UM-SJTU Joint Institute

Honors Mathematics III
(Vv285)

Term Project

Perfect Pendulum

Mathematical Pendulum and the Tautochrone Curve

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Abstract

Pendula play a very significant role in time-keeping and many other engineering areas. However, the period of general pendulum depends on the amplitude of the swing, which implies general pendula can not keep time accurately. Huygens solved this problem by devising a pivot that caused the suspended body, or bob, to swing along the arc of a cycloid so that the period of the pendulum will not change regardless of the amplitude [9]. This project aims to obtain a profound insight into pendula and their periods, along with which many interesting facts are discovered. To start with, we will investigate into the relationship between the period and the amplitude for a mathematical pendulum. Furthermore, detailed derivation of parametric equations of the tautochrone, which turns out to be a cycloid circle, will be presented later in our report. The construction of a tautochrone will also be demonstrated. Finally, we verified our results in the validation section. Based on the previous trial, the tautochrone problem is perfectly solved and the method of producing a clock that can keep time accurately is provided.

Contents

1 Introduction

1.1 Objective

This project will investigate into several crucial and interesting properties of pendulum and their period. The relationship between the period and the amplitude of a mathematical pendulum is supposed to be found. The project also aims at giving detailed derivation of parametric equations of the tautochrone and demostrating the construction of a tautochrone. The results obtained in this project should be verified using reasonable methods. In all, the objective of this project is to analyze the period of pendula and develop a tautochrone curve.

1.2 Background

Pendulum motion is one of the most typical cases of harmonic oscillations, which are of great significance in time-keeping (pendulum clock) and many other practical application in modern engineering. It has been so popular among scientists that many famous physicists and mathematicians, who are now considered pioneers and trailblazers of their days, have been indulged into the study of pendulum. "Physical pendulum" and "mathematical pendulum" are two common pendulum models. A mathematical pendulum is "a particle of mass m connected by a rigid and weightless thread to a base by means of a pin joint that can rotate in a plane"[7]. The motion of a rigid body "suspended on a horizontal axis and allowed to rotate about it" is called physical pendulum[7], which will not be discussed in this project.

Cycloid is a curve generated from the trajectory of a point fixed on a wheel that is rolling along a straight line without sliding, of which coordinates can be parametrized as

$$x = r(\theta - \sin\theta),\ \ y = r(1 - \cos\theta)$$

where r stands for the radius of the wheel and θ represents the angular displacemet. Though cycloid has a relatively simple expression, it is involved in some complex mathematical problems, namely the tautochrone problem and brachistochrone problem, that fascinated many famous mathematicians.

The tautochrone problem, also known as the isochrone problem [5], is a mathematical problem asking if there exists a pendulum of which period is independent of the amplitude. Surprisingly, the solution of the tautochrone is exactly a cycloid. Likewise, the solution to the brachistochrone problem, asking the path between two points through which a particle affected only by gravity travels within the least time, is also proved as a cycloid. This simple cycloid curve has so many sophisticated and beautiful properties, which never ceases to amaze every math enthusiast.

In this project we applied a variety of mathematical theorems, classified them for reference and provided ideas for further researching. To start with, we took advantage of integration calculus and derivatives, as well as partial derivatives in calculation.

$$f(x) = \frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} \frac{f(x)^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!\ 2^n} x^n.$$

Figure 1: The Pendulum Clock[3]

Laplace Transform also played an important role in helping interpret the potential equation, and there are still other important mathematics theorems are applied. Physically we have used the kinetics relationship between acceleration, velocity and displacement. The properties for Standard Harmonic Oscillation was also considered in our calculation.

1.3 Application

The relationship between pendulum's period of trajectory and its period is of crucial importance in the mechanism for pendulum clock to work. The method of producing a clock that can keep time accurately provided in this project will contribute to the industry of clocks with perfect time-keeping ability.

2 Mathematical Pendulum

2.1 Basic Equation for The Period of A Pendulum

The pendulum is widely considered as a perfect example for the conservation of mechanical energy. In this part, we first calculate the period of a simple system of one single pendulum. Notice that the pendulum's mechanical energy depend on the initial height, i.e. the initial angle θ, which lies between the rope line segment and vertical direction. From physical principles we know that an object with only conservative force exerted on possesses a constant amount of energy, which can be denoted by E.

$$E = U + K$$

where K denotes kinetic energy and U denotes potential energy. We can rewrite it as a function of θ that

$$E(\theta) = \frac{1}{2}ml^2(\frac{d\theta}{dt})^2 + mgl(1 - \cos\theta)$$

Since E remain constant for varying θ that change during the waving.

$$\frac{dE}{d\theta} = 0$$

Hence we attain

$$ml(l\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + g\sin\theta\frac{d\theta}{dt}) = 0$$

Since $\frac{d\theta}{dt} \neq 0$, we can obtain that

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{l}\sin(\theta(t)) = 0.$$

Suppose a pendulum is held at an angle $\theta(0) = \theta_0$ at time $t = 0$ and is then released. Since the magnitude of $\dot{\theta}$ is simply the angular frequency, the pendulum's velocity can be defined as

$$\frac{d\theta}{dt}l = v$$

Since the mechanic energy is conserved in this process, we can obtain that

$$\begin{aligned} \Delta E_k &= -\Delta E_p \\ \frac{1}{2}mv^2 - 0 &= -mg\Delta h \\ \frac{1}{2}mv^2 &= mgl(1 - \cos\theta_0 - (1 - \cos\theta)) \end{aligned}$$

We then attain

$$\dot{\theta}^2 l^2 = 2gl(\cos\theta - \cos\theta_0)$$

so

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0),$$

2.2 Advanced Calculation of A Pendulum's Period

From the previous equation 1, we can know that

$$|\dot{\theta}| = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)},$$

and it's obvious that

$$|\dot{\theta}| = |\frac{d\theta}{dt}| = |\frac{v}{l}| = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}.$$

As a result, the period T can be expressed as

$$T = \int_0^T dt = 4\int_0^{\theta_0} \frac{d\theta}{|\dot{\theta}|} = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

In this equation, we can change the formula by introducing a ϕ which satisfies the equation that

$$\sin\phi = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}, \qquad (\phi \in [0, \frac{\pi}{2}])$$

so about the ϕ, we can know that

$$\sin\frac{\theta_0}{2}\ d\sin\phi = d\sin\frac{\theta}{2},$$

$$\sin\frac{\theta_0}{2}\cos\phi\ d\phi = \frac{\cos\frac{\theta}{2}}{2}\ d\theta,$$

so

$$d\theta = \frac{2\sin\frac{\theta_0}{2}\cos\phi\ d\phi}{\cos\frac{\theta}{2}},$$

and at the same time, the value of relative expressions can be obtained as

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} = 1 - 2(\sin\phi\sin\frac{\theta_0}{2})^2,$$

$$\cos\frac{\theta}{2} = \sqrt{1 - \sin^2\frac{\theta}{2}} = \sqrt{1 - \sin^2\phi\sin^2\frac{\theta_0}{2}},$$

$$\cos\theta_0 = 1 - 2\sin^2\frac{\theta_0}{2}.$$

From the two equations above, we can get that

$$\sqrt{2(\cos\theta - \cos\theta_0)} = 2\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\sin^2\phi} = 2\sin\frac{\theta_0}{2}\cos\phi,$$

and the expression of T can be transferred into

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{2\sin\frac{\theta_0}{2}\cos\phi\ d\phi}{2\cos\phi\sin\frac{\theta_0}{2}\sqrt{1 - \sin^2\phi\sin^2\frac{\theta_0}{2}}} = 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2\phi\sin^2\frac{\theta_0}{2}}}. \tag{1}$$

So far, we have obtained all the equations needed for the period, but this integral is too complex. We will use two methods to represent the value of the period. The first method is to relate the period to the arithmetic-geometric mean.And the second method is to use appropriate series expansion to get its approximate value.

Arithmetic-geometric mean

To find the relation between the period and the arithmetic-geometric mean, we first consider the following case.

Let T(a,b) be defined as

$$T(a,b) := \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}, \quad a,b \in \mathbb{R}\backslash\{0\}$$

Substitute $t := b\tan\theta, 0 \le t \le \infty$, we can obtain

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}\theta} &= \frac{b}{\cos^2\theta}\\ \mathrm{d}\theta &= \frac{b\mathrm{d}t}{b^2+t^2}\end{aligned}$$

Thus

$$\begin{aligned}T(a,b) &= \frac{2}{\pi}\int_0^{+\infty} \frac{\frac{b}{b^2+t^2}\mathrm{d}t}{\frac{b}{\sqrt{b^2+t^2}}\sqrt{a^2+t^2}}\\ &= \frac{2}{\pi}\int_0^{+\infty} \frac{\mathrm{d}t}{\sqrt{(b^2+t^2)(a^2+t^2)}}\\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{\sqrt{(b^2+t^2)(a^2+t^2)}}\end{aligned}$$

Then substitute $u := \frac{1}{2}(t - \frac{ab}{t}), -\infty \le u \le +\infty$, we can obtain

$$\begin{aligned}t^2 &= 2ut + ab\\ t - u &= \pm\sqrt{u^2+ab}\\ \frac{\mathrm{d}u}{\mathrm{d}t} &= \frac{1}{2} + \frac{ab}{2t^2}\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\mathrm{d}t}{\sqrt{(b^2+t^2)(a^2+t^2)}} &= \frac{t^2\mathrm{d}u}{\frac{1}{2}(t^2+ab)\sqrt{a^2b^2+(a^2+b^2)t^2+t^4}} \\
&= \frac{\mathrm{d}u}{\frac{1}{2}(t^2+ab)\sqrt{\frac{a^2b^2}{t^4}-\frac{2ab}{t^2}+1+(\frac{a+b}{t})^2}} \\
&= \frac{\mathrm{d}u}{(t^2+ab)\sqrt{(\frac{1}{2}(\frac{ab}{t^2}-1))^2+(\frac{a+b}{2t})^2}}
\end{aligned}$$

As $1-\frac{ab}{t^2}=\frac{2u}{t}$ and, we obtain

$$\begin{aligned}
\frac{\mathrm{d}t}{\sqrt{(b^2+t^2)(a^2+t^2)}} &= \frac{\mathrm{d}u}{(t^2+ab)\sqrt{\frac{u^2}{t^2}+(\frac{a+b}{2t})^2}} \\
&= \frac{\mathrm{d}u}{\sqrt{(u^2+(\frac{a+b}{2})^2)(\frac{t^2+ab}{t})^2}} \\
&= \frac{\mathrm{d}u}{\sqrt{(u^2+(\frac{a+b}{2})^2)(2t-2u)^2}} \\
&= \frac{\mathrm{d}u}{\sqrt{(u^2+(\frac{a+b}{2})^2)(\pm 2\sqrt{u^2+ab})^2}} \\
&= \frac{\mathrm{d}u}{2\sqrt{(u^2+(\frac{a+b}{2})^2)(u^2+ab)}}
\end{aligned}$$

Thus

$$\begin{aligned}
T(a,b) &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{\sqrt{(b^2+t^2)(a^2+t^2)}} \\
&= \frac{2}{\pi}\int_{0}^{+\infty}\frac{\mathrm{d}t}{\sqrt{(b^2+t^2)(a^2+t^2)}} \\
&= \frac{2}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}u}{2\sqrt{(u^2+(\frac{a+b}{2})^2)(u^2+ab)}} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}u}{\sqrt{(u^2+(\frac{a+b}{2})^2)(u^2+ab)}} \\
&= T(\frac{a+b}{2}, ab)
\end{aligned}$$

We can iterate this equation until we get that

$$\begin{aligned} T(a,b) &= T(M(a,b),M(a,b)) \\ &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{M(a,b)^2+t^2} \\ &= \frac{1}{\pi M(a,b)}[\arctan\frac{t}{M(a,b)}]_{-\infty}^{+\infty} \\ &= \frac{1}{M(a,b)} \end{aligned}$$

where M(a,b) is the arithmetic-geometric mean of a and b.

So

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\frac{\pi}{2M(a,b)} \tag{2}$$

In this case,

$$\begin{aligned} T &= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}} \\ &= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\cos^2\phi+\sin^2\phi-\sin^2\frac{\theta_0}{2}\sin^2\phi}} \\ &= 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{\cos^2\phi+\cos^2\frac{\theta_0}{2}\sin^2\phi}} \end{aligned}$$

$$a=1, b=\cos\frac{\theta_0}{2}$$

So

$$T=4\sqrt{\frac{l}{g}}\frac{\pi}{2M(1,\cos\frac{\theta_0}{2})}=\frac{2\pi\sqrt{\frac{l}{g}}}{M(1,\cos\frac{\theta_0}{2})} \tag{3}$$

Series expansion

Before calculating this complicated integral, we can first observe the taylor series of a simpler formula $f(x)=\frac{1}{\sqrt{1-x}}$, and it can be given as

$$f(x)=\frac{1}{\sqrt{1-x}}=\sum_{n=0}^{\infty}\frac{f(x)^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!\,2^n}x^n.$$

From this equation, we can know that for the T we want to get,

$$\frac{1}{\sqrt{1-\sin^2\phi\sin^2\frac{\theta_0}{2}}}=\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!\,2^n}(\sin^2\phi\sin^2\frac{\theta_0}{2})^n.$$

What's more, we know that θ_0 is a pretty small value, so at that small angle, $\sin\theta_0\approx\theta_0$, and therefore we can obtain that

$$\begin{aligned}T&=4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(2n-1)!!}{n!\,2^n}(\sin\phi\sin\frac{\theta_0}{2})^{2n}d\phi\\&=4\sqrt{\frac{l}{g}}\,[\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!\,2^n}\sin\frac{\theta_0}{2}^{2n}\int_0^{\frac{\pi}{2}}\sin\phi^{2n}d\phi].\end{aligned}$$

Now, let's observe the integral $\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\;d\phi$, we can deduce that

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\sin^{2n}\phi\;d\phi&=[-\cos\phi\cdot\sin^{2n-1}\phi]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}(-\cos\phi)(2n-1)\sin^{2n-2}\phi\;\cos\phi\;d\phi\\&=(2n-1)\int_0^{\frac{\pi}{2}}(1-\sin^2\phi)\sin^{2n-2}\phi\;d\phi\\&=\frac{2n-1}{2n}\int_0^{\frac{\pi}{2}}\sin^{2n-2}\phi\;d\phi\\&=\frac{(2n-1)!!}{2^n\cdot n!}\cdot\frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}T&=4\sqrt{\frac{l}{g}}\,[\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{n!\,2^n}\cdot\sin\frac{\theta_0}{2}^{2n}\cdot\frac{(2n-1)!!}{2^n\cdot n!}\cdot\frac{\pi}{2}]\\&\approx2\pi\sqrt{\frac{l}{g}}\,[1+\sum_{n=1}^{\infty}(\frac{(2n-1)!!}{n!\,2^n})^2\cdot(\frac{\theta_0}{2})^{2n}]\\&\approx2\pi\sqrt{\frac{l}{g}}\,[1+(\frac{1}{2})^2\cdot(\frac{\theta_0}{2})^2]\\&=2\pi\sqrt{\frac{l}{g}}\,(1+\frac{\theta_0^2}{16})\\&\approx2\pi\sqrt{\frac{l}{g}}\end{aligned}\tag{4}$$

To demonstrate our approximation is appropriate enough, curves in term of θ_0 are relatively plotted

(Fig. 2), under the case $l = 1$ and $g = 9.8$, including

$$
\begin{aligned}
T_1(\theta_0) &= 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2\phi \sin^2\frac{\theta_0}{2}}} \\
T_2(\theta_0) &= 2\pi\sqrt{\frac{l}{g}}\,(1 + \frac{\theta_0^2}{16}) \\
T_3(\theta_0) &= 2\pi\sqrt{\frac{l}{g}}.
\end{aligned} \tag{5}
$$

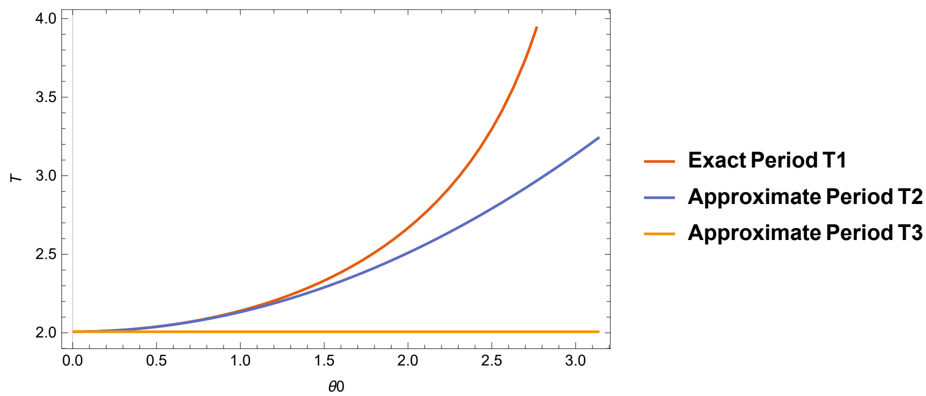


Figure 2: T vs θ_0 for different approximations

As shown in figure 2, the difference between T_1 and T_3 is neglectable in the range $\theta_0 \in [0, 0.5]$, and the difference between T_1 and T_2 remains relatively small until $\theta_0 > 1.5$. The approximation is overall acceptable on the small scales.

On the other hand, the discrepancy between the T_1 and T_3 is getting larger as θ_0 increases, indicating that the mathematical pendulum doesn't "have" same period at any amplitude. Therefore, an exactly "tautochronic" pendulum is needed for precise timing.

3 Tautochrone Curve

As Equation 1 shows, the period of a simple pendulum depends on amplitude of the swing θ_0, which means a clock using the simple pendulum is not really accurate. This section mainly aims to develop the equations of the tautochrone and shows how to construct a tautochronous pendulum. We will start with the physical model and its properties used in our derivation, through which the equation of motion of the tautochrone can be attained and we will eventually obtain the parametric equations of the tautochrone. Detailed method of constructing a tautochronous pendulum will be demonstrated and every approach we used will be verified. A validation section will justify our results in this part.

3.1 Physical model

Consider the one-dimensional motion of a point mass on the real axis, travelling along a trajectory $\gamma : \mathbb{R} \to \mathbb{R}^2$. According to Newton's law $F \circ \gamma(t) = m\gamma''(t)$. Suppose that there exists a function $U : \mathbb{R} \to \mathbb{R}$ such that $F(x) = U'(x)$ for all $x \in R$ (called the potential energy), $U(0) = 0$ and U is monotonically increasing on $[0, \infty)$. Since only the potential force does work to the point mass, according to the law of conservation of energy, the sum of the potential energy and the kinetic energy of the point mass is constant, which means that

$$U(\gamma(t)) + \frac{m}{2}\gamma'(t)^2 = \text{const}$$

holds at any moment in our situation. Considering the initial situation when $t = 0$ and $\gamma(0) = x_1$ and any instant of time t after it, the following equation holds,

$$U(\gamma(0)) + \frac{m}{2}\gamma'(0)^2 = U(x_1) + 0 = U(\gamma(t)) + \frac{m}{2}\gamma'(t)^2$$

$$U(x_1) - U(\gamma(t)) = \frac{m}{2}\gamma'(t)^2 = \frac{m}{2}(\frac{dx}{dt})^2 \tag{6}$$

Denote T(x_1) as the time need for the point mass to travel from γ=x_1 to γ=0, whose value depends on x_1 generally. Integrating on both sides of Equation 6, we can get

$$\int_0^{T(x_1)} dt = \sqrt{\frac{m}{2}} \int_0^{x_1} \frac{1}{\sqrt{U(x_1) - U(x)}} dx$$

$$T(x_1) = \sqrt{\frac{m}{2}} \int_0^{x_1} \frac{1}{\sqrt{U(x_1) - U(x)}} dx \tag{7}$$

In Equation 5, T(x_1) is related to the potential energy U(x). So next, we hope to find the potential energy U(x).

The idea of the following method was inspired by [6]. First we substitute in the integral as follows

$$y^2 = \frac{U(x)}{U(x_1)}, 0 \leq y \leq 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{U'(x)}{2\sqrt{U(x_1)U(x))}}$$

$$\frac{2\sqrt{U(x_1)U(x)}}{U'(x)} \mathrm{d}y = \mathrm{d}x$$

The original integral can be changed into

$$\begin{aligned} T(x_1) &= \sqrt{\frac{m}{2}} \int_0^{x_1} \frac{\frac{2\sqrt{U(x)U(x_1)}}{U'(x)} \mathrm{d}y}{\sqrt{U(x_1)(1 - y^2)}} \\ &= \sqrt{2m} \int_0^1 \frac{\sqrt{U(x)}\mathrm{d}y}{U'(x)\sqrt{1 - y^2}} \end{aligned}$$

Since the period is independent of the initial position x_1, thus

$$\begin{aligned}
0 &= \frac{\partial T}{\partial x_1} = \sqrt{2m}\int_0^1 \frac{\partial}{\partial x_1}\frac{\sqrt{U(x)}\mathrm{d}y}{U'(x)\sqrt{1-y^2}} \\
0 &= \int_0^1 \frac{\partial \frac{\sqrt{U(x)}}{U'(x)}}{\partial x_1}\frac{\mathrm{d}y}{\sqrt{1-y^2}}
\end{aligned} \tag{8}$$

Since $0 \le x \le x_1$ and x is related to x_1 and y, we just denote x as

$$x = f(y)x_1 + o(x_1)$$

Since Equation.(8) is true for any $x_1 > 0$, we can let x_1 be small enough such that $\frac{\partial}{\partial x_1}\frac{\sqrt{U(x)}}{U'(x)}$ is of the same sign in the small interval (0,x_1). Namely, it's either positive or negative in the whole interval. Since $\frac{1}{\sqrt{1-y^2}}$ is always positive in the interval (0,1), then $\frac{\partial \frac{\sqrt{U(x)}}{U'(x)}}{\partial x_1}\frac{1}{\sqrt{1-y^2}}$ also has the same sign in the whole interval (0,1). Since

$$0 = \int_0^1 \frac{\partial \frac{\sqrt{U(x)}}{U'(x)}}{\partial x_1}\frac{\mathrm{d}y}{\sqrt{1-y^2}}$$

We can deduce that

$$\frac{\partial}{\partial x_1}\frac{\sqrt{U(x)}}{U'(x)} = \frac{\partial}{\partial x_1}\frac{\sqrt{U(f(y)x_1 + o(x_1))}}{U'(f(y)x_1 + o(x_1))} = 0 \tag{9}$$

Thus

$$\frac{\sqrt{U(f(y)x_1 + o(x_1))}}{U'(f(y)x_1 + o(x_1))} = A = \frac{\sqrt{U(x)}}{U'(x)}$$

where A is a constant. It's equivalent to write the equation as

$$U^{'}(x) = c\sqrt{U(x)}$$

where $c = \frac{1}{A}$ is a constant.

Another approach can also deduce the target solution $U'(x) = c\cdot\sqrt{U(x)}$, which requires the Laplace transform and the convolution theorem.

According to [11], the definition of Laplace transformation is given by

"The *Laplace transform* of a function f, defined for $t \in [0,\infty)]$, is the function $F(s)$ defined as

$$F(s) = \mathcal{L}[f(t)](s) = \int_0^\infty e^{st}f(t)dt = \lim_{T\to\infty}\int_0^T e^{st}f(t)dt \tag{10}$$

provided that the limit exists for all sufficiently large s."

For example, let $f(t) = t^p$, with $p \ge 0$, we compute

$$F(s) = \int_0^\infty e^{-st}t^p dt = \frac{1}{s^{p+1}}\int_0^\infty e^{-r}r^p dr$$

with substitution $r = st$, which implies

$$\mathcal{L}\left(t^p\right) = \frac{\Gamma(p+1)}{s^{p+1}}$$

where the Gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-r} r^{x-1} dr$$

Especially for $x = \frac{1}{2}$, the proof in appendix A.3 shows that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, indicating for $p = -\frac{1}{2}$

$$\mathcal{L}(t^{-1/2}) = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \sqrt{\pi} s^{-\frac{1}{2}} \tag{11}$$

In order to utilize the *Laplace transform* to the best, we have to define convolution integral firstly ([11]) "The convolution of f and g is defined as

$$(f * g)(t) = \int_0^t f\left(t_1\right) g\left(t - t_1\right) dt_1$$

"

The convolution theorem is that "if f $f, g \in E$, then $f * g \in E$, and $\mathcal{L}(f * g) = (\mathcal{L}f)(\mathcal{L}g)$, where E denotes the set of functions of *expontential type*, and their *Laplace transform* exist for all s sufficiently large." The proof of the convolution theorem can also be found in the material [11].

Now we can apply the convolution theorem to solve the equation 3.1. Since the $T(x)$ is constant,

$$\begin{aligned}
T\left(x_1\right) &= \sqrt{\frac{m}{2}} \int_0^{x_1} \frac{1}{\sqrt{U\left(x_1\right) - U(x)}} dx \\
&= \sqrt{\frac{m}{2}} \int_0^{U\left(x_1\right)} \frac{1}{\sqrt{U\left(x_1\right) - U(x)}} \frac{dx}{du} du \\
&= \sqrt{\frac{m}{2}} \int_0^{U\left(x_1\right)} \frac{1}{\sqrt{U\left(x_1\right) - u}} t(u) du \\
&= c_0
\end{aligned}$$

where c_0 is constant and $u = U(x),\ t(u) = \frac{dx}{du} = \frac{1}{U'(x)}$. Now we can see the left side of the equation

$$\int_0^{U\left(x_1\right)} \frac{1}{\sqrt{U\left(x_1\right) - u}} t(u) du = \sqrt{\frac{2}{m}} c_0 = c_1$$

is the convolution of t and $u^{-1/2}$. Taking the *Laplace transfrom*, we can obtain

$$\mathcal{L}\left(u^{-1/2} * t\right) = (\mathcal{L}u^{-1/2})\left(\mathcal{L}t\right) = \mathcal{L}c_1 = \frac{c_1}{s}$$

Since $\mathcal{L}\left[u^{-1/2}\right] = \sqrt{\pi}s^{-1/2}$ we found previously, we will have

$$\mathcal{L}t = c_2 s^{-1/2}, \quad t(u) = c_3 u^{-1/2},$$

which applies transform 11 agian. So we can deduce the target equation

$$U'(x) = \frac{1}{t(u)} = \frac{\sqrt{u}}{c_3} = c \cdot \sqrt{U(x)}$$

which also means $\frac{du}{dx} = c\sqrt{u}$, and we can get

$$\begin{aligned}\frac{du}{\sqrt{u}} &= cdx \\ 2\sqrt{u} &= cx \\ U(x) = u &= \frac{c^2}{4}x^2,\end{aligned} \tag{12}$$

where $U(0) = 0$ is assumed.

Interestingly, the potential energy of the point mass is in tune with the potential energy of a spring oscillator (Figure 3) with potential energy U= $\frac{kx^2}{2}$ ($k = \frac{c^2}{2}$), whose motion is a simple harmonic oscillation. Since the period of a simple harmonic oscillation is a constant regardless of the amplitude of the oscillation, the consistency between the potential energy of the point mass and the spring oscillator somehow confirms that $U(x) = \frac{c^2}{4}x^2$ is under the condition that the time need for the point mass to travel from γ=x_1 to γ=0 is independent of x_1.

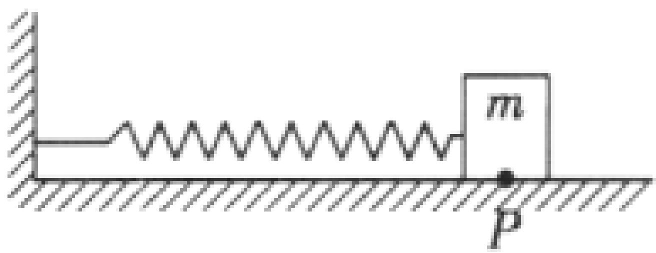


Figure 3: A spring oscillator [8]

3.2 Derivation of the parametric equations of the tautochrone

To find the tautochrone, we need to consider that the straight line (the one-dimensional motion on the real axis) is stretched into a curve (the path of the tautochrone), where the gravitational potential energy serve as the potential energy mentioned in the equation 3.1. To approach the tautochrone curve, the gravitational potential energy in term of the arc length $U(x)$ in this curve should satisfy the equation 12,

$$U(x) = \frac{c^2}{4}x^2, \tag{13}$$

where x is the arc length of the curve. Then, substituting φ in equation 9 with $s(t)$,

$$U(s(t)) + \frac{m}{2}s'(t)^2 = \text{const.} \tag{14}$$

Next, we substitute x in equation 14 with $s(t)$ and apply it into equation 14:

$$\frac{c^2}{4}s(t)^2 + \frac{m}{2}s'(t)^2 = \text{const} \tag{15}$$

Differentiating with respect to t on both sides of equation 15,

$$\frac{c^2}{2}s(t)s'(t) + ms'(t)s''(t) = 0$$

$$s''(t) = -\frac{c^2}{2m}s(t)$$

$$\frac{d^2s}{dt^2} = -ks, \tag{16}$$

where $k = \frac{c^2}{2m}$.

Hence, the tautochrone must be a path along which

$$\frac{d^2s}{dt^2} = -ks,$$

where t is time and s is the path length.

Interestingly, equation 16 coincides with the equation of of motion of a simple harmonic oscillation, which means the bob oscillates about the origin with maximum distance away from the origin along the curve $s(x_1)$. In fact, that's where Huygens got inspired to established the concept of tautochrone. Since the general solution for equation 16 is

$$s(t) = Acos(\omega_0 t + \varphi), \tag{17}$$

For the simple harmonic oscillation obeying equation 17, period of the motion is T= $2\pi\sqrt{\frac{1}{k}} = 2\pi\sqrt{\frac{2m}{c^2}}$, which is independent of the amplitude of this motion. Thus, it also confirms that all the paths satisfying equation 16 have the same period.

Notice that the path of a simple pendulum doesn't satisfy this property. For pendulums, denoting a as the tangential acceleration of the bob and θ as the angle formed by the taut thread and the y-axis, we have

$$ma = -mgsin\theta$$

$$a = -gsin\theta.$$

Since the tangential acceleration of the bob always points in the direction of decreasing magnitude for θ and sinθ is a monotonically increasing function when $\theta \in [0, \frac{\pi}{2}]$, the negative sign is used here to indicate that a is opposite to the direction in which sin θ is increasing. Tangential acceleration is the first derivative of speed v with respect to time and speed is the first derivative of the distance travelled s with respect to time. Thus,

$$a = \frac{dv}{dt}$$
$$v = \frac{ds}{dt}$$
$$\frac{d^2s}{dt^2} = -gsin\theta \tag{18}$$

According to Arc length formula, s= $l\theta$, we get,

$$\frac{d^2s}{dt^2} = -gsin(s/l),$$

which is different from

$$\frac{d^2s}{dt^2} = -ks.$$

For a simple pendulum, -g sin (s/l) can not be expressed as -ks for any constant k≥0 accurately. Thus, a simple pendulum dose not satisfy

$$\frac{d^2s}{dt^2} = -ks,$$

which implies a simple pendulum is not a tautochrone.

Comparing equation 18 with equation 16,

$$\frac{d^2s}{dt^2} = -ks = -gsin\theta$$
$$s = \frac{g}{k}sin\theta$$

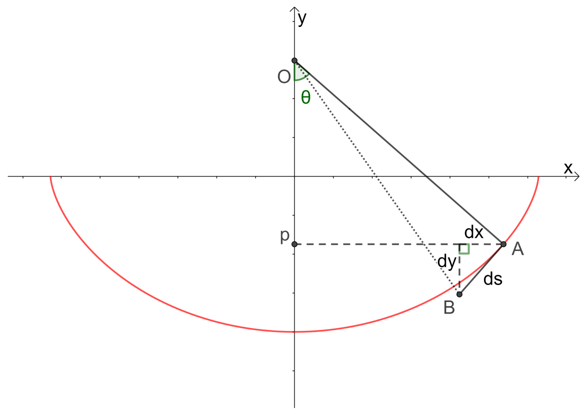
Differentiating both sides with respect to t,

$$\frac{ds}{dt} = (\frac{g}{k}cos\theta)\frac{d\theta}{dt} \tag{19}$$

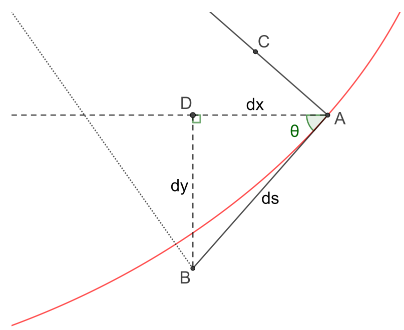
Now, we would like to find parametric equations for x and y in terms of θ.

Considering a instantaneous displacement ds of the bob, the relationship between ds, dx, dy can be obtained. In figure 4 (a), l_{AB} is the tangent line of the tautohrone curve, which means AB⊥AO. Since ∠ POA + ∠ PAO = ∠ DAB + ∠ DAB = $\frac{\pi}{2}$, ∠ AOB = ∠ DAB = θ. According to the geometrical relationship displayed in Figure 4 (b),

$$\begin{cases} dx = & cos\theta ds \\ dy = & sin\theta ds, \end{cases} \tag{20}$$



(a) A instant moment of motion



(b) Geometrical relationship

Figure 4: Relationship between ds, dx, and dy

Applying dx = $cos\theta$ ds to equation 19,

$$\frac{dx}{dt} = (\frac{g}{k}cos^2\theta)\frac{d\theta}{dt}$$
$$dx = (\frac{g}{k}cos^2\theta)d\theta$$

Integrating the resulting expressions on both sides,

$$\int_0^x dx = \int_0^\theta (\frac{g}{k}cos^2\theta)d\theta$$
$$\int_0^x dx = \int_0^\theta (\frac{g}{k}\frac{1+cos2\theta}{2})d\theta$$
$$x(\theta) - x(0) = \frac{g}{4k}(2\theta + sin2\theta).$$

Pluging the initial condition x(0) = 0,

$$x(\theta) = \frac{g}{4k}(2\theta + sin2\theta). \tag{21}$$

Similarly, applying dy = $sin\theta$ ds to equation 19,

$$\frac{dy}{dt} = (\frac{g}{k}sin\theta cos\theta)\frac{d\theta}{dt}$$
$$dy = (\frac{g}{2k}sin2\theta)d\theta$$

Integrating the resulting expressions on both sides,

$$\int_0^y dy = \int_0^\theta (\frac{g}{2k}sin2\theta)d\theta$$
$$y(\theta) - y(0) = -\frac{g}{4k}(cos2\theta - cos0).$$

Pluging the initial condition y(0) = $-\frac{g}{2k}$,

$$y(\theta) = -\frac{g}{4k}(1 + cos2\theta) \tag{22}$$

Therefore, the parametric equations of a tautochrone is

$$\begin{cases} x(\theta) = & \frac{g}{4k}(2\theta + sin2\theta) \\ y(\theta) = & -\frac{g}{4k}(1 + cos2\theta), \end{cases} \tag{23}$$

which is known as the cycloid curve. The graph when $\frac{g}{4k} = 1$ is shown in Figure 5, where we can see that the tautochrone is "a half–arch of a cycloid, with vertical axis of symmetry and concave upwards"[6].

Tautochronism of the cycloid

After we got the parametric equation of the tautochrone, we wanted to verify the isochronous property of the pendulum swinging along this trajectory, i.e. the cycloid. The method we took was similar to the one in [4].

Suppose the position function of a bob on the pendulum can be expressed as

$$\begin{aligned} x =& a(\theta + \sin\theta) \\ y =& -a(1 + \cos\theta) \end{aligned} \tag{24}$$

In this way, the trajectory of the bob is a cycloid(Figure 9). It can also be obtained that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a(1 + \cos\theta)$$
$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = a\sin\theta$$
$$(\frac{\mathrm{d}x}{\mathrm{d}\theta})^2 + (\frac{\mathrm{d}y}{\mathrm{d}\theta})^2 = (a(1 + \cos\theta))^2 + (a\sin\theta)^2 = 2a^2(1 + \cos\theta)$$

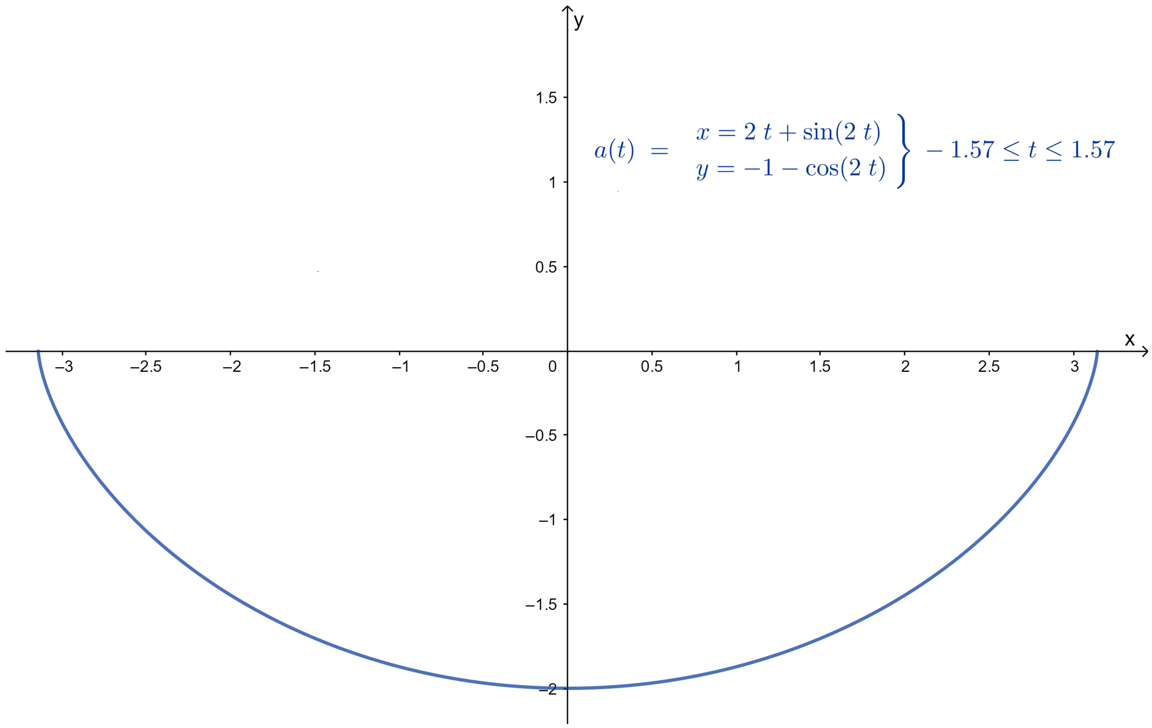


Figure 5: Graph of the tautochrone in the xy-plane when $\frac{g}{4k} = 1$.

Since in the process of swinging, the direction of tension in the string is perpendicular to the trajectory consistently. Thus only gravitation force does work. We set the line crossing the maximum height that the bob can reach as the zero potential line. Since the mechanic energy is conserved in this process, we can obtain that

$$\frac{1}{2}mv^2 = -mg(y - h_{max})$$
$$\frac{1}{2}mv^2 = -mgy$$
$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{-2gy}$$
$$\mathrm{d}t = \frac{\mathrm{d}s}{\sqrt{-2gy}}$$

$$\begin{aligned}\mathrm{d}t &= \frac{\sqrt{(\frac{\mathrm{d}x}{\mathrm{d}\theta})^2 + (\frac{\mathrm{d}y}{\mathrm{d}\theta})^2}\mathrm{d}\theta}{\sqrt{-2gy}} \\ &= \frac{a\sqrt{2(1+\cos\theta)}\mathrm{d}\theta}{\sqrt{-2g(-a(1+\cos\theta))}} \\ &= \sqrt{\frac{a}{g}}\mathrm{d}\theta\end{aligned}$$

Thus the time needed for the bob to travel from the bottom to the top of the cycloid is given by

$$T = \int_0^T dt = \int_0^\pi \sqrt{\frac{a}{g}} d\theta = \sqrt{\frac{a}{g}} \pi \tag{25}$$

If the bob is released from an intermediate point $(x(\theta_0), y(\theta_0))$,

$$\frac{1}{2} m v^2 = mg(y_0 - y)$$
$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{2g(y_0 - y)}$$
$$\mathrm{d}t = \frac{\mathrm{d}s}{\sqrt{2g(y_0 - y)}}$$

Thus

$$\begin{aligned} o\mathrm{d}t &= \frac{\sqrt{(\frac{\mathrm{d}x}{\mathrm{d}\theta})^2 + (\frac{\mathrm{d}y}{\mathrm{d}\theta})^2} \mathrm{d}\theta}{\sqrt{2g(y_0 - y)}} \\ &= \frac{a\sqrt{2(1+\cos\theta)}\mathrm{d}\theta}{\sqrt{2g(-a(1+\cos\theta_0) + a(1+\cos\theta))}} \\ &= \sqrt{\frac{a(1+\cos\theta)}{g(\cos\theta - \cos\theta_0)}} \mathrm{d}\theta \end{aligned} \tag{26}$$

Then use half angle formula

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

From the former, we can also get that

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

The Eq.(26) can be changed into

$$\mathrm{d}t = \sqrt{\frac{a}{g}} \frac{\cos\frac{\theta}{2}}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}} \mathrm{d}\theta$$

Then transform variable into

$$u = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$
$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \frac{\frac{1}{2}\cos\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}$$

$$\cos\frac{\theta}{2}\mathrm{d}\theta = 2\sin\frac{\theta_0}{2}\mathrm{d}u$$

Thus

$$\begin{aligned} T = &\int_0^1 \sqrt{\frac{a}{g}}\frac{2\mathrm{d}u}{\sqrt{1-u^2}} \\ =&2\sqrt{\frac{a}{g}}[\arcsin u]_0^1 \\ =&\sqrt{\frac{a}{g}}\pi \end{aligned} \tag{27}$$

This result conforms with the result got from Eq.(25), indicating that the period is the same for the bob to be released from any point on the cycloid. Thus the pendulum whose trajectory is a cycloid is isochronous.

3.3 Construction of a Tautochronous Pendulun

3.3.1 Curvature/Evolute/Involute

The curvature of a curve represents "the rate of bending as the point moves along the curve with unit speed. Let a curve $\gamma(s) = (x(s), y(s))$ parameterized by arc length s. Denote $\varphi(s)$ as the angle between the tangent line at the point $(x(s), y(s))$ and the positive direction of the x-axis. Then the magnitude of curvature at point $(x(s), y(s))$ is given by

$$k(s) = \frac{d\varphi(s)}{ds}$$

"[1]

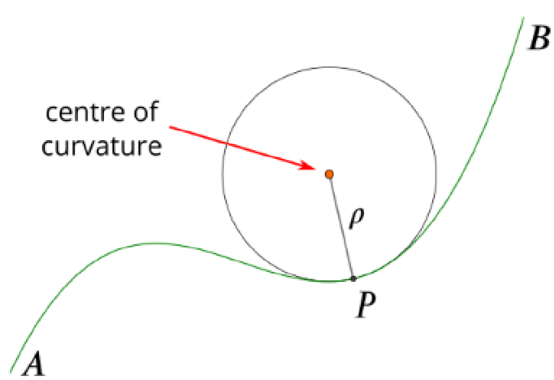


Figure 6: Curvature[2]

The radius of curvature is expressed as $R = \frac{1}{k}$. The center of curvature is on the normal vector of the curve at that point. Knowing these conditions, we can determine the center of curvature at any point of the curve.

"The locus of the centres of curvature of a given curve" is called the evolute of the curve[1]. "If E is the evolute of a curve I, then I is the involute of E"[12].

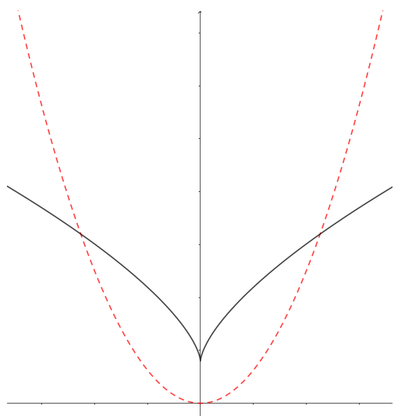


Figure 7: Parabola evolute

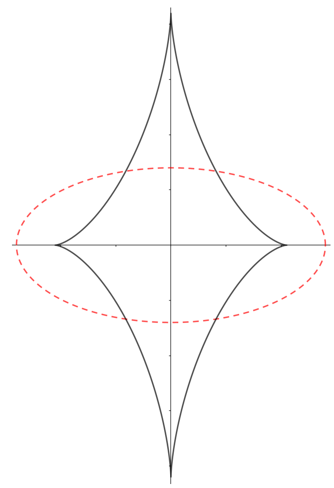


Figure 8: Ellipse evolute

3.3.2 Construction of the metal plates

We intended to construct metal plates in Huygens's pendulum based on a cycloid. According to Huygens, we just need to construct the evolute of this cycloid. This can be explained based on the geometric properties of the evolute. Since evolute is the locus of center of curvature C, for any point P on the curve, CP must be perpendicular to the curve at point P. If part of the pendulum's thread coincides with CP, the tension in the thread does no work on the bob, which satisfies a necessary condition for the isochronous pendulum. Another property of evolute is that the radius of curvature CP at any point P is tangential to the evolute. This property conforms to the actual motion of a bob when blocked by the metal plates. Thus the evolute of cycloid is the metal plate we seek.

Consider a cycloid whose function is given by

$$\varphi(\theta) = (a(\theta + \sin\theta), -a(1 + \cos\theta)), -\pi \leq \theta \leq \pi,$$

i.e., $x(\theta) = a(\theta + \sin\theta)$, $y(\theta) = -a(1 + \cos\theta)$. The evolute of $\varphi(\theta)$ is determined by the trajectory of the centres of the osculating circles for the curve $\varphi(\theta)$, which is actually the locus of its center of curvature. So we need to find the center of curvature corresponding to any point P on $\varphi(\theta)$. The center C is located on a line perpendicular to $\varphi(\theta)$ at P. The length of CP,i.e., the radius of is given by

$$R = \frac{1}{k}$$

where k is the curvature of at that point. The value of k can be calculated as

$$k = \frac{\left\|(T \circ \varphi)^{'}(\theta)\right\|}{\|\varphi^{'}(\theta)\|} \tag{28}$$

where $(T \circ \varphi)^{'}(\theta))$ is the normal vector of the curve at the point $\varphi(\theta)$. The $\varphi^{'}(\theta) = (x^{'}(\theta), y^{'}(\theta)) = (a(1+\cos\theta), a\sin\theta)$. So the $\left\|\varphi^{'}(\theta)\right\| = \sqrt{(a(1+\cos\theta))^2 + (a(\sin\theta))^2} = 2a\cos\frac{\theta}{2}$. The unit tangential vector $T \circ \varphi$ at this point is given by

$$T \circ \varphi(\theta) = \frac{\varphi^{'}(\theta)}{\|\varphi^{'}(\theta)\|} = \frac{(a(1+\cos\theta), a\sin\theta)}{2a\cos\frac{\theta}{2}} = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$$

Thus

$$(T \circ \varphi)^{'}(\theta) = (-\frac{1}{2}\sin\frac{\theta}{2}, \frac{1}{2}\cos\frac{\theta}{2})$$

and

$$\left\|(T \circ \varphi)^{'}(\theta)\right\| = \frac{1}{2}$$

The unit normal vector $(N \circ \varphi)(\theta)$ at the point $\varphi(\theta)$ is given by

$$(N \circ \varphi)(\theta) = \frac{(T \circ \varphi)'(\theta)}{\|(T \circ \varphi)'(\theta)\|} = (-\sin\frac{\theta}{2}, \cos\frac{\theta}{2})$$

Plugging the value we got in Eq.(28), we can get that

$$k = \frac{\frac{1}{2}}{2a\cos\frac{\theta}{2}} = \frac{1}{4a\cos\frac{\theta}{2}}$$

It follows that

$$R = \frac{1}{k} = 4a\cos\frac{\theta}{2}$$

As the unit direction vector and the radius is known, the point on the evolute $E(\theta)$ corresponding to the point $\varphi(\theta)$ can be calculated as

$$\begin{aligned} E(\theta) &= (x(\theta) - R\sin\frac{\theta}{2}, y(\theta) + R\cos\frac{\theta}{2}) \\ &= (a(\theta + \sin\theta) - 4a\cos\frac{\theta}{2}\sin\frac{\theta}{2}, -a - a\cos\theta + 4a\cos\frac{\theta}{2}\cos\frac{\theta}{2}) \\ &= (a(\theta - \sin\theta), a(1+\cos\theta)) \end{aligned} \tag{29}$$

In this way, the metal plate is constructed. Its function is expressed by Eq.(29). The shape of the metal plate turns out to be the cycloid.

Check the trajectory of the bob

But the construction of the metal plate does not ensure that the bob will follow the path of the cycloid $\varphi(\theta)$ because we have not verified that the unwound part of the thread will make the bob move along the cycloid. We are going to prove this.

Suppose that the metal plates satisfy the function $E(\theta)$ found above. It can be obtained that $E(0) = (0, 2a)$ and $E(\pi) = (a\pi, 0)$. Assume that the length of the pendulum thread is equal to the arc length of the evolute between $\theta = 0$ and $\theta = \pi$. If the thread is wound to the right at the beginning, then the bob is located at the point ($a\pi$,0). As the thread unwinds, it remains tangent to $E(\theta)$. The length of the unwound thread is equal to the arc length of the evolute between the point of tangency and ($a\pi$,0).

If the point of tangency C can be represented as $(x(\theta), y(\theta))$, the length of the unwounded thread $s(\theta)$ is given as

$$s(\theta) = \int_{\theta}^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

In this example, since $x^{'}(\theta) = a(1 - \cos\theta)$ and$y^{'}(\theta) = -a\sin\theta$. So $s(\theta)$ can be calculated as

$$\int_{\theta}^{\pi} \sqrt{(a(1-\cos\theta))^2 + (a(-\sin\theta))^2} d\theta = \int_{\theta}^{\pi} 2a\sin\frac{\theta}{2} d\theta = 4a\cos\frac{\theta}{2} \tag{30}$$

The unit tangential vector $T \circ E(\theta)$ at point C is given by

$$T \circ E(\theta) = \frac{E^{'}(\theta)}{\|E^{'}(\theta)\|} = \frac{(a(1-\cos\theta), -a\sin\theta)}{\sqrt{(a(1-\cos\theta))^2 + (-a\sin\theta)^2}} = (a\sin\frac{\theta}{2}, -a\cos\frac{\theta}{2})$$

Therefore, the point P on the bob's trajectory corresponds to C is given by

$$x_p = x(\theta) + s(\theta)\sin\frac{\theta}{2} = a(\theta - \sin\theta) + 4a\cos\frac{\theta}{2}\sin\frac{\theta}{2} = a(\theta + \sin\theta)$$

$$y_p = y(\theta) - s(\theta)\cos\frac{\theta}{2} = a(1 + \cos\theta) - 4a\cos\frac{\theta}{2}\cos\frac{\theta}{2} = -a(1 + \cos\theta)$$

which conforms to the $\varphi(\theta)$ in the previous part. Thus the evolute is the metal plate we want.

3.4 Validation

3.4.1 Validation on Construction

Here we will propose a concise method to prove that a shifted copy of the cycloid is its own evolute.

Given that a light string of length $4R$ attached with a ball P is hanged at B, which could swing in this plane around B. Two plates in the shape of cycloid curve parametrized by $(R(\theta + \sin(\theta - \pi), R(3 - \cos(\theta - \pi))$ are placed symmetrically to constrain the movement of the pendulum. The trajectory of P is shown in 9 as curve $A_{left}OA_{right}$, where O is the lowest point.

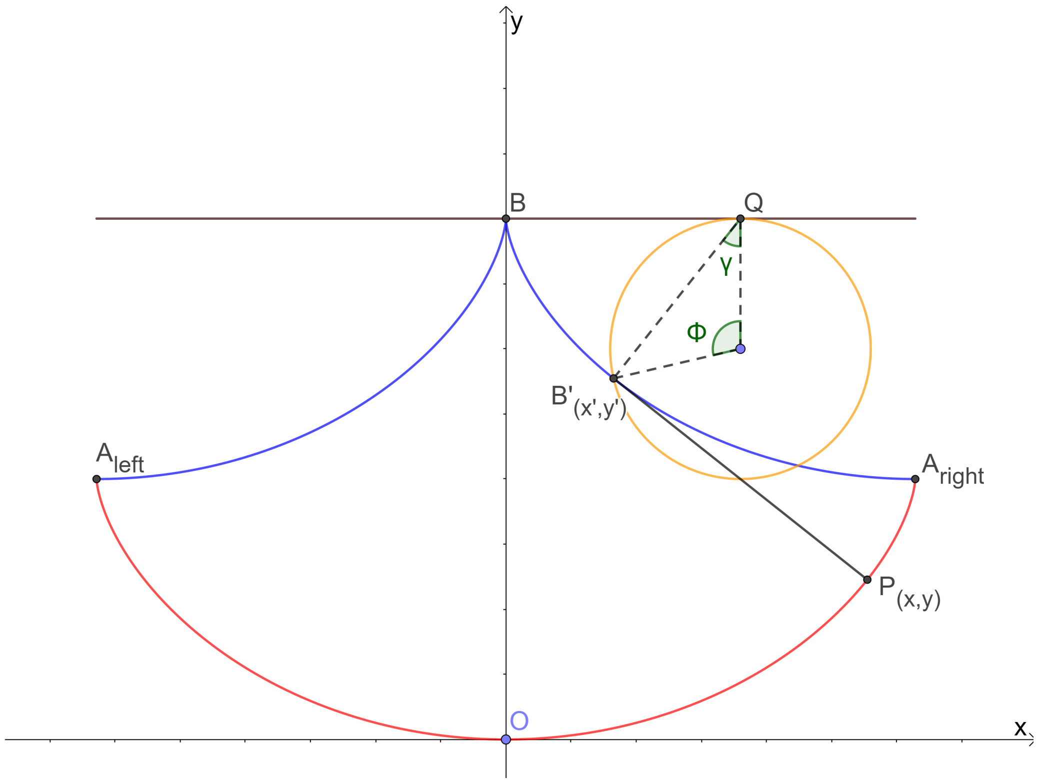


Figure 9: Validation on the evolute

When P is on the curve $A_{left}OA_{right}$, the string $BB'P$ is shown in the figure, where BB' is against the plate on the right, $B'P$ is the tangent segment pass through B'. Since the coordinate of B' is related to the central angle ϕ of the circle on cycloid, hence we can write

$$x' = R\phi - R\sin(\pi - \phi), \;\; y' = 3R - R\cos(\pi - \phi)$$

and

$$\begin{aligned} l_{BB'} &= \int_0^{\phi} \sqrt{(dx)^2 + (dy)^2} d\phi = 4R \sin\frac{\theta}{2} \\ l_{B'P} &= 4R \sin\frac{1}{2}(\pi - \phi) = l_{B'A_{right}} \end{aligned} \tag{31}$$

Then, due to $B'P \perp B'Q$ as we discussed in section 3.2, the coordinate of P is given by

$$
\begin{aligned}
x = x' + l_{B'p}\cos\gamma &= R_\phi - R\sin(\pi-\phi) + 4R\sin\frac{1}{2}(\pi-\phi)\cos\frac{1}{2}(\pi-\phi) \\
&= R\phi + R\sin\phi \\
y = y' - l_{B'P}\sin\gamma &= 3R - R\cos(\pi-\phi) - 4R\sin\frac{1}{2}(\pi-\phi)\sin\frac{1}{2}(\pi-\phi) \\
&= R - R\cos\phi
\end{aligned}
\tag{32}
$$

So the involute formed by the original cycloid is also a cycloid parametrized as

$$x = R(\phi + \sin\phi),\ \ y = R(1-\cos\phi)$$

Considering the conservation of mechanical energy, the equation of motion of P is equivalent to that in previous sections. So the time period of P in different amplitude is found by

$$
\begin{aligned}
mgy + \frac{1}{2}mv^2 &= const. \\
y &= R(1-\cos\theta) = 2R\sin^2\frac{\theta}{2} = \frac{s^2}{8R} \\
v &= \frac{ds}{dt} = \dot{s}^2 \\
\frac{mg}{8R}s^2 + \frac{1}{2}m\dot{s}^2 &= const \\
\ddot{s} + \frac{g}{4R}s &= 0 \\
T &= 2\pi\sqrt{\frac{4R}{g}}
\end{aligned}
\tag{33}
$$

3.4.2 Validation of Solution on Mathematica

Here we will use the Mathematica to validate the solution of the tautochrone curve in the following steps on [13].

By writing the fall time with respect to the speed v and path length s, we can yield the Abel integral equation $T = \int dt = \int \frac{ds}{v}$. The relation between the height of particle y and gravitational energy can be found in the conservation of energy equation

$$\frac{1}{2}mv^2 = mg(y_{max} - y)$$

and now we would like to define the unknown function h as $ds = h(y)dy$ and then use `DSolveValue` in Mathematica to solve h. The result turns out to be

$$h(y) = \frac{\sqrt{2}gT}{\pi\sqrt{gy}}$$

By applying the equation $ds = \sqrt{dx^2 + dy^2}$, we can get

$$\frac{dx}{dy} = \sqrt{-1 + \frac{2gT^2}{\pi^2 y}}$$

Then we can find the function x in terms of y by integrating the curve from the lowest point.

$$x[y] = \frac{\pi\sqrt{y(2gT^2 - \pi^2 y)} + gT^2(\pi - 2\arctan(\sqrt{-1 + \frac{2gT^2}{\pi^2 y}}))}{\pi^2}$$

Then we can plot the curve parametrized by y from the previous output.

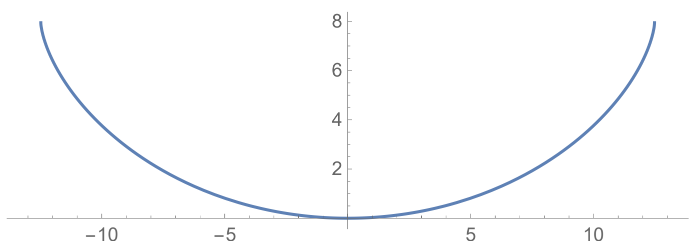


Figure 10: Validation of Trajectory

If we reparametrize the curve with $y = \frac{gT^2(1-\cos\theta)}{\pi^2}$, we can derive

$$x = \frac{gT^2(\pi - 2\arctan(\cot(\theta/2)) + \sin(\theta))}{\pi^2}$$

By noticing

$$\arctan(\cot\frac{\theta}{2}) = \frac{\pi}{2} - \frac{\theta}{2},$$

we can obtain

$$x = \frac{gT^2(\theta + \sin[\theta])}{\pi^2}$$

.

So now we get the parametrization of a cycloid,

$$x = \frac{gT^2(\theta + \sin\theta)}{\pi^2}$$
$$y = \frac{gT^2(1 - \cos\theta)}{\pi^2}$$

With the help of the chain rule

$$\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt},$$

we can obtain a differential equation $\theta(t)$.

$$\theta' = \pm\frac{\pi\sqrt{\cos\theta - \cos\theta_{Max}}}{T\sqrt{1+\cos\theta}}$$

The interactive input and output is shown in appendix A

3.4.3 Validation of Tautochronism on Mathematica

Then we use the Mathematica to plot the trajectory with respect to time of a the particle on a same tautochrone curve with different amplitude.

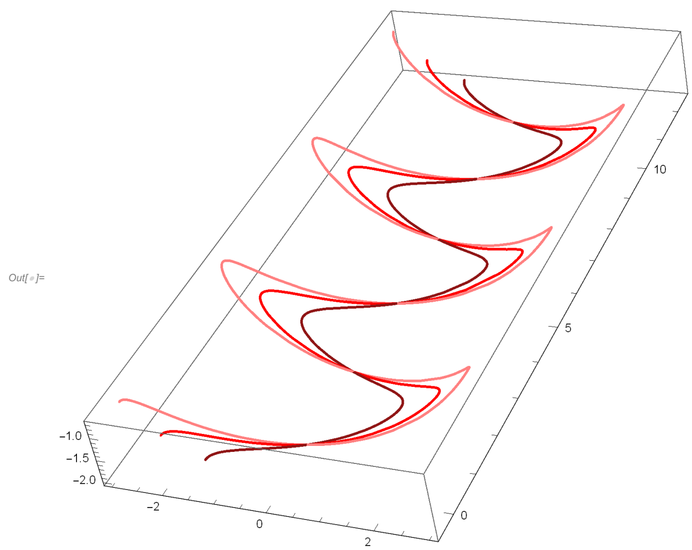


Figure 11: Visualization of coordinate in term of time

As shown in 11, the three trajectories always reach the lowest point simultaneously, which indicates the periods of different particles on the same tautochrone curve turn out to be same, demonstrating the isocronism of the tautochrone curve.

The interactive input and output is shown in appendix A.2

4 Conclusion

This project discussed mathematical pendulum and a modified pendulum, the tautochrone curve. We first calculated the period of a simple pendulum and found that a simple pendulum is not isochronous. As is shown in our result Eq.(3) and Eq.(4), the period of a simple pendulum actually depends on the its amplitude and the period will extend with the increase of the amplitude of the pendulum. In addition, the rate of the period's extension will be more remarkable as the amplitude increases. So pendula with other trajectory can be developed into excellent cases to solve the tautochrone problem. In the second part, we derived the parametric equations of the tautochrone by means of potential function and deduced that the tautochrone follows a cycloid curve. Then we managed to construct metal plates which could lead the pendulum to swing along the cycloid. By finding the locus of center of curvature, we finally got the evolute of cycloid curve, which is exactly the metal plate of the cycloid due to the geometric property of evolute. We interestingly found that metal plates are of the same shape as the its trajectory, i.e. the cycloid. Our solution of the tautochrone curve and the shape of the metal plates used to construct the tautochrone we concluded are validated by Mathematica.

To sum up, the tautochrone is a cycloid, whose period of the pendulum remains unchanged regardless of the amplitude of the oscillation. The tautochrone can be constructed by using two metal plates with the shape of a shifted copy of the tautochrone curve.

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A Mathematica Code

A.1 Validation of Solution

In[1]:=

```
abeleqn = T == 1/Sqrt[2 g]*
    Integral[h[z]/Sqrt[y-z],{z,0,y}];
dsdy=DSolveValue[abeleqn,h[y],y]
```

Out[1]=

$$\frac{\sqrt{2}\ g\ T}{\pi \sqrt{g\ y}}$$

In[2]:=

```
dxdy = Sqrt[dsdy^2 - 1]
```

Out[2]=

$$\sqrt{-1 + \frac{2\ g\ T^2}{\pi^2\ y}}$$

In[3]:=

```
x[y_] = Integrate[dxdy, {y, 0, y},
    Assumptions -> (2 g (T^2))/(Pi^2 y) > 1 && y > 0]
```

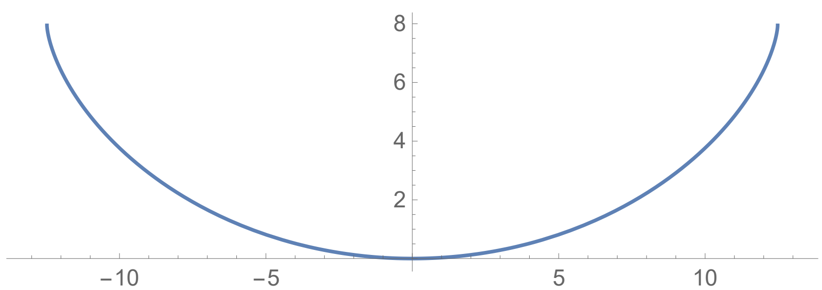
Out[3]=

$$\frac{\pi\ \sqrt{y\ (2\ g\ T^2 - \pi^2\ y)} + g\ T^2\ (\pi - 2\ \text{ArcTan}[\sqrt{-1 + \frac{2\ g\ T^2}{\pi^2\ y}}])}{\pi^2}$$

In[4]:=

```
Show[ParametricPlot[{{x[y], y}, {-x[y], y}} /. {g -> 9.8,
    T -> 2}, {y,0, (2 (9.8) 2^2)/\[Pi]^2}],
        ImageSize -> Medium]
```

Out[4]=



In[5]:=

```
FullSimplify[
    x[g T^2 (1 - Cos[\[Theta]])/\[Pi]^2], \[Pi]/
        4 > \[Theta] && \[Theta] > 0 && g > 0 && T > 0]
```

```
Out[5]=   g T^2 (π - 2 ArcTan[Cot[θ/2]] + Sin[θ])
          ---------------------------------------
                          π^2
```

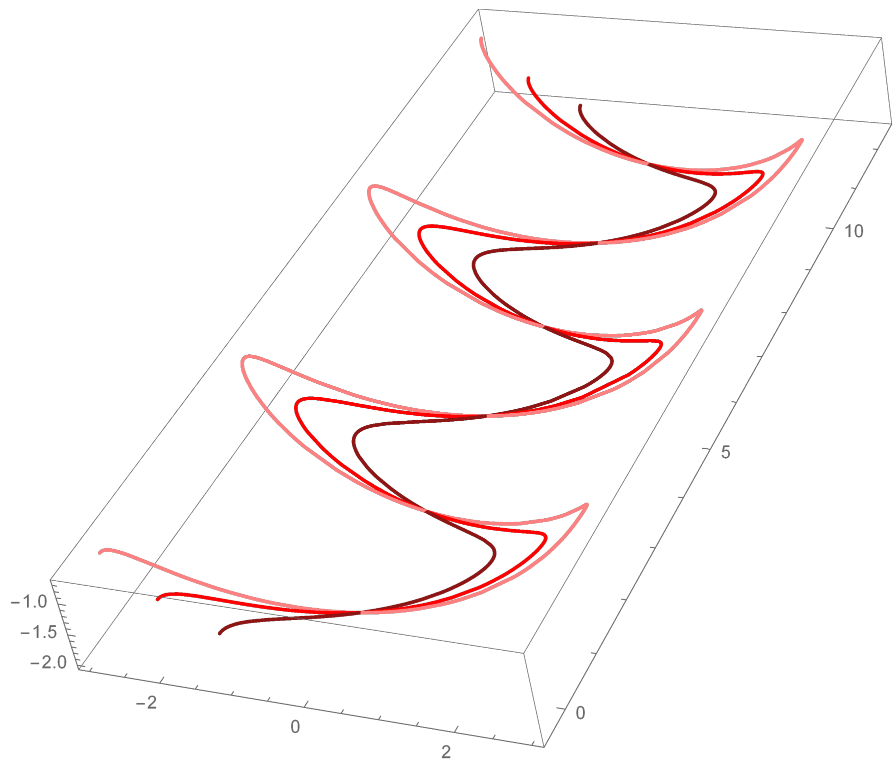
```
In[6]:=  c[\[Theta]_] = (g T^2)/\[Pi]^2 {Sin[\[Theta]] + \[Theta],
             1 - Cos[\[Theta]]};
         \[Theta]' == \[PlusMinus]FullSimplify[
         Sqrt[2 g (Last[c[\[Theta]Max]] - Last[c[\[Theta]]])]/
             Sqrt[c'[\[Theta]].c'[\[Theta]]], g > 0 && T > 0]
```

Out[6]= $\theta' == \pm\frac{\pi\sqrt{\mathrm{Cos}[\theta]-\mathrm{Cos}[\theta\mathrm{Max}]}}{\mathrm{T}\sqrt{1+\mathrm{Cos}[\theta]}}$

A.2 Validation of Tautochronism

```
In[7]:=  g = 9.81;
         eqn = Sin[s[t]] (g - Derivative[1][s][t]^2) +
             2 (-1 + Cos[s[t]]) (s^\[Prime]\[Prime])[t] == 0;
         sol[s0_] :=
             NDSolve[{eqn, s[1*^-3] == s0, Derivative[1][s][1*^-3]
                 == 0}, s, {t, 0, 100}, MaxSteps -> \[Infinity]][[1]
         p[A_, B_] :=
             ParametricPlot3D[{-\[Pi] + s[time] - Sin[s[time]],
                 time, -1 + Cos[s[time]]} /. sol[A], {time, 0, 12},
                     PlotStyle -> {B}]
         Show[p[1.25, Pink], p[1.9, Red],
             p[2.4, RGBColor[0.56, 0.08, 0.08]]]
```

Out[7]=



Graphics of this project are all plotted by Mathematica and Geogebra.

A.3 Proof $\Gamma(1/2)$

In previous, the Gamma function is defined as

$$\Gamma(x) = \int_0^\infty e^{-r} r^{x-1} dr$$

By Making the substitution $r = a^2$, we can derive the expression

$$\Gamma(x) = 2\int_0^\infty a^{2x-1} e^{-a^2} da$$

Plug in $x = 1/2$, and square the equation we can get

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= 2\int_0^\infty e^{-a^2}da \\
[\Gamma\left(\frac{1}{2}\right)]^2 &= [2\int_0^\infty e^{-a^2}da][2\int_0^\infty e^{-b^2}db] \\
&= 4\int_0^\infty\int_0^\infty e^{-\left(a^2+b^2\right)}dadb
\end{aligned}$$

Take the substitution $a = r\cos\theta,\ b = \sin\theta$, equation A.3 can be rewritten in

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= 4\int_0^\infty\int_0^\infty e^{-\left(a^2+b^2\right)}dadb \\
&= 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta \\
&= 2\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}dr^2d\theta \\
&= 2\int_0^{\frac{\pi}{2}} -e^{-t}\Big|_{t=0}^{t=\infty}d\theta \\
&= 2\int_0^{\frac{\pi}{2}} d\theta \\
&= \pi
\end{aligned}$$

Since $e^{-a^2} > 0$ always holds, indicating $\Gamma(\frac{1}{2}) \geq 0$, therefore,

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$