$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$$

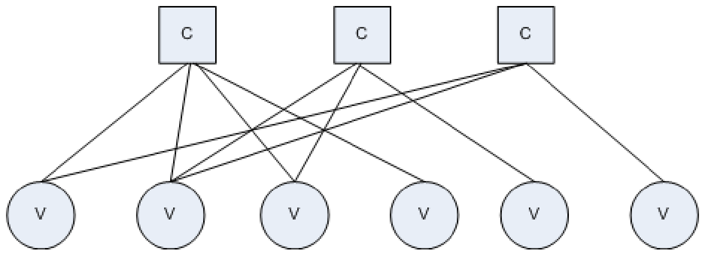
$$\mathbf{d} = \mathbf{c} \cdot \mathbf{G} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} = [\underbrace{1 \quad 0 \quad 1}_{\mathbf{c}} \quad \underbrace{0 \quad 1 \quad 1}_{\mathbf{w}}]$$

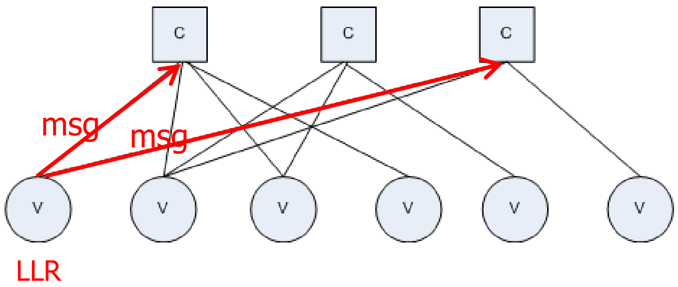
$$\mathbf{G} \cdot \boldsymbol{H}^{\boldsymbol{T}} = \mathbf{0}$$

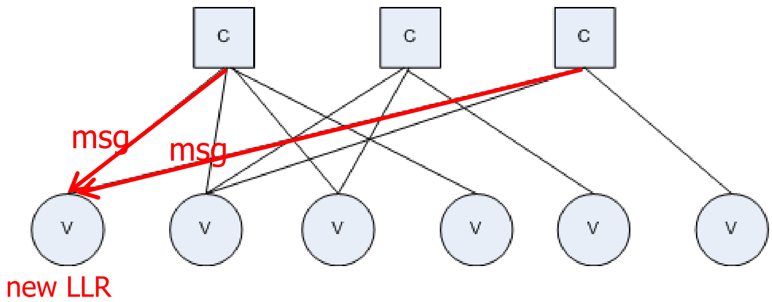
$$\boldsymbol{H}^{\boldsymbol{T}} \cdot \boldsymbol{c} = \mathbf{0}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_0 \oplus c_1 \oplus c_2 \oplus c_3 = 0$$
$$c_1 \oplus c_2 \oplus c_4 = 0$$
$$c_0 \oplus c_1 \oplus c_5 = 0$$


C
C
C
V
V
V
V
V
V


C
C
C
msg
msg
V
V
V
V
V
V
LLR


C
C
C
msg
msg
V
V
V
V
V
V
new LLR

(1) Initialization: For $i \in \{1, \dots, M\}, j \in \{1, \dots, N\}$

$$\alpha_j^0 = \ln \frac{P(VN_j = 0|y_j)}{P(VN_j = 1|y_j)} = \frac{2y_j}{\sigma^2} \quad ①$$

$$\alpha_{i,j}^0 = \alpha_j^0 \quad ②$$

(2) Update CN: $\forall\, CN_i, i \in \{1, \dots, M\}$ do

$$\beta_{i,j}^n = \mathrm{sgn}(\beta_{i,j}^n) \cdot |\beta_{i,j}^n| \quad ③$$

where

$$\mathrm{sgn}(\beta_{i,j}^n) = \prod_{j' \in \boldsymbol{\mathcal{N}}(i) \setminus j} \mathrm{sgn}(\alpha_{i,j'}^{n-1})$$

$$|\beta_{i,j}^n| = \bigwedge_{j' \neq j} (\alpha_{i,j'}^{n-1})$$

(3) Update VN: $\forall\, VN_j, j \in \{1, \dots, N\}$

$$\alpha_{i,j}^n = \alpha_{i,j}^0 + \sum_{i' \in \boldsymbol{\mathcal{M}}(j) \setminus i} \beta_{i',j}^n \quad ④$$

(4) Update a posteriori LLR:

$$\alpha_j^n = \alpha_j^0 + \sum_{i' \in \boldsymbol{\mathcal{M}}(j)} \beta_{i',j}^n \quad ⑤$$

(5) Repeat from (2) to (4) until the maximum number of iteration is reached, the estimated codeword is then given by:

$$\hat{c}_j = \begin{cases} 0, & if\ \alpha_j^n > 0 \\ 1, & else \end{cases} \quad ⑥$$

$$\alpha_{i,j}^{n} = \alpha_j^{\mathrm{n}} - \beta_{i,j}^{n}$$

(1) Initialization: For $i \in \{1, \dots, M\}, j \in \{1, \dots, N\}$

$$\alpha_j^0 = \ln \frac{P(VN_j = 0|y_j)}{P(VN_j = 1|y_j)} = \frac{2y_j}{\sigma^2} \qquad ⑦$$

$$\alpha_{i,j}^0 = \alpha_j^0 \qquad ⑧$$

$$\beta_{i,j}^0 = 0 \qquad ⑨$$

(2) $\forall\ CN_i, i \in \{1, \dots, M\}$ do

$$\beta_{i,j}^n = \mathrm{sgn}\left(\beta_{i,j}^n\right) \cdot \left|\beta_{i,j}^n\right| \qquad ⑩$$

and

$$\alpha_j^n = (\alpha_j^{\mathrm{n}-1} - \beta_{i,j}^{n-1}) + \beta_{i,j}^n \qquad ⑪$$

where

$$\mathrm{sgn}\left(\beta_{i,j}^n\right) = \prod_{j' \in \mathcal{N}(i) \backslash j} \mathrm{sgn}(\alpha_j^{n-1} - \beta_{i,j'}^{n-1})$$

$$\left|\beta_{i,j}^n\right| = \lambda \cdot \min_{j' \in \mathcal{N}(i) \backslash j} \left|\alpha_j^{n-1} - \beta_{i,j'}^{n-1}\right|$$

(3) Repeat (2) until the maximum number of iteration is reached, the estimated codeword is then given by ⑥

Algorithm	$\boldsymbol{\bigwedge_{j' \neq j} \left(\alpha_{i,j'}^{n-1} \right)}$
SP	$\Phi\left(\sum_{j' \in \mathcal{N}(i) \backslash j} \Phi(\left\| \alpha_{i,j'}^{n-1} \right\|) \right)$ $\Phi(x) = -\log\left(\tanh\left(\frac{x}{2}\right)\right)$
MS	$\min_{j' \in \mathcal{N}(i) \backslash j} \{\left\| \alpha_{i,j'}^{n-1} \right\|\}$
OMS	$\max\{\min_{j' \in \mathcal{N}(i) \backslash j} \{\left\| \alpha_{i,j'}^{n-1} \right\|\} - \beta, 0\}$ $\beta \geq 0$
NMS	$\lambda \cdot \min_{j' \in \mathcal{N}(i) \backslash j} \{\left\| \alpha_{i,j'}^{n-1} \right\|\}$ $\lambda < 1$