procedure and the MIP formulation based on Equation (5).

As the platform now has the flexibility to distribute the incoming demand unevenly to each product partition, each seller no longer faces M incoming customers when making its pricing decision. Thus, we first need to modify the tabularization procedure to incorporate flexibility in the number of customers that each seller could potentially face. To this end, we normalize the total demand to one and discretize the fraction of demand (i.e., market share) from 0% to 100%. Specifically, we denote the v-th entry of the discretized market share by P_v. For example, when discretizing with 10% separation, we ultimately have 11 market share levels in total, and $P_1 = 0\%$ and $P_{11} = 100\%$. We then replace M with MP_v as the demand level and replicate the tabulation process described in Algorithm 1. In doing so, the outputs of the tabularization, $E_{i,j,v}$ and $\Pi_{i,j,v}$, will now have an additional dimension reflecting the MIP inputs under different fractions of market shares.

We now extend formulation (6) to incorporate the platform's partition decision, which we denote by the subscript k. We first introduce the full formulation. Then, we will discuss relaxations and special cases of the full formulation to shorten computation time.

In the full formulation, let $x_{i,j,k,v}$ denote the binary decision variable indicating whether to allocate seller i to partition k, which has total partition attractiveness $\mathcal{Z}_j$ and market demand MP_v. Similarly, $z_{j,k,v}$ denotes the binary decision variable of whether to select the jth level of $\mathcal{Z}$ and vth level of market share for partition k. In this way, the full formulation can be expressed as:

$$
\begin{aligned}
\textbf{(FULL)} \qquad & \max_{x,z} \sum_k x_{i,j,k,v}\, \Pi_{i,j,v} && && (7)\\
& s.t. \sum_i x_{i,j,k,v} E_{i,j,v} = z_{j,k,v} \mathcal{Z}_j && \forall j,k,v \\
& \sum_{j,k,v} x_{i,j,k,v} = 1, && \forall i \\
& \sum_{j,v} z_{j,k,v} = 1, && \forall k \\
& \sum_{j,k,v} z_{j,k,v} P_v = 1, && \\
& x_{i,j,k,v} \text{ binary}, \quad z_{j,k,v} \text{ binary} &&
\end{aligned}
$$

Compared to formulation (6), which selects one feasible column and a subset of rows (sellers) in the table, the full formulation draws entries from the tabularized tensor. Again, the first constraint ensures that the exponential of individual attractiveness sums to $\mathcal{Z}$. The second constraint guarantees that every seller is assigned to one and only one partition. The third constraint ensures that for each partition k, only one $\mathcal{Z}_j$ and one market share P_v are selected. Finally, we add the fourth constraint to ensure that the market shares assigned to each partition sum to one. The additional dimensions incorporated to capture the partition of sellers and market demand make it computationally challenging for both the tabularization and optimization processes. To reduce the computation time, we propose two alternative formulations that compute the upper and lower bounds of formulation (7).

The idea behind the alternative formulations is to reduce the dimension of the problem. To achieve this, we remove the subscript k from formulation (7) and characterize each partition through a unique (j, v) dyad. Specifically, $z_{j,v}$ specifies whether the partition with total attractiveness $\mathcal{Z}_j$ is assigned with market share P_v. In addition, we denote by $x_{i,j,v}$ a binary decision variable indicating whether to allocate $\mathcal{Z}_j$ and market share P_v to seller i, and there are in total K pairs of (j, v) combinations. The value $x_{i,j,v}$ will then shape the first constraint and the total revenue through input tables $E_{i,j,v}$ and $\Pi_{i,j,v}$. The specific formulation is as follows:

$$
\begin{aligned}
\textbf{(LB)} \qquad & \max_{x,z} \sum_{k} x_{i,j,v}\ \Pi_{i,j,v} && && (8)\\
& s.t. \sum_{i} x_{i,j,v} E_{i,j,v} = z_{j,v} \mathcal{Z}_j && \forall j, v \\
& \sum_{j,v} x_{i,j,v} = 1, && \forall i \\
& \sum_{j,v} z_{j,v} = K, \\
& \sum_{j,v} z_{j,v} P_v = 1, \\
& x_{i,j,v} \text{ binary}, \quad z_{j,v} \text{ binary}
\end{aligned}
$$

In fact, formulation (7) (denoted FULL) is a relaxation of formulation (8) (denoted LB) and the optimal total revenue of LB serves as a lower bound of that for FULL. To see this, note that any feasible solution of LB is still feasible under FULL. However, a feasible solution in FULL in which two partitions receive the same (j, v) pair is infeasible in LB, as this will make the corresponding $z_{j,v} = 2$, thereby violating the binary constraint.

Using a similar idea, we construct another formulation (UB) whose objective is the upper bound of the FULL formulation. Specifically, the formulation of UB is identical to that of LB, except that we require $z_{j,v}$ to be nonnegative integers instead of binary variables. Again, using the same argument from the previous paragraph, it is clear that any feasible solution in FULL is also feasible for UB, but integer solutions such as $z_{j,v} = 2$ are feasible only in UB.

These two formulations, LB and UB, serve to provide the lower and upper bounds to the full optimization problem. Such approximations are particularly useful when solving the FULL turns out to be computationally challenging, which is the case when the number of listings becomes large and the discretization gap becomes sufficiently small. In Figure 3, we show the revenue and running time of the three formulations, together with a fourth option, namely, the LP relaxation of the FULL formulation. The instance is the same as in the previous numerical analysis. It is observed that the upper and lower bound and the LP relaxation yield almost the same outcome as the full formulation. However as shown in panel (a), we need finer discretization (smaller step sizes) of $\mathcal{Z}$ for the formulation to have a good approximation. However, panel (b) shows that when the step size is very small, the lower and upper bound formulations will have significantly shorter computational time.

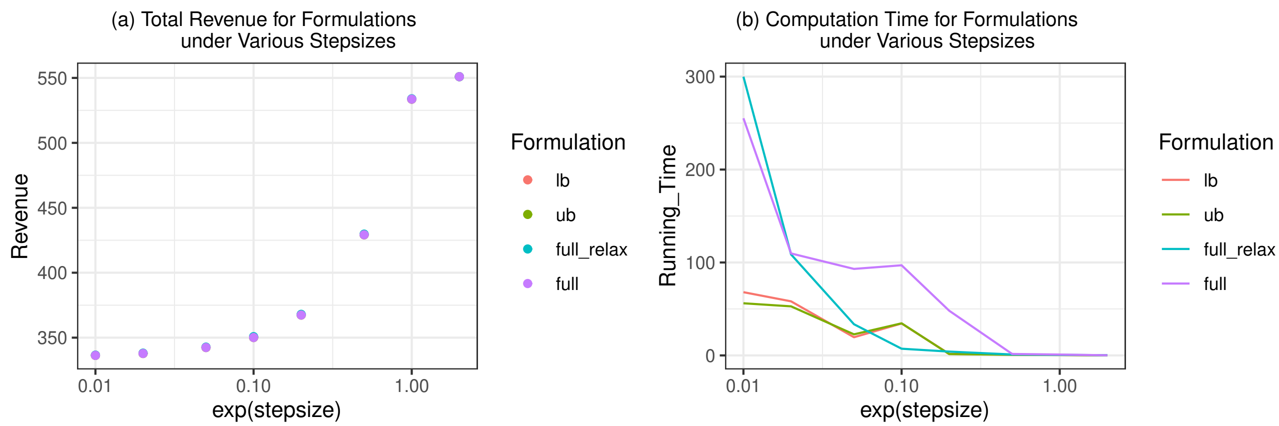

Figure 3: Revenue and Running Time under Partitioned Display

4.3 Partitioned Display with Fairness Constraints

As shown in Section 4.1, displaying the entire assortment may not always generate the highest revenue for the platform. As a result, the platform may choose to display only a subset of listings to achieve higher profitability. While such a policy might be practical in offline retail settings (Heese and Martínez-de Albéniz 2018), where the limited shelf space naturally prevents the store from showcasing all the products, this display policy potentially prevents a proportion of the listing owners from joining the platform and could spur concerns over an unfair business environment on online marketplaces, which the European Commission has regulated against (Bostoen 2018).

The literature has proposed a number of commonly used notions of fairness (e.g., Mo and Walrand 2000, Bertsimas et al. 2011); we nonetheless limit the scope of discussion within the fairness of the demand allocation and the partition competitiveness – two attributes that shape each seller's best response according to Equation (3). Specifically, each partition k is characterized by the incoming demand M_k and the total attractiveness in equilibrium $\mathcal{Z}^k = \sum_{\{i \in \mathcal{S}_k\}} \exp{(a_i - \beta p_i^*)}$. In what follows, we first propose a notion of fairness measured by the relative difference in attractiveness $\mathcal{Z}$ and the difference in the incoming demand that each of the K partitions receives. To this end, we introduce the definition of (α, δ)-fairness as follows

Definition 1. *A display policy is (α, δ)-fair, where $0 \leq \alpha \leq 1$ and $0 \leq \delta \leq 1$, if the total attractiveness of each group satisfies $Z^{k_i}/Z^{k_j} \geq \alpha$ and the total demand assigned to each group satisfies $M_{k_i}/M_{k_j} \geq \delta$ for all partitions k_i, k_j, $i, j \in \{1, ..., K\}$.*

Intuitively, α quantifies the level of unbalance in the total attractiveness among partitions. According to the definition above, $\alpha = 1$ refers to the fairest scenario because each partition has the same total attractiveness, while a smaller α allows certain partitions to be significantly better or worse off than the rest and corresponds to a less fair scenario. Similarly, the demand allocation is fairest when $\delta = 1$, as the demand assigned to each partition is identical and equal to M/K in this case. When $\delta = 0$, the fairness constraint ceases to matter, and the platform can freely allocate the market shares among partitions.

The notion of (α, δ)-fairness could be easily incorporated into our MIP Formulation 7 as additional constraints, which we will show in the following section. Intuitively, as α or δ approaches 1, the feasible region for the platform's optimization problem shrinks, which translates into a greater revenue loss. Nevertheless, the notion of (α, δ)-fairness is not directly linked to a single seller's revenue. To quantify the effect of various fairness policies from the perspective of an individual seller's revenue, we propose the following definition of the "envy level".

Definition 2. *Denote by Π_i^0 the revenue that seller i receives from the current partition. When seller i is moved to the k-th partition, $k = 1, 2, \ldots, K$, while the rest of the sellers remain unchanged, we denote the revenue that seller i will receive by Π_i^k. Then, we define the current "envy level" for seller i as:*

$$EN_i = \frac{\max_k\{\Pi_i^k\} - \Pi_i^0}{\Pi_i^0}.$$

The envy level, as a measure of the potential percentage of revenue gain, captures the incentive for one seller to move to another partition. A display policy is considered less fair if the envy level is large for some sellers under such policy, as these sellers would have strong a incentive to move to a more profitable partition. On the other hand, if $EN_i = 0$ for all $i = 1, 2, \ldots, N$, we consider the current partition to be fair because no seller has incentives to unilaterally switch partition groups. Notably, when a display policy is (α, δ)-fair, i.e., $\delta = \alpha = 1$, we also have $EN_i = 0$ for all $i = 1, 2, \ldots, N$, indicating that sellers would be indifferent to which partition they are assigned, as their equilibrium prices and revenues will be exactly the same across all partitions. In what follows, we augment Formulation (7) by including two additional constraints to reflect the restrictions imposed by (α, δ)-fairness. We also discuss in Appendix I the special case in which each partition receives equal demand ($\delta = 1$).

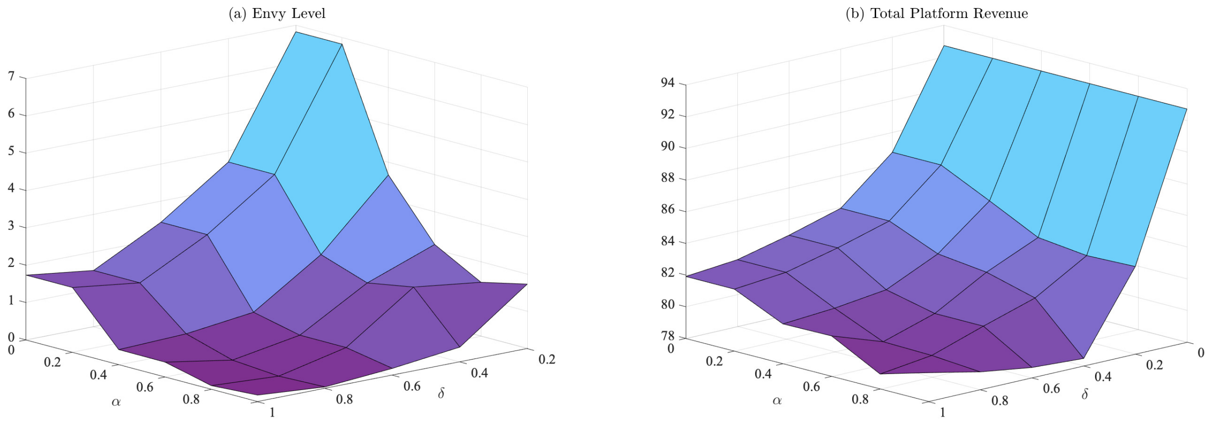

Figure 4: Envy Level and Total Platform Revenue under Different α and δ. The full assortment has 40 sellers with quality a_i following normal distribution $\mathcal{N}(4,1)$. Demand is set to be 60, and the partition number is set to be 3. Parameters α and δ are set from 0 to 1 with intervals of 0.2.

4.3.1 The General Case

Denoting the feasible region of Formulation (7) by $\mathcal{Q}$, we modify Formulation (7) to include the δ and α fairness constraints as

$$
\begin{aligned}
\max_{x,z} \quad & \sum_{k} x_{i,j,k,v}\, \Pi_{i,j,v} && && (9)\\
s.t. \quad & \sum_{j,v} z_{j,k_1,v} P_v \geq \delta \sum_{j,v} z_{j,k_2,v} P_v && \forall k_1, k_2 = 1,2,\ldots,K \\
& \sum_{j,v} z_{j,k_1,v} \mathcal{Z}_j \geq \alpha \sum_{j,v} z_{j,k_2,v} \mathcal{Z}_j && \forall k_1, k_2 = 1,2,\ldots,K \\
& x, z \in \mathcal{Q}
\end{aligned}
$$

The integration of fairness into the full formulation is straightforward. Recall that $\mathcal{Z}_j$ is the jth level of the tabularized Z value, and P_v is the vth level of the tabularized market share. The first two constraints ensure that the ratios of the market share and attractiveness between any two partitions are greater than δ and α, respectively. Moreover, note that Formulation (9) does not guarantee the existence of a feasible solution. In other words, Formulation (9) might be infeasible if we impose rather restrictive fairness constraints, i.e., both δ and α are required to be close to one.

Exploiting Formulation (9), we are able to derive the equilibrium price and the resulting seller revenue under different (α, δ) combinations. In Panel (a) of Figure 4, we demonstrate how the maximum of the envy level across sellers varies under different (α, δ) combinations. As expected, the envy level is the highest when the display policy is the least fair as measured by α- and δ-fairness, i.e., $(\alpha, \delta) = (0, 0.2)$. Moreover, Panel (b) of Figure 4 showcases how the total revenue as defined in the objective of Formulation 9 is affected by

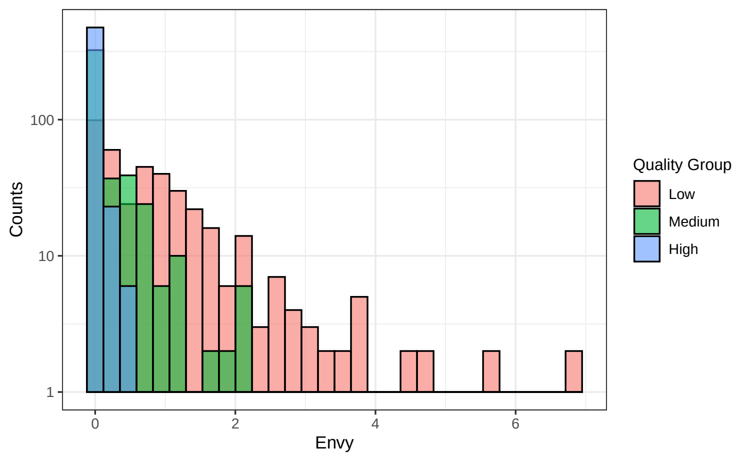

Figure 5: Envy Distribution for Different Quality Groups. The parameter setup is the same as that in Figure 4.

the display fairness. We observe that the total revenue is monotonically decreasing as either α or δ increases. The revenue under the fairest case in which $(\alpha, \delta) = (1, 1)$ is 86.1% of that when the platform is not concerned about fairness at all, i.e., $(\alpha, \delta) = (0, 0)$. Additionally, we scrutinize the relationship between seller quality and the envy level each seller exhibits. Specifically, we sort 40 sellers according to their product quality, classify them into three groups with sizes of 13, 13 and 14 and label them as 'Low', 'Medium' and 'High', respectively. We plot the envy distribution of these three groups in Figure 5. Notably, sellers in the low quality group exhibit a significantly higher envy level than their counterparts in the higher quality groups as shown in Figure 5, indicating that a seller with a lower quality product is more likely to receive an unfair assignment. In fact, the maximal envy level of the 'High' group is 0.58, while for the 'Low' group, it is as high as 6.85. Without resorting to Formulation (9), we can use the following proposition to bound the envy level of each individual seller.

Proposition 3. *For any display policies with fairness level $\frac{2}{3} < \alpha \leq 1$ and $0 < \delta \leq 1$, the envy level for each seller can be bounded by*

$$EN_i \leq \frac{2\alpha}{3\alpha - 2}\left(\frac{1}{\alpha} + \frac{1}{\delta} - 2\right), \quad \forall i = 1, 2, \ldots, N \tag{10}$$

The purpose of this proposition is to illustrate the connection between the individual envy level and fairness parameters (α, δ). It is obvious that the upper bound of the envy level is decreasing in both δ and α. Moreover, the rate of change is significantly higher when δ and α are small. This indicates that subject to the same decrease in the fairness parameters, the envy level is likely to increase more in less fair settings. Note that although the left-hand side EN_i is on individual sellers, the right-hand side upper bound is free of any individual inputs. In other words, the bound is tightest when benchmarked with the seller that has the maximum envy level.

5 Application

In this section, we present a case study using transaction data from Airbnb to demonstrate how our framework can be applied in practice. Our goal is to demonstrate the empirical strategy when fitting our model to the data and ultimately evaluate the impact of different display policies under different market conditions for the Airbnb platform. Our model, albeit stylized, closely reflects the gist of the decision-making processes of the hosts and the Airbnb platform for the following reasons. First, each host on the Airbnb platform supplies a listing with unit availability every day. Second, many hosts have disabled instant booking, indicating that they plan to evaluate the profiles of all the applicants before accepting one, thus translating the sequential arrival pattern of the customers to a simultaneous scenario. Finally, each host makes independent pricing decisions while observing the prices set by other listing owners.

In what follows, we first introduce the setting of Airbnb and provide a summary of our data. We then describe how to fit our model to the transaction data to estimate a_i, the quality of each Airbnb listing i. Finally, we tabularize the key inputs of our MIP, through which we examine the optimal display policy under different demand scenarios.

5.1 Data

Our Airbnb dataset covers the transaction history of listings in Manhattan, New York, in 2018. Airbnb offers different home types that include Private Room, Shared Room and Entire Home/Apartment. As different home types tend to target different customer segments, competition usually occurs only within each home type. Thus, we limit the scope of our analysis to include only listings labeled as Entire Home/Apartment. The data include the daily transaction history of 2,561 such listings. On each day, the data documents the status of each listing as either blocked (i.e., made unavailable by the owner), available or reserved. This way, we are able to recover the assortment of listings displayed to the customers that we use to form the consideration sets for the customers. The data also provide daily booking price for each listing, along with other listing characteristics such as overall ratings and number of reviews. We present the summary of the listing characteristics in Table 2.

5.2 Estimation

To derive the optimal display policy for Airbnb, we first need to estimate each listing's quality. According to our theoretical model in Section 3, the overall mean utility of listing i on day t can be expressed by the listing quality and price and as $a_i - \beta p_{it}$. Nevertheless, instead of fitting an individual a_i for each listing, we parametrize listing quality as a linear combination of listing-specific covariates to reduce the number of parameters to be estimated and avoid overfitting the model. Specifically, we write the utility that customer

Table 2: Summary of Listings Characteristics

Property characteristics	Mean	St. Dev.	Min	Max
Occupancy Rate	0.65	0.36	0.00	1.00
Price (in USD)	247.54	287.58	10.00	2,500
Number of Reviews	70.02	63.49	0.00	400.00
Overall Rating	4.59	0.61	0.00	5.00
Number of Bedrooms	1.24	0.89	0.00	6.00
Number of Bathrooms	1.13	0.41	0.00	5.00
Response Rate†	91.19	19.12	0.00	100.00
Superhost	0.23	0.42	0.00	1.00

† Note: Response rate is defined as the percentage of the time that a host responds to potential guests within 24 hours. A host becomes as a Superhost if the host satisfies a series of criteria set by Airbnb, such as a high overall rating and low cancellation rate.

m gains from booking listing i on day t as

$$\mu_{itm} = a_i - \beta p_{it} + \epsilon_{im} = \boldsymbol{X_i}\boldsymbol{\gamma} + FE_i - \beta p_{it} + \epsilon_{im} \,,$$

where ϵ_{im} follows i.i.d. $Gumbel(0,1)$. Quality is expressed as $a_i = \boldsymbol{X_i}\boldsymbol{\gamma} + FE_i$, where $\boldsymbol{X_i}$ contains the listing characteristics summarized in Table 2 (apart from occupancy rate and price). Additionally, following Li et al. (2019), we divide Manhattan into 10 regions, and assume that substitution occurs within each region. We use FE_i to represent the regional fixed effect that listing i shares with competing listings within the same region. For ease of notation, we write $\boldsymbol{\theta}\boldsymbol{\Xi_{it}} = \boldsymbol{X_i}\boldsymbol{\gamma} + FE_i - \beta p_{it}$. In this way, the theoretical probability that listing i is booked on day t is

$$1 - (1 - q_{it})^M = 1 - \left(1 - \frac{\exp(\boldsymbol{\theta}\boldsymbol{\Xi_{it}})}{1 + \sum_{j \in \mathcal{S}_{it}} \exp(\boldsymbol{\theta}\boldsymbol{\Xi_{it}})}\right)^M \,, \tag{11}$$

where $\mathcal{S}_{i,t}$ is the partition that listing i belongs to on day t. Since Airbnb by default displays all the available listings over a certain region to all customers, $\mathcal{S}_{i,t}$ in this case represents all the listings within the same neighborhood as listing i. We denote by the binary variable $Y_{i,t}$ the observed outcome from the transaction history indicating whether listing i is booked on day t. We adopt the nonlinear least squares framework to recover a consistent estimate of $\boldsymbol{\theta}$ that leads to the best fit of the demand realization:

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \sum_{t=1,\ldots,T} \sum_{i=1,\ldots,N} \left(1 - \left(1 - \frac{\exp\left(\boldsymbol{\theta}\boldsymbol{\Xi_{it}}\right)}{1 + \sum_{j \in \mathcal{S}_{i,t}} \exp\left(\boldsymbol{\theta}\boldsymbol{\Xi_{jt}}\right)}\right)^M - Y_{it}\right)^2 . \tag{12}$$

We would like to make several notes about this estimation equation. First, the listing-specific parameters are identifiable because we observe variations in these listing characteristics in the historical data. Moreover, the identifiability of the regional fixed effects is achieved through the variations in the overall occupancy rate across different regions. One estimation challenge is that Equation (12) requires the actual demand level M as a fixed input, which in reality reflects listing owners' common belief about future demand. However, as we

Table 3: Estimation Results

	Demand/Supply Ratio		
	(M/N =1)	(M/N = 1.5)	(M/N =2)
Price ($\times$ 100)	-0.193	-0.217	-0.291
	(-0.202 -0.185)	(-0.235 -0.205)	(-0.391 -0.25)
Overall Rating	0.355	0.382	0.499
	(0.332 0.378)	(0.355 0.413)	(0.410 0.691)
Number of Reviews	0.005	0.006	0.011
	(0.005 0.005)	(0.006 0.006)	(0.009 0.012)
Superhost	0.100	0.124	0.184
	(0.078 0.121)	(0.097 0.153)	(0.122 0.231)
Response Rate	0.007	0.007	0.007
	(0.006 0.007)	(0.006 0.008)	(0.006 0.008)
Number of Bedrooms	0.174	0.197	0.244
	(0.163 0.187)	(0.181 0.220)	(0.214 0.349)
Number of Bathrooms	0.205	0.231	0.311
	(0.179 0.232)	(0.191 0.304)	(0.232 0.425)
Observations	46,769	46,769	46,769
Regional Fixed Effects	Yes	Yes	Yes

do not directly observe M in the data, we conduct estimation under different demand scenarios. Specifically, we assume M to be proportional to the number of listings, with the demand-to-supply ratio set to be 1.0, 1.5 or 2.0. The estimation results are presented in Table 3.

As the demand level increases, Table 3 suggests that while the magnitude of these estimates increases, the relative magnitude across estimates largely persists within each demand scenario. The 95% confidence intervals of our estimation results obtained through bootstrap simulations are presented in brackets. Using the estimation results, we calculate the listing quality as $a_i = \boldsymbol{X_i \gamma} + FE_i$.

5.3 Counterfactual Analysis

Using our estimation results, we obtain three sets of listing quality, each corresponding to a demand level. For each demand scenario, we incorporate the listing quality into Algorithm 1 to tabularize $p^*_{i,j}$, $E_{i,j}$ and $\Pi_{i,j}$, which in turn are used as inputs for our MIP formulation, i.e., Equation (7). In this way, we are able to provide recommendations on the optimal partition number for each neighborhood under each demand scenario. We visualize the solutions from our MIP in Figure 6.

When listing owners expect the daily demand to be low or moderate, Panels (a) and (b) suggest that Airbnb should assign listings in each neighborhood to partitions. Nevertheless, Panel (c) of Figure 6 implies that it is best for Airbnb to use the current display-everything strategy in most neighborhoods when faced with sufficient demand, which is consistent with Theorem 1. Importantly, in addition to the optimal partition

(a) Demand Ratio = 1 (b) Demand Ratio = 1.5 (c) Demand Ratio = 2

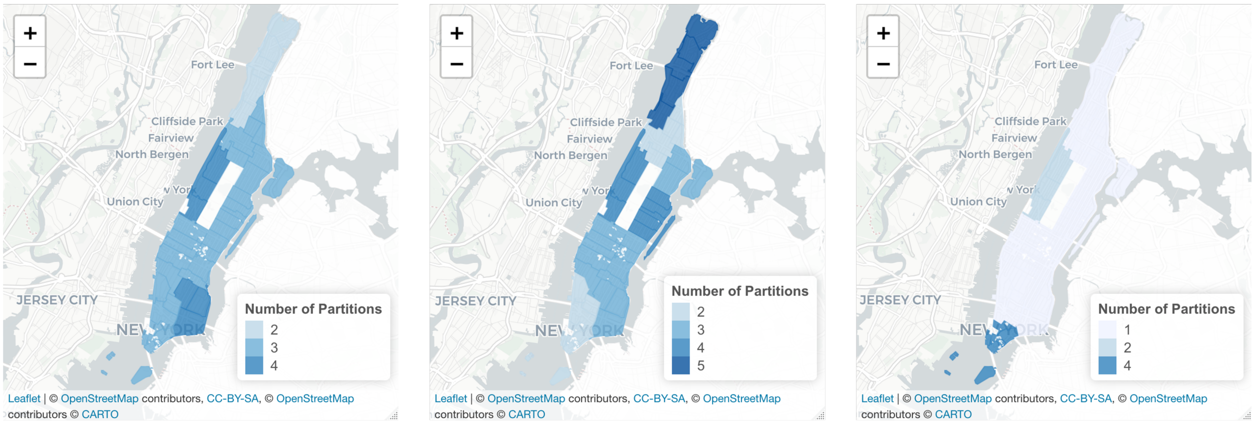

Figure 6: Optimal Number of Partitions in each Manhattan Neighborhood.

number, our MIP formulation also indicates explicitly which partition each listing is assigned to, making our results readily implementable by the platform.

Additionally, we also investigate the cost of fairness in the context of Airbnb. To this end, we compare the difference in the optimal partition number and revenue when the display policy is $(0, 0)$-fair and $(0, 1)$-fair. We present the results of our counterfactual analysis in Table 4. Under each demand scenario, we report the optimal partition numbers under the two abovementioned policies in parentheses. The resulting revenue gap generally falls within 20%. Notably, when demand is sufficiently high, the unconstrained optimal display policy is also $(0, 1)$-fair for most of the neighborhoods, as the optimal strategy is to display all listings to all the customers, which automatically satisfies the definition of $(0, 1)$-fairness.

Table 4: Optimal Partition Number (PN) when $(\alpha, \delta) = (0, 0)$ and $(0, 1)$ and the Revenue Gap.

District	$DR = 1.0$		$DR = 1.5$		$DR = 2.0$	
	PN	Gap	PN	Gap	PN	Gap
Central Harlem	(3, 3)	18.12%	(2, 4)	17.23%	(1, 1)	0.00%
Chelsea and Clinton	(3, 5)	19.84%	(3, 2)	21.84%	(1, 1)	0.00%
East Harlem	(3, 5)	15.53%	(3, 2)	12.84%	(1, 1)	0.00%
Gramercy Park and Murray Hill	(3, 5)	18.72%	(3, 2)	19.65%	(1, 1)	0.00%
Greenwich Village and Soho	(4, 5)	19.65%	(3, 4)	17.60%	(1, 1)	0.00%
Inwood and Washington Heights	(3, 5)	18.15%	(2, 3)	19.23%	(1, 1)	0.00%
Lower East Side	(3, 2)	9.26%	(2, 1)	14.14%	(4, 1)	55.02%
Lower Manhattan	(3, 4)	19.35%	(4, 3)	18.28%	(1, 1)	0.00%
Upper East Side	(4, 4)	17.76%	(4, 4)	16.22%	(2, 1)	5.79%
Upper West Side	(2, 3)	15.42%	(5, 4)	14.55%	(1, 1)	0.00%

In fact, Airbnb has been segmenting the listings into different partitions since launching 'Airbnb Plus', which, according to Airbnb, is a selection of only the highest quality homes with hosts known for great reviews and attention to detail. Our partition display policy can be implemented in a similar fashion. That is, customers searching for lodging options in a neighborhood will be presented with a selection of listings, with the probability of seeing each partition governed by the market share allocated to each partition.

6 Extension to Non-Unit Inventory

Different from Airbnb, sellers on platforms such as Amazon or Taobao may possess more than one unit of inventory for each product. We thus formulate the objective functions and derive the corresponding FOCs under multi-unit and infinite-unit scenarios. We theoretically derive the optimal display policy when the platform is sufficiently large. Notably, in the non-unit inventory case, we can still derive the optimal display policy under arbitrary market conditions and fairness constraints by incorporating the outputs from the tabularization procedure described in Table 1 into the MIP formulation.

6.1 Finite Inventory

First, we assume that seller i owns one SKU and has W_i units of inventory in stock. We assume the market size to be larger than the inventory level, i.e., $W_i < M$. The seller's objective function when facing demand M can be written as

$$\max_{p_i} \Pi_i(M, p_{-i}) = p_i \left(W_i - \sum_{j=0}^{W_i - 1} (W_i - j) \binom{M}{j} q_i^j (1 - q_i)^{M-j} \right) \tag{13}$$

where q_i is defined as in Equation (2). Note that, instead of expressing the total revenue as the sum of revenue from selling $1, 2, ...W_i$ products, Equation (13) captures the difference in the total revenue collected from selling all the products and sum of the revenue losses when there are $1, 2, ...W_i$ unsold products. Then, the FOC of Equation (13) can be expressed as

$$\sum_{j=0}^{W_i - 1} (W_i - j) \binom{M}{j} q_i^j (1 - q_i)^{M-j} (1 + M\beta p_i q_i - j\beta p_i) = W_i. \tag{14}$$

Given an arbitrary M, obtaining the closed-form solution for Equation (14) is analytically challenging due to the combinatorial and nonlinear nature of the equation. Nonetheless, when demand M approaches infinity, the probability that all the W_i products are purchased approaches 1. Thus, Equation (14) reduces to $(1 + M\beta p_i q_i - j\beta p_i) = W_i$, which allows us to obtain the following result:

Theorem 2. *Suppose that there are N products $\mathcal{S} = \{1, 2, \ldots, N\}$, each with finite inventory W_i. For any $\gamma > 1$, there exists a threshold $M(\gamma)$, such that when $M > M(\gamma)$, for any display policy $\{\mathcal{S}_k, M_k\}_{k=1}^K$ that satisfies $M_k < M/\gamma$, we have*

$$\sum_{i=1}^{N} \Pi_i^*(\mathcal{S}, M) > \sum_{k=1}^{K} \sum_{i \in \mathcal{S}_k} \Pi_i^*(\mathcal{S}_k, M_k).$$

Thus, when each product has finite units, it is still optimal for the platform to display the entire assortment to all customers when demand M is sufficiently large.

6.2 Infinite Inventory

Finally, we consider an extreme case in which each vendor on the platform holds infinite inventory for the listed product. In this case, the seller's objective function can be written as:

$$\max_{p_i} \Pi_i^s(p_{-i}) = p_i q_i = \frac{p_i \exp{(a_i - \beta p_i)}}{1 + \sum_1^N \exp{(a_j - \beta p_j)}}. \tag{15}$$

When all the demand can be satisfied, the purchasing probability in Equation (15) reduces to the standard MNL model (which is also equivalent to Equation (7) by setting $M = 1$). Then, the FOC for Equation (15) becomes

$$\frac{1}{\beta p_i} = 1 - q_i \tag{16}$$

In this case, the equilibrium price and revenue no longer depend on total demand. Thus, the platform's problem reduces to deciding only the number of listings in each partition. Intuitively, when the platform hosts a large number of sellers, with each possessing a high level of inventory, then it can be optimal to display only a subset of products. Proposition 4 formally characterizes the optimal display policy when the number of sellers is large:

Proposition 4. *Denote by* $\Pi(\mathcal{S})$ *the total revenue from displaying assortment* $\mathcal{S}$ *to customers. Suppose that product quality satisfies* $a_{ub} = a_1 \geq a_2 \geq \cdots \geq a_N = a_{lb}$*, where* $a_{lb} > 0, a_{ub} > 1/\beta$ *and* $a_{ub} - a_{lb} < \ln(\beta a_{ub} - 1)$*. Then, there exists a threshold* N_0 *such that when the cardinality of* $\mathcal{S}$ *satisfies* $|\mathcal{S}| > N_0$*, we have* $\Pi(\mathcal{S}) < \Pi(\{a_1\})$*.*

Theorem 4 indicates that when the market becomes sufficiently competitive and the products are similar in quality, the platform should only display the product with the highest quality. Notably, our numerical results of the unit-inventory case presented in Panel (b) of Figure 2 point to a similar conclusion: when the number of products offered is considerably larger than the market size, it is optimal to display only a small subset of the entire assortment.

7 Conclusion

In this paper, we investigate how platforms such as Airbnb should display their assortment to maximize total revenue. Specifically, we consider a stylized model where the profit-maximizing platform determines the partition of the products and traffic assigned to each partition. Each seller on the platform supplies a distinct product with one-unit inventory and sets the price of the product given the prices of the other sellers in the partition. We provide theoretical justification for Airbnb's current display strategy by showing that it is optimal to display the entire product assortment when a platform faces sufficiently large demand. Moreover, to derive the equilibrium price for each seller and ultimately offer recommendations on the display policy, we tabularize the FOC and formulate the platform's problem as an MIP. Exploiting the MIP, we are able to compute the equilibrium pricing for each seller and, on the basis of this, recommend the corresponding optimal display policy for the platform. Our theoretical and numerical results carry key managerial implications: depending on the market conditions characterized by the number of sellers and customers, the platform should Additionally, we introduce the concept of fairness in the display policy-making process. We leverage our MIP framework to further incorporate the fairness constraints on the attractiveness and the demand allocated to each partition. We also use data from Airbnb to demonstrate how our framework can be applied in reality and simultaneously demonstrate the revenue loss that Airbnb incurs to achieve a fair display. Finally, we extend our current model to the case in which each seller has more than one unit of the listed product and discuss the optimal display policy when demand is sufficiently high.

In general, our MIP framework can be applied to platforms on which the service provider supplies a single unit of product. Such cases include not only Airbnb but also the labor market, where each company posts a job with one vacancy and competes with other companies on salary, and eBay, where each seller posts one product and competes on price through the "buy it now" option. Nevertheless, our MIP framework is also flexible, as not only it is able to include other industry-specific constraints, but importantly, it can also be applied to characterize the equilibrium price and derive the optimal display policy for platforms on which sellers have more than one unit of inventory in stock.

References

Anderson, S. P., A. De Palma, and J.-F. Thisse (1992). *Discrete choice theory of product differentiation*. MIT press.

Bateni, M. H., Y. Chen, D. Ciocan, and V. Mirrokni (2018). Fair resource allocation in a volatile marketplace. *Available at SSRN 2789380*.

Bernstein, F., A. G. Kök, and L. Xie (2015). Dynamic assortment customization with limited inventories. *Manufacturing & Service Operations Management 17*(4), 538–553.

Bernstein, F. and V. Martínez-de Albéniz (2017). Dynamic product rotation in the presence of strategic customers. *Management Science 63*(7), 2092–2107.

Bertsimas, D., V. F. Farias, and N. Trichakis (2011). The price of fairness. *Operations research 59*(1), 17–31.

Bertsimas, D., V. F. Farias, and N. Trichakis (2012). On the efficiency-fairness trade-off. *Management Science 58*(12), 2234–2250.

Besbes, O. and D. Sauré (2016). Product assortment and price competition under multinomial logit demand. *Production and Operations Management 25*(1), 114–127.

Bostoen, F. (2018). Neutrality, fairness or freedom? principles for platform regulation. *Principles for Platform Regulation (March 31, 2018). Internet Policy Review 7*(1), 1–19.

Cachon, G. P. and S. Netessine (2006). Game theory in supply chain analysis. *Models, methods, and applications for innovative decision making*, 200–233.

Caro, F., V. Martínez-de Albéniz, and P. Rusmevichientong (2014). The assortment packing problem: Multiperiod assortment planning for short-lived products. *Management Science 60*(11), 2701–2721.

Cohen, M., A. N. Elmachtoub, and X. Lei (2019). Price discrimination with fairness constraints. *Available at SSRN 3459289*.

Ferreira, K. J. and J. Goh (2021). Assortment rotation and the value of concealment. *Management Science 67*(3), 1489–1507.

Fudenberg, D. and J. Tirole (1991). Game theory, 1991. *Cambridge, Massachusetts 393*(12), 80.

Gallego, G. and R. Wang (2014). Multiproduct price optimization and competition under the nested logit model with product-differentiated price sensitivities. *Operations Research 62*(2), 450–461.

Heese, H. S. and V. Martínez-de Albéniz (2018). Effects of assortment breadth announcements on manufacturer competition. *Manufacturing & Service Operations Management 20*(2), 302–316.

Kök, A. G., M. L. Fisher, and R. Vaidyanathan (2008). Assortment planning: Review of literature and industry practice. *Retail supply chain management*, 99–153.

Li, H., D. Simchi-Levi, M. X. Wu, and W. Zhu (2019). Estimating and exploiting the impact of photo layout in the sharing economy. *Available at SSRN 3470877*.

McCoy, J. H. and H. L. Lee (2014). Using fairness models to improve equity in health delivery fleet management. *Production and Operations Management 23*(6), 965–977.

Mo, J. and J. Walrand (2000). Fair end-to-end window-based congestion control. *IEEE/ACM Transactions on networking 8*(5), 556–567.

Talluri, K. and G. Van Ryzin (2004). Revenue management under a general discrete choice model of consumer behavior. *Management Science 50*(1), 15–33.

Wagner, L. and V. Martínez-de Albéniz (2020). Pricing and assortment strategies with product exchanges. *Operations Research 68*(2), 453–466.

Wang, R. (2012). Capacitated assortment and price optimization under the multinomial logit model. *Operations Research Letters 40*(6), 492–497.

Appendix for
Price Competition and Assortment Display in Online Marketplace

A Proof for Proposition 1

For existence, we can show the derivative of revenue for listing i with respect to p_i is

$$\frac{d\Pi_i}{dp_i} = 1 - (M\beta p_i q_i + 1)(1 - q_i)^M. \tag{A.1}$$

Clearly when $p_i = 0$, we have $\frac{d\Pi_i}{dp_i} = 1 - (1 - q_i)^M > 0$, and when p_i is very large, we can show that $\frac{d\Pi_i}{dp_i} \to 0$ and $\frac{d\Pi_i}{dp_i} < 0$. This is because as $p_i \to \infty$ we have $q_i \to 0$ and $p_i q_i = \frac{p_i}{1+S_{-i}e^{\beta p_i - a_i}} \to 0$, where $S_{-i} = 1 + \sum_{j \neq i} e^{a_j - \beta p_j}$ is a constant, so $\frac{d\Pi_i}{dp_i} \to 0$ and

$$\lim_{p_i \to \infty} \frac{M \ln(1 - q_i) + \ln(M\beta p_i q_i + 1)}{M\beta p_i q_i} = \lim_{p_i \to \infty} \frac{\ln(1 - q_i)}{\beta p_i q_i} + \frac{\ln(M\beta p_i q_i + 1)}{M\beta p_i q_i} = 1 > 0.$$

This implies $(M\beta p_i q_i + 1)(1 - q_i)^M > 1$ when p_i is large and we get $\frac{d\Pi_i}{dp_i} < 0$ immediately.

Now consider the second order derivative

$$\begin{aligned}
\frac{d^2\Pi_i}{dp_i^2} &= -(1 - q_i)^M (M\beta q_i - \beta^2 M p_i q_i (1 - q_i)) - M\beta q_i (1 - q_i)(1 - q_i)^{M-1}(M\beta p_i q_i + 1) && \text{(A.2)} \\
&= -M\beta q_i (1 - q_i)^M (M\beta p_i q_i - \beta p_i (1 - q_i) + 2). && \text{(A.3)}
\end{aligned}$$

When $p_i = 0$, we have $\frac{d^2\Pi_i}{dp_i^2} = -2\beta M q_i (1 - q_i)^M < 0$. From the expression above, we can see directly that $M\beta p_i q_i - \beta p_i (1 - q_i) + 2 \to -\infty$ and $q_i (M\beta p_i q_i - \beta p_i (1 - q_i) + 2) \to 0$. This means that when p_i is large, $\frac{d^2\Pi_i}{dp_i^2} > 0$ and approaches to zero. Because the expression is a continuous function of p_i and flips the sign on the domain $[0, +\infty)$, we know there exists at least one root. Setting the second order derivative to zero, we get $Mq_i + q_i - 1 = -\frac{2}{\beta p_i}$. The left hand side is a decreasing function in p_i and the right hand side is increasing in p_i, and there can be at most one intersect. Now we know that there exist a single non-saddle p^{**} such that $\frac{d^2\Pi_i}{dp_i^2} = 0$. It is then obvious that $\frac{d\Pi_i}{dp_i}$ is decreasing on $[0, p^{**}]$ and increasing on $[p^{**}, +\infty)$. Knowing that $\frac{d\Pi_i}{dp_i}$ is positive when $p_i = 0$, and it approaches to zero from the negative side when p_i is large, we conclude that there exists a single root p^* for the first derivative, so that $\frac{d\Pi_i}{dp_i} > 0$ on $[0, p^*]$ and $\frac{d\Pi_i}{dp_i} < 0$ on $[p^*, +\infty)$. This proves quasi-concavity of Π_i as a function of p_i.

Since $q_i = \frac{e^{a_i - \beta p_i}}{1+\sum_j e^{a_j - \beta p_j}} = \frac{1}{1+S_{-i}e^{\beta p_i - a_i}}$, where $S_{-i} = 1 + \sum_{j \neq i} e^{a_j - \beta p_j} \in (1, 1 + \sum_j e^{a_j}) \triangleq (1, S)$ is bounded, we have

$$(M\beta p_i q_i + 1)(1 - q_i)^M > (M\beta p_i \frac{1}{1 + Se^{\beta p_i - a_i}} + 1)(1 - \frac{1}{1 + e^{\beta p_i - a_i}})^M \triangleq f_i(p_i).$$

Following the same argument before, we can prove the existence of p_i^* such that $f_i(p_i) > 1$ when $p_i > p_i^*$.

This means $\frac{d\Pi_i}{dp_i} < 0$ when $p_i > p_i^*$. Note that p_i^* only depend on the qualities $\{a_j\}_{j=1}^n$, so we can restrict the price space to be $\Pi_i[0, p_i^*]$, which is a compact set. Fudenberg and Tirole (1991) implies existence of pure strategy Nash equilibrium under such condition.

For uniqueness, let $H = \{\frac{\partial^2\Pi_i}{\partial p_i \partial p_j}\}_{N\times N}$, from Cachon and Netessine (2006) we only need to prove that $(-1)^N|H|$ is positive at the equilibrium. From direct calculation, we get the following

$$\begin{aligned}
\frac{\partial^2\Pi_i}{\partial p_i^2} &= -M\beta q_i(1-q_i)^M(M\beta p_i q_i - \beta p_i(1-q_i)+2) \\
\frac{\partial^2\Pi_i}{\partial p_i \partial p_j} &= M\beta q_i q_j(1-q_i)^{M-1}(M\beta p_i q_i - \beta p_i(1-q_i)+1).
\end{aligned}$$

By applying the calculation rules for determinant, the sign of $|H|$ is the same as the sign of $|A|$ where

$$A_{ii} = -(\frac{1}{q_i}-1)(M\beta p_i q_i - \beta p_i(1-q_i)+2), \qquad A_{ij} = M\beta p_i q_i - \beta p_i(1-q_i)+1.$$

Denote $h_i = M\beta p_i q_i - \beta p_i(1-q_i)+1$ and $r_i = 1 - \frac{h_i+1}{q_i}$, we have

$$|A| = \begin{vmatrix} h_1+r_1 & h_1 & \dots & h_1 \\ h_2 & h_2+r_2 & \dots & h_2 \\ \vdots & \vdots & & \vdots \\ h_N & h_N & \cdots & h_N+r_N \end{vmatrix} = r_1 r_2 \dots r_N \left(1 + \frac{h_1}{r_1} + \frac{h_2}{r_2} + \cdots + \frac{h_N}{r_N}\right).$$

We next prove $r_i < 0$ at equilibrium point. Indeed, $r_i < 0 \iff h_i + 1 - q_i > 0 \iff M\beta p_i q_i + 1 - \beta p_i(1-q_i) + 1 - q_i > 0$. From the FOC condition 2, we get $\beta p_i = ((1-q_i)^{-M}-1)/(Mq_i)$, plug this into the expression of r_i, we get

$$\begin{aligned}
r_i < 0 &\iff (1-q_i)^{-M} - \frac{(1-q_i)^{-M}-1}{Mq_i}(1-q_i) + (1-q_i) > 0 \\
&\iff (M+1)q_i + (Mq_i+1)(1-q_i)^{M+1} > 1
\end{aligned}$$

By Bernoulli inequality and the fact that $M \geq 1$, we get

$$\begin{aligned}
\text{LHS of A} &\geq (M+1)q_i + (q_i+1)(1-q_i)^{M+1} \\
&= (M+1)q_i + (1-q_i^2)(1-q_i)^M \\
&> (M+1)q_i + (1-q_i^2)(1-Mq_i) \\
&= 1 + (q_i - q_i^2) + Mq_i^3 > 1
\end{aligned}$$

which proves $r_i < 0$.

Note that we have $h_i < 1 - q_i + h_i = -q_i r_i$, which implies $\frac{h_i}{r_i} > -q_i$ when $r_i < 0$. In this case, we can obtain $1 + \frac{h_1}{r_1} + \frac{h_2}{r_2} + \cdots + \frac{h_N}{r_N} > 1 - q_1 - q_2 - \cdots - q_N > 0$. So the sign of $|A|$ is $(-1)^N$ and we have

$(-1)^N|H|$ is positive at the equilibrium point. This combines with the fact that Π_i is quasi-concave implies the uniqueness of pure strategy Nash equilibrium by Cachon and Netessine (2006).

B Proof for Proposition 2

By applying Implicit Function Theorem to the ith FOC (2) we obtain

$$\begin{aligned}\frac{\partial FOC_i}{\partial a_i} &= -(1-q_i)^M(M\beta p_i q_i(1-q_i)) - Mq_i(1-q_i)(1-q_i)^{(M-1)}(M\beta p_i q_i+1)\\ &= -Mq_i(1-q_i)^M(M\beta p_i q_i - \beta p_i(1-q_i)+1)\end{aligned}$$

We already have the partial derivative with respect to p_i from Equation (A.2). Hence we are able to calculate $\frac{\partial p_i}{\partial a_i}$ as

$$\frac{\partial p_i}{\partial a_i} = -\frac{\partial FOC_i/\partial a_i}{\partial FOC_i/\partial p_i} = \frac{M\beta p_i q_i - \beta p_i(1-q_i)+1}{(M\beta p_i q_i - \beta p_i(1-q_i)+2)\beta}, \tag{A.4}$$

Note that by applying the Bernoulli inequality to the FOC equation (2) , we can get

$$1 = (1+M\beta p_i q_i)(1-q_i)^M > (1+M\beta p_i q_i)(1-Mq_i) \Longrightarrow 1+M\beta p_i q_i > \beta p_i. \tag{A.5}$$

Hence we can conclude that $\frac{\partial p_i}{\partial a_i} > 0$. Furthermore, when $M \to \infty$, we can have $Mq_i \to \infty$ (this is a direct result from Lemma 1), so the right hand side of (A.4) approaches to $\frac{1}{\beta}$ and we finish our proof.

C Proof for Lemma 1

For the first part of the lemma, if there exists p_U such that $0 < p_i < p_U$ for infinite many M, then there exists a lower bound q_L for q_i because $q_i = \frac{\exp(a_i-\beta p_i)}{1+\sum_j \exp(a_j-\beta p_j)} = \frac{1}{\exp(\beta p_i - a_i)+\sum_j \exp(a_j-a_i-\beta(p_j-p_i))} > \frac{1}{\exp(\beta p_U - a_i)+\sum_j \exp(a_j-a_i+2\beta p_U)}$ and this expression does not contain p_i. Then consider the FOC equation, we have

$$-M\ln(1-q_L) < -M\ln(1-q) = \ln(M\beta pq+1) < \ln(M\beta p_U+1). \tag{A.6}$$

Take $M \to \infty$ and we immediately get contradiction since the LHS of (A.6) is linear in M while the order of the RHS of (A.6) is logarithm in M. We can also have $q_i \to 0$ by the MNL equation since we have now already proven all $p_i \to \infty$.

For the second part, we first prove $M\beta p_i q_i \to \infty$ as $M \to \infty$. Since $q_i \to 0$, from the FOC equation, we have:

$$\ln(M\beta p_i q_i+1) = -M\ln(1-q_i) = Mq_i + o(Mq_i) \tag{A.7}$$

If there exists U^* such that $M\beta p_i q_i < U^*$ for infinite many M, then because $p_i \to \infty$, we must have $Mq_i \to 0$.

This means the right side of (A.7) goes to 0, hence we get $M\beta p_i q_i \to 0$. However, in this case we can get $\ln(M\beta p_i q_i + 1) \sim M\beta p_i q_i$, which is a contradiction to (A.7) because we would have $M\beta p_i q_i \sim Mq_i$ and this cannot be true since $p_i \to \infty$.

Then we prove the second statement in the lemma. Note that from the MNL model, we have $\beta p_i = a_i - \ln q_i + \ln(\frac{1-q_i}{1+S_{-i}})$ where $S_{-i} = \sum_{j \neq i} e^{a_j - \beta p_j}$. Since $p_j \to \infty$, we have $S_{-i} \to 0$ and $\ln(\frac{1-q_i}{1+S_{-i}}) \to 0$. Because $\lim_{M\to\infty} M\beta p_i q_i = \infty$, plug this into the FOC, we get

$$\ln Mq_i + \ln(a_i - \ln q_i + \ln(\frac{1-q_i}{1+S_{-i}})) + o(1) = -M\ln(1-q_i)$$

Since $q_i \to 0$ as $M \to \infty$, we further get $\ln(a_i - \ln q_i + \ln(\frac{1-q_i}{1+S_{-i}})) \to \infty$ and $\lim_{M\to\infty}(\ln(a_i - \ln q_i + \ln(\frac{1-q_i}{1+S_{-i}})) - \ln\ln\frac{1}{q_i}) = 0$, so we have

$$\ln Mq_i + \ln\ln\frac{1}{q_i} + o(1) = -M\ln(1-q_i) = Mq_i + o(Mq_i).$$

Note that $\ln\ln\frac{1}{q_i} \to \infty$, so $Mq_i \to \infty$. In this case, we can have the order $\ln Mq_i = o(Mq_i)$. By merging the term $\ln Mq_i$ to $o(Mq_i)$ we finally achieve $Mq_i \sim \ln\ln\frac{1}{q_i}$ as desired.

D Proof for Lemma 2

Denote $q_1 = q_i^*(M, \mathcal{S}_1), q_2 = q_i^*(M/\gamma, \mathcal{S}_2)$, which is the probability of item a_i in the two sub-markets (we drop index i for simplicity). Then from Lemma 1 we have: $Mq_1 \sim \ln\ln\frac{1}{q_1}$, $\frac{M}{\gamma}q_2 \sim \ln\ln\frac{1}{q_2}$. Take the ratio of these two equations, we get

$$\lim_{M\to\infty} \frac{\gamma q_1}{q_2}\frac{\ln(-\ln q_2)}{\ln(-\ln q_1)} = 1 \tag{A.8}$$

We next prove by contradiction that the above equation (A.8) implies

$$\lim_{M\to\infty} \frac{\gamma q_1}{q_2} = 1.$$

Let $x = \ln\gamma q_1, y = \ln q_2$, then by taking logarithm of both sides of (A.8), we have

$$\lim_{M\to\infty} x - y + \ln\ln(-y) - \ln\ln(\ln\gamma - x) = 0$$

If $\lim_{M\to\infty} x - y \neq 0$, then without loss of generality, suppose there exists $\epsilon > 0$ such that $x - y > \epsilon$ for infinite many times. Take M large enough, we can require x, y to satisfy

$$x - y + \ln\ln(-y) - \ln\ln(\ln\gamma - x) < \epsilon/2$$

This means $\ln\ln(-y) - \ln\ln(\ln\gamma - x) < -\epsilon/2$, which implies $\ln(-y)/\ln(\ln\gamma - x) < e^{-\epsilon/2}$. Take exponential and we get

$$\epsilon - x < -y < (\ln\gamma - x)^{e^{-\frac{\epsilon}{2}}}.$$

This cannot hold when M is large enough because $e^{-\epsilon/2} < 1$ and we have $x \to -\infty$ when $M \to \infty$. As a result, we can conclude that

$$\lim_{M\to\infty} x - y = \lim_{M\to\infty} \ln(\frac{\gamma q_1}{q_2}) = 0,$$

which gives us $\lim_{M\to\infty} \frac{\gamma q_1}{q_2} = 1$ as desired.

Finally, we consider the revenue Π_i. From Lemma 1, we have $\lim_{M\to\infty} Mq_1 = \lim_{M\to\infty}(M/\gamma)q_2 = \infty$, so we are able to calculate the revenue difference as follows,

$$\begin{aligned}
\lim_{M\to\infty} \Pi_i(\mathcal{S}_1, M) - \Pi_i(\mathcal{S}_2, M/\gamma) &= \lim_{M\to\infty} (p_1 - p_2) - (\frac{p_1}{M\beta p_1 q_1 + 1} - \frac{p_2}{\frac{M}{\gamma}\beta p_2 q_2 + 1}) \\
&= \lim_{M\to\infty} p_1 - p_2 \\
&= \lim_{M\to\infty} \ln(\frac{q_2}{q_1}) + \ln(\frac{1-q_1}{1+S_1}) - \ln(\frac{1-q_2}{1+S_2}) \qquad \text{(By the FOC)} \\
&= \lim_{M\to\infty} \ln(\frac{q_2}{q_1}) \\
&= \ln\gamma.
\end{aligned}$$

E Proof for Lemma 3

We first prove p_i is monotone increasing with respect to M, i.e., when the demand expands, the seller should charge higher price. We prove this by decoupling demand M and total attractiveness z. Specifically, by writing the FOC equation $(1 + M\beta p_i q_i)(1 - q_i)^M = 1$ as

$$\left(1 + M\beta p_i \frac{\exp(a_i - \beta p_i)}{1+z}\right)\left(1 - \frac{\exp(a_i - \beta p_i)}{1+z}\right)^M = 1, \tag{A.9}$$

we can define $p_i(z, M)$ to be the solution of the above equation (A.9). In this way, the real attractiveness z^* would be the solution of $\sum_{i=1}^N \exp(a_i - \beta p_i(z, M)) = z$. For any $M_1 < M_2$, suppose z_1 and z_2 to be the total attractiveness at equilibrium under demand M_1 and M_2, i.e., $\sum_{i=1}^N \exp(a_i - \beta p_i(z_1, M_1)) = z_1$ and $\sum_{i=1}^N \exp(a_i - \beta p_i(z_2, M_2)) = z_2$, then we only need to prove $p_i(z_1, M_1) < p_i(z_2, M_2)$.

By implicit function theorem, we have

$$\begin{aligned}
\frac{\partial p_i(z, M)}{\partial z} &= \frac{\beta p_i(1-q_i) - (1 + M\beta p_i q_i)}{\beta(1+z)\left((1-\beta p_i)(1-q_i) + 1 + M\beta p_i q_i\right)}, \\
\frac{\partial p_i(z, M)}{\partial M} &= \frac{(\beta p_i q_i + \ln(1-q_i)(1 + M\beta p_i q_i))(1-q_i)}{M\beta q_i((\beta p_i - 1)(1-q_i) - (1 + M\beta p_i q_i))}.
\end{aligned}$$

Note that we already prove in (A.5) that $M\beta p_i q_i + 1 > \beta p_i$. Also we have $\ln(1 - q_i) < -q_i$ by simple calculus. As a result, we obtain $\frac{\partial p_i}{\partial M} > 0$ and $\frac{\partial p_i}{\partial z} < 0$. Hence, we have $p_i(z_1, M_1) < p_i(z_1, M_2)$ as $M_1 < M_2$.

We next prove $z_1 > z_2$. If we do have $z_1 > z_2$, this would result in $p_i(z_1, M_2) < p_i(z_2, M_2)$ and we have our desired result. In fact, let $f(z) = \sum_{i=1}^N \exp(a_i - \beta p_i(z, M_2)) - z$, then z_2 is the root of $f(z)$. Define $q_i(z, M) = \exp(a_i - \beta p_i)/(1 + z)$, then we can calculate its derivative with respect to z as

$$\frac{\partial q_i(z, M)}{\partial z} = -\frac{q_i}{1+z} \cdot \frac{1 - q_i}{((1 - \beta p_i)(1 - q_i) + 1 + M\beta p_i q_i)}. \tag{A.10}$$

For attractiveness level z_1, we have $q_i(z_1, M_2) < q_i(z_1, M_1) < 1$ because p_i is monotone increasing with respect to M. Also note that $q_i(z, M_2) \equiv 0$ and $q_i(z, M_2) \equiv 1$ is two solutions to the above ODE (A.10). Then by classical ODE theory, because $0 < q_i(z_1, M_2) < 1$, then $0 < q_i(z, M_2) < 1$ for any $z > 0$. As a result, we would have $\frac{\partial q_i(z,M)}{\partial z} < 0$. Hence, for any $z > z_1$, we can get $q_i(z, M_2) < q_i(z_1, M_2) < q_i(z_1, M_1)$ and we can calculate $f'(z)$ as

$$\begin{aligned}
f'(z) &= -\sum_{i=1}^N \exp(a_i - \beta p_i)\beta\frac{\partial p_i}{\partial z} - 1 \\
&= -\sum_{i=1}^N (1+z)q_i\beta\frac{\beta p_i(1 - q_i) - (1 + M\beta p_i q_i)}{\beta(1+z)\left((1 - \beta p_i)(1 - q_i) + 1 + M\beta p_i q_i\right)} - 1 \\
&= \sum_{i=1}^N q_i\frac{(1 + M\beta p_i q_i) - \beta p_i(1 - q_i)}{(1+z)\left((1 - \beta p_i)(1 - q_i) + 1 + M\beta p_i q_i\right)} - 1 \\
&\le \sum_{i=1}^N q_i - 1 \\
&< \sum_{i=1}^N q_i(z_1, M_1) - 1 < 0.
\end{aligned}$$

Because we already have $f(z_1) = \sum_{i=1}^N \exp(a_i - \beta p_i(z_1, M_2)) - z_1 < \sum_{i=1}^N \exp(a_i - \beta p_i(z_1, M_1)) - z_1 = 0$, hence $f(z) < 0$ for every $z > z_1$ and we must have $z_2 < z_1$ since z_2 is a root of $f(z)$. As a result, p_i is increasing in M.

Then calculate $\frac{\partial \Pi_i}{\partial M}$ without decoupling z and M. In this case, M would be the only variable and z is determined by the system of FOC equations. By taking derivative with respect to M at both sides of the MNL equation, we have (write $q_i' = \frac{\partial q_i}{\partial M}$ and $p_i' = \frac{\partial p_i}{\partial M}$):

$$q_i' = -q_i\beta p_i' + q_i\left(\sum_{j=1}^N q_j\beta p_j'\right).$$

So we can further calculate $\frac{\partial \Pi_i}{\partial M}$ as

$$\begin{aligned}
\frac{\partial \Pi_i}{\partial M} &= \frac{\beta p_i}{(M\beta p_i q_i + 1)^2}\left(p_i q_i + (M\beta p_i q_i + 1 + q_i) M q_i p_i' + M p_i q_i'\right) \\
&= \frac{\beta p_i}{(M\beta p_i q_i + 1)^2}\left(p_i q_i + (M\beta p_i q_i + 1 + q_i) M q_i p_i' + M\beta p_i(-q_i p_i' + q_i(\sum_{j=1}^{N} q_j p_j'))\right) \\
&\geq \frac{\beta p_i}{(M\beta p_i q_i + 1)^2}\left(p_i q_i + (1 + q_i) M q_i p_i' + M\beta p_i q_i(\sum_{j=1}^{N} q_j p_j')\right) \\
&> 0.
\end{aligned}$$

The first inequality is because of (A.5) and the second inequality is because we already prove $p_i' > 0$.

F Proof for Theorem 2

By similar argument in Lemma 1, we can still have $\lim_{M\to\infty} q_i = 0$ and $\lim_{M\to\infty} p_i = \infty$. We then prove that $Mq_i \to \infty$ when $M \to \infty$. If this is not true, there must exists $U > 0$ such that $Mq_i < U$ for infinite many M, with out loss of generality, we assume this is true for all M. Denote $A_j(M) = (W_i - j)\binom{M}{j} q_i^j (1-q_i)^{M-j}(1 + M\beta p_i q_i - j\beta p_i)$, then the FOC can be written as $\sum_{j=0}^{W_i-1} A_j(M) = W_i$, and we can have

$$\begin{aligned}
\frac{A_j(M)}{A_{j-1}(M)} =& \frac{(W_i - j)\binom{M}{j} q_i^j (1-q_i)^{M-j}(1 + M\beta p_i q_i - j\beta p_i)}{(W_i - j + 1)\binom{M}{j-1} q_i^{j-1}(1-q_i)^{M-j+1}(1 + M\beta p_i q_i - (j-1)\beta p_i)} \\
=& \frac{W_i - j}{W_i - j + 1}\frac{\binom{M}{j}}{\binom{M}{j-1}}\frac{q_i}{1-q_i}\frac{1 + M\beta p_i q_i - j\beta p_i}{1 + M\beta p_i q_i - (j-1)\beta p_i}
\end{aligned} \tag{A.11}$$

which is bounded under our hypothesis. (We use the fact that $\binom{M}{j}/\binom{M}{j-1} = O(M)$ when $M \to \infty$.) Note that $\sum_{j=0}^{W_i-1} A_j(M) = W_i$, so each term $A_j(M)$ is bounded, which is true for $j = 0$. Then by the same argument in Lemma 1, we can have $Mq_i \to \infty$, which is a contradiction.

Since we now $\lim_{M\to\infty} Mq_i = \infty$, we can deduce that $\frac{A_j(M)}{A_{j-1}(M)} \to \infty$ by (A.11). However, the FOC requires $\sum_{j=0}^{W_i-1} A_j(M) = W_i$, so we must have $\lim_{M\to\infty} A_{W_i-1}(M) = W_i$ and $\lim_{M\to\infty} A_j(M) = 0, \forall j < W_i - 1$. This implies

$$\binom{M}{W_i - 1} q_i^{W_i-1}(1-q_i)^{M-W_i+1}(1 + (Mq_i - W_i + 1)\beta p_i) \to W_i, \quad (M \to \infty) \tag{A.12}$$

We can analyze the order of the left hand side of equation (A.12) as we did in Lemma 2:

$$\text{LHS of A.12} = O\left((Mq_i)^{W_i-1}(1-q_i)^{M-W_i+1} M q_i p_i\right).$$

By taking logarithm, we can further have

$$(W_i - 1)\ln(Mq_i) + (M - W_i + 1)\ln(1 - q_i) + \ln(Mq_i) + \ln(p_i) = O(1).$$

Because $\ln(1 - q_i) = O(q_i)$, we then achieve

$$Mq_i + \ln(p_i) = Mq_i + \ln\left(a_i - \ln q_i + \ln\left(\frac{1 - q_i}{1 + S_{-i}}\right)\right) = O(1).$$

Hence, we conclude $Mq_i \sim \ln\ln\frac{1}{q_i}$ by the fact that $S_{-i} \to 0$ and $q_i \to 0$. Then it follows from the proof of Lemma 2 that

$$\lim_{M\to\infty} p_i(\mathcal{S}_1, M) - p_i(\mathcal{S}_2, M/\gamma) = \lim_{M\to\infty} \ln\left(\frac{q_i(\mathcal{S}_1, M)}{q_i(S_2, M/\gamma)}\right) = \ln\gamma.$$

Note that the FOC requires

$$\sum_{j=0}^{W_i-1}(W_i - j)\binom{M}{j}q_i^j(1 - q_i)^{M-j}(1 + (Mq_i - j)\beta p_i) = W_i.$$

this combines with the fact that $Mq_i \to \infty$ give us

$$\lim_{M\to\infty}\sum_{j=0}^{W_i-1}(W_i - j)\binom{M}{j}q_i^j(1 - q_i)^{M-j}p_i = 0.$$

So the difference in revenue can be calculated as

$$\lim_{M\to\infty} \Pi_i(\mathcal{S}_1, M) - \Pi_i(\mathcal{S}_2, M/\gamma) = \lim_{M\to\infty} W_i(p_i(\mathcal{S}_1, M) - p_i(\mathcal{S}_2, M/\gamma)) = W_i \ln\gamma.$$

From here, the rest of the proof is similar to Theorem 1.

G Proof for Proposition 3

We will prove this bound by using the decoupling technique as we did in the proof of Lemma (3). This means we will consider the total attractiveness and demand separately. In fact, by writing the FOC equation $(1 + M\beta p_i q_i)(1 - q_i)^M = 1$ as

$$\left(1 + M\beta p_i\frac{\exp(a_i - \beta p_i)}{1 + z}\right)\left(1 - \frac{\exp(a_i - \beta p_i)}{1 + z}\right)^M = 1, \tag{A.13}$$

we can define $p_i(z, M)$ to be the solution of the above equation (A.13) and define the corresponding $q_i(z, M) = \exp(a_i - \beta p_i(z, M))/(1 + z)$ and $\Pi_i(z, M) = p_i(1 - (1 - q_i)^M)$. Then we have the following two lemmas:

Lemma A.1. *For any demand level M and total attractiveness $z_1 > z_2 > z_i^*$, where z_i^* is the attractiveness* *level when we only display the i-th seller under demand M, we get*

$$\frac{|\Pi_i(z_1,M)-\Pi_i(z_2,M)|}{\Pi_i(z_2,M)} \le \frac{1+q_i(z_2,M)}{\beta p_i(z_2,M)}\frac{z_1-z_2}{1+z_2} \le \frac{2(z_1-z_2)}{1+z_2}. \tag{A.14}$$

Here, $q_i(z_2,M)$ and $p_i(z_2,M)$ are the corresponding probability and price under z_2 and M.

Lemma A.2. *For any demand level $M_1 > M_2$ and total attractiveness level $z > z_i^*$, where z_i^* is the attractiveness level when we only display the i-th seller under demand M_1, we get*

$$\frac{|\Pi_i(z,M_1)-\Pi_i(z,M_2)|}{\Pi_i(z,M_2)} \le \left(\frac{1}{\beta p_i(z,M_2)}+q_i(z,M_2)\right)\frac{M_2-M_1}{M_2} \le \frac{2(M_2-M_1)}{M_2}. \tag{A.15}$$

Here, $p_i(z,M_2)$ and $q(z,M_2)$ are the corresponding price and probability under z and M_2.

Proof for Proposition 3: We will use these two lemmas to prove the statement in Proposition 3. Suppose seller i is in a partition of attractiveness z_1 and demand M_1 and he/she hope to switch to another partition with attractiveness z_2 and demand M_2. We further assume after the switching, the attractiveness of the new partition becomes z_2^*. Obviously, we get $z_2^* > z_2$. So the envy level can be written as

$$EN_i = \frac{\Pi_i(z_2^*,M_2)-\Pi_i(z_1,M_1)}{\Pi_i(z_1,M_1)}.$$

If $z_1 < z_2$ or $M_1 > M_2$, then we can directly apply Lemma A.2 or A.1 with replacing z_2 to be z_1 or M_2 to be M_1. Hence, we only need to prove the case where $z_1 \ge z_2$ and $M_1 \le M_2$. Denote $\Pi_i^1 = \Pi_i(z_1,M_1), \Pi_i^2 = \Pi_i(z_2^*,M_1), \Pi_i^3 = \Pi_i(z_2^*,M_2)$, then from Lemma A.1:

$$\frac{\Pi_i^2-\Pi_i^1}{\Pi_i^2} \le 2\frac{z_1-z_2^*}{1+z_2^*} \le 2\frac{z_1-z_2}{z_2} \le 2(\frac{1}{\alpha}-1), \tag{A.16}$$

where the last inequality follows from the definition of α-fair policy. Similarly, by Lemma A.2, we get

$$\frac{\Pi_i^3-\Pi_i^2}{\Pi_i^2} \le 2\frac{M_2-M_1}{M_1} \le 2(\frac{1}{\delta}-1), \tag{A.17}$$

where the last inequality follows from the definition of δ-fair policy. Furthermore, if $2/3 < \alpha$, then $2(\frac{1}{\alpha}-1) < 1$, so by using (A.16), we could obtain

$$\frac{\Pi_i^2}{\Pi_i^1} \le \frac{\alpha}{3\alpha-2}.$$

Finally, by adding (A.16) and (A.17), and noticing the inequality above, we would have

$$\frac{\Pi_i^3-\Pi_i^1}{\Pi_i^1} \le \frac{2\alpha}{3\alpha-2}\left(\frac{1}{\alpha}+\frac{1}{\delta}-2\right).$$

Proof for Lemma A.1: We will use mean value theorem to estimate the difference in revenue when the attractiveness z moves from z_1 to z_2. In this section, we omit the same demand level M for simplicity. We first consider $\frac{\partial p_i}{\partial z}$. From the definition of $q_i = \frac{e^{a_i - \beta p_i}}{1+z}$, we get $\frac{\partial q_i}{\partial z} = q_i(-\beta \frac{\partial p_i}{\partial z} - \frac{1}{1+z})$. By differentiating both side of the FOC equation and plugging in this relationship, we get

$$\frac{\partial p_i}{\partial z} = \frac{\beta p_i(1-q_i) - (1 + M\beta p_i q_i)}{\beta(1+z)\left((1-\beta p_i)(1-q_i) + 1 + M\beta p_i q_i\right)}. \tag{A.18}$$

Then, because $\Pi_i(z) = p_i - \frac{p_i}{M\beta p_i q_i + 1}$, we further achieve

$$\frac{\partial \Pi_i(z)}{\partial z} = -\frac{\Pi_i}{\beta p_i(1+z)}\left(1 + \frac{q_i}{2 + M\beta p_i q_i - \beta p_i + \beta p_i q_i - q_i}\right). \tag{A.19}$$

Note that by applying the Bernoulli inequality to the FOC, we can get

$$1 = (1 + M\beta p_i q_i)(1-q_i)^M > (1 + M\beta p_i q_i)(1 - Mq_i) \Longrightarrow 1 + M\beta p_i q_i > \beta p_i. \tag{A.20}$$

And by taking logarithm of the FOC and using the fact that $\ln(1+x) \le x$ for $x > -1$, we will get a lower bound for p_i:

$$0 = \ln(1 + M\beta p_i q_i) + M\ln(1-q_i) \le M\beta p_i q_i - Mq_i \Longrightarrow \beta p_i \ge 1. \tag{A.21}$$

Now that we have (A.20) and (A.21), we can estimate the denominator in equation (A.19) that $2 + M\beta p_i q_i - \beta p_i + \beta p_i q_i - q_i > 1$. Hence,

$$-\frac{\Pi_i}{\beta p_i(1+z)}(1+q_i) < \frac{\partial \Pi_i(z)}{\partial z} < -\frac{\Pi_i}{\beta p_i(1+z)}. \tag{A.22}$$

From (A.20) and (A.18), we can conclude that $\frac{\partial p_i}{\partial z} < 0$, which means as the total attractiveness increases, the price p_i will decrease in return. And for probability q_i, we can calculate

$$\frac{\partial q_i}{\partial z} = -\frac{q_i}{1+z} * \frac{1-q_i}{((1-\beta p_i)(1-q_i) + 1 + M\beta p_i q_i)} < 0. \tag{A.23}$$

The term $\frac{\Pi_i}{p_i} = 1 - (1-q_i)^M$ is therefore decreasing as z increases as well. Hence this is also the case for $\frac{\Pi_i}{p_i(1+z)}$ and $\frac{\Pi_i(1+q_i)}{p_i(1+z)}$. Then by mean value theorem, for any $z_1 > z_2$, there exists a $z_2 < z^* < z_1$ such that

$$\Pi_i(z_1) - \Pi_i(z_2) = \frac{\partial \Pi_i(z^*)}{\partial z^*}(z_1 - z_2) > -\frac{\Pi(z^*)(1 + q_i(z^*))}{\beta p_i(z^*)(1+z^*)}(z_1 - z_2) > -\frac{\Pi_i(z_2)(1 + q_i(z_2))}{\beta p_i(z_2)(1+z_2))}(z_1 - z_2).$$

which further gives (note that $z_1 > z_2$ means $\Pi_i(z_1) < \Pi_i(z_2)$)

$$\frac{|\Pi_i(z_1) - \Pi_i(z_2)|}{\Pi_i(z_2)} < \frac{1 + q_i(z_2)}{\beta p_i(z_2)}\frac{z_1 - z_2}{1 + z_2} < \frac{2(z_1 - z_2)}{1 + z_2}, \tag{A.24}$$

where the last inequality follows from $\beta p_i > 1$ and $q_i < 1$.

Proof for Lemma A.2: For the demand M, we use similar approach as in lemma A.1. First, by the implicit function theorem, we are able to calculate $\frac{\partial p}{\partial M}$ as:

$$\frac{\partial p_i}{\partial M} = \frac{(\beta p_i q_i + \ln(1-q_i)(1+M\beta p_i q_i))(1-q_i)}{M\beta q_i((\beta p_i - 1)(1-q_i) - (1+M\beta p_i q_i))}. \tag{A.25}$$

By using (A.20), we can easily see that $\frac{\partial p_i}{\partial M} > 0$. Meanwhile, we can also derive $\frac{\partial \Pi_i}{\partial M}$ by implicit function theorem:

$$\frac{\partial \Pi_i}{\partial M} = -\frac{(\beta p_i q_i + (1+M\beta p_i q_i)\ln(1-q_i))(1-q_i)(M\beta p_i q_i + 2 - \beta p_i)}{(M\beta p_i q_i + 1)^2[(1-\beta p_i)(1-q_i) + 1 + M\beta p_i q_i]} + \frac{p_i^2 q_i}{(1+M\beta p_i q_i)^2} \tag{A.26}$$

$$= p_i \frac{-(1-q_i)\ln(1-q_i)(M\beta p_i q_i + 2 - \beta p_i) + \beta p_i q_i^2}{(M\beta p_i q_i + 1)[(1-\beta p_i)(1-q_i) + 1 + M\beta p_i q_i]}. \tag{A.27}$$

Because we have $M\beta p_i q_i + 1 > \beta p_i$ and $\ln(1-q_i) < -q_i$, then the first term in (A.26) is positive, which gives us $\frac{\partial \Pi_i}{\partial M} > 0$.

For the fact that $0 < q < 1$, then by simple calculus, we can show that $|(1-q)\ln(1-q)| \le q$, so we are able to estimate $|\frac{\partial \Pi_i}{\partial M}|$ as

$$\begin{aligned}
|\frac{\partial \Pi_i}{\partial M}| &= p_i \frac{|(1-q_i)\ln(1-q_i)(M\beta p_i q_i + 2 - \beta p_i)| + \beta p_i q_i^2}{(M\beta p_i q_i + 1)[(1-\beta p_i)(1-q_i) + 1 + M\beta p_i q_i]} \\
&< p_i \frac{q_i(M\beta p_i q_i + 2 - \beta p_i) + \beta p_i q_i^2}{(M\beta p_i q_i + 1)[(1-\beta p_i)(1-q_i) + 1 + M\beta p_i q_i]} \\
&< \frac{p_i}{M\beta p_i q_i + 1}[q_i + \frac{q_i(q_i(1-\beta p_i))}{(1-\beta p_i)(1-q_i) + 1 + M\beta p_i q_i} + \beta p_i q_i^2] \\
&< \frac{p_i}{M\beta p_i q_i + 1}[q_i + \beta p_i q_i^2] = \frac{\Pi_i}{M}(\frac{1}{\beta p_i} + q_i)
\end{aligned} \tag{A.28}$$

We next prove the term $\frac{p_i}{M\beta p_i q_i + 1}$ is decreasing with respect to M. If we have the mononicity condition, note that $p_i q_i^2$ is also decreasing with respect to M by the fact that $\beta p_i > 1$ in (A.21), then the term $\frac{\Pi_i}{M}(\frac{1}{\beta p_i} + q_i)$ is decreasing with respect to M. By (A.25) and (A.26), we can calculate the derivative as

$$\frac{\partial(\frac{p_i}{M\beta p_i q_i + 1})}{\partial M} = \frac{(1 + M\beta p_i^2 q_i)\frac{\partial p_i}{\partial M} - \beta^2 p_i^2 q_i}{\beta(M\beta p_i q_i + 1)^2}. \tag{A.29}$$

Hence $\frac{\partial(\frac{p_i}{M\beta p_i q_i + 1})}{\partial M} < 0 \iff \frac{\partial p_i}{\partial M} < \frac{\beta^2 p_i^2 q_i}{1 + M\beta^2 p_i^2 q_i}$. By plugging in (A.25), this condition is further equivalent to

$$\beta p_i q_i(1-q_i) + M\beta^2 p_i^2 q_i^2 + \ln(1-q_i)(1 + M\beta^2 p_i^2 q_i)(1-q_i) > 0.$$

By simple calculus, we can prove $x\ln(x) > x - 1$ for all $1 > x > 0$. Hence, $\ln(1-q_i)(1-q_i) > -q_i$. As a result, we only need to prove $\beta p_i q_i + \ln(1-q_i) > 0$. This can be derived from the FOC equation. Recall that the FOC is $\ln(1 + M\beta p_i q_i) + M\ln(1-q_i) = 0$, so $\ln(1-q_i) = -\frac{\ln(1+M\beta p_i q_i)}{M} > -\beta p_i q_i$ where we use the fact that $\ln(1+x) < x$ for $x > -1$.

Hence, for any $M_1 > M_2$, by mean value theorem, there exists $M_1 > M^* > M_2$ such that

$$\Pi_i(M_1)-\Pi_i(M_2) = |\frac{\partial \Pi_i(M^*)}{\partial M}|(M_1-M_2) \leq |\frac{\partial \Pi_i(M_1)}{\partial M}|(M_1-M_2) \leq \Pi_i(M_1)\left(\frac{1}{\beta p_i(M_1)} + q_i(M_1)\right)\frac{M_1 - M_2}{M_1}$$

where we used the monotonicity condition for the partial derivative $\frac{\partial \Pi}{\partial M}$. Finally, by rearranging terms, we have

$$\frac{\Pi_i(M_1) - \Pi_i(M_0)}{\Pi_i(M_i)} \leq \left(\frac{1}{\beta p_i(M_1)} + q_i(M_1)\right)\frac{M_1 - M_2}{M_1} \leq 2\frac{M_1 - M_2}{M_1}.$$

Here, the last inequality follows from $\beta p_i > 1$ and $q_i < 1$.

H Proof for Proposition 4

We first prove that under the equilibrium, we have $p_1 \geq p_2 \geq \cdots \geq p_N > 1$, $q_1 \geq q_2 \geq \cdots \geq q_N$ and $e^{a_1-\beta p_1} \geq e^{a_2-\beta p_2} \geq \cdots \geq e^{a_N-\beta p_N}$, i.e., products with higher quality should charge higher prices and have larger attractiveness. From the first order condition, we have $\frac{1}{\beta p_i} = 1 - q_i$, which indicates $\beta p_i > 1$ immediately. Then for any $a_i \geq a_j$

$$\frac{1 - 1/(\beta p_i)}{1 - 1/(\beta p_j)} = \frac{q_i}{q_j} = e^{(a_i-\beta p_i)-(a_j-\beta p_j)} \geq e^{\beta(p_j-p_i)}$$

if $p_i < p_j$, then the left side $\frac{1-1/(\beta p_i)}{1-1/(\beta p_j)} < 1$ and the right side $e^{\beta(p_j-p_i)} > 1$, which is a contradiction. Hence we must have $p_i \geq p_j$. From the FOC, this indicates $q_i \geq q_j$ which further shows $e^{a_i-\beta p_i} \geq e^{a_j-\beta p_j}$.

Then for any $i = 1, 2, \ldots, N$, we have

$$\frac{1}{\beta p_i} = 1 - \frac{e^{a_i-\beta p_i}}{1 + \sum_k e^{a_k-\beta p_k}} \geq 1 - \frac{e^{a_i-\beta p_i}}{1 + Ne^{a_N-\beta p_N}}. \tag{A.30}$$

Denote p_N^* to be the solution of equation

$$\frac{1}{\beta p_N^*} = 1 - \frac{e^{a_N-\beta p_N^*}}{1 + Ne^{a_N-\beta p_N^*}},$$

then we have $p_N < p_N^*$ by (A.30). We can also easily see that $\lim_{N\to\infty} \beta p_N^* = 1$

For the seller $i = 1$, we further have

$$\frac{1}{\beta p_1} \geq 1 - \frac{e^{a_1-\beta p_1}}{1 + Ne^{a_N-\beta p_N}} \geq 1 - \frac{e^{a_1-\beta p_1}}{1 + Ne^{a_N-\beta p_N^*}}.$$

Let p_1^* be the solution that satisfies

$$\frac{1}{\beta p_1^*} = 1 - \frac{e^{a_1-\beta p_1^*}}{1 + Ne^{a_N-\beta p_N^*}}, \tag{A.31}$$

then we have $p_N \leq p_{N-1} \leq p_{N-2} \leq \cdots \leq p_1 \leq p_1^*$. As a result, $\Pi(\mathcal{S}) = \sum_{i=1}^N p_i q_i = \sum_{i=1}^N p_i - N/\beta \leq$

$N(p_1^* - 1/\beta)$. And note that $\lim_{N\to\infty} p_1^* = \lim_{N\to\infty} p_N^* = 1/\beta$, this enable us to derive an upper bound on the revenue $\Pi(\mathcal{S})$

$$\Pi(\mathcal{S}) < N(p_1^* - 1/\beta) = p_1^* \frac{N e^{a_1 - \beta p_1^*}}{1 + N e^{a_N - \beta p_N^*}} \to e^{a_1 - a_N}/\beta, \quad (N \to \infty).$$

For a single display with product a_1, the revenue is $\Pi(\{a_1\}) = p'q' = p' - 1/\beta$, where p' satisfies

$$\frac{1}{\beta p'} = 1 - \frac{e^{a_1 - \beta p'}}{1 + e^{a_1 - \beta p'}} \iff p' - \frac{1}{\beta} = \frac{e^{a_1 - p'}}{\beta}.$$

So $\Pi(a_1) > \lim_{N\to\infty} \Pi(\mathcal{S}) \iff a_N > p' \iff a_N > (1 + e^{a_1 - a_N})/\beta$, which is satisfied by the condition $u - l < \ln(\beta u - 1)$.

I The Equal Demand Case

One of the commonly used traffic splitting practices is to assign each partition the same market share (i.e., $\delta = 1$). In this case, the tabulation process remains identical to that in Section 4, except that now the market size each seller faces is M/k instead of MP_v. Consequently, we drop the market share subscript v and the corresponding constraints in our MIP formulation, which can be written as:

$$\begin{aligned}
\max_{x,z} \; & \sum_k x_{i,j,k} \, \Pi_{i,j} && && \text{(A.32)} \\
s.t. \; & \sum_i x_{i,j,k} E_{i,j} = z_{j,k} \mathcal{Z}_j && \forall j, k \\
& \sum_{j,k} x_{i,j,k} = 1, && \forall i \\
& \sum_j z_{j,k} = 1, && \forall k \\
& \sum_j z_{j,k_1} \mathcal{Z}_j \geq \alpha \sum_j z_{j,k_2} \mathcal{Z}_j && \forall k_1, k_2 = 1, 2, \ldots, K \\
& x_{i,j,k} \text{ binary}, \quad z_{j,k} \text{ binary}
\end{aligned}$$

Formulation (A.32) is a simplification of formulation (7) as each seller faces equal demand. In this case, only the α-fair constraint remains. Similar to Proposition 3, we can bound seller i's revenue difference between different levels of α through the following proposition:

Proposition I.1. *When $\delta = 1$, consider two display policies that are α_0-fair and α_1-fair, respectively. When $\alpha_0 > \alpha_1$ and the corresponding revenue satisfies $\Pi_i^{\alpha_0} < \Pi_i^{\alpha_1}$, then the revenue difference between these two* *display policies can be bounded by*

$$\frac{\Pi_i^{\alpha_1} - \Pi_i^{\alpha_0}}{\Pi_i^{\alpha_1}} \leq \frac{1 + q_i^{\alpha_1}}{\beta p_i^{\alpha_1}} \left(\frac{1}{\alpha_0 \alpha_1} - 1 \right) \leq 2 \left(\frac{1}{\alpha_0 \alpha_1} - 1 \right). \tag{A.33}$$

Moreover, under the α_1-fair display policy, for any two sellers i, j that belong to the same partition and have quality a_i, a_j where $a_i < a_j$, we have $\frac{1+q_i^{\alpha_1}}{\beta p_i^{\alpha_1}} > \frac{1+q_j^{\alpha_1}}{\beta p_j^{\alpha_1}}$.

The left hand side of Equation (A.33) represents the percentage revenue gain for seller i if the display rule is relaxed from α_0- to α_1-fair. As α_1 decreases, the platform allows for greater imbalance of the attractiveness $\mathcal{Z}$ among different partitions, thus widening the potential revenue gap as captured by the term $1/(\alpha_0\alpha_1) - 1$ on the right hand side. Moreover, this proposition indicates that a less fair display rule does not affect every seller in a homogeneous fashion. Instead, sellers with lower product quality are prone to suffer from bigger revenue losses under unfair partition rules, which is manifested by the fact that sellers with lower quality a_i faces a larger potential revenue gap due to greater $1 + q_i^{\alpha_1}/\beta p_i^{\alpha_1}$.

We next use numerical examples to demonstrate the impact of an α-fair policy on the revenue of sellers. In Panel (a) of Figure I.1, we demonstrate the tightness of the bound proposed in Proposition I.1 by plotting the revenue change when $(\alpha_0, \alpha_1) = (1, 0.8)$. We use the same instance as before, with $K = 3$ partitions. The dots in each color represent the actual revenue change for listings in each partition. We observe that, consistent with Proposition I.1, products with higher quality have lower bound of potential revenue loss, which we plot as dashed line. In addition, the bound is relatively close to the actual data with the maximal ratio around 50%. Interestingly, the relationship between the optimal partition, the quality and the revenue loss is not straightforward. The green partition suffers the most revenue loss on average, yet it contains both the very high end and low end listings. Such complexity also illustrates the importance of the MIP framework, since simple display policies such as dividing into quality tiers are clearly not optimal. Meanwhile, in Panel (b) of Figure I.1, we show how sellers with different levels of quality get affected by different α-fair policies. Instead of focusing on the optimal partitions, we plot the average revenue for high, middle or low end sellers under various levels of α values. Specifically, we sort 100 sellers according to their product quality a_i, split into five groups each with size 20, label them from low to high quality group, and then calculate the average of their revenue under a range of α-fair policies. The result indicates that product with high quality benefit the most when there is no fairness constraint but will lose revenue as the partition becomes more fair. On the contrary, low quality products generally benefit from a more fair display policy, highlighting the welfare transfer mechanism the α-fairness achieves.

J Proof for Proposition I.1

Suppose for α_i-fair policy $(i = 0, 1)$, the corresponding attractiveness for each partition is $z_1^i \leq z_2^i \leq \cdots \leq z_K^i$. We then state the following lemma for the relationship between these attractiveness and leave the proof later:

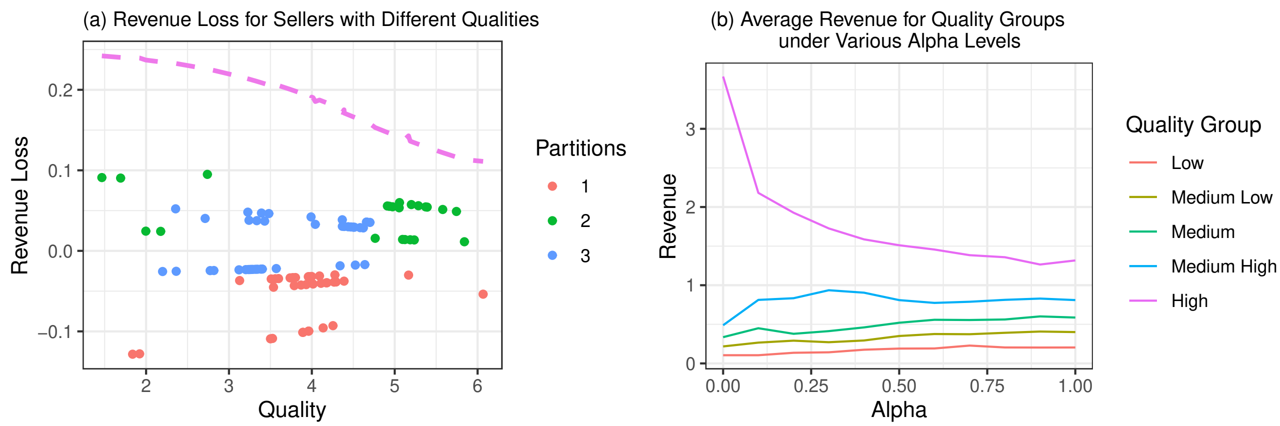

Figure I.1: Revenue difference and average partitioned revenue under equal demand case. The market combines of 100 sellers with quality a_i following normal distribution $\mathcal{N}(3, 1)$. We set the partition number and demand to be 3 and 50 respectively. In panel (a), the revenue difference is calculated when α changes from 0.8 to 1 and the partition is for the case $\alpha = 1$. In panel (b), the group size is set to be 20.

Lemma A.3. *We have*

$$\frac{\max\{z_K^0, z_K^1\}}{\min\{z_1^0, z_1^1\}} < \frac{1}{\alpha_0\alpha_1}.$$

Proof for Proposition I.1: Suppose for the i-th seller, it faces total attractiveness of z^{α_0} and z^{α_1} in the α_0-fair and α_1-fair case respectively. Because we assume $\Pi_i^{\alpha_0} < \Pi_i^{\alpha_1}$, we get $z^{\alpha_0} > z^{\alpha_1}$, then from Lemma A.1, we have

$$\frac{\Pi_i^{\alpha_1} - \Pi_i^{\alpha_0}}{\Pi_i^{\alpha_1}} \leq \frac{1 + q_i^{\alpha_1}}{\beta p_i^{\alpha_1}} \frac{z^{\alpha_0} - z^{\alpha_1}}{z^{\alpha_1}} \leq \frac{1 + q_i^{\alpha_1}}{\beta p_i^{\alpha_1}} \left(\frac{1}{\alpha_0\alpha_1} - 1\right),$$

where the last inequality follows from our Lemma A.3, and we are done with our proof.

As for the monotonicity of $\frac{1+q_i(z_j)}{\beta p_i(z_j)}$, suppose both product i and j is in a partition of total attractiveness z and demand M. Then by implicit function theorem, we have

$$\frac{\partial p}{\partial a} = \frac{1}{\beta} - \frac{1 - q}{\beta[(1 - \beta p)(1 - q) + 1 + M\beta pq]}. \tag{A.34}$$

For $q = \frac{\exp(a - \beta p)}{1+z}$, hence $\frac{\partial q}{\partial a} = q(1 - \beta\frac{\partial p}{\partial a})$, and we can further have

$$\frac{\partial\left(\frac{1+q}{\beta p}\right)}{\partial a} = \frac{pq - (\beta pq + q + 1)\frac{\partial p}{\partial a}}{\beta p^2}.$$

Then by (A.34) and (A.20), we can directly prove that $\frac{\partial(\frac{1+q}{\beta p})}{\partial a} < 0$, which leads to our statement in the proposition.

Proof for Lemma A.3: To prove the result, we only need to justify that $z_1^0 \leq z_K^1$ and $z_1^1 \leq z_K^0$. In this way, the interval $[z_1^0, z_K^0]$ and $[z_1^1, z_K^1]$ will always overlap, and we will have the desired result naturally. In the following text, we implicitly assume each partition receives M/k demand level and omit the notation for demand for simplicity.

If $z_1^1 > z_K^0$, then for each seller, it faces a higher attractiveness level under α_1-fair policy, hence $\Pi_i^{\alpha_1} < \Pi_i^{\alpha_0}$ for every $i = 1, 2, \ldots, N$. However, this can not hold since the partition policy that is α_0-fair is always α_1-fair, so the result of the optimization problem will always yield $\sum_i \Pi_i^{\alpha_1} > \sum_i \Pi_i^{\alpha_0}$, which is a contradiction.

If $z_1^0 > z_K^1$, that is, each seller will face a lower attractiveness level under α_1-fair policy. Denote $E_i(z) = \exp(a_i - p_i(z))$ to be the attractiveness level of the i-th seller facing total attractiveness z. Write A_k^i to be the set of all the seller that belongs to the k-th partition under α_i-fair partition policy. Then by the definition of attractiveness level z_k^i, we have

$$z_k^i = \sum_{j \in A_k^i} E_j(z_k^i).$$

Then we prove we cannot have the above equations hold for both $i = 0$ and 1. We first prove that for a single partition group A_k^1, for any $z' > z_k^1$, we have $z' > \sum_{j \in A_k^1} E_j(z')$. Write the corresponding function for the group A_k^1 as $f(z) = z' - \sum_{j \in A_k^1} E_j(z')$, then from the calculation in the previous proof G, we have

$$f'(z) = 1 - \sum_{j \in A_k^1} E_j(z) \frac{(1 + M\beta p_j q_j) - \beta p_j (1 - q_j)}{(1 + z)((1 - \beta p_j)(1 - q_j) + 1 + M\beta p_j q_j)} > 1 - \sum_{j \in A_k^1} q_j(z) \tag{A.35}$$

where $q_j(z) = \frac{E_j(z)}{1+z}$. Also from the proof G, we get $q_j(z)$ is decreasing with respect to z. So for any $z' > z_k^1$, we have $\sum_{j \in A_k^1} q_j(z') < \sum_{j \in A_k^1} q_j(z_k^1) = \frac{z_k^1}{1+z_k^1} < 1$, which indicates that $f'(z) > 0$ for any $z > z_k^1$. Note that $f(z_k^1) = 0$, which indicates that $f(z') > 0$ for any $z' > z_k^1$.

Because $z_k^1 = \sum_{j \in A_k^1} E_j(z_k^1)$, we would get $z_1^0 > \sum_{j \in A_k^1} E_j(z_1^0)$ for the fact that $z_1^0 > z_K^1 \geq z_k^1$, $k = 1, 2, \ldots, K$. By summing over each partition group, we further have $K z_1^0 > \sum_{j=1}^N E_j(z_1^0)$. Then by subtracting the sellers in A_1^0, which satisfies $z_1^0 = \sum_{j \in A_1^0} E_j(z_1^0)$, we obtain $(K-1) z_1^0 > \sum_{k=2}^K \sum_{j \in A_k^0} E_j(z_1^0)$. Consider the function $g_K(z) = (k-1)z - \sum_{k=2}^K \sum_{j \in A_k^0} E_j(z)$, then by similar calculation as A.35, we have for any $z_2^0 \geq z \geq z_1^0$,

$$g_K'(z) > (K-1) - \sum_{k=2}^K \sum_{j \in A_k^0} q_j(z) > (K-1) - \sum_{k=2}^K \sum_{j \in A_k^0} q_j(z_k^0) = \sum_{k=2}^K (1 - \sum_{j \in A_k^0} q_j(z_k^0)) > 0.$$

So $(K-1) z_2^0 > \sum_{k=2}^K \sum_{j \in A_k^0} E_j(z_2^0)$, and we then drop all the seller in partition group A_2^0 and proceed the previous argument. By iterating through this for K steps, we finally have $z_K^1 > \sum_{j \in A_K^1} E_j(z_K^1)$, which is a contradiction. We hence have $z_1^0 \leq z_K^1$ and the result follows by previous discussion.