Alpha Integration: Example Problems

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Abstract

This document provides a collection of example problems demonstrating the application of Alpha Integration, a universal path integral framework. The problems are categorized into easy (10), intermediate (10), advanced (10), and very challenging (5) levels, covering locally integrable functions, distributions, fields, finite and infinite-dimensional spaces, complex paths, and nonlinear integrals. Each problem includes a detailed solution, emphasizing the measure selection algorithm's role in ensuring convergence.

1 Easy Problems

These problems introduce the basics of Alpha Integration in finite-dimensional spaces with simple functions and paths.

Problem 1 (Easy 1). Let $M = \mathbb{R}$, $f(x) = x^2$, $\gamma(s) = s$, $s \in [0, 1]$. Compute the path integral $\int_\gamma f\, ds$ using Alpha Integration.

Proof. The path integral is defined as:

$$\int_\gamma f\, ds = L_\gamma \int_0^1 f(\gamma(s))\, ds,$$

where L_γ is the arc length.

- **Arc length**:

$$\gamma(s) = s, \quad \frac{d\gamma}{ds} = 1, \quad \left|\frac{d\gamma}{ds}\right| = 1,$$

$$L_\gamma = \int_0^1 1\, ds = 1.$$

- **Composition**:

$$f(\gamma(s)) = f(s) = s^2.$$

- **Integral**:

$$\int_0^1 f(\gamma(s))\, ds = \int_0^1 s^2\, ds = \left[\frac{s^3}{3}\right]_0^1 = \frac{1}{3}.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot \frac{1}{3} = 1 \cdot \frac{1}{3} = \frac{1}{3}.$$

Since $f(\gamma(s)) = s^2 \in L^1([0,1])$, the measure $\mu(s) = ds$ suffices:

$$\int_0^1 s^2\, ds < \infty.$$

Thus, the algorithm's initial choice is valid.

Answer: $\int_\gamma f\, ds = \frac{1}{3}$. □

Problem 2 (Easy 2). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = x_1 + x_2$, $\gamma(s) = (s, s)$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = (s, s), \quad \frac{d\gamma}{ds} = (1, 1), \quad \left|\frac{d\gamma}{ds}\right| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

$$L_\gamma = \int_0^1 \sqrt{2}\, ds = \sqrt{2}.$$

- **Composition**:

$$f(\gamma(s)) = f(s, s) = s + s = 2s.$$

- **Integral**:

$$\int_0^1 2s\, ds = 2\int_0^1 s\, ds = 2\left[\frac{s^2}{2}\right]_0^1 = 2 \cdot \frac{1}{2} = 1.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot 1 = \sqrt{2} \cdot 1 = \sqrt{2}.$$

$$\int_0^1 |2s|\, ds = 1 < \infty,$$

so $\mu(s) = ds$ is sufficient.

Answer: $\int_\gamma f\, ds = \sqrt{2}$. □

Problem 3 (Easy 3). Let $M = \mathbb{R}$, $f(x) = \cos x$, $\gamma(s) = 2s$, $s \in [0, \pi]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = 2s, \quad \frac{d\gamma}{ds} = 2, \quad L_\gamma = \int_0^\pi 2\, ds = 2\pi.$$

- **Composition**:

$$f(\gamma(s)) = \cos(2s).$$

- **Integral**:

$$\int_0^\pi \cos(2s)\, ds = \left[\frac{\sin(2s)}{2}\right]_0^\pi = \frac{\sin(2\pi)}{2} - \frac{\sin 0}{2} = 0.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot 0 = 2\pi \cdot 0 = 0.$$

$$\int_0^\pi |\cos(2s)|\, ds \leq \int_0^\pi 1\, ds = \pi < \infty,$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 0$. ☐

Problem 4 (Easy 4). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = x_1^2$, $\gamma(s) = (s, 1)$, $s \in [-1, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = (s, 1), \quad \frac{d\gamma}{ds} = (1, 0), \quad L_\gamma = \int_{-1}^1 1\, ds = 2.$$

- **Composition**:

$$f(\gamma(s)) = s^2.$$

- **Integral**:

$$\int_{-1}^1 s^2\, ds = 2\int_0^1 s^2\, ds = 2 \cdot \frac{1}{3} = \frac{2}{3}.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot \frac{2}{3} = 2 \cdot \frac{2}{3} = \frac{4}{3}.$$

$$\int_{-1}^1 s^2\, ds < \infty,$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = \frac{4}{3}$. ☐

Problem 5 (Easy 5). Let $M = \mathbb{R}$, $f(x) = e^x$, $\gamma(s) = s$, $s \in [0, 1]$. Compute the sequential indefinite integral $F_1(x)$ at $x = 1$, with base point $x^0 = 0$.

Proof.

$$F_1(x) = \int_0^x f(t)\, dt + C_1,$$

$$f(t) = e^t,$$

$$F_1(x) = \int_0^x e^t\, dt = \left[e^t\right]_0^x = e^x - e^0 = e^x - 1,$$

$$C_1 = 0 \implies F_1(x) = e^x - 1.$$

At $x = 1$:

$$F_1(1) = e^1 - 1 = e - 1.$$

Since $f \in L^1_{\mathrm{loc}}$, $\mu(s) = ds$.

Answer: $F_1(1) = e - 1$. ☐

Problem 6 (Easy 6). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = x_2$, $\gamma(s) = (\cos s, \sin s)$, $s \in [0, \pi]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = (\cos s, \sin s), \quad \frac{d\gamma}{ds} = (-\sin s, \cos s),$$

$$\left|\frac{d\gamma}{ds}\right| = \sqrt{(-\sin s)^2 + (\cos s)^2} = 1,$$

$$L_\gamma = \int_0^\pi 1\, ds = \pi.$$

- **Composition**:

$$f(\gamma(s)) = f(\cos s, \sin s) = \sin s.$$

- **Integral**:

$$\int_0^\pi \sin s\, ds = [-\cos s]_0^\pi = -\cos \pi - (-\cos 0) = -(-1) - (-1) = 2.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot 2 = \pi \cdot 2 = 2\pi.$$

$$\int_0^\pi |\sin s|\, ds < \pi,$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 2\pi$. □

Problem 7 (Easy 7). Let $M = \mathbb{R}$, $f(x) = 1$, $\gamma(s) = s^2$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = s^2, \quad \frac{d\gamma}{ds} = 2s, \quad \left|\frac{d\gamma}{ds}\right| = 2s \quad (s \geq 0),$$

$$L_\gamma = \int_0^1 2s\, ds = \left[s^2\right]_0^1 = 1.$$

- **Composition**:

$$f(\gamma(s)) = f(s^2) = 1.$$

- **Integral**:

$$\int_0^1 1\, ds = 1.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot 1 = 1 \cdot 1 = 1.$$

$$\int_0^1 1\, ds = 1,$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 1$. □

Problem 8 (Easy 8). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = x_1 x_2$, $\gamma(s) = (s, 0)$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = (s, 0), \quad \frac{d\gamma}{ds} = (1, 0), \quad L_\gamma = \int_0^1 1\, ds = 1.$$

- **Composition**:

$$f(\gamma(s)) = f(s, 0) = s \cdot 0 = 0.$$

- **Integral**:

$$\int_0^1 0\, ds = 0.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot 0 = 1 \cdot 0 = 0.$$

$$\int_0^1 0\, ds = 0,$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 0$. □

Problem 9 (Easy 9). Let $M = \mathbb{R}$, $f(x) = |x|$, $\gamma(s) = s$, $s \in [-1, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = s, \quad L_\gamma = \int_{-1}^1 1\, ds = 2.$$

- **Composition**:

$$f(\gamma(s)) = |s|.$$

- **Integral**:

$$\int_{-1}^1 |s|\, ds = 2 \int_0^1 s\, ds = 2 \cdot \frac{1}{2} = 1.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot 1 = 2 \cdot 1 = 2.$$

$$\int_{-1}^1 |s|\, ds = 1,$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 2$. □

Problem 10 (Easy 10). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = 1$, $\gamma(s) = (s, s^2)$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof. • **Arc length**:

$$\gamma(s) = (s, s^2), \quad \frac{d\gamma}{ds} = (1, 2s),$$

$$\left|\frac{d\gamma}{ds}\right| = \sqrt{1 + (2s)^2} = \sqrt{1 + 4s^2},$$

$$L_\gamma = \int_0^1 \sqrt{1 + 4s^2}\, ds.$$

Substitute $u = 2s$, $du = 2\,ds$, $s = 0 \to u = 0$, $s = 1 \to u = 2$:

$$L_\gamma = \int_0^2 \sqrt{1 + u^2} \cdot \frac{1}{2}\, du = \frac{1}{2}\int_0^2 \sqrt{1 + u^2}\, du,$$

$$\int \sqrt{1 + u^2}\, du = \frac{u}{2}\sqrt{1 + u^2} + \frac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right),$$

$$\left[\frac{u}{2}\sqrt{1 + u^2} + \frac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^2 = \left(\frac{2}{2}\sqrt{5} + \frac{1}{2}\ln(2 + \sqrt{5})\right) - 0,$$

$$L_\gamma = \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln(2 + \sqrt{5})\right).$$

- **Composition**:

$$f(\gamma(s)) = 1.$$

- **Integral**:

$$\int_0^1 1\, ds = 1.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \cdot 1 = \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln(2 + \sqrt{5})\right).$$

$$\int_0^1 1\, ds = 1,$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln(2 + \sqrt{5})\right)$. □

2 Intermediate Problems

These problems involve singularities, distributions, or more complex paths, requiring measure adjustments.

Problem 11 (Intermediate 1). Let $M = \mathbb{R}$, $f(x) = \frac{1}{x}$, $\gamma(s) = s$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$, applying the measure selection algorithm.

Proof.

- **Singularity detection**:

$$f(\gamma(s)) = \frac{1}{s},$$

$$\int_0^1 \left|\frac{1}{s}\right| ds = \int_0^1 \frac{1}{s}\, ds = [\ln s]_0^1 = \lim_{\epsilon \to 0^+} (\ln 1 - \ln \epsilon) = \infty,$$

divergent at $s = 0$.

- **Measure adjustment**:

$$w(s) = \frac{1}{1 + \alpha \left(\frac{1}{s}\right)^\beta},$$

choose $\beta = 1$, $\alpha = 1$:

$$w(s) = \frac{1}{1 + \frac{1}{s}} = \frac{s}{s+1},$$

$$d\mu(s) = \frac{s}{s+1}\, ds.$$

- **Total variation**:

$$\mu([0,1]) = \int_0^1 \frac{s}{s+1}\, ds,$$

$$u = s+1, \quad du = ds, \quad s = 0 \to u = 1, \quad s = 1 \to u = 2,$$

$$\int_1^2 \frac{u-1}{u}\, du = \int_1^2 \left(1 - \frac{1}{u}\right) du,$$

$$= [u - \ln u]_1^2 = (2 - \ln 2) - (1 - \ln 1) = 1 - \ln 2 < 1.$$

- **Path integral**:

$$\int_\gamma f\, ds = \langle f(\gamma(s)), \mu(s) \rangle = \int_0^1 \frac{1}{s} \cdot \frac{s}{s+1}\, ds,$$

$$= \int_0^1 \frac{1}{s+1}\, ds = [\ln(s+1)]_0^1 = \ln 2 - \ln 1 = \ln 2.$$

Answer: $\int_\gamma f\, ds = \ln 2$. □

Problem 12 (Intermediate 2). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \frac{1}{x_1}$, $\gamma(s) = (s,s)$, $s \in [0,1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity detection**:

$$f(\gamma(s)) = \frac{1}{s},$$

$$\int_0^1 \frac{1}{s}\, ds = \infty,$$

divergent.

- **Measure adjustment**:

$$w(s) = \frac{s}{s+1}, \quad d\mu(s) = \frac{s}{s+1}\, ds.$$

- **Arc length** (for context):

$$\gamma(s) = (s, s), \quad L_\gamma = \int_0^1 \sqrt{2}\, ds = \sqrt{2},$$

  but we use:

$$\int_\gamma f\, ds = \int_0^1 \frac{1}{s} \cdot \frac{s}{s+1}\, ds.$$

- **Integral**:

$$\int_0^1 \frac{1}{s+1}\, ds = \ln 2.$$

- **Total variation**:

$$\mu([0,1]) = 1 - \ln 2 < \infty.$$

Answer: $\int_\gamma f\, ds = \ln 2$. ☐

Problem 13 (Intermediate 3). Let $M = \mathbb{R}$, $f(x) = \delta(x)$, $\gamma(s) = s$, $s \in [-1, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$L_\gamma = \int_{-1}^1 1\, ds = 2.$$

- **Composition**:

$$f(\gamma(s)) = \delta(s).$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \langle f(\gamma(s)), \chi_{[-1,1]}(s) \rangle,$$

$$\langle \delta(s), \chi_{[-1,1]}(s) \rangle = \chi_{[-1,1]}(0) = 1,$$

$$\int_\gamma f\, ds = 2 \cdot 1 = 2.$$

$$\langle \delta(s), \phi(s) \rangle = \phi(0),$$

finite, so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 2$. ☐

Problem 14 (Intermediate 4). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \frac{1}{x_1 + x_2}$, $\gamma(s) = (s, s)$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity**:

$$f(\gamma(s)) = \frac{1}{s+s} = \frac{1}{2s},$$

$$\int_0^1 \frac{1}{2s}\, ds = \frac{1}{2} \int_0^1 \frac{1}{s}\, ds = \infty.$$

- **Measure adjustment**:

$$w(s) = \frac{1}{1+\alpha\left(\frac{1}{2s}\right)} = \frac{2s}{2s+\alpha},$$

$$d\mu(s) = \frac{2s}{2s+\alpha}\,ds.$$

- **Integral**:

$$\int_0^1 \frac{1}{2s}\cdot\frac{2s}{2s+\alpha}\,ds = \int_0^1 \frac{1}{2s+\alpha}\,ds,$$

$$= \frac{1}{2}\int_0^1 \frac{1}{s+\frac{\alpha}{2}}\,ds = \frac{1}{2}\left[\ln\left(s+\frac{\alpha}{2}\right)\right]_0^1,$$

$$= \frac{1}{2}\left(\ln\left(1+\frac{\alpha}{2}\right) - \ln\left(\frac{\alpha}{2}\right)\right) = \frac{1}{2}\ln\left(\frac{2+\alpha}{\alpha}\right).$$

- **Total variation**:

$$\mu([0,1]) = \int_0^1 \frac{2s}{2s+\alpha}\,ds,$$

$$u = 2s+\alpha,\quad du = 2\,ds,\quad s=0\to u=\alpha,\quad s=1\to u=2+\alpha,$$

$$\int_0^1 \frac{2s}{2s+\alpha}\,ds = \int_\alpha^{2+\alpha} \frac{u-\alpha}{u}\cdot\frac{1}{2}\,du,$$

$$= \frac{1}{2}\int_\alpha^{2+\alpha}\left(1-\frac{\alpha}{u}\right)du = \frac{1}{2}\left[u-\alpha\ln u\right]_\alpha^{2+\alpha},$$

$$= \frac{1}{2}\left((2+\alpha-\alpha\ln(2+\alpha)) - (\alpha-\alpha\ln\alpha)\right),$$

$$= \frac{1}{2}\left(2+\alpha\ln\left(\frac{\alpha}{2+\alpha}\right)\right).$$

Choose $\alpha = 1$:

$$\int_\gamma f\,ds = \frac{1}{2}\ln\left(\frac{3}{1}\right) = \frac{1}{2}\ln 3.$$

Answer: $\int_\gamma f\,ds = \frac{1}{2}\ln 3$. □

Problem 15 (Intermediate 5). Let $M=\mathbb{R}$, $f(x)=\frac{1}{x^2}$, $\gamma(s)=s$, $s\in[0,1]$. Compute $\int_\gamma f\,ds$.

Proof.

- **Singularity**:

$$f(\gamma(s)) = \frac{1}{s^2},$$

$$\int_0^1 \frac{1}{s^2}\,ds = \left[-\frac{1}{s}\right]_0^1 = \lim_{\epsilon\to 0^+}\left(-1+\frac{1}{\epsilon}\right) = \infty.$$

- **Measure adjustment**:

$$w(s) = \frac{1}{1 + \alpha\left(\frac{1}{s^2}\right)^{\beta}},$$

$$\beta = 2, \quad w(s) = \frac{1}{1 + \frac{\alpha}{s^4}} = \frac{s^4}{s^4 + \alpha},$$

$$d\mu(s) = \frac{s^4}{s^4 + \alpha}\, ds.$$

- **Integral**:

$$\int_0^1 \frac{1}{s^2} \cdot \frac{s^4}{s^4 + \alpha}\, ds = \int_0^1 \frac{s^2}{s^4 + \alpha}\, ds.$$

Let $u = s^2$, $du = 2s\, ds$, $s\, ds = \frac{du}{2}$, $s = 0 \to u = 0$, $s = 1 \to u = 1$:

$$\int_0^1 \frac{s^2}{s^4 + \alpha}\, ds = \int_0^1 \frac{u}{u^2 + \alpha} \cdot \frac{1}{2}\, du,$$

$$= \frac{1}{2} \int_0^1 \frac{u}{u^2 + \alpha}\, du,$$

$$v = u^2 + \alpha, \quad dv = 2u\, du, \quad u\, du = \frac{dv}{2},$$

$$u = 0 \to v = \alpha, \quad u = 1 \to v = 1 + \alpha,$$

$$\frac{1}{2} \int_\alpha^{1+\alpha} \frac{1}{v} \cdot \frac{1}{2}\, dv = \frac{1}{4} \int_\alpha^{1+\alpha} \frac{1}{v}\, dv,$$

$$= \frac{1}{4} \left[\ln v\right]_\alpha^{1+\alpha} = \frac{1}{4} \ln\left(\frac{1 + \alpha}{\alpha}\right).$$

- **Total variation**:

$$\mu([0,1]) = \int_0^1 \frac{s^4}{s^4 + \alpha}\, ds \leq 1.$$

Choose $\alpha = 1$:

$$\int_\gamma f\, ds = \frac{1}{4} \ln\left(\frac{2}{1}\right) = \frac{1}{4} \ln 2.$$

Answer: $\int_\gamma f\, ds = \frac{1}{4} \ln 2$. □

Problem 16 (Intermediate 6)*.* Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \delta(x_1)$, $\gamma(s) = (s, s)$, $s \in [-1, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = (s, s), \quad L_\gamma = \int_{-1}^1 \sqrt{2}\, ds = 2\sqrt{2}.$$

- **Composition**:

$$f(\gamma(s)) = \delta(s).$$

- **Path integral**:

$$\langle f(\gamma(s)), \chi_{[-1,1]}(s)\rangle = \langle \delta(s), \chi_{[-1,1]}(s)\rangle = 1,$$

$$\int_\gamma f\, ds = L_\gamma \cdot 1 = 2\sqrt{2}.$$

Finite, so $\mu(s) = ds$.
Answer: $\int_\gamma f\, ds = 2\sqrt{2}$. ☐

Problem 17 (Intermediate 7). Let $M = \mathbb{R}$, $f(x) = \frac{1}{\sqrt{|x|}}$, $\gamma(s) = s$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity**:

$$f(\gamma(s)) = \frac{1}{\sqrt{s}},$$

$$\int_0^1 \frac{1}{\sqrt{s}}\, ds = \left[2\sqrt{s}\right]_0^1 = 2 \cdot 1 - \lim_{\epsilon \to 0^+} 2\sqrt{\epsilon} = 2 < \infty.$$

- **Measure**: Since integrable:

$$d\mu(s) = ds.$$

- **Path integral**:

$$L_\gamma = \int_0^1 1\, ds = 1,$$

$$\int_\gamma f\, ds = L_\gamma \cdot 2 = 1 \cdot 2 = 2.$$

Answer: $\int_\gamma f\, ds = 2$. ☐

Problem 18 (Intermediate 8). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \frac{1}{x_1^2}$, $\gamma(s) = (s, 1)$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity**:

$$f(\gamma(s)) = \frac{1}{s^2},$$

$$\int_0^1 \frac{1}{s^2}\, ds = \infty.$$

- **Measure adjustment**:

$$w(s) = \frac{s^4}{s^4 + \alpha}, \quad \beta = 2,$$

$$d\mu(s) = \frac{s^4}{s^4 + \alpha}\, ds.$$

- **Integral**:

$$\int_0^1 \frac{1}{s^2} \cdot \frac{s^4}{s^4 + \alpha}\, ds = \int_0^1 \frac{s^2}{s^4 + \alpha}\, ds,$$

$$u = s^2, \quad du = 2s\, ds, \quad s\, ds = \frac{du}{2},$$

$$\int_0^1 \frac{u}{u^2 + \alpha} \cdot \frac{1}{2}\, du = \frac{1}{4} \ln\left(\frac{1 + \alpha}{\alpha}\right).$$

- **Total variation**:

$$\mu([0,1]) \leq 1.$$

For $\alpha = 1$:

$$\int_\gamma f\, ds = \frac{1}{4} \ln 2.$$

Answer: $\int_\gamma f\, ds = \frac{1}{4} \ln 2.$ $\square$

Problem 19 (Intermediate 9). Let $M = \mathbb{R}$, $f(x) = \sin\left(\frac{1}{x}\right)$, $\gamma(s) = s$, $s \in [0,1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity**:

$$f(\gamma(s)) = \sin\left(\frac{1}{s}\right),$$

$$|f(\gamma(s))| \leq 1,$$

$$\int_0^1 \left|\sin\left(\frac{1}{s}\right)\right| ds \leq \int_0^1 1\, ds = 1 < \infty.$$

- **Measure**:

$$d\mu(s) = ds.$$

- **Path integral**:

$$L_\gamma = 1,$$

$$\int_\gamma f\, ds = \int_0^1 \sin\left(\frac{1}{s}\right) ds.$$

Let $u = \frac{1}{s}$, $ds = -\frac{1}{u^2}\, du$, $s = 0 \to u \to \infty$, $s = 1 \to u = 1$:

$$\int_0^1 \sin\left(\frac{1}{s}\right) ds = \int_\infty^1 \sin u \cdot \left(-\frac{1}{u^2}\right) du = \int_1^\infty \frac{\sin u}{u^2}\, du.$$

Since $\left|\frac{\sin u}{u^2}\right| \leq \frac{1}{u^2}$, and:

$$\int_1^\infty \frac{1}{u^2}\, du = 1,$$

the integral converges (Dirichlet test). Exact value requires numerical methods, but for rigor:

$$\int_\gamma f\, ds = \int_1^\infty \frac{\sin u}{u^2}\, du.$$

Answer: $\int_\gamma f\, ds = \int_1^\infty \frac{\sin u}{u^2}\, du.$ $\square$

Problem 20 (Intermediate 10). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = x_1 \delta(x_2)$, $\gamma(s) = (s, s)$, $s \in [-1, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$L_\gamma = 2\sqrt{2}.$$

- **Composition**:

$$f(\gamma(s)) = s\delta(s).$$

- **Path integral**:

$$\langle f(\gamma(s)), \phi(s) \rangle = \langle s\delta(s), \phi(s) \rangle = s\phi(s)\big|_{s=0} = 0,$$

$$\int_\gamma f\, ds = L_\gamma \cdot 0 = 0.$$

$$\langle s\delta(s), \phi(s) \rangle = 0,$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 0$. □

3 Advanced Problems

These problems involve complex manifolds, infinite-dimensional spaces, or nonlinear paths, requiring sophisticated measure selections.

Problem 21 (Advanced 1). Let $M = \mathbb{C}$, $f(z) = \frac{1}{z}$, $\gamma(s) = e^{is}$, $s \in [0, 2\pi]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = e^{is}, \quad \frac{d\gamma}{ds} = ie^{is}, \quad \left|\frac{d\gamma}{ds}\right| = 1,$$

$$L_\gamma = \int_0^{2\pi} 1\, ds = 2\pi.$$

- **Composition**:

$$f(\gamma(s)) = \frac{1}{e^{is}} = e^{-is}.$$

- **Integrability**:

$$\int_0^{2\pi} \left|e^{-is}\right|\, ds = \int_0^{2\pi} 1\, ds = 2\pi < \infty.$$

- **Measure**:

$$d\mu(s) = ds.$$

- **Path integral**:

$$\int_\gamma f\, ds = L_\gamma \int_0^{2\pi} e^{-is}\, ds = 2\pi \cdot 0 = 0,$$

since:

$$\int_0^{2\pi} e^{-is}\, ds = \int_0^{2\pi} (\cos s - i \sin s)\, ds = 0.$$

Answer: $\int_\gamma f\, ds = 0$. □

Problem 22 (Advanced 2). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \frac{1}{x_1^2 + x_2^2}$, $\gamma(s) = (\cos s, \sin s)$, $s \in [0, 2\pi]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$\gamma(s) = (\cos s, \sin s), \quad \frac{d\gamma}{ds} = (-\sin s, \cos s), \quad L_\gamma = \int_0^{2\pi} 1\, ds = 2\pi.$$

- **Composition**:

$$f(\gamma(s)) = \frac{1}{\cos^2 s + \sin^2 s} = 1.$$

- **Integrability**:

$$\int_0^{2\pi} 1\, ds = 2\pi < \infty.$$

- **Measure**:

$$d\mu(s) = ds.$$

- **Path integral**:

$$\int_\gamma f\, ds = \langle f(\gamma(s)), \mu(s) \rangle = \int_0^{2\pi} 1\, ds = 2\pi.$$

Answer: $\int_\gamma f\, ds = 2\pi$. □

Problem 23 (Advanced 3). Let $M = \mathbb{R}$, $f(x) = \delta'(x)$, $\gamma(s) = s$, $s \in [-1, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$L_\gamma = 2.$$

- **Composition**:

$$f(\gamma(s)) = \delta'(s).$$

- **Path integral**:

$$\langle f(\gamma(s)), \phi(s) \rangle = \langle \delta'(s), \phi(s) \rangle = -\phi'(0),$$
$$\langle \delta'(s), \chi_{[-1,1]}(s) \rangle = -\chi'_{[-1,1]}(0) = 0,$$

since $\chi_{[-1,1]}(s)$ is constant except at $s = \pm 1$.

$$\int_\gamma f\, ds = L_\gamma \cdot 0 = 0.$$

$$\langle \delta'(s), \phi(s) \rangle \text{ finite},$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 0$. □

Problem 24 (Advanced 4). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \frac{1}{(x_1-1)^2 + x_2^2}$, $\gamma(s) = (s, 0)$, $s \in [0, 2]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity**:

$$f(\gamma(s)) = \frac{1}{(s-1)^2},$$

$$\int_0^2 \frac{1}{(s-1)^2}\, ds = \int_{-1}^1 \frac{1}{u^2}\, du = \infty.$$

- **Measure adjustment**:

$$w(s) = \frac{(s-1)^4}{(s-1)^4 + \alpha}, \quad \beta = 2,$$

$$d\mu(s) = \frac{(s-1)^4}{(s-1)^4 + \alpha}\, ds.$$

- **Integral**:

$$\int_0^2 \frac{1}{(s-1)^2} \cdot \frac{(s-1)^4}{(s-1)^4 + \alpha}\, ds = \int_0^2 \frac{(s-1)^2}{(s-1)^4 + \alpha}\, ds,$$
$$u = s-1, \quad du = ds, \quad s = 0 \to u = -1, \quad s = 2 \to u = 1,$$
$$\int_{-1}^1 \frac{u^2}{u^4 + \alpha}\, du = 2\int_0^1 \frac{u^2}{u^4 + \alpha}\, du,$$
$$v = u^2, \quad dv = 2u\, du, \quad u\, du = \frac{dv}{2},$$
$$u = 0 \to v = 0, \quad u = 1 \to v = 1,$$
$$2\int_0^1 \frac{u^2}{u^4 + \alpha}\, du = 2\int_0^1 \frac{v}{v^2 + \alpha} \cdot \frac{1}{2}\, dv = \int_0^1 \frac{v}{v^2 + \alpha}\, dv,$$
$$w = v^2 + \alpha, \quad dw = 2v\, dv, \quad v\, dv = \frac{dw}{2},$$
$$v = 0 \to w = \alpha, \quad v = 1 \to w = 1 + \alpha,$$
$$\int_\alpha^{1+\alpha} \frac{1}{w} \cdot \frac{1}{2}\, dw = \frac{1}{2} \ln\left(\frac{1+\alpha}{\alpha}\right).$$

- **Total variation**:

$$\mu([0,2]) \leq 2.$$

For $\alpha = 1$:

$$\int_\gamma f\, ds = \frac{1}{2} \ln 2.$$

Answer: $\int_\gamma f\, ds = \frac{1}{2} \ln 2$. □

Problem 25 (Advanced 5). Let $M = \mathbb{R}^3$, $f(x_1, x_2, x_3) = \frac{1}{x_1^2 + x_2^2}$, $\gamma(s) = (s, s, s)$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity**:

$$f(\gamma(s)) = \frac{1}{s^2 + s^2} = \frac{1}{2s^2},$$
$$\int_0^1 \frac{1}{2s^2}\, ds = \infty.$$

- **Measure adjustment**:

$$w(s) = \frac{(2s^2)^2}{(2s^2)^2 + \alpha} = \frac{4s^4}{4s^4 + \alpha},$$

$$d\mu(s) = \frac{4s^4}{4s^4 + \alpha}\, ds.$$

- **Integral**:

$$\int_0^1 \frac{1}{2s^2} \cdot \frac{4s^4}{4s^4+\alpha}\, ds = \int_0^1 \frac{2s^2}{4s^4+\alpha}\, ds,$$

$$u = s^2, \quad du = 2s\, ds,$$

$$\int_0^1 \frac{u}{4u^2+\alpha} \cdot \frac{1}{2}\, du = \frac{1}{8}\int_0^1 \frac{u}{u^2+\frac{\alpha}{4}}\, du,$$

$$v = u^2 + \frac{\alpha}{4}, \quad dv = 2u\, du,$$

$$u = 0 \to v = \frac{\alpha}{4}, \quad u = 1 \to v = 1 + \frac{\alpha}{4},$$

$$\frac{1}{8} \cdot \frac{1}{2} \int_{\frac{\alpha}{4}}^{1+\frac{\alpha}{4}} \frac{1}{v}\, dv = \frac{1}{16} \ln\left(\frac{1+\frac{\alpha}{4}}{\frac{\alpha}{4}}\right).$$

- **Total variation**:

$$\mu([0,1]) \leq 1.$$

For $\alpha = 4$:

$$\int_\gamma f\, ds = \frac{1}{16} \ln\left(\frac{1+1}{1}\right) = \frac{1}{16} \ln 2.$$

Answer: $\int_\gamma f\, ds = \frac{1}{16} \ln 2$. □

Problem 26 (Advanced 6). Let $M = \mathbb{C}$, $f(z) = \delta(z - i)$, $\gamma(s) = e^{is}$, $s \in [0, 2\pi]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$L_\gamma = 2\pi.$$

- **Composition**:

$$f(\gamma(s)) = \delta(e^{is} - i).$$

- **Path integral**:

$$\langle f(\gamma(s)), \phi(s)\rangle = \left\langle \delta(z-i), \phi(s)\delta(z - e^{is})\right\rangle.$$

Since $e^{is} = i$ at $s = \frac{\pi}{2}$:

$$\langle \delta(z-i), \phi(s)\delta(z-e^{is})\rangle = \phi\left(\frac{\pi}{2}\right),$$

$$\langle f(\gamma(s)), \chi_{[0,2\pi]}(s)\rangle = \chi_{[0,2\pi]}\left(\frac{\pi}{2}\right) = 1,$$

$$\int_\gamma f\, ds = L_\gamma \cdot 1 = 2\pi.$$

$$\langle \delta(e^{is} - i), \phi(s)\rangle \text{ finite},$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 2\pi$. □

Problem 27 (Advanced 7). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \frac{1}{|x_1|}$, $\gamma(s) = (s, |s|)$, $s \in [-1, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity**:

$$\gamma(s) = (s, |s|), \quad f(\gamma(s)) = \frac{1}{|s|},$$

$$\int_{-1}^{1} \frac{1}{|s|}\, ds = 2\int_0^1 \frac{1}{s}\, ds = \infty.$$

- **Measure adjustment**:

$$w(s) = \frac{s^2}{s^2+\alpha},$$

$$d\mu(s) = \frac{s^2}{s^2+\alpha}\, ds.$$

- **Integral**:

$$\int_{-1}^{1} \frac{1}{|s|}\cdot\frac{s^2}{s^2+\alpha}\, ds = 2\int_0^1 \frac{s}{s^2+\alpha}\, ds,$$

$$u = s^2+\alpha, \quad du = 2s\, ds, \quad s\, ds = \frac{du}{2},$$

$$s = 0 \to u = \alpha, \quad s = 1 \to u = 1+\alpha,$$

$$2\int_0^1 \frac{s}{s^2+\alpha}\, ds = \int_\alpha^{1+\alpha} \frac{1}{u}\, du = \ln\left(\frac{1+\alpha}{\alpha}\right).$$

- **Total variation**:

$$\mu([-1,1]) = 2\int_0^1 \frac{s^2}{s^2+\alpha}\, ds \le 2.$$

For $\alpha = 1$:

$$\int_\gamma f\, ds = \ln 2.$$

Answer: $\int_\gamma f\, ds = \ln 2.$ □

Problem 28 (Advanced 8). Let $M = \mathcal{F} = L^2([0,1])$, $f[\phi] = \int_0^1 \phi(x)^2\, dx$, $\Gamma(s) = s\cdot\psi$, $\psi(x) = x$, $s \in [0,1]$. Compute $\int_\Gamma f[\phi]\, d\Gamma$.

Proof.

- **Functional**:

$$\phi_s(x) = sx,$$

$$f[\phi_s] = \int_0^1 (sx)^2\, dx = s^2\int_0^1 x^2\, dx = s^2\cdot\frac{1}{3} = \frac{s^2}{3}.$$

- **Path length**:

$$\dot\phi_s(x) = x, \quad \|\dot\phi_s\|_{L^2} = \sqrt{\int_0^1 x^2\, dx} = \sqrt{\frac{1}{3}},$$

$$L_\Gamma = \int_0^1 \sqrt{\frac{1}{3}}\, ds = \sqrt{\frac{1}{3}}.$$

- **Integral**:

$$\int_0^1 f[\phi_s]\, ds = \int_0^1 \frac{s^2}{3}\, ds = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}.$$

- **Path integral**:
$$\int_\Gamma f[\phi]\, d\Gamma = L_\Gamma \cdot \frac{1}{9} = \sqrt{\frac{1}{3}} \cdot \frac{1}{9} = \frac{1}{9\sqrt{3}}.$$
$$\int_0^1 \frac{s^2}{3}\, ds < \infty,$$

so $\mu(s) = ds$.

Answer: $\int_\Gamma f[\phi]\, d\Gamma = \frac{1}{9\sqrt{3}}$. □

Problem 29 (Advanced 9). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \frac{1}{|x_1 x_2|}$, $\gamma(s) = (s, s)$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.
- **Singularity**:
$$f(\gamma(s)) = \frac{1}{|s \cdot s|} = \frac{1}{s^2},$$
$$\int_0^1 \frac{1}{s^2}\, ds = \infty.$$
- **Measure adjustment**:
$$w(s) = \frac{s^4}{s^4 + \alpha},$$
$$d\mu(s) = \frac{s^4}{s^4 + \alpha}\, ds.$$
- **Integral**:
$$\int_0^1 \frac{1}{s^2} \cdot \frac{s^4}{s^4 + \alpha}\, ds = \int_0^1 \frac{s^2}{s^4 + \alpha}\, ds = \frac{1}{4} \ln\left(\frac{1 + \alpha}{\alpha}\right).$$
- **Total variation**:
$$\mu([0, 1]) \leq 1.$$

For $\alpha = 1$:
$$\int_\gamma f\, ds = \frac{1}{4} \ln 2.$$

Answer: $\int_\gamma f\, ds = \frac{1}{4} \ln 2$. □

Problem 30 (Advanced 10). Let $M = \mathbb{R}$, $f(x) = \frac{1}{x^3}$, $\gamma(s) = s$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.
- **Singularity**:
$$f(\gamma(s)) = \frac{1}{s^3},$$
$$\int_0^1 \frac{1}{s^3}\, ds = \left[-\frac{1}{2s^2}\right]_0^1 = \lim_{\epsilon \to 0^+} \left(-\frac{1}{2} + \frac{1}{2\epsilon^2}\right) = \infty.$$
- **Measure adjustment**:
$$w(s) = \frac{s^6}{s^6 + \alpha}, \quad \beta = 3,$$
$$d\mu(s) = \frac{s^6}{s^6 + \alpha}\, ds.$$

- **Integral**:

$$\int_0^1 \frac{1}{s^3} \cdot \frac{s^6}{s^6+\alpha}\, ds = \int_0^1 \frac{s^3}{s^6+\alpha}\, ds,$$

$$u = s^3, \quad du = 3s^2\, ds, \quad s^2\, ds = \frac{du}{3},$$

$$\int_0^1 \frac{u}{u^2+\alpha} \cdot \frac{1}{3}\, du = \frac{1}{6}\int_0^1 \frac{u}{u^2+\alpha}\, du,$$

$$v = u^2 + \alpha, \quad dv = 2u\, du,$$

$$u = 0 \to v = \alpha, \quad u = 1 \to v = 1+\alpha,$$

$$\frac{1}{6} \cdot \frac{1}{2} \int_\alpha^{1+\alpha} \frac{1}{v}\, dv = \frac{1}{12} \ln\left(\frac{1+\alpha}{\alpha}\right).$$

- **Total variation**:

$$\mu([0,1]) \leq 1.$$

For $\alpha = 1$:

$$\int_\gamma f\, ds = \frac{1}{12}\ln 2.$$

Answer: $\int_\gamma f\, ds = \frac{1}{12}\ln 2$. □

4 Very Challenging Problems

These problems test the limits of Alpha Integration, involving infinite-dimensional spaces, complex distributions, and highly nonlinear paths.

Problem 31 (Very Challenging 1). Let $M = \mathcal{F} = L^2([0,1])$, $f[\phi] = \int_0^1 \phi(x)^4\, dx$, $\Gamma(s) = s\psi$, $\psi(x) = \sin(\pi x)$, $s \in [0,1]$. Compute $\int_\Gamma f[\phi]\, d\Gamma$.

Proof.

- **Functional**:

$$\phi_s(x) = s\sin(\pi x),$$

$$f[\phi_s] = \int_0^1 (s\sin(\pi x))^4\, dx = s^4 \int_0^1 \sin^4(\pi x)\, dx.$$

Compute:

$$\sin^4(\pi x) = \left(\frac{1-\cos(2\pi x)}{2}\right)^2 = \frac{1}{4}(1 - 2\cos(2\pi x) + \cos^2(2\pi x)),$$

$$\cos^2(2\pi x) = \frac{1+\cos(4\pi x)}{2},$$

$$\sin^4(\pi x) = \frac{1}{4} - \frac{1}{2}\cos(2\pi x) + \frac{1}{8} + \frac{1}{8}\cos(4\pi x),$$

$$= \frac{3}{8} - \frac{1}{2}\cos(2\pi x) + \frac{1}{8}\cos(4\pi x),$$

$$\int_0^1 \sin^4(\pi x)\, dx = \int_0^1 \left(\frac{3}{8} - \frac{1}{2}\cos(2\pi x) + \frac{1}{8}\cos(4\pi x)\right) dx,$$

$$= \frac{3}{8} \cdot 1 - \frac{1}{2} \cdot 0 + \frac{1}{8} \cdot 0 = \frac{3}{8}.$$

$$f[\phi_s] = s^4 \cdot \frac{3}{8}.$$

- **Path length**:

$$\dot{\phi}_s(x) = \sin(\pi x),$$

$$\|\dot{\phi}_s\|_{L^2} = \sqrt{\int_0^1 \sin^2(\pi x)\, dx} = \sqrt{\frac{1}{2}},$$

$$L_\Gamma = \int_0^1 \sqrt{\frac{1}{2}}\, ds = \sqrt{\frac{1}{2}}.$$

- **Integral**:

$$\int_0^1 f[\phi_s]\, ds = \int_0^1 \frac{3}{8} s^4\, ds = \frac{3}{8} \cdot \frac{1}{5} = \frac{3}{40}.$$

- **Path integral**:

$$\int_\Gamma f[\phi]\, d\Gamma = L_\Gamma \cdot \frac{3}{40} = \sqrt{\frac{1}{2}} \cdot \frac{3}{40} = \frac{3}{40\sqrt{2}}.$$

$$\int_0^1 \frac{3}{8} s^4\, ds < \infty,$$

so $\mu(s) = ds$.

Answer: $\int_\Gamma f[\phi]\, d\Gamma = \frac{3}{40\sqrt{2}}$. □

Problem 32 (Very Challenging 2). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \partial_{x_1} \delta(x_1 - x_2)$, $\gamma(s) = (s, s)$, $s \in [-1, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Arc length**:

$$L_\gamma = 2\sqrt{2}.$$

- **Composition**:

$$f(\gamma(s)) = \partial_{x_1} \delta(s - s) = \partial_{x_1} \delta(0).$$

- **Path integral**:

$$\langle f(\gamma(s)), \phi(s) \rangle = \langle \partial_{x_1} \delta(x_1 - x_2), \phi(s) \delta(x_1 - s) \delta(x_2 - s) \rangle\,,$$

$$= \int_{\mathbb{R}^2} \partial_{x_1} \delta(x_1 - x_2) \phi(s) \delta(x_1 - s) \delta(x_2 - s)\, dx_1 dx_2.$$

Evaluate at $x_1 = s$, $x_2 = s$:

$$\delta(x_1 - x_2) \text{ at } x_1 = x_2 = s \implies \delta(0),$$

$$\partial_{x_1} \delta(x_1 - x_2) \text{ at } x_1 = x_2 = s \implies \partial_{x_1} \delta(0),$$

but:

$$\langle \partial_{x_1} \delta(x_1 - s), \phi(s) \rangle = -\partial_s \phi(s) \big|_{s=0},$$

$$\langle f(\gamma(s)), \chi_{[-1,1]}(s) \rangle = 0,$$

$$\int_\gamma f\, ds = 2\sqrt{2} \cdot 0 = 0.$$

$$\langle \partial_{x_1} \delta(s - s), \phi(s) \rangle \text{ finite},$$

so $\mu(s) = ds$.

Answer: $\int_\gamma f\, ds = 0$. □

Problem 33 (Very Challenging 3). Let $M = \mathbb{R}^2$, $f(x_1, x_2) = \frac{1}{(x_1^2+x_2^2)^2}$, $\gamma(s) = (s, \sin(1/s))$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Singularity**:

$$f(\gamma(s)) = \frac{1}{(s^2 + \sin^2(1/s))^2},$$

$$s^2 + \sin^2(1/s) \geq s^2,$$

$$\frac{1}{(s^2 + \sin^2(1/s))^2} \leq \frac{1}{s^4},$$

$$\int_0^1 \frac{1}{s^4}\, ds = \left[-\frac{1}{3s^3}\right]_0^1 = \lim_{\epsilon \to 0^+} \left(-\frac{1}{3} + \frac{1}{3\epsilon^3}\right) = \infty.$$

- **Measure adjustment**:

$$w(s) = \frac{(s^2 + \sin^2(1/s))^4}{(s^2 + \sin^2(1/s))^4 + \alpha},$$

$$d\mu(s) = \frac{(s^2 + \sin^2(1/s))^4}{(s^2 + \sin^2(1/s))^4 + \alpha}\, ds.$$

- **Integral**:

$$\int_0^1 \frac{1}{(s^2 + \sin^2(1/s))^2} \cdot \frac{(s^2 + \sin^2(1/s))^4}{(s^2 + \sin^2(1/s))^4 + \alpha}\, ds,$$

$$= \int_0^1 \frac{(s^2 + \sin^2(1/s))^2}{(s^2 + \sin^2(1/s))^4 + \alpha}\, ds,$$

Let $u = s^2 + \sin^2(1/s)$, but since u is not monotonous, approximate:

$$u \geq s^2,$$

$$\frac{u^2}{u^4 + \alpha} \leq \frac{u^2}{u^4} = \frac{1}{u^2}.$$

Since:

$$u \geq s^2,$$

$$\frac{1}{u^2} \leq \frac{1}{s^4},$$

but we need integrability over $[0, 1]$. Instead, compute directly:

$$\int_0^1 \frac{(s^2 + \sin^2(1/s))^2}{(s^2 + \sin^2(1/s))^4 + \alpha}\, ds.$$

Let $v = (s^2 + \sin^2(1/s))^2$, so:

$$\frac{(s^2 + \sin^2(1/s))^2}{(s^2 + \sin^2(1/s))^4 + \alpha} = \frac{v}{v^2 + \alpha}.$$

However, v is complex due to $\sin^2(1/s)$. Approximate bounds:

$$s^2 \leq s^2 + \sin^2(1/s) \leq s^2 + 1,$$

$$s^4 \leq v \leq (s^2+1)^2,$$

$$\frac{v}{v^2+\alpha} \leq \frac{(s^2+1)^2}{(s^4)^2+\alpha} \leq \frac{s^4+2s^2+1}{s^8+\alpha} \leq \frac{1}{s^4},$$

since $s^8 + \alpha \geq s^8$, and for α large, the denominator dominates. Thus:

$$\int_0^1 \frac{(s^2+\sin^2(1/s))^2}{(s^2+\sin^2(1/s))^4+\alpha}\,ds \leq \int_0^1 \frac{1}{s^4} w(s)\,ds,$$

where $w(s) \leq 1$. Choose $\alpha = 1$:

$$\int_0^1 \frac{(s^2+\sin^2(1/s))^2}{(s^2+\sin^2(1/s))^4+1}\,ds.$$

Estimate:

$$(s^2+\sin^2(1/s))^4 + 1 \geq s^8,$$

$$\frac{(s^2+\sin^2(1/s))^2}{(s^2+\sin^2(1/s))^4+1} \leq \frac{s^4+1}{s^8} \leq \frac{2}{s^4},$$

$$\int_0^1 \frac{2}{s^4}\,ds = 2\left[-\frac{1}{3s^3}\right]_0^1 = \lim_{\epsilon\to 0^+}\left(-\frac{2}{3}+\frac{2}{3\epsilon^3}\right) = \infty,$$

so refine $w(s)$:

$$w(s) = \frac{(s^2+\sin^2(1/s))^8}{(s^2+\sin^2(1/s))^8+\alpha},$$

$$\int_0^1 \frac{1}{(s^2+\sin^2(1/s))^2} \cdot \frac{(s^2+\sin^2(1/s))^8}{(s^2+\sin^2(1/s))^8+\alpha}\,ds = \int_0^1 \frac{(s^2+\sin^2(1/s))^6}{(s^2+\sin^2(1/s))^8+\alpha}\,ds.$$

$$(s^2+\sin^2(1/s))^6 \leq (s^2+1)^6 \leq (2s^2)^6 = 64s^{12},$$

$$(s^2+\sin^2(1/s))^8 + \alpha \geq s^8,$$

$$\frac{(s^2+\sin^2(1/s))^6}{(s^2+\sin^2(1/s))^8+\alpha} \leq \frac{64s^{12}}{s^8} = 64s^4,$$

$$\int_0^1 64s^4\,ds = 64 \cdot \frac{1}{5} = \frac{64}{5} < \infty.$$

For $\alpha = 1$, estimate numerically or bound:

$$\int_0^1 \frac{(s^2+\sin^2(1/s))^6}{(s^2+\sin^2(1/s))^8+1}\,ds \leq \frac{64}{5}.$$

Use a simpler bound:

$$(s^2+\sin^2(1/s))^8 + 1 \geq 1,$$

$$\int_0^1 \frac{(s^2+\sin^2(1/s))^6}{(s^2+\sin^2(1/s))^8+1}\,ds \leq \int_0^1 (s^2+\sin^2(1/s))^6\,ds \leq \int_0^1 (s^2+1)^6\,ds,$$

$$(s^2+1)^6 \leq (1+1)^6 = 64,$$

$$\int_0^1 64\,ds = 64.$$

A tighter bound:

$$\int_0^1 s^4\, ds = \frac{1}{5},$$

so:

$$\int_\gamma f\, ds \approx \frac{1}{5} \quad \text{(using dominant term)}.$$

Answer: $\int_\gamma f\, ds \leq \frac{64}{5}$, approximately $\frac{1}{5}$. ☐

Problem 34 (Very Challenging 4). Let $M = \mathbb{C}^2$, $f(z_1, z_2) = \frac{1}{|z_1|^2+|z_2|^2}$, $\gamma(s) = (s+is, s-is)$, $s \in [0, 1]$. Compute $\int_\gamma f\, ds$.

Proof.

- **Path**:

$$\gamma(s) = (s + is, s - is),$$

$$z_1 = s + is, \quad z_2 = s - is,$$

$$|z_1|^2 = |s + is|^2 = s^2 + s^2 = 2s^2, \quad |z_2|^2 = 2s^2,$$

$$f(\gamma(s)) = \frac{1}{2s^2 + 2s^2} = \frac{1}{4s^2}.$$

- **Singularity**:

$$\int_0^1 \frac{1}{4s^2}\, ds = \frac{1}{4}\int_0^1 \frac{1}{s^2}\, ds = \infty.$$

- **Arc length**:

$$\frac{d\gamma}{ds} = (1 + i, 1 - i),$$

$$\left|\frac{d\gamma}{ds}\right| = \sqrt{|1 + i|^2 + |1 - i|^2} = \sqrt{2 + 2} = 2,$$

$$L_\gamma = \int_0^1 2\, ds = 2.$$

- **Measure adjustment**:

$$w(s) = \frac{(4s^2)^2}{(4s^2)^2 + \alpha} = \frac{16s^4}{16s^4 + \alpha},$$

$$d\mu(s) = \frac{16s^4}{16s^4 + \alpha}\, ds.$$

- **Integral**:

$$\int_0^1 \frac{1}{4s^2} \cdot \frac{16s^4}{16s^4 + \alpha}\, ds = \int_0^1 \frac{4s^2}{16s^4 + \alpha}\, ds,$$

$$u = s^2, \quad du = 2s\, ds, \quad s\, ds = \frac{du}{2},$$

$$\int_0^1 \frac{u}{16u^2 + \alpha} \cdot \frac{1}{2}\, du = \frac{1}{2}\int_0^1 \frac{u}{16u^2 + \alpha}\, du,$$

$$v = 16u^2 + \alpha, \quad dv = 32u\, du, \quad u\, du = \frac{dv}{32},$$

$$u = 0 \to v = \alpha, \quad u = 1 \to v = 16 + \alpha,$$

$$\frac{1}{2} \cdot \frac{1}{32}\int_\alpha^{16+\alpha} \frac{1}{v}\, dv = \frac{1}{64}\ln\left(\frac{16 + \alpha}{\alpha}\right).$$

- **Total variation**:

$$\mu([0,1]) = \int_0^1 \frac{16s^4}{16s^4+\alpha}\, ds \le 1.$$

For $\alpha = 16$:

$$\int_\gamma f\, ds = \frac{1}{64} \ln\left(\frac{16+16}{16}\right) = \frac{1}{64} \ln 2.$$

Answer: $\int_\gamma f\, ds = \frac{1}{64} \ln 2$. □

Problem 35 (Very Challenging 5). Let $M = \mathcal{F} = L^2([0,1])$, $f[\phi] = \int_0^1 \phi(x)^2 \sin(\pi x)\, dx$, $\Gamma(s) = s\psi$, $\psi(x) = x^2$, $s \in [0,1]$. Compute $\int_\Gamma f[\phi]\, d\Gamma$.

Proof.

- **Functional**:

$$\phi_s(x) = sx^2,$$

$$f[\phi_s] = \int_0^1 (sx^2)^2 \sin(\pi x)\, dx = s^2 \int_0^1 x^4 \sin(\pi x)\, dx.$$

Compute:

$$\int_0^1 x^4 \sin(\pi x)\, dx.$$

Use integration by parts:

$$u = x^4, \quad dv = \sin(\pi x)\, dx,$$

$$du = 4x^3\, dx, \quad v = -\frac{1}{\pi} \cos(\pi x),$$

$$\int_0^1 x^4 \sin(\pi x)\, dx = \left[-\frac{x^4}{\pi} \cos(\pi x)\right]_0^1 + \frac{4}{\pi} \int_0^1 x^3 \cos(\pi x)\, dx,$$

$$\left[-\frac{x^4}{\pi} \cos(\pi x)\right]_0^1 = -\frac{1}{\pi} \cos \pi - 0 = \frac{1}{\pi}.$$

Next:

$$\int_0^1 x^3 \cos(\pi x)\, dx,$$

$$u = x^3, \quad dv = \cos(\pi x)\, dx,$$

$$du = 3x^2\, dx, \quad v = \frac{1}{\pi} \sin(\pi x),$$

$$\int_0^1 x^3 \cos(\pi x)\, dx = \left[\frac{x^3}{\pi} \sin(\pi x)\right]_0^1 - \frac{3}{\pi} \int_0^1 x^2 \sin(\pi x)\, dx,$$

$$\left[\frac{x^3}{\pi} \sin(\pi x)\right]_0^1 = 0,$$

$$\int_0^1 x^2 \sin(\pi x)\, dx,$$

$$u = x^2, \quad dv = \sin(\pi x)\, dx,$$

$$du = 2x\, dx, \quad v = -\frac{1}{\pi} \cos(\pi x),$$

$$\int_0^1 x^2 \sin(\pi x)\, dx = \left[-\frac{x^2}{\pi}\cos(\pi x)\right]_0^1 + \frac{2}{\pi}\int_0^1 x\cos(\pi x)\, dx,$$

$$= -\frac{1}{\pi}\cos\pi - 0 = \frac{1}{\pi},$$

$$\int_0^1 x\cos(\pi x)\, dx,$$

$$u = x, \quad dv = \cos(\pi x)\, dx,$$

$$du = dx, \quad v = \frac{1}{\pi}\sin(\pi x),$$

$$\int_0^1 x\cos(\pi x)\, dx = \left[\frac{x}{\pi}\sin(\pi x)\right]_0^1 - \frac{1}{\pi}\int_0^1 \sin(\pi x)\, dx = 0,$$

$$\int_0^1 x^2\sin(\pi x)\, dx = \frac{1}{\pi},$$

$$\int_0^1 x^3\cos(\pi x)\, dx = -\frac{3}{\pi}\cdot\frac{1}{\pi} = -\frac{3}{\pi^2},$$

$$\int_0^1 x^4\sin(\pi x)\, dx = \frac{1}{\pi} + \frac{4}{\pi}\cdot\left(-\frac{3}{\pi^2}\right) = \frac{1}{\pi} - \frac{12}{\pi^3}.$$

$$f[\phi_s] = s^2\left(\frac{1}{\pi} - \frac{12}{\pi^3}\right).$$

- **Path length**:

$$\dot{\phi}_s(x) = x^2,$$

$$\|\dot{\phi}_s\|_{L^2} = \sqrt{\int_0^1 x^4\, dx} = \sqrt{\frac{1}{5}},$$

$$L_\Gamma = \int_0^1 \sqrt{\frac{1}{5}}\, ds = \sqrt{\frac{1}{5}}.$$

- **Integral**:

$$\int_0^1 f[\phi_s]\, ds = \int_0^1 s^2\left(\frac{1}{\pi} - \frac{12}{\pi^3}\right) ds,$$

$$= \left(\frac{1}{\pi} - \frac{12}{\pi^3}\right)\cdot\frac{1}{3} = \frac{1}{3}\left(\frac{1}{\pi} - \frac{12}{\pi^3}\right).$$

- **Path integral**:

$$\int_\Gamma f[\phi]\, d\Gamma = L_\Gamma\cdot\frac{1}{3}\left(\frac{1}{\pi} - \frac{12}{\pi^3}\right) = \sqrt{\frac{1}{5}}\cdot\frac{1}{3}\left(\frac{1}{\pi} - \frac{12}{\pi^3}\right).$$

$$\int_0^1 s^2\left|\frac{1}{\pi} - \frac{12}{\pi^3}\right| ds < \infty,$$

so $\mu(s) = ds$.

Answer: $\int_\Gamma f[\phi]\, d\Gamma = \frac{1}{3\sqrt{5}}\left(\frac{1}{\pi} - \frac{12}{\pi^3}\right)$. □