Open Problems in Mathematical Physics

Grok 3

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Abstract

This document presents rigorous mathematical and physical definitions of three fundamental open problems in mathematical physics: the Yang-Mills Mass Gap Hypothesis, the Navier-Stokes Existence and Smoothness Problem, and the Quantum Gravity Problem. Each problem is stated with precision, reflecting the current understanding in their respective fields.

1 Problem Statements

Problem 1 (Yang-Mills Mass Gap Hypothesis). *Let G be a compact, simple Lie group with Lie algebra $\mathfrak{g}$, equipped with an invariant inner product. Consider a pure Yang-Mills quantum field theory defined on Minkowski spacetime $\mathbb{R}^{1,3}$ with the action functional*

$$S(A) = -\frac{1}{4}\int_{\mathbb{R}^{1,3}} Tr(F \wedge \star F),$$

where A is a $\mathfrak{g}$-valued connection 1-form, $F = dA + A \wedge A$ is the curvature 2-form, Tr denotes the trace induced by the inner product on $\mathfrak{g}$, and $\star$ is the Hodge star operator with respect to the Minkowski metric $\eta = diag(-1,1,1,1)$. The quantum Yang-Mills theory is defined via a hypothetical functional integral

$$Z = \int \mathcal{D}A\, e^{iS(A)/\hbar},$$

satisfying the axioms of quantum field theory, including locality, unitarity, and Lorentz invariance.

Prove that for any such G, the quantum Yang-Mills theory possesses a mass gap*, i.e., there exists a positive constant $\Delta > 0$ such that the spectrum of the Hamiltonian H in the physical Hilbert space has no states with energy in the interval $(0, \Delta)$, except for the vacuum state at energy $E = 0$. Specifically:*

(a) Construct a mathematically rigorous quantum field theory for the Yang-Mills action, defining the Hilbert space $\mathcal{H}$, the algebra of observables, and the Hamiltonian H.

(b) Demonstrate that the correlation functions of the theory decay exponentially at large distances, consistent with a positive mass gap Δ.

(c) Show that the particle-like excitations (e.g., glueballs) have positive rest mass $m \geq \Delta$, and that the vacuum is unique.

Problem 2 (Navier-Stokes Existence and Smoothness). *Consider the incompressible Navier-Stokes equations on* $\mathbb{R}^3 \times [0,\infty)$ *for a velocity field* $\mathbf{u} : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}^3$ *and pressure* $p : \mathbb{R}^3 \times [0,\infty) \to \mathbb{R}$*, given by*

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \nabla p,$$
$$\nabla \cdot \mathbf{u} = 0,$$

where $\nu > 0$ *is the kinematic viscosity,* Δ *is the Laplacian, and* $\nabla \cdot \mathbf{u} = 0$ *enforces incompressibility. Let the initial condition be* $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$*, where* $\mathbf{u}_0 \in C^\infty(\mathbb{R}^3)$ *is a divergence-free vector field with compact support.*

Address one of the following for the Cauchy problem:

(a) Existence and Smoothness*: Prove that for any such* $\mathbf{u}_0$*, there exists a unique global solution* $\mathbf{u} \in C^\infty(\mathbb{R}^3 \times [0,\infty); \mathbb{R}^3)$*,* $p \in C^\infty(\mathbb{R}^3 \times [0,\infty); \mathbb{R})$ *satisfying the Navier-Stokes equations, with* $\mathbf{u}$ *remaining bounded and smooth for all* $t \geq 0$*.*

(b) Breakdown*: Prove that there exists a* $\mathbf{u}_0 \in C^\infty(\mathbb{R}^3)$ *such that no smooth solution exists for all* $t \geq 0$*, i.e., there is a finite time* $T < \infty$ *such that* $\sup_{\mathbf{x} \in \mathbb{R}^3} |\nabla \mathbf{u}(\mathbf{x}, t)| \to \infty$ *as* $t \to T^-$*.*

In either case, assume no external forcing ($\mathbf{f} = 0$)*, and quantify the behavior of solutions in terms of energy norms, e.g.,* $\int_{\mathbb{R}^3} |\mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x}$*, and enstrophy,* $\int_{\mathbb{R}^3} |\nabla \times \mathbf{u}(\mathbf{x}, t)|^2 d\mathbf{x}$*.*

Problem 3 (Quantum Gravity). *Let* M *be a 4-dimensional Lorentzian manifold with metric* $g_{\mu\nu}$ *of signature* $(-1, 1, 1, 1)$*, representing spacetime. Consider a quantum theory of gravity that combines general relativity, with action*

$$S_{EH} = \frac{1}{16\pi G} \int_M R\sqrt{-g}\, d^4x,$$

where R *is the Ricci scalar,* $g = \det(g_{\mu\nu})$*, and* G *is Newton's constant, with a quantum field theory of matter fields* ϕ *(e.g., scalar, spinor, or gauge fields) coupled to* $g_{\mu\nu}$*, satisfying the Einstein field equations in the classical limit:*

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where $T_{\mu\nu}$ *is the stress-energy tensor of* ϕ*.*

Construct a mathematically consistent quantum gravity theory by addressing the following:

(a) Define a Hilbert space $\mathcal{H}$ *and a state space for quantum geometries, possibly via a functional integral*

$$Z = \int \mathcal{D}g \mathcal{D}\phi\, e^{i(S_{EH} + S_{matter})/\hbar},$$

that is finite and well-defined, handling the non-renormalizability of perturbative quantum gravity.

(b) Ensure the theory is diffeomorphism-invariant, preserving general covariance at the quantum level, and specify how observables (e.g., correlation functions of $T_{\mu\nu}$ *or curvature invariants) are computed.*

(c) *Prove that the theory reproduces classical general relativity in the limit $\hbar \to 0$, and predict testable quantum corrections, such as deviations from Newtonian gravity at small scales or cosmological signatures.*

The formulation may draw upon approaches like loop quantum gravity, string theory, or asymptotic safety, but must be mathematically rigorous, specifying the domain of integration, measure, and regularization scheme.

If you solve at least one of these three problems, you will be a successful mathematician and physicist.