Orthogonal Decomposition Technique for Ionospheric Tomography

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ABSTRACT

The possibility of reconstructing two-dimensional electron-density profiles in the ionosphere with ionospheric tomography is significant. However, due to the nature of the imaging system, there are several resolution degradation parameters. In order to compensate for these degradation parameters, *a priori* information must be used. This article introduces the orthogonal decomposition algorithm for image reconstruction, which uses the *a priori* information to generate a set of orthogonal basis functions for the source domain. This algorithm consists of two simple steps: orthogonal decomposition and recombination. In the development of the algorithm, it is shown that the degradation parameters of the imaging system result in correlations among projections of orthogonal functions. Gram–Schmidt orthogonalization is used to compensate for these correlations, producing a matrix that measures the degradation of the system. Any set of basis functions can be used, and depending upon this choice, the nature of the algorithm varies greatly. Choosing the basis functions of the source domain to be the Fourier kernels produces an algorithm capable of isolating individual frequency components of individual projections. This particular choice of basis functions also results in an algorithm that strongly resembles the direct Fourier method, but without requiring the use of inverse Fourier transforms.

I. INTRODUCTION

The problem of imaging the electron-density profiles in the ionosphere has been of interest for many years and recently has been examined from a tomographic viewpoint [1, 2]. A reconstruction algorithm based upon the filtered backprojection algorithm of x-ray tomography has been used to reconstruct images of the electron density. The resolving capability of this algorithm, however, is limited due to finite receiving apertures, uneven sample spacing, low sampling rate, and angular quantization error [3]. Ambiguities in the reconstructions that result from these physical constraints can also be regarded as the effect of correlations between projections obtained from orthogonal functions. In an ideal imaging system, projections from orthogonal functions would themselves be orthogonal. These correlations can, however, be compensated for by using Gram–Schmidt orthogonalization. This article introduces an algorithm that uses Gram–Schmidt orthogonalization to generate an orthonormal basis for the domain of projections. Image components can then be extracted and reconstruction simplifies to decomposing projection data into orthogonal components and calculating weighting for corresponding basis functions in the source domain. Resolution enhancement is achieved in this algorithm by the incorporation of *a priori* information into both the definition of source and projection domains, and the selection of basis functions. This is especially appropriate for ionospheric tomography as considerable information about the ionosphere exists, and accurate basis functions can be determined. This article will first review the principles of ionospheric tomography, then develop the generalized orthogonal decomposition algorithm. In the development of this algorithm, correlations among the projections of orthogonal source functions will be analyzed. The development will be made for any arbitrary set of basis functions to demonstrate the general structure of the algorithm. Since the nature of the algorithm is highly dependent upon the set of basis functions chosen, an interesting special case will be examined which uses the Fourier kernel to form the orthogonal basis functions of the source domain. This is an important special case because it allows individual frequency components of each projection to be isolated. An interesting relationship between the central slice theorem and this special case of the orthogonal decomposition algorithm is also presented. Finally, some discussion of the effects of individual degradation parameters on this algorithm, for any choice of basis functions, is given.

Received 30 June 1991; revised manuscript received 15 October 1991

II. IONOSPHERIC TOMOGRAPHY

Ionospheric tomography is a relatively new area that uses tomographic imaging techniques to reconstruct distributions in the ionosphere. The goal of ionospheric tomography is to reconstruct images of the electron-density distributions in the F region of the ionosphere, which is the region that lies roughly between 140 km and 1000 km above the Earth's surface. Knowledge of the distribution of the electron density in this region is important for understanding the geophysical forces that create them. Because the ionosphere consists of a layer of free electrons in the upper atmosphere that act as a waveguide for radio signals, such knowledge is also used to determine the propagation conditions for these signals.

Current remote sensing methods are capable of accurately determining electron densities at specific locations in the ionosphere or measuring total electron content along paths in the ionosphere. Considerable work has been done over the years to improve the accuracy of remote sensing methods such as the Faraday rotation and differential Doppler techniques

International Journal of Imaging Systems and Technology, Vol. 3, 354–365 (1991)

CCC 0899-9457/91/040354-12$04.00

[4–6]. In 1986, Austen *et al.* discovered that the data obtained from these two techniques could be interpreted as tomographic data [2]. This brought about the possibility of reconstructing two-dimensional images of the electron density in each slice of the ionosphere. Consider the data acquisition system in the differential Doppler method. A satellite orbits the Earth at an altitude of 1000 kilometers above the Earth's surface and a set of receivers is located in a straight line on the Earth's surface, as shown in Fig. 1. At each satellite position, two coherent signals of harmonic frequencies are transmitted. The data received by each ground station is proportional to the total electron content along the propagation path. This total electron content represents the integral of the electron density along the propagation path. In x-ray tomography, each sample on a projection is the line integral of the distribution to be reconstructed, along the path of the x-ray. Therefore the total electron content data can be interpreted, in tomographic terms, as a sample on a projection taken at the angle of the propagation path. This is shown in Fig. 2. The collection of data can be reindexed according to their "projection angles" and used to form projections of the electron density as shown in Fig. 3. A filtered backprojection algorithm can be used to reconstruct images from the projection data created in this manner.

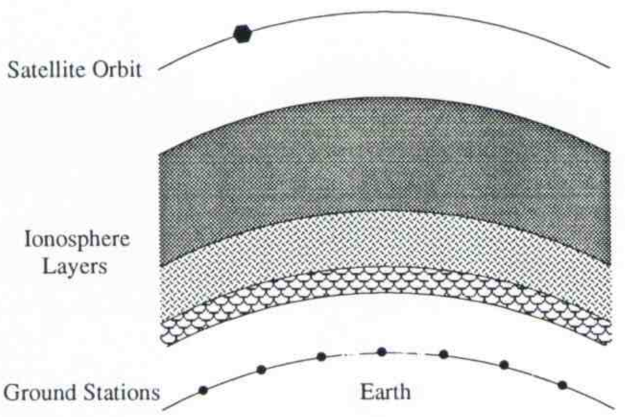

Figure 1. Data acquisition system for ionospheric tomography.

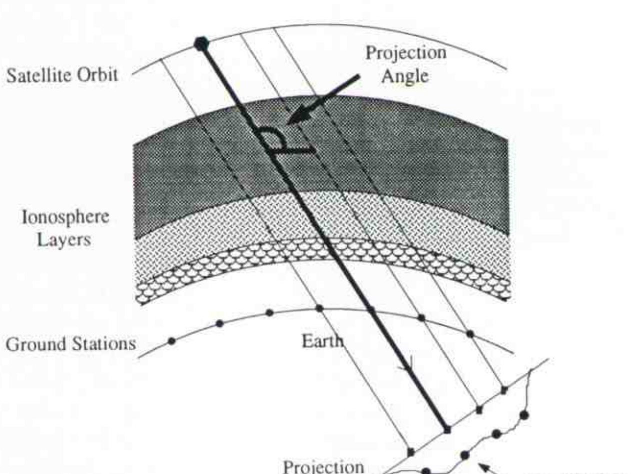

Figure 2. Acquisition of one piece of data.

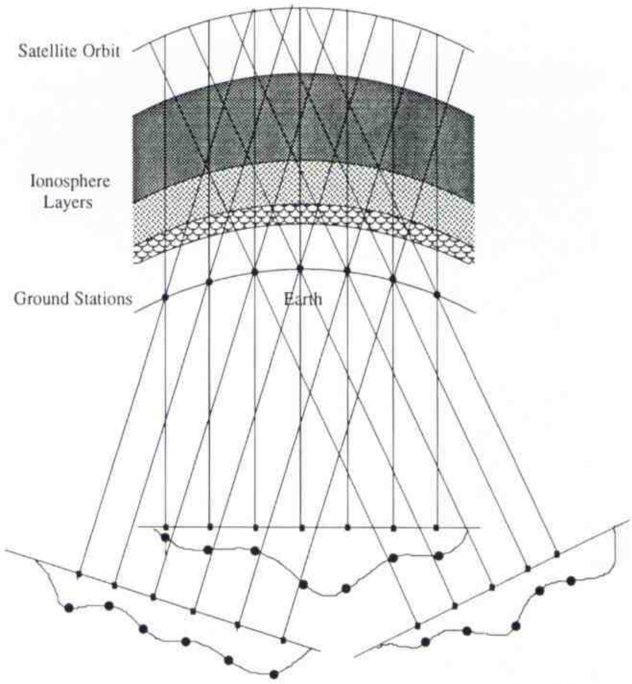

Figure 3. Reindexing of data to form projections.

There are several characteristics of both the physical system and the tomographic reconstruction algorithm that limit the resolving capability of such algorithms. The physical constraints arise from the limitations of the ground stations. Most obviously, there can only be a finite number of ground stations, so the receiving apertures are finite for all projections. The positions of the ground stations are also fixed, causing the samples in various projections to be unevenly spaced. In the best case, these ground stations will be evenly situated on the Earth. However, since the projections are formed by grouping data based upon their projection angles, these ground stations will appear unevenly spaced in most projections, as shown in Fig. 4. The number of ground stations in practice is also relatively small, hence, there will be low sampling rate in the projections. Aside from the physical constraints, the algorithm also imposes some resolution limits. In reindexing the data to form projections, there will be angular quantization error. Since the angles of the propagation paths will cover a range of values, it may be difficult to classify the data exactly according to projection angles. To do so exactly may result in many projections, each with only one sample. A more practical approach is to allow some error in the classification of these data points, for example, projection angles of 89.93° and 90.14° could both be classified as having a projection angle of 90°. While this is a more reasonable approach, this introduces angular quantization error. All of these limitations cause ambiguities in the reconstructed images.

In order to compensate for these degradation factors, *a priori* information can be used to improve the reconstructions. Since there is considerable information about the ionosphere, its formation, and its structure, it is possible to use this *a priori* information to determine a set of basis functions for the source domain. The orthogonal decomposition algorithm developed in this article will use *a priori* information in this manner to enhance the resolution of the reconstructions, and

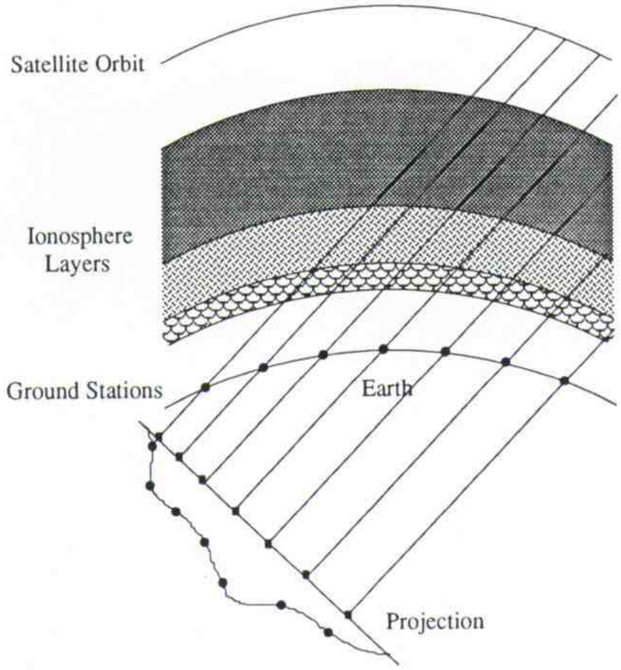

Figure 4. Uneven spacing of samples in a projection.

will show that the ambiguities caused by the resolution degradation parameters are manifested as correlations in the projections of orthogonal functions.

III. ORTHOGONAL DECOMPOSITION ALGORITHM

The orthogonal decomposition algorithm examines the reconstruction problem from a linear system point of view. This is possible because the process of forming projections from source distributions can be represented as a linear operation. This algorithm has many benefits over the traditional filtered backprojection approach. Unlike the filtered backprojection algorithm, which is based upon a continuous formulation and requires discrete approximations in the backprojection of each value, the orthogonal decomposition algorithm uses a discrete formulation that allows all approximations to be made in the selection of discrete basis functions. No approximations are needed in the actual reconstruction steps. Since there is no filtering step in the algorithm, evenly spaced samples of the projections are not necessary, thereby removing the requirement for interpolation or extrapolation. In addition to not using backprojection, this algorithm also does not involve matrix inversion, which is widely used in other algorithms. Instead, the implementation is straightforward, with a natural method of incorporating the *a priori* information. Since basis functions are used, specific components of the image can also be extracted. Several steps of the procedure can be performed in advance for each particular imaging system and the ensuing algorithm consists only of two simple steps: decomposition and recombination with respect to the basis functions.

From a linear system point of view, the acquisition of projections from a source distribution is a linear mapping between two domains. The first domain encompasses all source distributions while the second domain consists of sets of projections from each of these source distributions. Each element in the domain of projections contains a set of projections taken from a source distribution, not simply individual projections. Image reconstruction can then be considered to be an inverse mapping from the projection domain to the domain of source distributions. In the ideal case of complete and accurate projections, this inverse mapping will reconstruct the original source distribution. However, when the projection data is incomplete, the range of the inverse mapping is restricted to a subset of the source distribution domain and hence an ideal reconstruction may not be possible. The reconstruction algorithm presented in this article defines basis functions for each of the two domains. The inverse mapping is then based upon the relation between these two sets of basis functions. Since all elements in the projection domain arise from source distributions in the other domain, the basis functions of the two domains are related and the inverse mapping is possible.

The basic idea behind this algorithm is that once a set of orthogonal basis functions is found for the source domain, then reconstruction simplifies to determining the coefficients of the original source distribution when expressed as a linear combination of these basis functions. Since the data is obtained in the projection domain, an orthogonal set of basis functions for the projection domain must be found that is related to the original set of basis functions for the source domain. If such a set of basis functions for the projection domain can be found, then the reconstruction is straightforward.

In order to develop this algorithm, some definitions must first be made. To begin with, the source domain that has been referred to is defined as the set of all possible reconstructed source distributions. Each of these distributions is to be reconstructed from a set of data consisting of projections taken at various angles as shown in Fig. 5. All possible data sets or sets of these projections then form what will be termed the projection domain. If P is defined to be the projection operator, V the source domain, and W the projection domain, then it can be said that P maps source distributions into projections,

$$P: V \rightarrow W . \tag{1}$$

In this algorithm, basis functions for these two domains must first be identified and related. These basis functions are determined by the imaging system and will be reused in every reconstruction process. The physical constraints of the imaging system and the types of images to be reconstructed define these domains. For example, the number of projections taken, the aperture size, the number of ground stations, the extent of the region being reconstructed, and the nonnegativity of the images and projections are all constraints on the domains. Once a set of basis functions for the source domain is chosen, the basis functions for the projection domain can be determined from those of the source domain.

Given these two sets of basis functions and the relationship between them, the reconstruction of source distributions from their projections is significantly simplified. Since the data is a set of projections, it can be decomposed into a collection of orthogonal components, giving a set of coefficients for the linear combination of the basis functions in the projection domain. The desired reconstruction is a linear combination of the basis functions in the source domain. The coefficients of

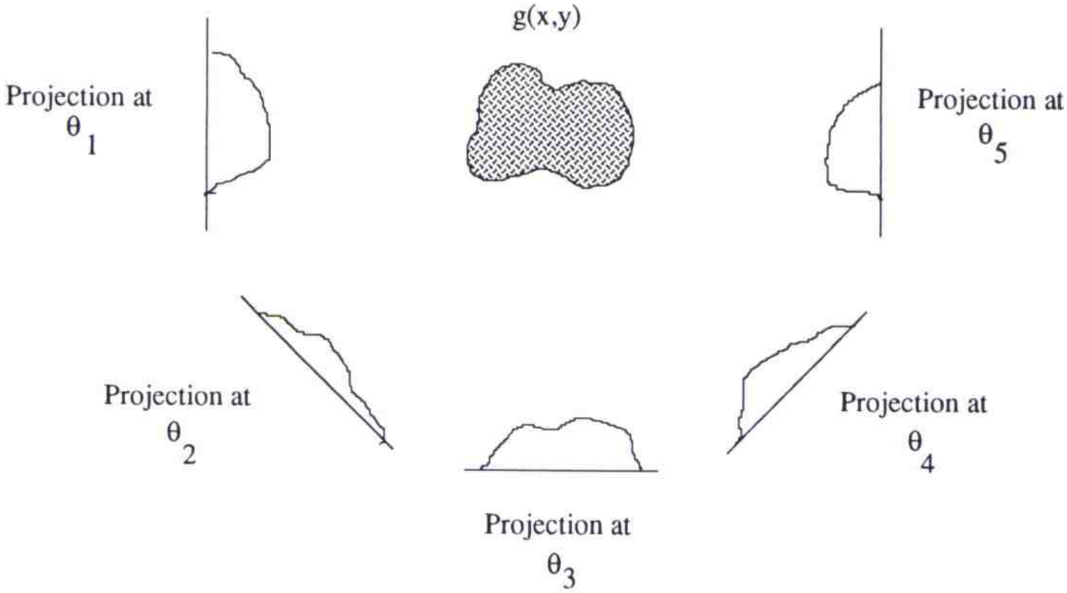

Figure 5. Projections taken at various angles of an object.

this linear combination can be found from the coefficients in the projection domain since the relationship between the two sets of basis functions is known. Therefore the algorithm consists only of the two steps of decomposition and recombination.

To derive this algorithm mathematically, we first consider a set of orthogonal basis functions for the source domain $\{\phi_i(x, y)\}$. Each $\phi_i(x, y)$ will itself be a source function, and every function $g(x, y)$ belonging to the source domain can be expressed as a linear combination of these basis functions,

$$g(x, y) = \sum_{i=1}^{N} a_i \phi_i(x, y)\,, \quad a_i \in \Re\,. \tag{2}$$

To express this in vector notation, let ϕ_i be a column vector representing samples or pixels of $\phi_i(x, y)$. Then a matrix Φ can be formed to include all of these basis functions,

$$[\Phi] = [\phi_1|\phi_2|\cdots|\phi_N]\,. \tag{3}$$

If the coefficients a_i are combined into a coefficient vector $\mathbf{a}$ and $g(x, y)$ is sampled and written as a column vector $\mathbf{g}$, then Eq. (2) can be rewritten

$$\mathbf{g} = \mathbf{\Phi a}\,. \tag{4}$$

Each source function $\mathbf{g}$ is then completely determined by knowledge of the coefficient vector $\mathbf{a}$.

Reconstruction of the source distribution from projection data is then equivalent to determining the coefficient vector. In order to do this, we must form a set of orthogonal basis functions for the projection domain and identify their relationships to the basis functions $\{\phi_i(x, y)\}$ of the source domain. Since each basis function is a source function, projections of each basis function may be taken. Therefore consider K projections of the basis function ϕ_i, taken at angles $\theta_1, \ldots, \theta_K$. Let $s_k(j)$ represent the jth sample of the projection taken at the angle θ_k, as shown in Fig. 6. Assume that

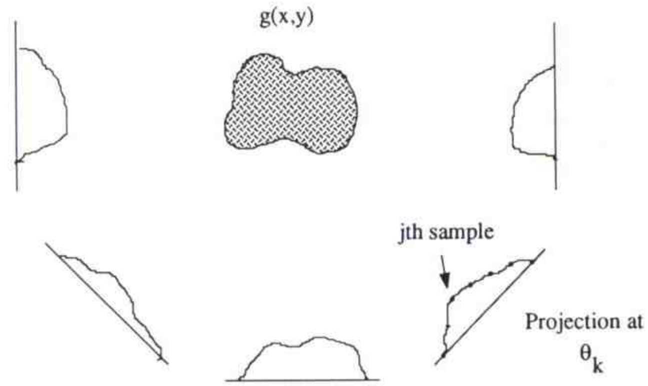

Figure 6. One sample on one projection.

there are n samples per projection. Then, the samples from the projection taken at angle θ_k can be represented as a vector $\mathbf{s}_k$,

$$\mathbf{s}_k = \begin{bmatrix} s_k(1) \\ s_k(2) \\ \vdots \\ s_k(n) \end{bmatrix}. \tag{5}$$

If projections are taken at K different angles for each basis function ϕ_i, and each has n samples, then an $nK \times 1$ vector $\mathbf{q}_i$ can be defined which contains all of the samples from the projections of ϕ_i. This vector consists of the samples from each projection stacked upon those of the next projection, and will be referred to as representing a set of projections,

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \vdots \\ \mathbf{s}_K \end{bmatrix}. \tag{6}$$

The linear operator P gives the relationship between each of the basis functions and their projections,

$$\mathbf{q}_i = P\phi_i \,. \tag{7}$$

Define Q to be a matrix composed of these vectors of projections,

$$[\mathbf{Q}] = [\mathbf{q}_1|\mathbf{q}_2|\cdots|\mathbf{q}_N] \,. \tag{8}$$

A linear relationship between the matrix of basis functions and the matrix of corresponding projections can also be stated, extending Eq. (7),

$$\mathbf{Q} = P\Phi \,. \tag{9}$$

Consider the set of vectors $\{\mathbf{q}_i\}$ in the projection domain. If $\{\mathbf{q}_i\}$ forms an orthogonal basis for the projection domain, then the reconstruction of a source distribution from a set of projections is simple. The data can be decomposed into their orthogonal components and written as a sum of the basis functions $\{\mathbf{q}_i\}$,

$$\text{data} = \sum_i a_i \mathbf{q}_i \,. \tag{10}$$

Since $\{\mathbf{q}_i\}$ are projections of the basis functions $\{\phi_i\}$, Eq. (7) can be substituted into Eq. (10), and the data can be expressed as a linear combination of the source domain basis vectors,

$$\text{data} = \sum_i a_i P\phi_i \,. \tag{11}$$

But P is a linear operator and therefore can be taken outside the summation,

$$\text{data} = P\Big[\sum_i a_i \phi_i\Big] \,. \tag{12}$$

Hence, the data can be expressed as the projection of $\Sigma_i\,(a_i\phi_i)$ and therefore the desired reconstructed image is simply $\Sigma_i\,(a_i\phi_i)$. For this special case in which the $\{\mathbf{q}_i\}$ form an orthogonal basis for the projection domain, the orthogonal decomposition of the data directly gives the coefficients $\{a_i\}$ needed for the reconstruction. A flowchart of the algorithm in this case is shown in Fig. 7.

The question then becomes: under what conditions are the vectors $\{\mathbf{q}_i\}$ an orthogonal basis for the projection domain? Since the sets of projections and the source distributions are both expressed as real vectors, both domains are real vector spaces. For $\{\mathbf{q}_i\}$ to form an orthogonal basis for the projection space, the vectors $\{\mathbf{q}_i\}$ must be orthogonal and must span the projection space W. If the vectors span W but are not orthogonal, they can still be used to form an orthogonal basis, as will be discussed below. Therefore it is sufficient that the vectors span W. It can be shown that if P is a linear mapping *onto* W, then $\{\mathbf{q}_i\}$ will always span W (see Appendix for proof). In other words, every set of projections in W must be the projection of some source distribution. This criteria is always satisfied since that is the definition of the projection space. The vectors $\{\mathbf{q}_i\}$ therefore always span W. For the vectors to be orthogonal, the inner product between any two sets of projections $\mathbf{q}_i$ and $\mathbf{q}_j$ must equal zero whenever $i \neq j$. Using the theory of adjoints and the special case of linear operators in real spaces, the condition for orthogonality can

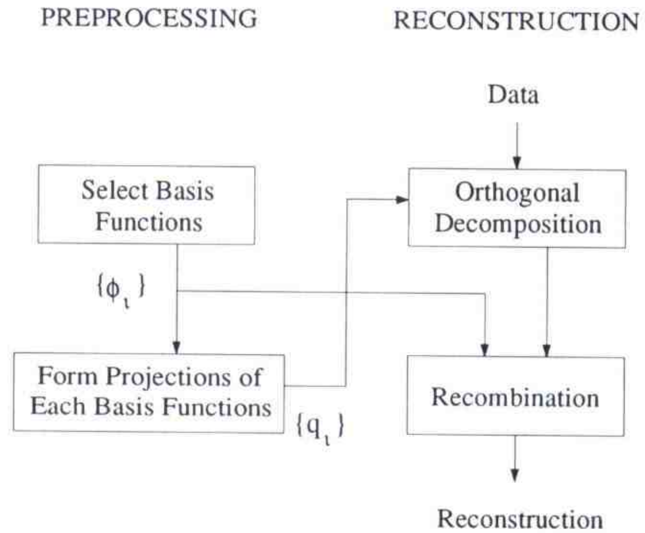

Figure 7. Flowchart of algorithm where projections are orthogonal.

be shown to be equivalent to $P^*P\phi_j = \phi_j$, $\forall j$ where P^* is the adjoint of P and can be interpreted as the reconstruction operation. The physical meaning of this condition will be discussed after its derivation. To begin with, the inner product of the two sets of projections $\mathbf{q}_i$ and $\mathbf{q}_j$ can be expressed in terms of their corresponding source domain basis functions, ϕ_i and ϕ_j,

$$\langle \mathbf{q}_i, \mathbf{q}_j \rangle_W = \langle P\phi_i, P\phi_j \rangle_W \,. \tag{13}$$

Given the linear projection operator $P: V \to W$, the definition of the adjoint of P is that P^* is the mapping from the dual space of W to the dual space of V that satisfies the following equation. The subscripts on the inner products indicate the space in which the inner product is to be taken and hence, the particular inner product definition used:

$$\begin{gathered} P^*: W^* \to V^* \,, \\ \langle Pv, w \rangle_W = \langle v, P^*w \rangle_V \,, \quad v \in V, w \in W \,. \end{gathered} \tag{14}$$

Since V and W are real vector spaces, the duals of these spaces are just the same spaces, and the adjoint of P is then simply a linear operator that maps W into V. In other words, P^* will map sets of projections into source functions. Using the adjoint definition with ϕ_i as v and $P\phi_j$ as w, the inner product of $\mathbf{q}_i$ and $\mathbf{q}_j$ in Eq. (13) can be rewritten as an inner product in the source domain,

$$\langle \mathbf{q}_i, \mathbf{q}_j \rangle_W = \langle P\phi_i, P\phi_j \rangle_W = \langle \phi_i, P^*P\phi_j \rangle_V \,. \tag{15}$$

Clearly, if $P^*P\phi_j = \phi_j$, $\forall j$, then $\{\mathbf{q}_i\}$ are orthogonal since the source domain basis functions $\{\phi_i\}$ are orthogonal.

$$\langle \mathbf{q}_i, \mathbf{q}_j \rangle_W = \langle \phi_i, \phi_j \rangle_V \,, \tag{16}$$

$$\langle \mathbf{q}_i, \mathbf{q}_j \rangle_W = \delta(i, j) \,. \tag{17}$$

Here $P^*P\phi_j$ represents the reconstruction of the source function ϕ_j from its projections $P\phi_j$. In the cases for which complete and accurate projection data are available, the exact reconstruction of the source function from its projections is possible. This is the fundamental idea of tomography. However, often the available data do not contain complete information due to factors such as limited aperture size or low sampling rate. As a result, ambiguity will exist in the reconstructed source function. Hence, the reconstructed source function will not be the original source distribution,

$$P^* P\phi_j \neq \phi_j \ . \tag{18}$$

As a result, $\{\mathbf{q}_i\}$ will not be orthogonal and correlations between the projection vectors will exist. Mathematically, this corresponds to the case in which the projection operator is not a one-to-one operator. The ambiguity exists because two different source functions may have the same projected values when incomplete projections are taken. The degradation caused by the physical limitations of the imaging system is thus directly manifested in the projection domain as correlations between projection vectors obtained from orthogonal source functions. It is important to note that $P^* P\phi_j = \phi_j$, $\forall j$, is a sufficient condition but is not a necessary condition. There are special choices of the basis functions $\{\phi_i\}$ that always result in orthogonal sets of projections $\{\mathbf{q}_i\}$. An example of such a basis will be described in the following section. Although such basis functions exist, they are not always the most appropriate choice for a particular imaging system. In most cases, the basis functions will not fall into this special category and will result in nonorthogonal projections.

In the cases where the vectors $\{\mathbf{q}_i\}$ are not orthogonal, an orthogonal set of basis vectors for the projection space can be obtained from $\{\mathbf{q}_i\}$ by using Gram–Schmidt orthogonalization [7]. Let $\{\mathbf{e}_i\}$ be a set of orthogonal basis vectors for the projection space that are generated from $\{\mathbf{q}_i\}$ using Gram–Schmidt orthogonalization. Let $\mathbf{E}$ be a matrix formed from this set of basis vectors,

$$[\mathbf{E}] = [\mathbf{e}_1|\mathbf{e}_2|\cdots|\mathbf{e}_N] \ . \tag{19}$$

Then a relationship between the basis vectors and the original projection vectors can be found where $\mathbf{C}$ is a matrix of coefficients determined by Gram–Schmidt orthogonalization,

$$\mathbf{E} = \mathbf{QC} \ . \tag{20}$$

By Gram–Schmidt orthogonalization, it is known that the set of vectors $\{\mathbf{e}_i\}$ will span the same space which $\{\mathbf{q}_i\}$ span. In this case, they both span the projection space W.

The reconstruction of a source distribution from a set of projection data is now possible. Since both $\{\mathbf{q}_i\}$ and $\{\mathbf{e}_i\}$ span the same projection space W, a set of data represented by the vector $\mathbf{r}$ can be expressed as a linear combination of either set of vectors,

$$\mathbf{r} = \sum_i d_i \, \mathbf{q}_i \ , \tag{21}$$

$$\mathbf{r} = \sum_i d'_i \, \mathbf{e}_i \ . \tag{22}$$

These equations can be written in vector form as well by defining $\mathbf{d}$ and $\mathbf{d}'$ to be the vectors composed of the coefficients $\{d_i\}$ and $\{d'_i\}$, respectively,

$$\mathbf{r} = \mathbf{Qd} \ , \tag{23}$$

$$\mathbf{r} = \mathbf{Ed}' \ . \tag{24}$$

The coefficients of the decompositions with respect to the orthogonal basis for the projection space $\{d'_i\}$ can be found easily by taking the inner product of $\mathbf{r}$ and $\mathbf{e}_i$, because the vectors $\{\mathbf{e}_i\}$ are orthogonal,

$$d'_i = \langle \mathbf{r}, \mathbf{e}_i \rangle \ . \tag{25}$$

This is known as orthogonal decomposition. From the Gram–Schmidt orthogonalization, the relationship between the two sets of vectors $\{\mathbf{q}_i\}$ and $\{\mathbf{e}_i\}$ can be computed. Substituting the Gram–Schmidt relation in Eq. (20) into the orthogonal decomposition in Eq. (24), the coefficients of the decomposition with respect to the original set of projection vectors $\{d_i\}$ can be determined by comparing the result with Eq. (23),

$$\mathbf{r} = \mathbf{Ed}' = \mathbf{QCd}' \ , \tag{26}$$

$$\mathbf{d} = \mathbf{Cd}' \ . \tag{27}$$

The matrix $\mathbf{C}$ can be interpreted here as the degradation factor arising from the imaging system limitations since, without these degradations, the vectors $\{\mathbf{q}_i\}$ would be orthogonal and hence, $\mathbf{d} = \mathbf{d}'$. Now following the same reasoning as in the previously discussed special case, the reconstructed image can be found. Since each $\mathbf{q}_i$ is the projection of one of the original basis functions of the source domain ϕ_i, the decomposition expression in Eq. (21) can be rewritten in terms of the source domain basis vectors instead of the projection space basis vectors,

$$\mathbf{r} = \sum_i d_i \, P\phi_i \ . \tag{28}$$

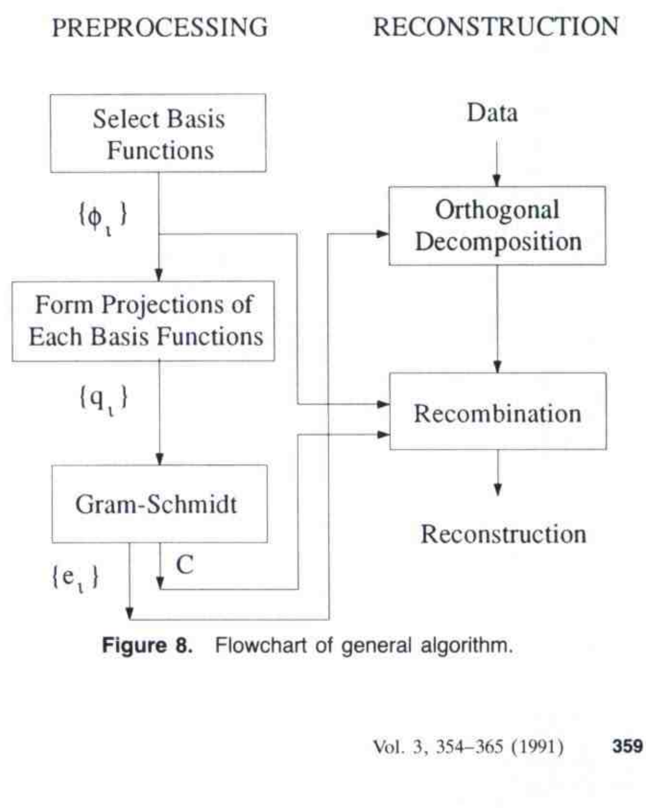

Figure 8. Flowchart of general algorithm.

Since P is linear it can be taken outside the summation, as before:

$$\mathbf{r} = P\Big[\sum_i d_i\, \phi_i\Big] . \qquad (29)$$

The vector $\mathbf{r}$ is then the projection of $\Sigma_i\,(d_i\,\phi_i)$ which is thus the desired reconstruction. The image can also be expressed in vector notation,

$$\text{image} = \mathbf{\Phi d} = \mathbf{\Phi C d}' . \qquad (30)$$

The algorithm then consists of the two steps of orthogonal decomposition and weighted summation of the basis vectors in the source domain. A flowchart of the algorithm is shown in Fig. 8.

Orthogonal Decomposition Algorithm

1. Find $\{d_i'\}$ by evaluating the inner products $d_i' = \langle \mathbf{r}, \mathbf{e}_i \rangle$.
2. Reconstruct the image by matrix multiplication: image = $\mathbf{\Phi C d}'$.

In this algorithm, the data are first decomposed into their orthogonal components in the projection domain, generating a set of coefficients. These coefficients are then transformed into their source domain counterparts. These new coefficients represent the orthogonal decomposition of the desired reconstruction with respect to the set of basis vectors of the source domain. The desired source distribution can then be reconstructed by using these coefficients in a linear combination of the source domain basis vectors. This algorithm succeeds not only in demonstrating the degradation of the system in terms of the correlations of projections, but also in providing a straightforward approach to reconstructing source functions while incorporating *a priori* information. The difficulty in this algorithm is selecting appropriate basis functions. However, since the selection of basis functions depends upon the source and projection domains, *a priori* information such as spatial limits, band limits, specific values, range of possible values, and positivity can be used to define these domains more precisely. The choice of basis functions is then affected by this *a priori* information. Once these basis functions have been chosen for the system, all subsequent reconstructions reuse these same functions. For a particular imaging system, the matrix $\mathbf{C}$ can be determined ahead of time by finding the set of vectors $\{\mathbf{q}_i\}$ and performing Gram–Schmidt orthogonalization. Since this matrix will not change for any particular set of basis functions, it can be stored and reused.

IV. SPECIAL CASE: FOURIER KERNELS

The orthogonal decomposition algorithm is general in nature. The choice of the source domain basis functions shapes the algorithm, and many interesting cases can be found. Depending upon this choice the algorithm may have a variety of unique properties that simplify the reconstruction process for a specific imaging system, or allow special manipulation of the projection data. Of particular interest in the case of choosing the Fourier kernels as the set of source domain basis functions. The uniqueness of choosing the Fourier kernels lies in the ability of these basis functions to separate projections and furthermore, to isolate individual frequency components of each projection.

This unique property of the orthogonal decomposition algorithm with Fourier kernel basis can be described pictorially. To begin with, let the source domain basis functions originate from the Fourier kernel $e^{j\omega s}$. Each basis vector is chosen to be this Fourier kernel at a specific frequency, rotated at a certain angle. The frequency will be denoted ω_m for $m = 1, \ldots, M$, where M is the number of frequencies. The angle of rotation will be denoted θ_r, for $r = 1, \ldots, R$, where R is the number of projection angles. The rotated axes are u' and s' as shown in Fig. 9. The basis functions can then be written

$$\{\phi_i(x, y)\} = \{e^{j\omega_m s'}\} , \qquad (31)$$

where

$$s' = x \sin\theta_r - y \cos\theta_r . \qquad (32)$$

Consider the real portion of a single basis function. It will be a cosine function in one direction and constant in the other, resembling a wave. It will also be rotated at an angle as shown in Fig. 10. The projection of this source function taken at the angle of rotation θ_r will simply be a cosine function, with the same frequency as the Fourier kernel, but with a higher amplitude due to the integration, as shown in Fig. 11. Projections at any other angle will be zero, provided the aperture is a multiple of the period of the cosine function. This is because the source function along any line of integration will also be a cosine function but with a period different than that of the basis function. Integrating along such a line will result in zero since the negative portions of the cosine function are identical to the positive portions. This is shown in Fig. 12. The imaginary portion of the basis function acts in exactly the same manner. It is simply a sine function instead of a cosine function. As a result, each basis function in the source domain has a nonzero projection only for the one taken at the angle of rotation. This projection is thus isolated. In addition, the real portion of the projection is a cosine wave and the imaginary portion is a sine wave, both of the same fixed frequency. Therefore this basis function isolates the contribution of a single frequency component of a single projection. Collectively, the source domain basis functions correspond to a set of basis functions in the projection domain that are quite

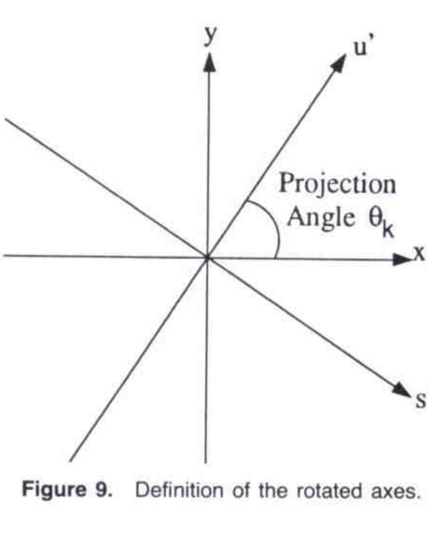

Figure 9. Definition of the rotated axes.

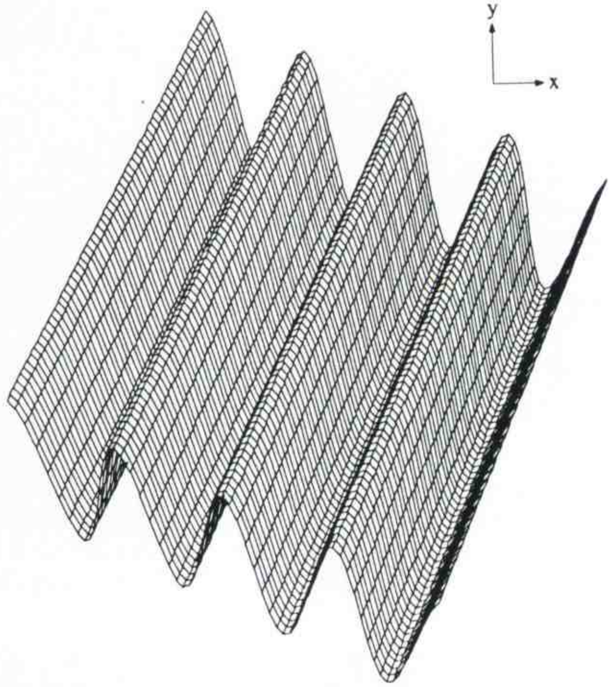

Figure 10. Real portion of a single image space basis function.

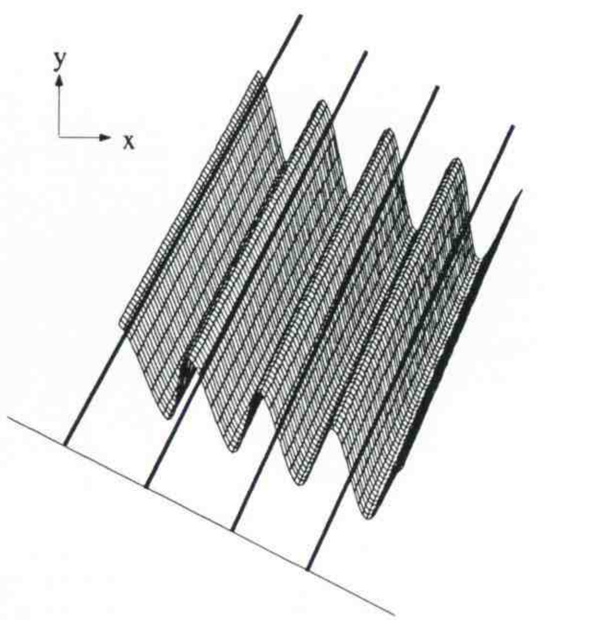

Figure 11. Integration paths for a projection at the same angle.

obviously orthogonal. This will be proved in detail below, however, this is quite apparent since each basis function in the projection domain corresponds to a single projection and a single frequency. Any basis functions that have overlapping nonzero projections have different frequencies. Any basis functions that have the same frequency will have different nonzero projections.

Although the orthogonality of the projections of the source domain basis functions is intuitive, a more formal derivation of these orthogonal basis functions in the projection domain is

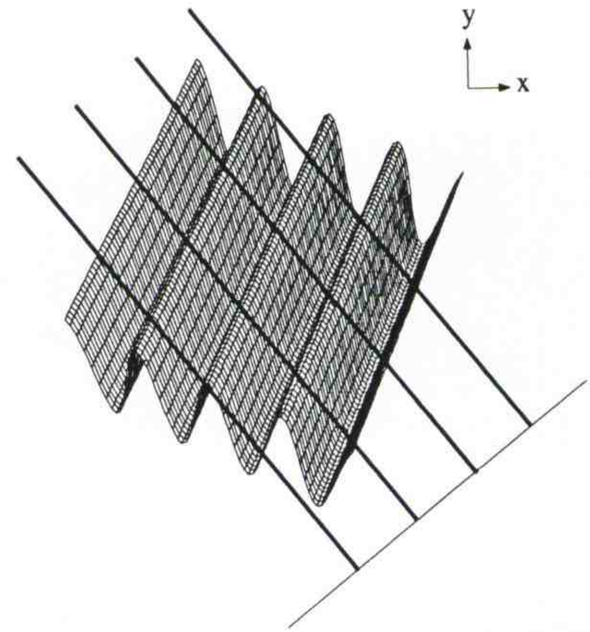

Figure 12. Integration paths for a projection at a different angle.

also presented for clarity. Consider the basis functions in the source domain,

$$\phi_i(x, y) = e^{j\omega_m s'} = \cos(\omega_m s') + j\sin(\omega_m s')\,, \tag{33}$$

where

$$s' = x\sin\theta_r - y\cos\theta_r\,. \tag{34}$$

Although the analysis can be made by treating the entire basis function at once, dealing with the real and imaginary parts separately is more intuitive and follows the pictorial explanation. First, examine the real portion, which is a cosine wave,

$$\mathrm{Re}\{\phi_i(x, y)\} = \cos(\omega_m s')\,. \tag{35}$$

A projection taken at angle θ_i is found by integrating the source function along lines perpendicular to the projection. These lines of integration are shown in Fig. 13 and can be expressed by the following equation, where s is the axis of the projection,

$$y = (\tan\theta_i)x - \frac{s}{\cos\theta_i}\,. \tag{36}$$

Each sample along a projection taken at angle θ_i is a line integral of the basis function over the aperture defined by the limits $-A$ and A,

$$\mathrm{Re}\{s_i(s)\} = \int_{-A}^{A} \cos(\omega_m s')\,dl\,. \tag{37}$$

By substituting in for s', the integrand can be expressed in terms of the original coordinate system of x and y. Furthermore, by substituting the expression for the line of integration, found in Eq. (36), for y, the integration can be performed over one variable, x,

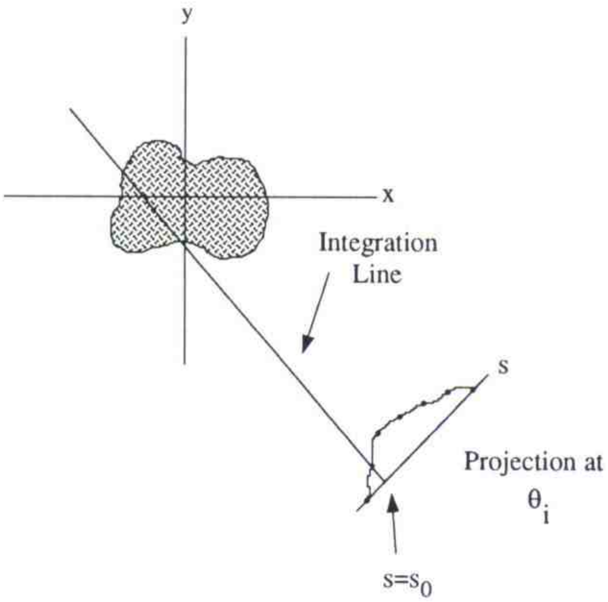

Figure 13. Line of integration for one point on a projection.

$$\mathrm{Re}\{s_i(s)\} = \int_{-A}^{A} \cos\left(\omega_m (\sin\theta_r - \tan\theta_i \cos\theta_r)x + \frac{\omega_m s \cos\theta_r}{\cos\theta_i} \right) dx \,. \tag{38}$$

The equation of a projection can then be expressed as follows:

$$\mathrm{Re}\{s_i(s)\} = \frac{2\cos(\omega_m s \cos\theta_r / \cos\theta_i)}{\omega_m(\sin\theta_r - \tan\theta_i \cos\theta_r)}$$
$$\sin[\omega_m(\sin\theta_r - \tan\theta_i\cos\theta_r)A]\,, \quad \theta_i \neq \theta_r\,, \tag{39}$$
$$\mathrm{Re}\{s_i(s)\} = 2A\cos\omega_m s\,, \quad \theta_i = \theta_r\,.$$

In the case in which the projection angle and the angle of rotation are identical, the projection is simply a shadow of the cosine function and is itself a cosine function. In the case in which $\theta_i \neq \theta_r$, any sample on a projection is simply a constant α, times a sine function which depends upon the aperture A. For any fixed s, both α and β are constants,

$$\mathrm{Re}\{s_i(s)\} = \alpha \sin(\beta A)\,. \tag{40}$$

This function is zero if the aperture is an integer multiple of π/β. Here, Z denotes the set of all integers

$$\mathrm{Re}\{s_i(s)\} = 0 \quad \text{for} \quad A = \frac{n\pi}{\beta}\,, \quad n \in Z\,. \tag{41}$$

Since the aperture is usually much greater than the period of the cosine function, the projections are still negligible even if the aperture is not exactly an integer multiple of π/β. Therefore the basis function triggers only one projection of any significance, and, that projection is a cosine wave of the same frequency as the original basis function.

$$\mathrm{Re}\{s_i(s)\} = \begin{cases} 2A\cos(\omega_m s)\,, & \theta_i = \theta_r\,, \\ 0\,, & \theta_i \neq \theta_r\,. \end{cases} \tag{42}$$

Similarly, in the imaginary case, the projection is a line integral of a sine function,

$$\mathrm{Im}\{s_i(s)\} = \int_{-A}^{A} \sin(\omega_m s')\, dl\,. \tag{43}$$

Evaluating the integral in the same manner produces the projection as follows:

$$\mathrm{Im}\{s_i(s)\} = \frac{2\sin(\omega_m s\cos\theta_r/\cos\theta_i)}{\omega_m(\sin\theta_r - \tan\theta_i\cos\theta_r)}$$
$$\sin[\omega_m(\sin\theta_r - \tan\theta_i\cos\theta_r)A]\,, \quad \theta_i \neq \theta_r\,, \tag{44}$$
$$\mathrm{Im}\{s_i(s)\} = 2A\sin(\omega_m s)\,, \quad \theta_i = \theta_r\,.$$

As expected, in the case in which the projection angle and rotation angle are identical, the projection is a sine function. For any sample in a projection where $\theta_i \neq \theta_r$, the value has the same format as in the real case, the only difference lying in the constant α,

$$\mathrm{Im}\{s_i(s)\} = \alpha' \sin\beta A\,. \tag{45}$$

Therefore, as before, the projections will be zero for the proper-sized aperture and negligible otherwise,

$$\mathrm{Im}\{s_i(s)\} = 0 \quad \text{for} \quad A = \frac{n\pi}{\beta}\,, \quad n \in Z\,. \tag{46}$$

Hence, only one projection is triggered again. Here, in the imaginary case, the triggered projection is a sine function,

$$\mathrm{Im}\{s_i(s)\} = \begin{cases} 2A\sin(\omega_m s)\,, & \theta_i = \theta_r\,, \\ 0\,, & \theta_i \neq \theta_r\,. \end{cases} \tag{47}$$

Combining the real and imaginary parts, it can be seen that the projection is simply a weighted version of the Fourier kernel when the rotation and projection angles are equal and is negligible otherwise,

$$\begin{aligned} s_i(s) &= 2Ae^{j\omega_m s'}\,, & \theta_i = \theta_r\,, \\ s_i(s) &\approx 0\,, & \theta_i \neq \theta_r\,. \end{aligned} \tag{48}$$

Note that when the angle of the projection equals the angle of rotation, $\theta_i = \theta_r$, then the axis of the projection is equivalent to one of the axes of the rotated basis function,

$$s = s'\,. \tag{49}$$

The projections of the source domain basis functions are sampled and used to form the vectors $\{\mathbf{q}_i\}$. Recall that each $\mathbf{q}_i$ is a vector formed from the samples of all of the projections of the basis function ϕ_i. It can be expressed in vector notation in terms of the vectors $\{s_k\}$, each of which contains the samples of the projection taken at angle θ_k,

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \vdots \\ \mathbf{s}_K \end{bmatrix} \quad \text{where} \quad s_k = \begin{bmatrix} s_k(1) \\ s_k(2) \\ \vdots \\ s_k(n) \end{bmatrix}. \tag{50}$$

The $\{\mathbf{q}_i\}$ formed from the Fourier kernels are especially useful because they are orthogonal. A closer examination of the vectors $\{\mathbf{q}_i\}$ makes this apparent. Each source domain basis function has been shown to trigger only one frequency in one projection. This corresponds in vector notation to the $\mathbf{q}_i$ having nonzero values only in the portion $\mathbf{s}_k$, where the angle of rotation in the source domain basis function is also the angle at which the projection $\mathbf{s}_k$ is taken. Consider a basis function rotated at θ_r. The corresponding vector of projection samples would have the form

$$\mathbf{q}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{s}_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{51}$$

The frequency that is triggered determines the frequency of the zeroes within the vector $\mathbf{s}_r$. Hence each $\mathbf{q}_i$ will have a unique set of nonzero values. Obviously, the $\mathbf{q}_i$ which result from source domain basis functions rotated at different angles are orthogonal since they have entirely different nonzero regions. The $\mathbf{q}_i$ which result from source domain basis functions rotated at the same angle are also orthogonal since, within the same nonzero region, each has zeroes at unique intervals. Since the $\{\mathbf{q}_i\}$ are orthogonal, the reconstruction does not even require the Gram–Schmidt orthogonalization step, as previously discussed. Once the data are decomposed into their orthogonal components, the same coefficients can be used with the corresponding source domain basis functions to reconstruct the desired source function. Given the projection data $\mathbf{r}$ the reconstruction is completed in two steps.

Orthogonal Decomposition Algorithm: Fourier Kernel Basis

1. Find $\{d_i\}$ by evaluating the inner products $d_i = \langle \mathbf{r}, \mathbf{q}_i \rangle$.
2. Reconstruct the source function by matrix multiplication: image $= \mathbf{\Phi d}$.

V. THE ODA–DIRECT FOURIER METHOD RELATIONSHIP

The choice of the Fourier kernel as the source domain basis functions is interesting not only because it separates the contribution from each frequency of each projection, but also because it provides a relationship between the orthogonal decomposition algorithm and the direct Fourier method. In the orthogonal decomposition algorithm, the projection data is first decomposed into orthogonal components. The coefficients of the decomposition are then used in the recombination of source domain basis functions. By choosing the Fourier kernels as the basis functions for the source domain, the coefficients of the orthogonal decomposition have additional physical interpretations. This physical interpretation is the basis of the link to the direct Fourier method.

In the direct Fourier method, projections are isolated and individually transformed. These projection spectra are filtered, then collected and, based on the central slice theorem, are used to form the spectrum of the image. A two-dimensional inverse Fourier transform then gives the desired reconstruction. In the orthogonal decomposition algorithm, the Fourier kernel basis functions isolate each frequency in each projection. Collecting all frequency components from each projection would give the spectrum of that projection. Unlike the direct Fourier method, which uses a two-dimensional inverse Fourier transform on the collection of these projection spectra, the orthogonal decomposition algorithm uses the relationship between the two sets of basis functions to generate the desired reconstruction directly.

In order to understand the relation between the coefficients in the orthogonal decomposition algorithm and the projection spectra, consider a single spectrum corresponding to a projection taken at angle θ_k. A single frequency sample represents the contribution of that frequency to this particular projection. Now recall that the coefficient associated with a particular $\mathbf{q}_i$ represents the contribution from one frequency of one projection, determined by the frequency and angle of rotation of the corresponding source domain basis function ϕ_i. Then this coefficient is simply a frequency sample of the spectrum of that projection. Conversely, the value of the spectrum at a particular frequency f_0 is simply the coefficient generated in the orthogonal decomposition algorithm associated with the vector $\mathbf{q}_i$ corresponding to the image basis function with an angle of rotation equal to the angle of the projection θ_k and having the same frequency f_0. Therefore the collection of coefficients corresponding to basis functions rotated at that angle, gives a collection of frequency samples of the spectrum of the projection taken at the same angle. In other words, grouping the coefficients by the angle of their associated basis function forms projection spectra.

The central slice theorem states that the spectrum of the projection is a slice of the object spectrum at the projection angle. Collectively, the coefficients from the orthogonal decomposition represent all of the projection spectra. Therefore the orthogonal decomposition of the projection data generates a set of coefficients that actually represent the spectrum of the object. In the direct Fourier method, each projection spectrum is filtered then placed at the appropriate rotated angle in the object spectrum, and then the two-dimensional object spectrum is inverse Fourier transformed. In the orthogonal decomposition algorithm, the inverse Fourier transform is not needed. Figure 14 compares the flow charts of these two algorithms.

The reason a two-dimensional inverse Fourier transform is not needed is that the source domain basis functions $\{\phi_i\}$ already contain the two-dimensional inverse Fourier transforms of each sample of the object spectrum In other words, the inverse Fourier transform of each point on each rotated slice corresponds to one of the source domain basis functions. Since inverse Fourier transforms are linear, then the summation of the inverse Fourier transforms of each point, that is, the summation of the source domain basis functions, is equivalent to the inverse Fourier transform of the entire spectrum. Therefore only the weighting of each point in the spectrum, or each basis function, needs to be determined. However, due to the orthogonality of the projection domain basis vectors $\{\mathbf{q}_i\}$, the coefficients associated with a particular $\mathbf{q}_i$ are also the coefficients in the reconstruction associated with the corresponding ϕ_i. Taking a linear combination of the source domain basis vectors with these coefficients then gives the

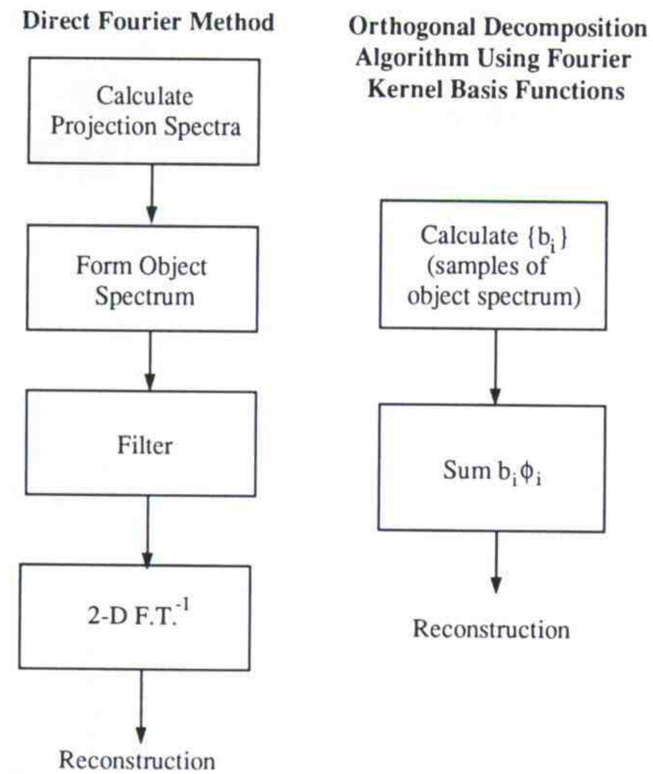

Figure 14. Comparison of direct Fourier method and orthogonal decomposition algorithm with Fourier kernel basis.

reconstruction. Using the Fourier kernels to form the source domain basis functions thus relates the orthogonal decomposition algorithm to the direct Fourier method.

VI. DISCUSSION

In this article, an algorithm for reconstructing images based upon orthogonal basis functions has been developed. This algorithm clearly demonstrates the degradation of the reconstructions due to the physical limitations of the imaging system. *A priori* information can be incorporated into this algorithm for resolution enhancement and is used to define the source and projection domains, thereby affecting the selection of the basis functions that span these domains. Unlike many other reconstruction algorithms, this algorithm does not require even sample spacing, matrix inversion, or backprojection. Once a set of basis functions is determined for an imaging system, the algorithm consists of two simple steps: orthogonal decomposition and recombination of source domain basis functions. In addition to reconstruction of the source distributions, these orthogonal basis functions can be used to examine specific components of the source distribution. The nonorthogonality of the projections of these basis functions can be used to determine the degradation caused by the various physical limitations such as angular quantization error, limited aperture size, and low sampling rate. A special case of this algorithm has also been examined. By selecting the Fourier kernel to form the source domain basis functions, this algorithm can be interpreted in terms of the direct Fourier method. This choice of basis functions allows each frequency component of each projection to be isolated. However, unlike the direct Fourier method, it avoids the use of either Fourier transforms or inverse Fourier transforms in the reconstruction of the desired image.

This algorithm has many features that make it appropriate for ionospheric tomography. Since even sample spacing is not required, the extrapolation or interpolation of projections is unnecessary. In addition, the correlation of the projections of the basis functions provides a method of analyzing the system's degradation factors. In particular, both the low sampling rate and the restricted receiver positions of the ionospheric imaging system affect the dimensions of the algorithm. The limited information content causes a problem of nonunique solutions, or ambiguity in the resulting reconstruction. Uneven sample spacing is no longer an issue and angular quantization error can be removed. Since the projections are not backprojected, it is not necessary to group the data into projections. Recall the definitions of the vectors $\mathbf{s}_k$, which denotes a projection, and $\mathbf{q}_i$, which denotes a set of projections for a single basis function, given in Eqs. (5) and (6), respectively. The $\mathbf{q}_i$ vector is formed by stacking the projections $\mathbf{s}_k$. The data vector $\mathbf{r}$ is never separated into projections, it is simply decomposed based upon either the $\{\mathbf{q}_i\}$ vectors or the $\{\mathbf{e}_i\}$ vectors, depending upon orthogonality. As long as the elements of the $\mathbf{q}_i$ vector are obtained in a manner consistent with the data in the vector $\mathbf{r}$, no reindexing is actually required and therefore no angular quantization error will occur. Thus both algorithmic degradation factors can be removed using the orthogonal decomposition algorithm.

ACKNOWLEDGMENTS

This research was supported by the AT&T Bell Laboratories Graduate Research Program for Women and National Science Foundation Grant No. BCS-9196020.

APPENDIX

It can be shown that the set of vectors of projections $\{\mathbf{q}_i\}$ formed by projecting the orthogonal basis of the image space $\{\phi_i\}$ span the projection space if and only if the linear mapping P projects *onto* the image space. Begin with the definition of a set of vectors spanning a space. $\{\mathbf{q}_i\}$ spans W if and only if every vector in W can be expressed as a linear combination of $\{\mathbf{q}_i\}$

$$\{\mathbf{q}_i\} \text{ spans } W \Leftrightarrow \forall \mathbf{x} \in W\,, \quad \exists a_i\,, \qquad i = 1, \ldots, N \quad \text{such that } \sum_i a_i \mathbf{q}_i = \mathbf{x}\,.$$

□ First, prove that if P projects onto W, then $\{\mathbf{q}_i\}$ spans W.

- Consider any vector $\mathbf{x}$ in the projection space W. If P is onto, then every vector in the projection space is the set of projections of some image in the image space V,

$$\forall z \in W\,, \quad \exists \mathbf{y} \in V \text{ such that } P\{\mathbf{y}\} = \mathbf{z}\,.$$

- Then let $\mathbf{v}$ be the image in the image space V whose projections form $\mathbf{x}$,

$$P\{\mathbf{v}\} = \mathbf{x}\,.$$

- Since $\{\phi_i\}$ is a basis for the image space V, the image $\mathbf{v}$ can be expressed as a linear combination of $\{\phi_i\}$,

$$\exists b_i \quad i = 1, \ldots, N \text{ such that } \mathbf{v} = \sum_i b_i \phi_i .$$

- Substituting for $\mathbf{v}$,

$$\mathbf{x} = P\{\mathbf{v}\} = P\left\{\sum_i b_i \phi_i\right\} .$$

- Since P is a linear operator,

$$\mathbf{x} = \sum_i b_i P\{\phi_i\} .$$

- Since $\mathbf{q}_i = P\{\phi_i\}$,

$$\mathbf{x} = \sum_i b_i \mathbf{q}_i .$$

- Because $\mathbf{x}$ was any arbitrary vector in W and it can be expressed as a linear combination of $\{\mathbf{q}_i\}$, then by the definition of vectors spanning sets, $\{\mathbf{q}_i\}$ spans W.
- Therefore if P projects *onto* W then $\{\mathbf{q}_i\}$ spans W.

□ Now prove the converse.

- Assume $\{\mathbf{q}_i\}$ spans W. Then by the definition of spanning sets, every vector in W can be expressed as a linear combination of $\{\mathbf{q}_i\}$,

$$\forall \mathbf{x} \in \mathrm{W} \ \exists a_i \quad i = 1, \ldots, N \text{ such that } \mathbf{x} = \sum_i a_i \mathbf{q}_i .$$

- Since $\mathbf{q}_i = P\{\phi_i\}$,

$$\mathbf{x} = \sum_i a_i P\{\phi_i\} .$$

- Since P is a linear operator,

$$\mathbf{x} = P\left\{\sum_i a_i \phi_i\right\} .$$

- Let $\mathbf{v} = \Sigma_i a_i \phi_i$, then

$$\mathbf{x} = P\{\mathbf{v}\} .$$

- Since $\{\phi_i\}$ is a basis for the image space V, and $\mathbf{v}$ is a linear combination of $\{\phi_i\}$, $\mathbf{v}$ must be in the image space.
- Therefore every vector $\mathbf{x}$ in the projection space can be expressed as the projection of an image in V,

$$\forall \mathbf{x} \in W , \quad \exists \mathbf{v} \in V \quad \text{such that } \mathbf{x} = P\{\mathbf{v}\} .$$

- P is then said to project *onto* W.
- Therefore if $\{\mathbf{q}_i\}$ spans W then P projects *onto* W.

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