2.1.

(a)

$$AX + B = A^2X$$
$$A^2X - AX = B$$
$$(A^2 - A)X = B$$
$$X = (A^2 - A)^{-1}B$$
$$X = (A - I)^{-1}A^{-1}B$$

(b)

$$X - A = XB$$
$$X - XB = A$$
$$X(I - B) = A$$
$$X = A(I - B)^{-1}$$

(c)

$$2X - AX + 2A = 0$$
$$2A = (A - 2I)X$$
$$X = (A - 2I)^{-1}2A$$
$$X = (A/2 - I)^{-1}A$$

2.3

$$Ax + (y^T B)^T = \alpha 1$$
$$Ay + c = 0$$
$$\Downarrow$$
$$\begin{aligned} Ax + B^T y - \alpha 1 &= 0 \\ Ay &= -c \end{aligned}$$
$$\Downarrow$$
$$\underbrace{\begin{pmatrix} A & B^T & -1 \\ 0 & A & 0 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} x \\ y \\ \alpha \end{pmatrix}}_{u} = \underbrace{\begin{pmatrix} 0 \\ -c \end{pmatrix}}_{q}$$

3.1. (a)

$\{x \in \mathbb{R}^n \mid a^T x = 0\}$

lineární podprostor $\rightarrow$ prochází nulou

dimenze: $n-1$ pro $a \neq 0$

n pro $a = 0$

(b)

$\{x \in \mathbb{R}^n \mid a^T x = b\}$

afinní podprostor

dimenze $n-1$ pro $a \neq 0$

n pro $a = 0$ $b = 0$

(c)

$\{x \in \mathbb{R}^n \mid x^T x = 1\}$

není lineární ani afinní podprostor

3.2.

$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_3 = 0\}$

ANO, JE TO LINEÁRNÍ PODPROSTOR

$a^T x = 0$ $\qquad a^T = (1, 0, 1, 0)$

$a^T x = x_1 + x_3$

báze:

$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \end{array}\right) \rightarrow$ rank $= 1$

dim null $= 3$

$x_1 = \begin{pmatrix} a \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad a = 0 \quad x_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$x_2 = \begin{pmatrix} a \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad a = -1 \quad x_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$x_3 = \begin{pmatrix} a \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad a = 0 \quad x_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$X = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

3.7. $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f(x,y) = (x+y, 2x-1, x-y)$

afinní zobrazení

$$f: \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} x+y \\ 2x-1 \\ x-y \end{pmatrix}$$

3.8.

$x + 2y + z = 1$

$-x + y + 2z = 2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

hledání báze:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$x = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$ $\quad a \cdot 0 + 1 \cdot b + 1 \cdot 1 = 0 \quad b = -1$

$\quad a \cdot 1 - 2 \cdot 1 + 1 \cdot 1 = 0 \quad a = 1$

$x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\quad X = \text{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\}$

posun:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$x_0 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ $\quad 0 \cdot a + 1 \cdot b = 1 \quad b = 1$

$\quad 1 \cdot a + 2 \cdot 1 = 1 \quad a = -1$

$x_0 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

3.10. (a)

$f(x_1,x_2,x_3)=(x_1-x_2,\ x_2-x_3+2x_1)$

$$A \nearrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right)$$

báze nulového prostoru:

$$x=\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \qquad \begin{array}{ll} -3b+1=0 & b=1/3 \\ a-1/3=0 & a=1/3 \end{array} \qquad x=\begin{pmatrix} 1/3 \\ 1/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\underline{\underline{X=\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right\}}}$$

báze obrazů:

$f: \mathbb{R}^3 \to \mathbb{R}^2 \qquad A \in \mathbb{R}^{2\times 3}$

$$\left.\begin{array}{l} \dim \operatorname{null} A = 1 \\ \operatorname{rank} A = 2 \end{array}\right\} 3$$

↳ hodnost matice je dva, takže obrazy jsou v celém prostor $\mathbb{R}^2$, proto báze může být klasicky

$$\underline{\underline{X=\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}}}$$