Zhou Yi, Signals and Systems
Have Fun!

1 LTI

CT DE

$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$
$\sum_{k=0}^{N} a_k x^k = a_N \prod_{i=1}^{p} (x-x_i)^{n_i}$
$y_h(t) = \sum_{i=1}^{p} (\sum_{j=0}^{n_i-1} A_{i,j} t^j) e^{x_i t}$
If $x(t) = \sum_{k=0}^{m} C_k t^k e^{x_i t}$
$y_p(t) = (\sum_{k=0}^{m} B_k t^k) t^{n_i} e^{x_i t}, n_i \ge 0$
Determine constants.

DT DE

$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
$\sum_{k=0}^{N} a_{N-k} x^k = a_0 \prod_{i=1}^{p} (x-x_i)^{n_i}$
$y_h[n] = \sum_{i=1}^{p} (\sum_{j=0}^{n_i-1} A_{i,j} n^j) x_i^n$
if $x[n] = \sum_{i=0}^{m} C_i n^i s^n$
$y_p[n] = (\sum_{k=0}^{m} B_k n^k) n^{n_i} x_i^n, n_i \ge 0$
Special: $x_i = 1$

Unit Impulse Resp

$h_p(t) = 0$
Matching $\delta^{(k)}(t), \delta^{(k-1)}(t), \delta^{(k-2)}(t)\ldots$ (Only singularity part)
$\int_{0-}^{0+}$ to determine coefficient
$y(t) = \int_\tau x(\tau) h(t-\tau)$

2 Fourier Analysis

CT FS

$x(t) = x(t+T), \omega_0 = \frac{2\pi}{T}$
$c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} \mathrm{d}t$
$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} = \begin{cases} x(t) \\ \frac{x(t+)+x(t-)}{2} \end{cases}$

Real Signals

When $c_k* = c_{-k}$

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$
$$= A_0 + \sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \phi_k)$$

$|c_k|^2 = |c_{-k}|^2 = \frac{a_k^2 + b_k^2}{4}$

Examples

$x(t) = a[u(t+\frac{\tau}{2}) - u(t-\frac{\tau}{2})]$
$c_0 = a\frac{\tau}{T}$
$c_k = \frac{a\tau}{T} Sa(\frac{k\omega_0 \tau}{2})$
$Sa(t) = \begin{cases} \frac{\sin(t)}{t}, t \neq 0 \\ 1, t = 0 \end{cases}$

Parseval Theorem

$\frac{1}{T}\int_T |x(t)|^2 \mathrm{d}t = \sum_k |c_k|^2 = A_0^2 + \frac{1}{2}\sum_{k\ge 1} A_k^2$

Properties

$\mathcal{FS}\{x(t)\} = c_k, \mathcal{FS}\{x(t-t_0)\} = c_k e^{-jk\omega_0 t_0}$
$\mathcal{FS}\{x(t)\} = c_k, \mathcal{FS}\{x^*(t)\} = c_{-k}*$
$x(t) = \sum \frac{c_k + c_{-k}*}{2} e^{jk\omega_0 t} + j\sum \frac{c_k - c_{-k}*}{2j} e^{jk\omega_0 t}$
$\mathcal{FS}\{x(t)y(t)\} = \sum c_l d_{k-l} = c_k * d_k$
$\mathcal{FS}\{x'(t)\} = jk\omega_0 c_k$
$\mathcal{FS}\{y(t)\} = \int_0^t x(\tau)\mathrm{d}\tau, (0<t<T), y(t) = y(t+T)$
$y(t) = \sum_{k\neq 0} \frac{c_k}{jk\omega_0}(e^{jk\omega_0 t} - 1) + c_0 t$

2.1 DT FS

$x[n] = x[n+N]$
$x[n] = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{N}}, 0 \le n < N$
$c_k = \frac{1}{N}\sum_{l=0}^{N-1} x[l] e^{-\frac{j2\pi kl}{N}} = c_{k+N}$

Examples

$x[n] = \begin{cases} 1, |n| <= N_1 \\ 0, N_1 < |n| <= \frac{N}{2} \end{cases},$
$\mathcal{FS}\{x[n]\} = \begin{cases} \frac{\sin[(2\pi k/N)(N_1+\frac{1}{2})]}{N\sin(2\pi k/2N)}, k \neq tN \\ \frac{2N_1+1}{N}, k = tN \end{cases}$
$\mathcal{FS}\{e^{j\omega_0 n}\} = \begin{cases} 1, k = m+tN \\ 0, otherwise \end{cases}, \omega_0 = \frac{2\pi m}{N}$

2.2 CT FT

$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\mathrm{d}t$
$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\mathrm{d}\omega$

Examples

$\mathcal{F}\{x(t) = a[u(t+\frac{\tau}{2}) - u(t-\frac{\tau}{2})]\} = a\tau Sa(\frac{\omega\tau}{2})$
$\mathcal{F}\{x(t) = e^{-\alpha t}u(t), (\alpha > 0)\} = \frac{1}{\alpha + j\omega}$
$\mathcal{F}\{\delta(t)\} = 1, \mathcal{F}\{1\} = 2\pi\delta(\omega)$
$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$
$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\omega}$

Periodic Signals

$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$
$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t} e^{-j\omega t}\mathrm{d}t$
$= 2\pi \sum_{k=-\infty}^{+\infty} c_k \delta(\omega - k\omega_0)$

Parseval Theorem

$\int_{-\infty}^{+\infty} |x(t)|^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 \mathrm{d}\omega$

Properties

$\mathcal{F}\{x(t)\} = X(j\omega), \mathcal{F}\{X(jt)\} = 2\pi x(j^2\omega)$
$\mathcal{F}\{x(t-\tau)\} = X(j\omega) e^{-j\omega\tau}$
$\mathcal{F}\{x(\alpha t)\} = \frac{1}{|\alpha|} X(\frac{j\omega}{\alpha})$
$\mathcal{F}\{x^*(t)\} = X^*(-j\omega)$
$\mathcal{F}\{x'(t)\} = j\omega X(j\omega), \mathcal{F}\{tx(t)\} = j\frac{\mathrm{d}X(j\omega)}{\mathrm{d}\omega}$
$\mathcal{F}\{\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau\} = \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$
$\mathcal{F}\{-\frac{x(t)}{jt} + \pi x(0)\delta(t)\} = \int_{-\infty}^{\omega} X(j\eta)\mathrm{d}\eta$
$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\omega) X_2(j\omega)$,
$\mathcal{F}\{x_1(t) x_2(t)\} = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$

2.3 DT FT

$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} = X(e^{j(\omega + 2\pi)})$ (*)
$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}\mathrm{d}\omega$

Examples

$\mathcal{F}\{[|n| \le N_1]\} = \frac{\sin\omega(N_1 + \frac{1}{2})}{\sin(\frac{\omega}{2})}$
$\mathcal{F}\{\frac{\sin(Wn)}{\pi n}\} = [u(\omega + W) - u(\omega - W)]$(Periodic, $T = 2\pi$)
$x[n] = \begin{cases} 1, n \ge 0 \\ -1, n < 0 \end{cases}$
$\mathcal{F}\{x[n]\} = \frac{2}{1 - e^{-j\omega}}$
$\mathcal{F}\{u[n]\} = \frac{1}{1 - e^{-j\omega}} + \pi\sum_{k=-\infty}^{+\infty} \delta(\omega - 2k\pi)$
$\mathcal{F}\{e^{j\omega_0 n}\} = \sum_{n=-\infty}^{+\infty} e^{-j(\omega - \omega_0)n} = 2\pi\sum\delta(\omega - \omega_0 - 2k\pi)$
$\mathcal{F}\{\binom{n+r-1}{n} a^n u[n]\} = (1 - ae^{-j\omega})^{-r}, |a| < 1$

Periodic Signals

$x[n] = x[n+N]$, Fourier Series $x[n] = \sum_{k=0}^{N-1} c_k e^{\frac{j2\pi kn}{N}}$
$c_k = \frac{1}{N}\sum x[l] e^{\frac{-j2\pi kl}{N}} = c_{k+N}$
$\mathcal{F}\{x(t)\} = \sum_{k=0}^{N-1} c_k \mathcal{F}\{e^{\frac{j2\pi kn}{N}}\} = \sum_{k=-\infty}^{+\infty} 2\pi c_k \delta(\omega - \frac{2k\pi}{N})$

CT FS and DT FT

$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n} = X(e^{j(\omega + 2\pi)})$
$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}\mathrm{d}\omega$
$c_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\mathrm{d}t$
$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{jk\omega_0 t}$
$x[n] = c_{-n} \to x(t) = X(e^{j\omega_0 t})$

Properties

$x_k[n] = \begin{cases} x[n/k], n/k \in Z \\ 0, otherwise \end{cases}$
$\mathcal{F}\{x_k[n]\} = X(e^{jk\omega})$
$\mathcal{F}\{x[n] - x[n-1]\} = (1 - e^{-j\omega}) X(e^{jk\omega})$
$\mathcal{F}\{nx[n]\} = j\frac{\mathrm{d}X(e^{j\omega})}{\mathrm{d}\omega}$
$\mathcal{F}\{\sum_{k\le n} x[k]\} = \frac{X(e^{j\omega})}{(1 - e^{-j\omega})} + \pi X(e^{j0})\sum\delta(\omega - 2k\pi)$
$\mathcal{F}\{x[n]y[n]\} = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta}) Y(e^{j(\omega - \theta)})\mathrm{d}\theta$

2.4 DFT

$X[k] = X(e^{\frac{j2\pi k}{N}})$ (sampling DT FT)

Distortionless

$Y(j\omega) = X(j\omega)H(j\omega) = AX(j\omega)e^{-j\omega\tau}$
$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = AX(e^{j\omega})e^{-j\omega n_0}$

Group Delay

$\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\phi(H(j\omega))$

2.5 Sampling

Impulse

$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$
$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s), \omega_s = \frac{2\pi}{T}$
$x_p(t) = x(t)p(t), X_p(j\omega) = \frac{1}{T}\sum_k X(j(\omega - k\omega_s))$ (copy and paste)
If $X(j\omega) = 0(|\omega| > \omega_M)$, No overlap in freq domain if $\omega_s > 2\omega_M$
lowpass filter $\omega_c \in (\omega_M, \omega_s - \omega_M)$ to restore signal

Zero-Order Holding

$x_0(t) = \sum_{k=-\infty}^{+\infty} x(kT) p_0(t - kT) = x(t)p(t) * p_0(t)$
$p_0(t) = u(t) - u(t - T)$
$P(j\omega) = Te^{-\frac{j\omega T}{2}} Sa(\frac{\omega T}{2})$
$X_0(j\omega) = \sum_{k=-\infty}^{+\infty} e^{-\frac{j\omega T}{2}} Sa(\frac{\omega T}{2}) X(j(\omega - k\omega_s))$
$H_r(j\omega) = \frac{e^{\frac{j\omega T}{2}}}{Sa(\frac{\omega T}{2})}, |\omega| < |\omega_c|$

Linear Interpolation

$x_r(t) = x_p(t) * h(t) = \sum_{n=-\infty}^{+\infty} x_p(nT) h(t - nT)$
$= \sum_{n=-\infty}^{+\infty} x(nT)$