Electromagnetic Field Theory

Complete Course

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Outline

Introduction

Electromagnetic field is everywhere and touches every aspect of our everyday life.

- This course is about learning the basics of electromagnetic field theory and its mathematical description.

About the Course

- Lectures: Wednesday from 4:15 PM, T2:C3-132.
- Exercises: Thursday, T2:C4-156.
- Web: Courseware
- Students are expected to make their own notes!

Contact

`em@fel.cvut.cz`

Team of Teachers

Miloslav Čapek
Course guarantor
(Lectures, Exam)

Vojtěch Neuman
Course assistant
(Exercises, Tests)

Jakub Liška
Course assistant
(Exercises, Tests)

Štěpán Bosák
Course assistant
(Codes, Homeworks)

Contact

`em@fel.cvut.cz`

Requirements

To get the points during the semester:

- $4\times$ homework ($4 \times 8 = 32$ pts).
- $2\times$ semester test ($2 \times 9 = 18$ pts).
- Exam (written part) 40 pts.
- Exam (oral part) 20 pts.

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Notes:

- All the points are the subject of final classification.
- At least 15 pts from the semester.
- At least 10 pts from the written part of the exam.
- The oral part of the exam is optional.

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ECTS	**Body**	**Známka**
A	90-110	1
B	80-89	1,5
C	70-79	2
D	60-69	2,5
E	50-59	3
F	0-49	4

Notes:

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- At least 15 pts from the semester.
- At least 10 pts from the written part of the exam.
- The oral part of the exam is optional.

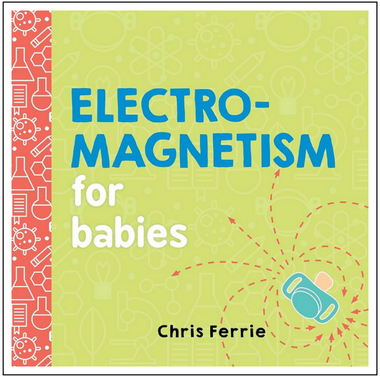
Syllabus (The best guess...)

1. Introduction to the topic, terminology, scope, history
2. Electrostatics, Part #1, Homework 1 (Vector algebra)
3. Electrostatics, Part #2
4. Conductors and Dielectrics, Homework 2
5. Forces and Energy in Electrostatics, Test 1
6. Steady-state Magnetic Field, Part #1
7. Steady-state Magnetic Field, Part #2, Homework 3
8. Magnetic Materials, Electromagnetic Force
9. Time-Varying Fields, Maxwell's Equations, Test 2
10. Uniform Plane Wave (in Frequency Domain)
11. Reflection and Dispersion, Laser
12. Guided Waves, Homework 4
13. Transmission Lines
14. Radiation, biological Aspects of EM Field

    Exam: Written part + Oral part

Literature

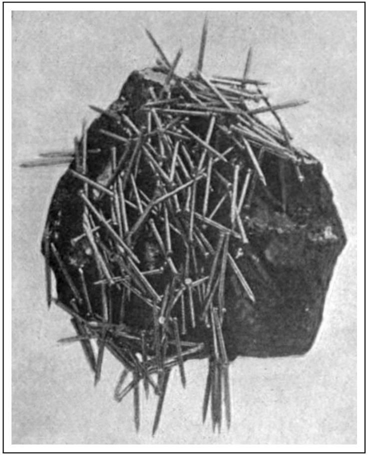
1. Novotný, K.: *Teorie elektromagnetického pole*
2. Feynman, R. P.: *Feynmanovy přednášky z fyziky*, 2. díl
3. Sedlák, B., Štoll, I.: *Elektřina a magnetismus*
4. Štoll, I.: *Dějiny fyziky*
5. Griffiths, D. J.: *Introduction to electrodynamics*
6. Hayt, W.: *Engineering Circuit Analysis*
7. Purcell, E. M., Morin, D. J.: *Electricity and Magnetism*
8. Nasar, S. A.: *2008+ Solved Problems in Electromagnetics*
9. Pierrus, J.: *Solved Problems in Classical Electromagnetism: Analytical and Numerical Solutions with Comments*
10. Franklin, J.: *Solved Problems in Classical Electromagnetism*
11. Bloise, F. S., *et al.*: *Solved Problems in Electromagnetics*
12. Zangwill, A.: *Modern Electrodynamics*



Ancient Time

$\approx$ 600 BCE: Sparking Amber in Ancient Greece (Thales of Miletus)

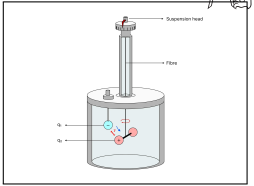
$\approx$ 221–206 BCE: Chinese Lodestone Compass (Qin dynasty)



Lodestone attracting nails.

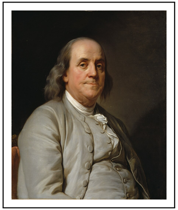
17th and 18th Centuries

- William Gilbert (1600s) Earth's Magnetism and Electrical Attraction
- Benjamin Franklin (1752) Lightning Rod
- Charles-Augustin de Coulomb (1785) Coulomb's Law
- Luigi Galvani (1789) Animal Electricity
- Alessandro Volta (1790) Electric Battery



de Coulomb's experiment

William Gilbert



Benjamin Franklin

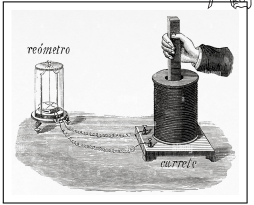
Charles-A. de Coulomb

Luigi Galvani

Alessandro Volta

19th Century, Part #1

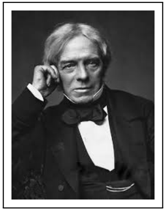
Hans Christian Ørsted (1820) Electromagnetism
Andre Marie Ampère (1821) Ampère's Law
Michael Faraday (1831) Electromagnetic Induction
James Clerk Maxwell (1873) Unification of electricity and magnetism
Heinrich Hertz (1887) Electromagnetic Waves



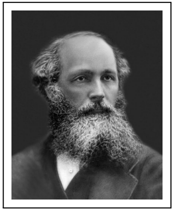
Faraday's experiment

Hans Ch. Ørsted

Andre Marie Ampère



Michael Faraday



James Clerk Maxwell

Heinrich Hertz

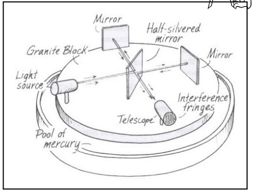
19th Century, Part #2

Nikola Tesla (1888) Inventor of AC Technology

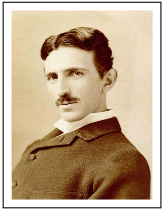
Albert A. Michelson and Edward W. Morley (1887) Absence of "Luminiferous Aether"

Guglielmo Marconi (1895) Long-distance Radio Transmission

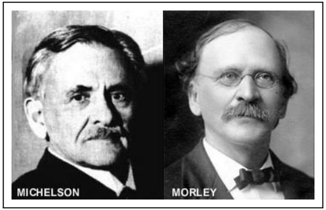
Joseph John Thomson (1897) Electron and Charge-to-Mass Ratio



Experiment of Michelson and Morley



Nikola Tesla



Albert A. Michelson and Edward W. Morley

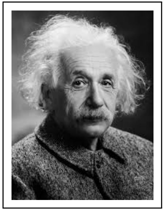
Guglielmo Marconi

Joseph J. Thomson

20th Century

Albert Einstein (1905) Quantum Nature of Electromagnetic Radiation

Richard Feynman (1965) Quantum Electrodynamics



Albert Einstein

Richard Feynman

21th Century. . .

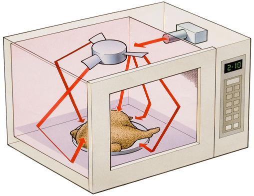
It is your turn. . . ! :)

Fundamental Interactions

S	"color"	quarks/gluons	gluons	$\approx 10^{-15}\,\mathrm{m}$	1-10
E	el. charge	e^{-}, p^{+}	photons	∞	10^{-2}
W	"flavour"	quarks/leptons	intermed. bossons	$10^{-18}\,\mathrm{m}$	10^{-10}
G	matter	all	(gravitons?)	∞	10^{-38}

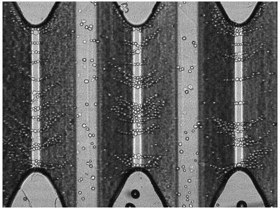
Fundamental interactions – comparison.

A Few Applications of EM



Microwave oven.

Magnetic Resonance Imaging (MRI).



Dielectrophoresis assembling cancer cells in a 3D microfluidic model.

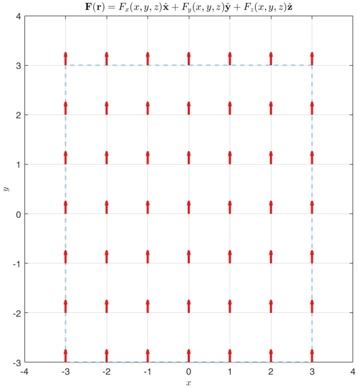
Green laser.

Vector Functions

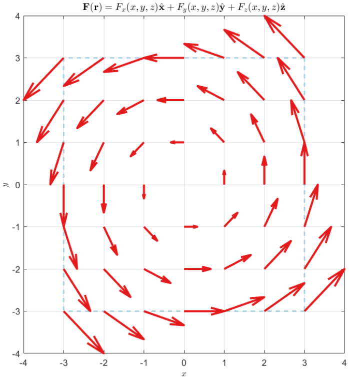
$$\boldsymbol{F}(x, y, z, t) = F_x(x, y, z, t)\hat{\boldsymbol{x}} + F_y(x, y, z, t)\hat{\boldsymbol{y}} + F_z(x, y, z, t)\hat{\boldsymbol{z}}$$

$$\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

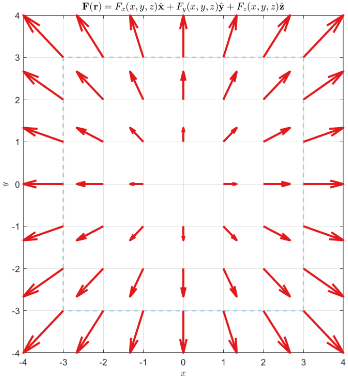
Vector Functions – Examples



$\boldsymbol{F}_1(\boldsymbol{r}) = x/4\hat{\boldsymbol{y}}$



$\boldsymbol{F}_2(\boldsymbol{r}) = -y/3\hat{\boldsymbol{x}} + x/3\hat{\boldsymbol{y}}$



$\boldsymbol{F}_3(\boldsymbol{r}) = x/3\hat{\boldsymbol{x}} + y/3\hat{\boldsymbol{y}}$

Basic EM Quantities and Their Units

Quantity	**Symbol**	**Units**	**SI Units**
Charge	q	Coulombs (C)	$\mathrm{A\,s}$
Current	I	Amperes (A)	A
Potential	φ, V	Volts (V)	$\mathrm{kg\,m^2\,A^{-1}\,s^{-3}}$
Capacitance	C	Farads (F)	$\mathrm{A^2\,s^4\,kg^{-1}\,m^{-2}}$
Resistance	R	Ohms (Ω)	$\mathrm{kg\,m^2\,A^{-2}\,s^{-3}}$
Inductance	L	Henrys (H)	$\mathrm{kg\,m^2\,A^{-2}\,s^{-2}}$
Magnetic Flux	Φ	Webers (Wb)	$\mathrm{kg\,m^2\,A^{-1}\,s^{-2}}$
Electric Field	$\boldsymbol{E}$	Volts per meter (V/m)	$\mathrm{kg\,m\,A^{-1}\,s^{-3}}$
Magnetic Field	$\boldsymbol{B}$	Teslas (T)	$\mathrm{kg\,A^{-1}\,s^{-2}}$
EM Force	$\boldsymbol{F}$	Netwon (N)	$\mathrm{kg\,m\,s^{-2}}$

Several fundamental electromagnetic quantities. More detailed table is in supplementary material “Formulas”.

Electrostatic Field

Maxwell's equations (in differential form)

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J}$$

Electrostatics ($\partial/\partial t = 0$)

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}$$

- Electric charge is displaced and fixed.
- All phenomena are time-independent.

Coulomb's Law

Coulomb's law

$$\boldsymbol{F}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q_1(\boldsymbol{r}) q_2(\boldsymbol{r}') \left(\boldsymbol{r} - \boldsymbol{r}'\right)}{\left|\boldsymbol{r} - \boldsymbol{r}'\right|^3}, \quad [\mathrm{N}]$$

Permittivity of vacuum

$$\varepsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.854 \cdot 10^{-12}\,\mathrm{F m^{-1}}$$

Superposition

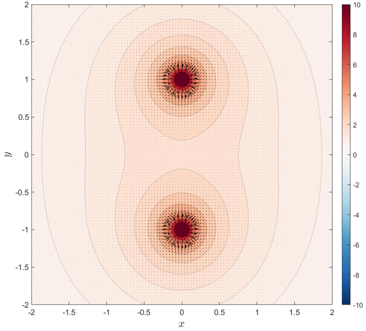
$$\boldsymbol{F}(\boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0} \sum_{n=1}^{N} q_n \frac{\boldsymbol{r} - \boldsymbol{r}'_n}{|\boldsymbol{r}'_n - \boldsymbol{r}|^3}$$

Electric Field Intensity $\boldsymbol{E}$

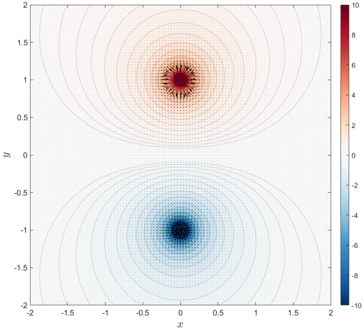
$$\boldsymbol{F}(\boldsymbol{r}) = q\boldsymbol{E}(\boldsymbol{r}), \quad [\mathrm{N}]$$

Electric field intensity $\boldsymbol{E}$ is a vector function in V/m.

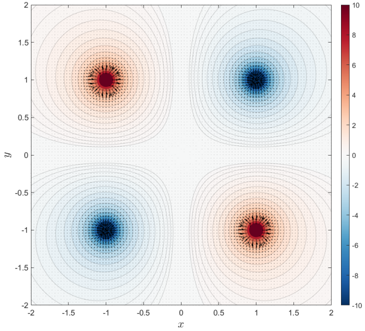
Electric Field of Several Point Charge Configurations



Two point charges of the same polarity.

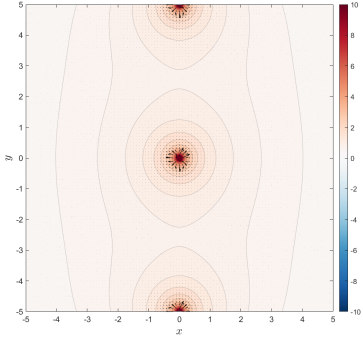


Two point charges of the opposite polarity.

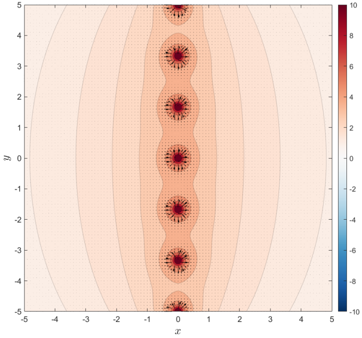


Four point charges – quadrupole configuration.

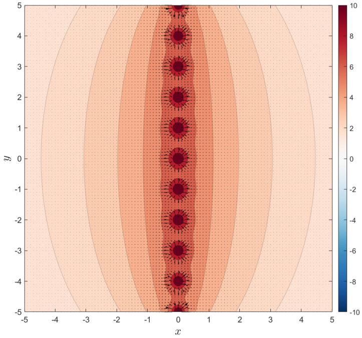
Electric Field of Several Point Charge Configurations



3 point charges along straight line.



7 point charges along straight line.



11 point charges along straight line.

Electric Field of Continuous Charge Distributions

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_{l} \tau(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \mathrm{d}l$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \iint\limits_{S} \sigma(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \mathrm{d}S$$

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \iiint\limits_{V} \rho(\boldsymbol{r}') \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \mathrm{d}V$$

Gauss's Law

- integral form

$$\oiint\limits_{S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0} \int\limits_{V'} \rho(\boldsymbol{r}')\,\mathrm{d}V' = \frac{Q}{\varepsilon_0}$$

- differential form

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

Work in Electrostatic Field

$$W = -\int_a^b \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l}, \quad [\mathrm{J}]$$

Electric potential

$$-\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \varphi(b) - \varphi(a)$$

$$\varphi = -\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} + K, \quad [\mathrm{V}]$$

Electric Field Over Closed Path

- integral form

$$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$$

- differential form

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}$$

Relation Between Electric Field and Potential

$$\boldsymbol{E} = -\nabla \varphi$$

voltage

$$U_{\mathrm{AB}} = \varphi(\mathrm{B}) - \varphi(\mathrm{A}), \quad [\mathrm{V}]$$

Summary

Coulomb's law

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int\limits_{V'} \rho(\boldsymbol{r}') \frac{(\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \, \mathrm{d}V$$

Integral form

$$\oiint\limits_{S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{\varepsilon_0}$$

$$\oint\limits_{l} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Differential form

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \boldsymbol{E} = \boldsymbol{0}$$

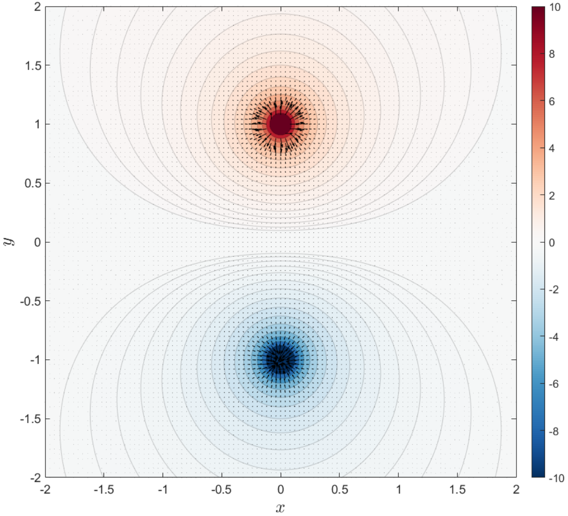
Elementary Electric Dipole

The potential generated by elementary dipole

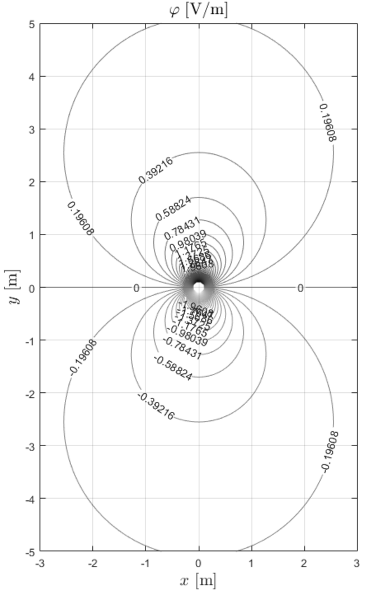
$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos(\theta)}{R^2} = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{R^2}, \quad [\mathrm{V}]$$

Electric field of elementary dipole? Solved in the exercises!

Electric Field of Two Opposite Charges



- Electric potential of the elementary dipole:



Conductors

Inside the conductor

$$\boldsymbol{E} = \mathbf{0},$$
$$\varphi = K.$$

Boundary condition

$$\boldsymbol{E}(\boldsymbol{r} \in \partial V) = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}.$$

Parallel-Plate Capacitor

Potential from the charged plane

$$\varphi = -\int \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} + K = -\frac{\sigma}{2\varepsilon_0}|z| + K.$$

Electric potential from two planes charged with opposite surface charge density

$$\varphi(z) = \varphi_+(z) + \varphi_-(z) - = \frac{\sigma}{2\varepsilon_0}\left(\left|z + \frac{d}{2}\right| - \left|z - \frac{d}{2}\right|\right).$$

Voltage and capacity

$$U = Q\frac{d}{\varepsilon_0 S} \quad \Rightarrow \quad CU = Q.$$

Poisson's and Laplace's Equation

Poisson's equation

$$\Delta\varphi = -\frac{\rho}{\varepsilon_0}$$

Laplace's equation

$$\Delta\varphi = 0,$$

Dirichlet boundary condition

$$\varphi(\mathit{\Gamma}) = K$$

Neumann boundary condition

$$\frac{\partial\varphi(\mathit{\Gamma})}{\partial\hat{\boldsymbol{n}}} = K$$

Fundamental Solution: Dirac Delta

Dirac delta $\delta(x)$ if defined by its properties:

$$\int\limits_{-\infty}^{+\infty} \delta(x)\,\mathrm{d}x = 1,$$

$$\int\limits_{-\infty}^{+\infty} f(x)\delta(x-x')\,\mathrm{d}x = f(x').$$

Fundamental Solution: Green's Function

Consider linear integro-differential equation

$$\mathcal{L}\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{r}),$$

and attempt to solve

$$\mathcal{L}\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}').$$

Then, the solution can be written as

$$\boldsymbol{u}(\boldsymbol{r}) = \int \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}')\boldsymbol{v}(\boldsymbol{r}') \, \mathrm{d}V'.$$

Polarization

Surface polarization

$$\boldsymbol{P} = \sigma_\mathrm{b}\,\hat{\boldsymbol{p}}, \quad [\mathrm{Cm}^{-2}]$$

Volumetric polarization

$$\rho_\mathrm{b} = -\nabla \cdot \boldsymbol{P}$$

Electric Susceptibility

Relation between electric field and polarization:

$$\boldsymbol{P} = \varepsilon_0 \boldsymbol{\chi}_\mathrm{e} \boldsymbol{E},$$

where $\boldsymbol{\chi}_\mathrm{e}$ is susceptibility

$$\boldsymbol{\chi}_\mathrm{e} = \begin{bmatrix} \chi_{\mathrm{e},xx} & \chi_{\mathrm{e},xy} & \chi_{\mathrm{e},xz} \\ \chi_{\mathrm{e},yx} & \chi_{\mathrm{e},yy} & \chi_{\mathrm{e},yz} \\ \chi_{\mathrm{e},zx} & \chi_{\mathrm{e},zy} & \chi_{\mathrm{e},zz} \end{bmatrix}, \quad [-]. \tag{1}$$

Material classification:

Linearity × Nonlinearity
Homogeneity × Inhomogeneity
Isotropy × Anisotropy

Electric Displacement, Permittivity

Total charge

$$\rho = \rho_0 + \rho_{\mathrm{b}}.$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_0 + \rho_{\mathrm{b}}}{\varepsilon_0}.$$

$$\nabla \cdot \boldsymbol{D} = \rho_0.$$

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

$$\boldsymbol{P} = \varepsilon_0 \chi_{\mathrm{e}} \boldsymbol{E}$$

Electric displacement field

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \varepsilon_0 \chi_{\mathrm{e}} \boldsymbol{E} = \varepsilon_0 \left(1 + \chi_{\mathrm{e}}\right) \boldsymbol{E} = \varepsilon_0 \varepsilon_{\mathrm{r}} \boldsymbol{E} = \varepsilon \boldsymbol{E}.$$

Boundary Conditions in Dielectrics

Tangential components:

$$E_{1,\mathrm{tan}} = E_{2,\mathrm{tan}}.$$

Normal components:

$$D_{1,\mathrm{norm}} - D_{2,\mathrm{norm}} = \sigma_0.$$

Condition	General case	Two dielectrics	Dielectric-conductor
Tan. comp.	$E_{1,\mathrm{tan}} = E_{2,\mathrm{tan}}$	$E_{1,\mathrm{tan}} = E_{2,\mathrm{tan}}$	$E_{\mathrm{tan}} = 0$
Norm. comp.	$D_{1,\mathrm{norm}} - D_{2,\mathrm{norm}} = \sigma_0$	$D_{1,\mathrm{norm}} = D_{2,\mathrm{norm}}$	$D_{\mathrm{norm}} = \sigma_0$
Electric pot.		$\varphi_1 = \varphi_2$	

Capacity

Capacity is defined as

$$C = \frac{Q_0}{U}, \quad [\mathrm{F}].$$

Serial connection

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \dots.$$

Parallel connection

$$C = C_1 + C_2 + \dots.$$

Energy of a Set of Charges

Electrostatic energy of a group of charges

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j \\ i \neq j}} \frac{q_i q_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

Energy

Electrostatic energy evaluated for a charge density distribution.

- From charge density

$$W = \frac{1}{8\pi\varepsilon_0} \int\limits_V \int\limits_{V'} \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \,\mathrm{d}V \,\mathrm{d}V'$$

- from potential

$$W = \frac{1}{2} \int\limits_V \rho(\boldsymbol{r})\varphi(\boldsymbol{r}) \,\mathrm{d}V,$$

item from electric field

$$W = \frac{1}{2} \int\limits_V \varepsilon(\boldsymbol{r})|\boldsymbol{E}(\boldsymbol{r})|^2 \,\mathrm{d}V.$$

Energy in Capacitor

$$W = \frac{1}{2}CU^2$$

Virtual Work Method

Force acting in electrostatics

$$F = -\frac{\mathrm{d}W}{\mathrm{d}x}.$$

Forces Acting On a Dipole

Uniform field: Torque

$$\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}.$$

Non-uniform field: Torque & drift

$$\boldsymbol{N} = (\boldsymbol{p} \times \boldsymbol{E}) + (\boldsymbol{r} \times \boldsymbol{F}).$$

Electrets and Ferroelectricity

- Electrets (Permanent polarization)
- Ferroelectricity (Spontaneous polarization)
  - Ferroelectrics
  - Pyroelectrics
  - Piezoelectrics
  - ...

Magnetostatic field

Magnetostatic field

$$\nabla \cdot \boldsymbol{B} = 0,$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

Magnetic force

$$\boldsymbol{F}_{\mathrm{m}} = q(\boldsymbol{v} \times \boldsymbol{B}), \quad [\mathrm{N}].$$

Lorentz force

$$\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}} + \boldsymbol{F}_{\mathrm{m}} = q\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right), \quad [\mathrm{N}].$$

Steady current field

$$I = \iint\limits_S Nq\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \iint\limits_S \rho\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \iint\limits_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}.$$

Different current types:

- conductive currents (in conductors and semiconductors),
- convective currents (e^- or ions in vacuum),
- in electrolyte (ions, *e.g.*, in battery).

Steady current condition:

$$\frac{\partial \boldsymbol{J}}{\partial t} = \mathbf{0}.$$

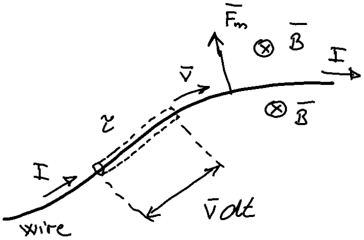
Current Continuity Equation – Steady Currents

$$\oiint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = 0$$

$$\nabla \cdot \boldsymbol{J} = 0$$

(valid only in for steady currents, *i.e.*, $\mathrm{d}d\rho/\,\mathrm{d}t = 0$)

Magnetic Force Defined by Current



$$\boldsymbol{F}_{\mathrm{m}} = \int\limits_{l} I\,(\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}) = I \int\limits_{l} (\mathrm{d}\boldsymbol{l} \times \boldsymbol{B})$$

Ohm's Law in Differential Form

$$\boldsymbol{J} = \sigma \boldsymbol{E}, \quad [\mathrm{A/m^2}]$$

with conductivity σ in [S/m]and resistivity

$$\rho = \frac{1}{\sigma}, \quad [\Omega\,\mathrm{m}].$$

Ohm's Law in Integral Form

$$R = \int_0^l \frac{\mathrm{d}l}{\sigma S}, \quad [\Omega].$$

$$U = RI, \quad [\mathrm{V}, \Omega, \mathrm{A}],$$

with R being resistance and

$$G = \frac{1}{R}, \quad [\mathrm{S}]$$

being conductance.

Connection of Resistors

Serial connection

$$R = \sum_i R$$

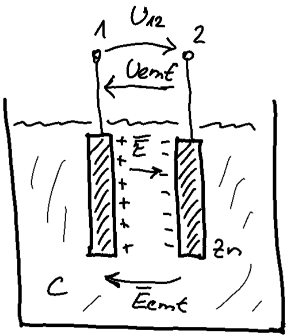
Parallel connection

$$\frac{1}{R} = \sum_i \frac{1}{R_i}$$

Electromotive Force

Many possible mechanisms of emf

- chemical force,
- mechanical pressure,
- temperature gradient,
- light,
- running belt,
- moving charges in the magnetic field.



$$U_{\mathrm{emf}} = \int_2^1 \boldsymbol{E}_{\mathrm{emf}} \cdot \mathrm{d}\boldsymbol{l} = -\int_2^1 \boldsymbol{E}_{\mathrm{c}} \cdot \mathrm{d}\boldsymbol{l} = \int_1^2 \boldsymbol{E}_{\mathrm{c}} \cdot \mathrm{d}\boldsymbol{l} = U_{12}$$

Boundary Conditions for Current Density

Normal Component

$$
\begin{aligned}
J_{1,\mathrm{norm}} &= J_{2,\mathrm{norm}} \\
\sigma_1 E_{1,\mathrm{norm}} - \sigma_2 E_{2,\mathrm{norm}} &= 0 \\
\varepsilon_1 E_{1,\mathrm{norm}} - \varepsilon_2 E_{2,\mathrm{norm}} &= \sigma_0
\end{aligned}
$$

Tangential Component

$$
\begin{aligned}
\frac{J_{2,\tan}}{\sigma_2} &= \frac{J_{1,\tan}}{\sigma_1} \\
E_{2,\tan} &= E_{1,\tan}
\end{aligned}
$$

Kirchhoff's Circuit Laws

Kirchhoff's Current Law

- The current entering a note escapes it:

$$\oiint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} = 0$$

$$\sum_i I_i = 0$$

Kirchhoff's Voltage Law

- The closed-loop integral of the electric field is zero everywhere except EMF sources:

$$\oint_l \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = 0$$

$$\sum_i U_i = 0$$

Analogies Between Electrostatic and Steady Current Field

Electrostatics	$\boldsymbol{E}$	φ	U	$\boldsymbol{D}$	ε	Q	C
Steady current	$\boldsymbol{E}$	φ	U	$\boldsymbol{J}$	σ	I	G

What about differences?

Stationary Magnetic Field

Conditions

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = 0.$$

The equivalent terms in the stationary magnetic field are

$$\mathrm{d}Q\,\boldsymbol{v}, \quad I\,\mathrm{d}\boldsymbol{l}, \quad \boldsymbol{J}\,\mathrm{d}V, \quad [\mathrm{A\,m}].$$

Biot-Savart Law

For linear current

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int\limits_{l'} I(\boldsymbol{r}') \frac{\mathrm{d}\boldsymbol{l}' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}, \quad [\mathrm{T}],$$

or for the conductor's volume

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{\boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}V' \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}, \quad [\mathrm{T}].$$

where

μ_0 is vacuum permeability,
$\boldsymbol{r}$ points to the observation region,
$\boldsymbol{r}'$ points to the source region,
T is physical unit Tesla, equivalent to $\mathrm{N}/(\mathrm{A\,m})$.

Magnetic Field of a Straight Wire

Start from

$$\boldsymbol{B}(s) = \frac{\mu_0}{4\pi} I \int \frac{\mathrm{d}\boldsymbol{l}' \times \hat{\boldsymbol{R}}}{R^2}$$

with $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$, $R = |\boldsymbol{R}|$, $\hat{\boldsymbol{R}} = \boldsymbol{R}/R$ to get

$$\boldsymbol{B}(s) = \frac{\mu_0 I}{2\pi s} \hat{\boldsymbol{\phi}}.$$

Force between two parallel wires:

$$\frac{\boldsymbol{F}_\mathrm{m}}{l} = -\frac{\mu_0 I_1 I_2}{2\pi d} \hat{\boldsymbol{x}}.$$

Ampère's Law

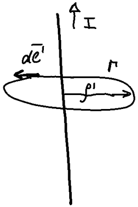
Ampère's law dictates that

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I_{\mathrm{tot}},$$

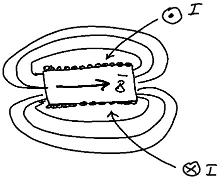
or equivalently

$$\oint_l \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 \iint_S \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}.$$

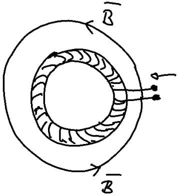
Coils: Basic Arrangements



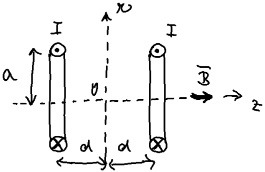
A loop.



A solenoid.



A toroid.



A loop.

Divergence of Magnetic Field

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0.$$

Magnetic Flux

The flux of magnetic field through a closed surface

$$\oiint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0.$$

Magnetic flux

$$\Phi = \iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}, \quad [\mathrm{Wb}].$$

(Wb is the physical unit Weber, equivalent to V s.)

Vector Potential

Use

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

where $\boldsymbol{A}$ is the magnetic vector potential, with the choice

$$\nabla \cdot \boldsymbol{A} = 0$$

which gives

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J},$$

with

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int\limits_{V'} \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \, \mathrm{d}V', \quad [\mathrm{Wb/m^2}].$$

Magnetic Flux

$$\Phi = \iint\limits_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \quad [\mathrm{Wb}]$$

$$\Phi = \oint\limits_l \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l}$$

Comparison of Electrostatics and Magnetostatics

Electrostatic field	Magnetostatic field
$\nabla \cdot \boldsymbol{E} = \frac{1}{\varepsilon_0}\rho$	$\nabla \cdot \boldsymbol{B} = 0$
$\nabla \times \boldsymbol{E} = \boldsymbol{0}$	$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$
BC $\boldsymbol{E} \to \boldsymbol{0}$ far from sources	BC $\boldsymbol{B} \to \boldsymbol{0}$ far from sources
$\boldsymbol{E}(\boldsymbol{r}_2) = \frac{1}{4\pi\varepsilon_0}\int\limits_{V_1} \frac{\rho(\boldsymbol{r}_1)(\boldsymbol{r}_2 - \boldsymbol{r}_1)}{\lvert\boldsymbol{r}_2 - \boldsymbol{r}_1\rvert^3}\,\mathrm{d}V_1$	$\boldsymbol{B}(\boldsymbol{r}_2) = \frac{\mu_0}{4\pi}\int\limits_{V_1} \frac{\boldsymbol{J}(\boldsymbol{r}_1) \times (\boldsymbol{r}_2 - \boldsymbol{r}_1)}{\lvert\boldsymbol{r}_2 - \boldsymbol{r}_1\rvert^3}\,\mathrm{d}V_1$
$\boldsymbol{F}_\mathrm{e} = q\boldsymbol{E}$	$\boldsymbol{F}_\mathrm{m} = q\left(\boldsymbol{v} \times \boldsymbol{B}\right)$

Magnetic Dipole Moment

Multipole expansion for

$$\frac{1}{R} = \frac{1}{r} + \frac{\cos(\alpha) r'}{r^2} + \left(-\frac{1}{2} + \frac{3}{2}\cos^2(\alpha)(r')^2\right)\frac{1}{r^3} + \mathcal{O}((r')^3).$$

Magnetic dipole moment

$$\boldsymbol{A}_{\mathrm{d}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{r^2}, \quad [\mathrm{Wb/m}]$$

with

$$\boldsymbol{m} = I \iint\limits_{S'} \mathrm{d}\boldsymbol{S}', \quad [\mathrm{A\,m^2}].$$

The magnetic field of small loop of current is

$$\boldsymbol{B}_{\mathrm{d}} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos(\theta)\hat{\boldsymbol{r}} + \sin(\theta)\hat{\boldsymbol{\theta}}\right).$$

Atomic Model and Material Classification

1. Electron in orbit.
2. Spin of the electron.
3. Nuclear spin.

The magnetic materials are classified as

- diamagnetic,
- paramagnetic,
- ferromagnetic,
- antiferromagnetic,
- ferrimagnetic,
- superparamagnetic.

Magnetization

Magnetization as the average of loop magnetic moments

$$\boldsymbol{M} = \lim_{\Delta V \to 0} \frac{\sum_i \boldsymbol{m}_i}{\Delta V} \quad [\mathrm{A/m}].$$

Ampère's law including bound current I_b

$$\oint_{l'} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}' = \mu_0 I_\mathrm{tot} = \mu_0 \left(I_0 + I_\mathrm{b} \right).$$

We have

$$\boldsymbol{m}_\mathrm{b} = I_\mathrm{b} \int \mathrm{d}\boldsymbol{S} = I_b \boldsymbol{S},$$

and

$$I_\mathrm{b} = hM.$$

Intensity of Magnetic Field

Ampère's law for free and bound current expressed as magnetization

$$\oint_{l'} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}' = \mu_0 I_0 + \mu_0 \int_{l'} \boldsymbol{M} \cdot \mathrm{d}\boldsymbol{l}'.$$

Definition of the intensity of magnetic field

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}, \quad [\mathrm{A/m}].$$

The integral form of Ampère's law for free current only

$$\oint_{l'} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = I_0 = \iint_{S'} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S},$$

and its differential form

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_0.$$

Magnetic Susceptibility

Linear (linearized) materials

$$\boldsymbol{M} = \chi_{\mathrm{m}} \boldsymbol{H},$$

where χ_{m} is magnetic susceptibility.

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \chi_{\mathrm{m}} \boldsymbol{H} = \mu \boldsymbol{H},$$

Magnetic Materials

$$\boldsymbol{M} = \chi_{\mathrm{m}} \boldsymbol{H}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \chi_{\mathrm{m}} \boldsymbol{H} = \mu_0 \left(1 + \chi_{\mathrm{m}}\right) \boldsymbol{H} = \mu_0 \mu_{\mathrm{r}} = \mu \boldsymbol{H}$$

Classification	**Magnetic Moments**	$\boldsymbol{B}$ **Values**	**Comments**
Diamagnetic	$\lvert \boldsymbol{m}_{\mathrm{o}} + \boldsymbol{m}_{\mathrm{s}} \rvert = 0$	$B_{\mathrm{int}} < B_0$	Nonmagnetic matrix
Paramagnetic	$\lvert \boldsymbol{m}_{\mathrm{o}} + \boldsymbol{m}_{\mathrm{s}} \rvert \to 0$	$B_{\mathrm{int}} > B_0$	Low magnetic response
Ferromagnetic	$\lvert \boldsymbol{m}_{\mathrm{s}} \rvert \gg \lvert \boldsymbol{m}_{\mathrm{o}} \rvert$	$B_{\mathrm{int}} \gg B_0$	Domains
Antiferromagnetic	$\lvert \boldsymbol{m}_{\mathrm{s}} \rvert \gg \lvert \boldsymbol{m}_{\mathrm{o}} \rvert$	$B_{\mathrm{int}} = B_0$	Adjacent moments oppose
Ferrimagnetic	$\lvert \boldsymbol{m}_{\mathrm{s}} \rvert \gg \boldsymbol{m}_{\mathrm{o}} \rvert$	$B_{\mathrm{int}} > B_0$	Unequal adjacent moments oppose
Superparamagnetic	$\lvert \boldsymbol{m}_{\mathrm{s}} \rvert \gg \lvert \boldsymbol{m}_0 \rvert$	$B_{\mathrm{int}} > B_0$	Nonmagnetic matrix

Classification of magnetic materials (the table is adopted from Hyat & Buck), $\boldsymbol{m}_{\mathrm{o}}$ stands for an orbital moment, m_{s} for a spin moment, B_{int} stands for the magnitude of magnetic field inside the material, and B_0 stands for the magnitude of external magnetic field.

Diamagnetic Materials

- $\chi_\mathrm{m} \approx -10^{-5}$, $\mu_\mathrm{r} < 1$.
- Weakly repelled by a magnetic field.
- No permanent magnetic moment in the absence of a magnetic field.

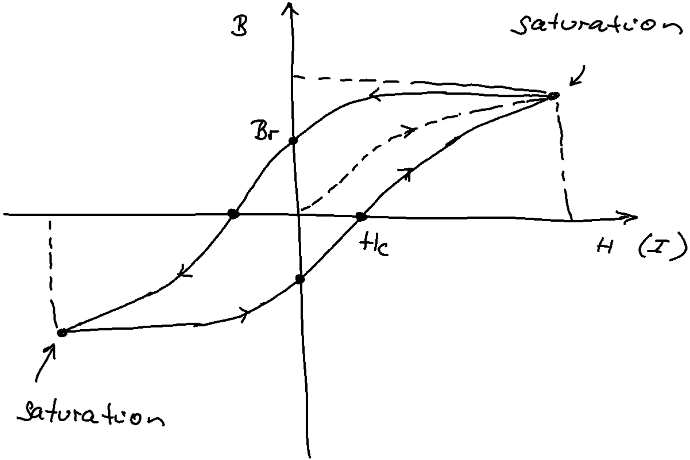
Paramagnetic Materials

- $\chi_\mathrm{m} > 0$, $\mu_\mathrm{r} > 1$.
- Weakly attracted to a magnetic field.
- Magnetic moments align with the external field but only in its presence; the alignment is random when the field is removed.

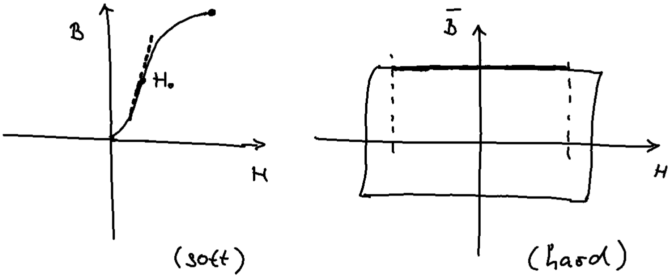
Ferromagnetic Materials

- $\chi_{\mathrm{m}} \approx 10^3$-$10^5$
- Strongly attracted to a magnetic field.
- Highly nonlinear materials.
- Exhibit hysteresis in their magnetization curve.

Hysteresis

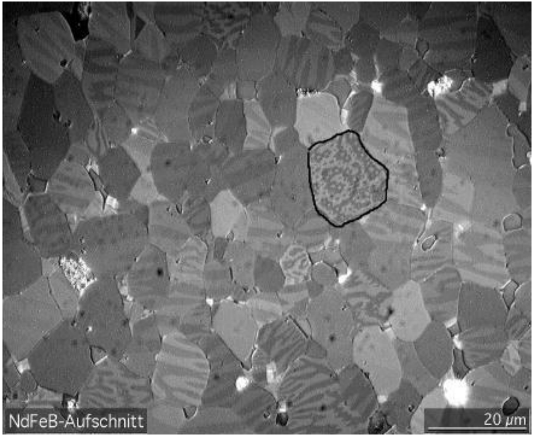


Soft and Hard Materials



Soft materials are usually used around a point where the permeability can be linearized. Hard materials (permanent magnets) are used with magnetic field intensity so they are never de-magnetized.

Soft and Hard Materials



Microcrystalline grains within a piece of $Nd_2Fe_{14}B$ (neodymium magnet). The domains are the light and dark stripes visible within each grain. The outlined grain has an almost vertical magnetocrystalline axis, so the domains are seen end-on. Credits: Wikipedia.

Boundary Conditions in Magnetic Field

$$\mu_1 H_{1,\mathrm{norm}} = \mu_2 H_{2,\mathrm{norm}}$$

$$\frac{B_{1,\tan}}{\mu_1} = \frac{B_{2,\tan}}{\mu_2}$$

$$H_{1,\tan} - H_{2,\tan} = \frac{NI}{\mathrm{d}l} = nI$$

Magnetic Circuits

Electric circuit	**Magnetic circuit**
electromotive voltage U	magnetomotive voltage NI
electric current I	magnetic flux Φ
resistance R	reluctance R_{m}
conductance G	permeance P_{m}
conductivity σ	permeability μ

Analogies between electric and magnetic circuits.

Hopkinson's law

$$U_{\mathrm{m}} = \Phi R_{\mathrm{m}}$$

Faraday's Law of Induction

Motional electromotive (emf) force

$$U_{\mathrm{emf}} = \oint\limits_{l} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}.$$

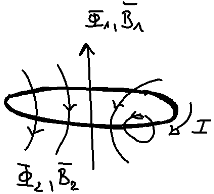
Faraday's law of induction

$$\oint \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{l}.$$

Faraday's law of induction in differential form

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}.$$

Lenz's Law



Ideal Transformer

Current

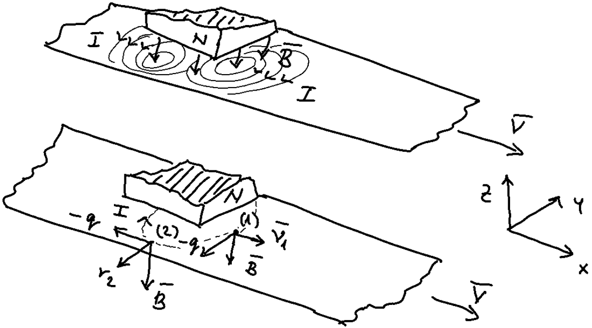
$$\frac{i_1}{i_2} = \frac{N_2}{N_1}$$

and voltage

$$\frac{u_1}{u_2} = \frac{N_1}{N_2}$$

transformation of ideal transformer.

Eddy Currents



Static and Dynamic Definition

Static definition

$$L = \frac{\Phi_\mathrm{t}}{I} \quad [\mathrm{H}].$$

Dynamic definition

$$u(t) = L\frac{\mathrm{d}i}{\mathrm{d}t}.$$

Mutual Inductance

Mutual inductance M_{21}

$$\Phi_2 = M_{21} I_1.$$

Neumann Formula

$$\Phi_2 = \frac{\mu_0}{4\pi} I_1 \oint_{l_1} \oint_{l_2} \frac{\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2}{R} = M_{21} L_1.$$

Energy in Magnetic Field

Using currents and potential

$$W = \frac{1}{2} \iiint_V \boldsymbol{A} \cdot \boldsymbol{J} \, \mathrm{d}V, \quad [\mathrm{J}].$$

Using magnetic field

$$W = \frac{1}{2} \iiint_V \boldsymbol{H} \cdot \boldsymbol{B} \, \mathrm{d}V, \quad [\mathrm{J}].$$

Need For Displacement Current

Maxwell's equations before Maxwell

$$
\begin{aligned}
\nabla \cdot \boldsymbol{E} &= \frac{\rho}{\varepsilon_0}, \\
\nabla \cdot \boldsymbol{B} &= 0, \\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\
\nabla \times \boldsymbol{B} &= \mu_0 \boldsymbol{J}.
\end{aligned}
$$

Displacement Current

Complete Ampère's law:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

with

$$\boldsymbol{J}_\mathrm{d} = \frac{\partial \boldsymbol{D}}{\partial t}$$

being the displacement current.

Energy in Magnetic Field

Complete Maxwell's equations in differential and integral form:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c_0^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\iint\limits_{S'} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{1}{\varepsilon_0} \iiint\limits_{V'} \rho \, \mathrm{d}V'$$

$$\oint\!\!\!\oint\limits_{S'} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0$$

$$\oint\limits_{l'} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}' = -\frac{\mathrm{d}}{\mathrm{d}t} \iint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}$$

$$\oint\limits_{l'} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}' = \mu_0 \iint\limits_{S'} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} + \frac{1}{c_0^2}\frac{\mathrm{d}}{\mathrm{d}t} \iint\limits_{S'} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S}$$

Questions?

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