KANA Důkazy – přednášky

co mám zapsaný xd

Harmonická fce

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$f(z) = u(x,y) + i\,v(x,y)$ $\qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}$$

zaměnné

$$\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v}{\partial y \partial x} = 0 \qquad \partial y \;\; \partial x$$

C-R podmínky

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$f(z_0) = u(x_0,y_0) + i\,v(x_0,y_0)$ $\qquad f = u + iv$

$f(z_0+h) = u(x_0+h_1, y_0+h_2) + i\,v(x_0+h_1, y_0+h_2)$ $\qquad h = h_1 + ih_2$

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{h\to 0} \frac{u(x_0+h_1; y_0+h_2) - u(x_0,y_0) + i\left[v(x_0+h_1, y_0+h_2) + v(x_0,y_0)\right]}{h_1 + i\,h_2}$$

1) $$\lim_{\substack{h_1\to 0\\ h_2=0}} \frac{u(x_0+h_1; y_0) - u(x_0,y_0)}{h_1} + i\,\frac{v(x_0+h_1, y_0) + v(x_0,y_0)}{h_1} = \frac{\partial u}{\partial x}(x_0,y_0) + i\,\frac{\partial v}{\partial x}(x_0,y_0)$$

2) $$\lim_{\substack{h_1=0\\ h_2\to 0}} \frac{u(x_0; y_0+h_2) - u(x_0,y_0)}{i\,h_2} + i\,\frac{v(x_0,y_0+h_2) + v(x_0,y_0)}{i\,h_2} = -i\,\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0)$$

Derivace exp

$$(e^z)' = e^z$$

$e^z = e^x(\cos y + i \sin y)$

$u(x,y) = e^x \cos y$

$v(x,y) = e^x \sin y$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -e^x \sin y$$

$$(e^z)' = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = e^x \cos y + i\,e^x \sin y = e^z$$

Sin z a Cos z

$\sin z = 0 \iff \exists k \in \mathbb{Z} : z = k\pi$

$\cos z = 0 \iff \exists k \in \mathbb{Z} : z = \frac{\pi}{2} + k\pi$

$\forall z \in \mathbb{C} : (\sin z)' = \cos z \wedge (\cos z)' = \sin z$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = 0$

$e^{iz} - e^{-iz} = 0 \quad / \cdot e^{iz}$

$e^{2iz} - 1 = 0$

$e^{2iz} = 1 = e^0$

lišší se pouze o $u k\pi i$

$2iz = 0 + 2k\pi i$

$z = k\pi$

$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 0$

$e^{iz} + e^{-iz} = 0 \quad / \cdot e^{iz}$

$e^{2iz} + 1 = 0$

$e^{2iz} = -1 = e^{i\pi}$

$2iz = i\pi + 2k\pi i$

$z = \frac{\pi}{2} + k\pi$

$$(\sin z)' = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)' = \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos z$$

$$(\cos z)' = \left(\frac{e^{iz} + e^{-iz}}{2}\right)' = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) = \frac{i}{2}\left(e^{iz} - e^{-iz}\right) = -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

Logaritmus

$\ln z = \ln|z| + i \arg z \qquad z \in \mathbb{C} \setminus \{0\}$

$z \in \mathbb{C} \setminus \{0\}$

$e^w = z = |z|(\cos\varphi + i \sin\varphi)$

$e^{\mathrm{Re}(w)}\left(\cos(\mathrm{Im}\, w) + i \sin(\mathrm{Im}\, w)\right) = |z|(\cos\varphi + i \sin\varphi)$

$e^{\mathrm{Re}(w)} = |z| \quad \& \quad \mathrm{Im}\, w = \varphi + 2k\pi$

$|z| = e^{\ln|z|}$

$\mathrm{Re}\, w = \ln|z| \quad \& \quad \mathrm{Im}\, w = \varphi + 2k\pi$

$w = \ln|z| + i(\varphi + 2k\pi)$

Integrování člen po členu

$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \;,\; \hat{R} > 0 \;,\; U(z_0, R)$

$F(z) = \sum_{n=0}^{\infty} a_n \frac{(z - z_0)^{n+1}}{n+1}$

$F'(z) = f(z) \;,\; R = \hat{R}$

Geometrická řada

$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots + z^n$

$\sum_{n=0}^{\infty} z^n = s_n$

podmínka $\lim_{n \to \infty} |z^n| = 0$

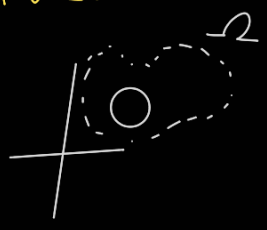
$(1 - z)s_n = (1 + z + z^2 + \ldots + z^n) - (z + z^2 + z^3 + \ldots + z^n + z^{n+1})$

$(1 - z)s_n = 1 - z^{n+1} \qquad |z| < 1$

$s_n = \frac{1 - z^{n+1}}{1 - z} \qquad \lim_{n \to \infty} \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z}$

Cauchyova věta

Je-li $\Omega \subset \mathbb{C}$ jednoduše souvislá oblast a f je holomorfní na Ω, pak $\oint_C f(z)\,dz = 0$ pro každou uzavřenou křivku C ležící v Ω



C je kladně orientovaná kružnice uvnitř Ω

$z_0 \in \Omega$; $R > 0$

$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ na $U(z_0, \tilde{R})$, kde $\tilde{R} > R$

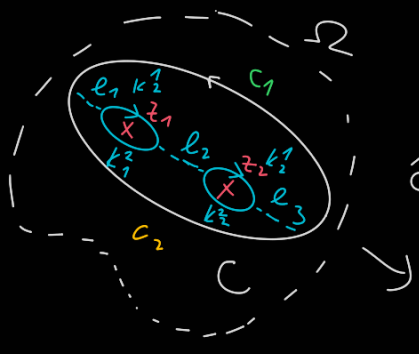
$$\int_C f(z)\,dz = \int_C \left(\sum_{n=0}^{\infty} a_n (z-z_0)^n \right) dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,dz = 0$$

Reziduová věta

$\Omega \subset \mathbb{C}$ jednoduše souvislá oblast, C je kladně orientovaná Jordanova křivka ležící v Ω
f je holomorfní na $\Omega \setminus S$, $S = \{ z \in \mathrm{Int}\, C : z \text{ je isol. sing. } f \}$ S je konečná množina

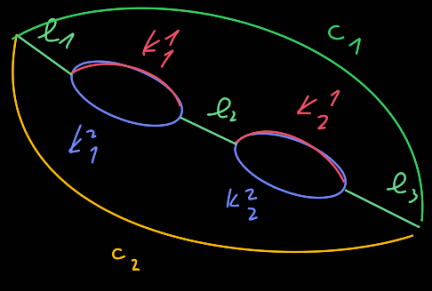
potom $$\int_C f(z)\,dz = 2\pi i \sum_{w \in S} \mathrm{res}_w f(z)$$

mějme isol. sing. $z_1, z_2, \dots, z_n$
individuálně obalíme kružnicemi k_1, k_2
a spojíme úsečkami ℓ_1, ℓ_2, ℓ_3
Rozdělíme na dílky



$\tilde{c}_1 = c_1 + \ell_1 + k_1^1 + \ell_2 + k_2^1 + \ell_3$

$\tilde{c}_2 = c_2 + \ell_1 + k_1^2 + \ell_2 + k_2^2 + \ell_3$



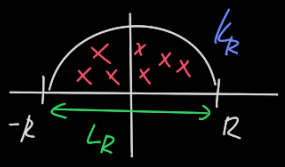
$$\int_{\tilde{c}_1} f(z) = 0 \qquad \int_{\tilde{c}_2} f(z) = 0$$

$$0 = \left(\int_{c_1} + \int_{c_2} + \int_{k_1^1} + \int_{k_1^2} + \int_{k_2^1} + \int_{k_2^2} \right) f(z) = \int_C f(z)\,dz - 2\pi i \left(\mathrm{res}_{z_1} + \mathrm{res}_{z_2} \right)$$

Just use the pic too diff to write xd

Liouvillova věta — Nekonstantní celistvá fce je neomezená

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx$$



$C_R = k_R + L_R$

$$\int_{C_R} = \int_{L_R} + \int_{k_R} = \int_{-R}^{R} + \int_{k_R}$$

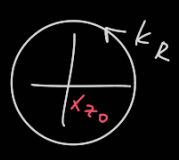
$$\int_{C_R} \frac{P(z)}{Q(z)}\,dz = 2\pi i \sum_{w\in S_+} \operatorname{res}_w \frac{P(z)}{Q(z)} \qquad \int_{L_R} \frac{P(z)}{Q(z)}\,dz = \int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx$$

$$\left|\int_{k_R} \frac{P(z)}{Q(z)}\,dz\right| \le \int_{k_R} \left|\frac{P(z)}{Q(z)}\right| dz \le \pi R \max_{z\in k_R} \left|\frac{P(z)}{Q(z)}\right| = \pi R \max_{z\in k_R} \frac{|z^2 P(z)|}{|z|^2\,|Q(z)|} = \frac{\pi R}{R^2} \max_{z\in k_R} \left|\frac{z^2 P(z)}{Q(z)}\right|$$

$$\lim_{|z|\to\infty} \left|\frac{z^2 P(z)}{Q(z)}\right| \in [0,\infty) \le M$$

Cauchyho integrální vzorec

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$



$$|f'(z_0)| = \left|\frac{1}{2\pi i} \int_{k_R} \frac{f(z)}{(z-z_0)^2}\,dz\right| \le \frac{1}{2\pi} 2\pi R \max_{z\in k_R} \left|\frac{f(z)}{(z-z_0)^2}\right| = R \max_{z\in k_R} \frac{|f(z)|}{|z-z_0|^2} =$$

$$= \frac{R}{R^2} \max_{z\in k_R} |f(z)| = \xrightarrow{R\to\infty} 0 \qquad f'(z_0) = 0 \Rightarrow f = \text{konst.}$$

$\mathcal{F}$ - Derivace obrazu

$$\mathcal{F}[t\,f(t)](w) = i \frac{d}{dw} \mathcal{F}[f(t)](w)$$

$$\frac{d}{dw} \int_{-\infty}^{\infty} f(t)\,e^{-iwt}\,dt = \int_{-\infty}^{\infty} f(t)\,(-it)\,e^{-iwt}\,dt = -i \int_{-\infty}^{\infty} t f(t)\,e^{-iwt}\,dt = -i\,\mathcal{F}[t f(t)](w)$$

$\mathcal{F}$ - obraz derivace

$$\mathcal{F}[f^{(n)}(t)](w) = (iw)^n\,\mathcal{F}[f(t)](w)$$

$$\mathcal{F}[f'(t)](w) = \int_{-\infty}^{\infty} f'(t)\,e^{-iwt}\,dt \overset{PP}{=} \left[f(t)\,e^{-iwt}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t)\,(-iw)\,e^{-iwt}\,dt =$$

$$= \underbrace{\left[f(t)\,e^{-iwt}\right]_{-\infty}^{\infty}}_{=0} + iw \int_{-\infty}^{\infty} f(t)\,e^{-iwt}\,dt = iw\,\hat{f}(w)$$

$\mathcal{L}$ - transformace

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a}$$

$$= \int_0^\infty e^{at} \cdot e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \lim_{t\to\infty} \frac{e^{-(s-a)t}}{-(s-a)} + \frac{1}{s-a} =$$

$$\left|\frac{e^{-(s-a)t}}{-(s-a)}\right| = \frac{1}{|s-a|}\, e^{-(\mathrm{Re}\,s - \mathrm{Re}\,a)t} \xrightarrow[\text{pokud } \mathrm{Re}\,s > \mathrm{Re}\,a,\ t\to\infty]{} 0$$

$$= \frac{1}{s-a}$$

$$\mathcal{L}[\sin(\omega t)](s) = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left[\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right](s) = \frac{1}{2i}\left[\mathcal{L}[e^{i\omega t}](s) - \mathcal{L}[e^{-i\omega t}](s)\right] = \frac{1}{2i}\left(\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right) = \frac{1}{2i}\,\frac{s+i\omega-s+i\omega}{(s-i\omega)(s+i\omega)} = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[f(t-a)\,\mathbb{1}(t-a)](s) = e^{-as}\,\mathcal{L}[f(t)](s)$$

$$= \int_0^\infty f(t-a)\,\mathbb{1}(t-a)\,e^{-st}\,dt = \int_a^\infty f(t-a)\,e^{-st}\,dt = \left|\begin{matrix}\tau = t-a \\ d\tau = dt\end{matrix}\right| = \int_0^\infty f(\tau)\,e^{-s(\tau+a)}\,d\tau = e^{-as}\int_0^\infty f(\tau)\,e^{-s\tau}\,d\tau = e^{-as}\,\mathcal{L}[f(t)](s)$$

$$\mathcal{L}[f(t)\,\mathbb{1}(t-a)](s) = e^{-as}\,\mathcal{L}[f(t+a)](s)$$

$$\mathcal{L}[f(t)\,\mathbb{1}(t-a)](s) = \mathcal{L}[f(t-a+a)\,\mathbb{1}(t-a)](s) = \left|\,g(t) = f(t+a)\,\right| = \mathcal{L}[g(t-a)\,\mathbb{1}(t-a)](s) = e^{-as}\,\mathcal{L}[f(t+a)](s)$$

$\mathcal{L}$ - derivace obrazu

$$\mathcal{L}[t\,f(t)](s) = -\frac{d}{ds}\,\mathcal{L}[f(t)]ds$$

$$-\frac{d}{ds}\,\mathcal{L}[f(t)](s) = -\frac{d}{ds}\int_0^\infty f(t)\,e^{-st}\,dt = -\int_0^\infty f(t)\,(-t)\,e^{-st}\,dt = \int_0^\infty t f(t)\,e^{-st}\,dt = \mathcal{L}[t\,f(t)](s)$$

$\mathcal{L}$ - Obraz derivace

$$\mathcal{L}[f^{(n)}](s) = s^n\,\mathcal{L}[f(t)](s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0)$$

$$\mathcal{L}[f'(t)](s) = \int_0^\infty f'(t)\,e^{-st}\,dt \overset{PP}{=} \left[f(t)\,e^{-st}\right]_0^\infty - \int_0^\infty f(t)\,(-s)\,e^{-st}\,dt = -f(0) + s\int_0^\infty f(t)\,e^{-st}\,dt =$$

$$= sF(s) - f(0)$$

$$\lim_{t\to\infty} f(t)\,e^{-st} - f(0)$$

$$|f(t)\,e^{-st}| \le M\,|e^{\alpha t}|\,|e^{-it}| = M\,e^{\alpha t}\cdot e^{-(\mathrm{Re}\,s)t} = M\,e^{-(\mathrm{Re}\,s - \alpha)t}$$

$\mathrm{Re}\,s > \alpha$

Obraz posunuté posloupnosti

$$\mathcal{Z}[a_{n+k}](z) = z^k \mathcal{Z}[a_n](z) - \sum_{j=0}^{k-1} a_j z^{k-j}$$

$$\mathcal{Z}[a_{n+k}](z) = \sum_{n=0}^{\infty} \frac{a_{n+k}}{z^n} = \sum_{n=k}^{\infty} \frac{a_n}{z^{n-k}} = \left| m = n+k \right| = z^k \sum_{m=k}^{\infty} \frac{a_m}{z^m} = z^k \left(\sum_{m=0}^{\infty} \frac{a_m}{z^m} - \sum_{m=0}^{k-1} \frac{a_m}{z^m} \right) =$$

$$= z^k \left(\mathcal{Z}[a_n](z) - \frac{a_1}{z} - \frac{a_2}{z^2} - \dots - \frac{a_{k-1}}{z^{k-1}} \right)$$