① a) $$\nabla \times F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-\alpha z e^{2x} & 3y^2 & z^2-e^{2x} \end{vmatrix}$$

$$= \left(0-0, -\alpha e^{2x}+2e^{2x}, 0-0\right) = (0,0,0) \text{ na } \mathbb{R}^3 \Leftrightarrow$$

$\Leftrightarrow \alpha = 2$.

b) $$\frac{\partial f}{\partial x} = x - 2ze^{2x} \qquad (1)$$

$$\frac{\partial f}{\partial y} = 3y^2 \qquad (2)$$

$$\frac{\partial f}{\partial z} = z^2 - e^{2x} \qquad (3)$$

$$(1) \Rightarrow f(x,y,z) = \frac{x^2}{2} - ze^{2x} + \tilde{C}(y,z)$$

Dosazením do (2) máme $\frac{\partial \tilde{C}}{\partial y} = 3y^2 \Rightarrow \tilde{C}(y,z) = y^3 + C(z)$

Z (3) poté plyne, že

$$-e^{2x} + C'(z) = z^2 - e^{2x}$$

Proto $C(z) = \frac{z^3}{3} + K$, $K \in \mathbb{R}$

Z $f(0,0,1) = 0$ dostaneme $0 = -1 + \frac{1}{3} + K$. Odtud

$$f(x,y,z) = \frac{x^2}{2} - ze^{2x} + y^3 + \frac{z^3}{3} + \frac{2}{3}$$

c) Pro $\alpha = 2$ je F potenciálové na $\mathbb{R}^3$ a $C \subseteq \mathbb{R}^3$ je uzavřená křivka. Proto $\int_C F = 0$.

② a) Hledáme body minima funkce

$$f(k,q)=(-2k+q+1)^2+(q-1)^2+(2k+q-2)^2$$

na množině $M=\mathbb{R}^2$.

b) Protože f je konvexní, je každý stacionární bod bodem minima.

$$\frac{\partial f}{\partial k}=-4(-2k+q+1)+4(2k+q-2)=0 \qquad (1)$$

$$\frac{\partial f}{\partial q}=2(-2k+q+1)+2(q-1)+2(2k+q-2)=0 \qquad (2)$$

$$(1)\iff 4k-3=0\iff k=\frac{3}{4}$$

$$(2)\iff 3q-2=0\iff q=\frac{2}{3}$$

Hledaná přímka má proto rovnici $y=\frac{3}{4}x+\frac{2}{3}$.

[Alternativně můžeme v řešení použít maticový zápis úlohy spolu s tvrzením z přednášky: minimalizujeme $f(k,q)=\left\|A\begin{pmatrix}k\\q\end{pmatrix}-b\right\|$ na $\mathbb{R}^2$, kde

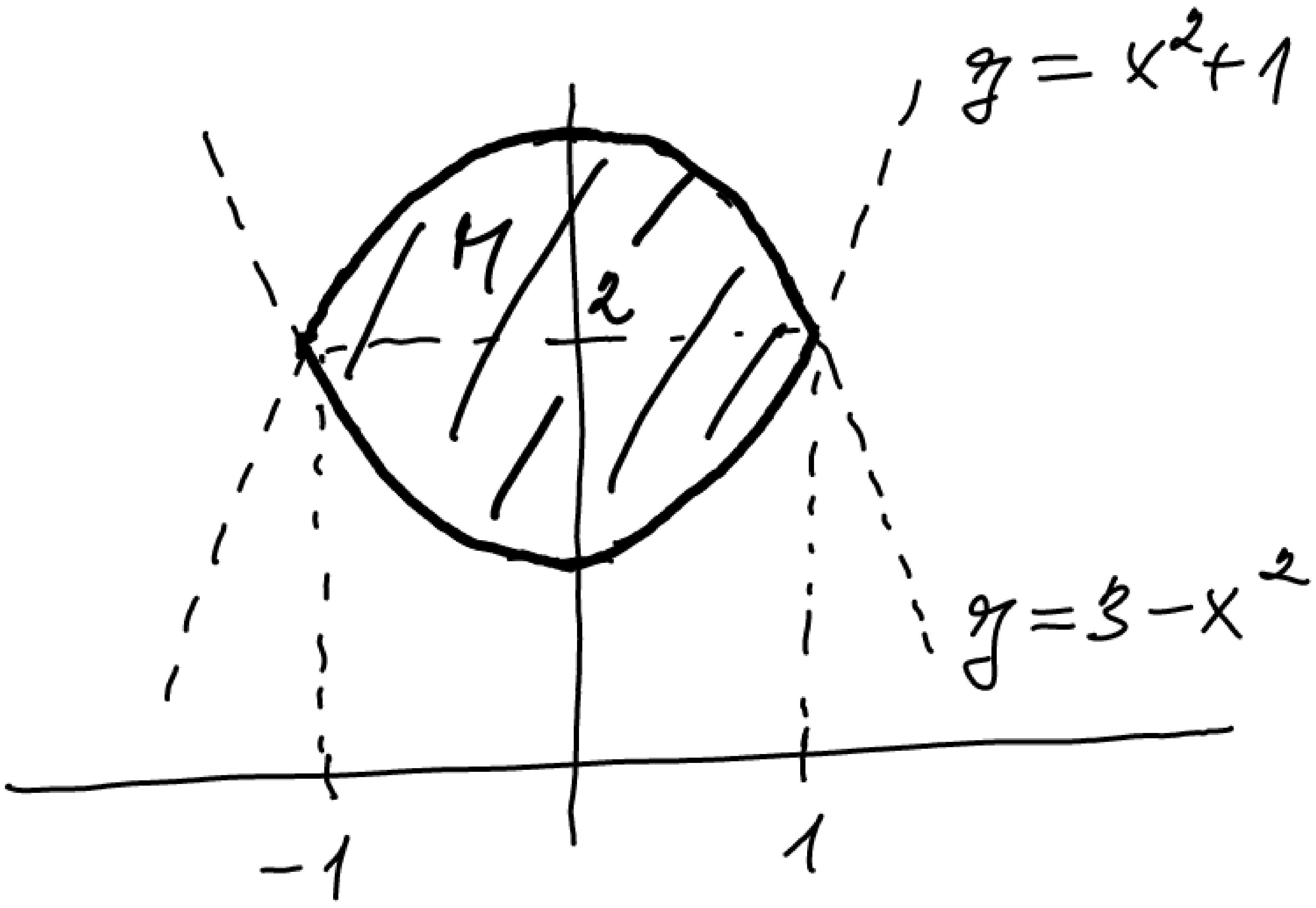
$$A=\begin{pmatrix}-2&1\\0&1\\2&1\end{pmatrix},\quad b=\begin{pmatrix}-1\\1\\2\end{pmatrix}.$$

Protože $A^TA=\begin{pmatrix}8&0\\0&3\end{pmatrix}$, je $(A^TA)^{-1}=\frac{1}{24}\begin{pmatrix}3&0\\0&8\end{pmatrix}$.

Odtud

$$\begin{pmatrix}k\\q\end{pmatrix}=(A^TA)^{-1}A^Tb=\frac{1}{24}\begin{pmatrix}3&0\\0&8\end{pmatrix}\begin{pmatrix}6\\2\end{pmatrix}=\frac{1}{24}\begin{pmatrix}18\\16\end{pmatrix}=\begin{pmatrix}\frac{3}{4}\\\frac{2}{3}\end{pmatrix}.$$]

③

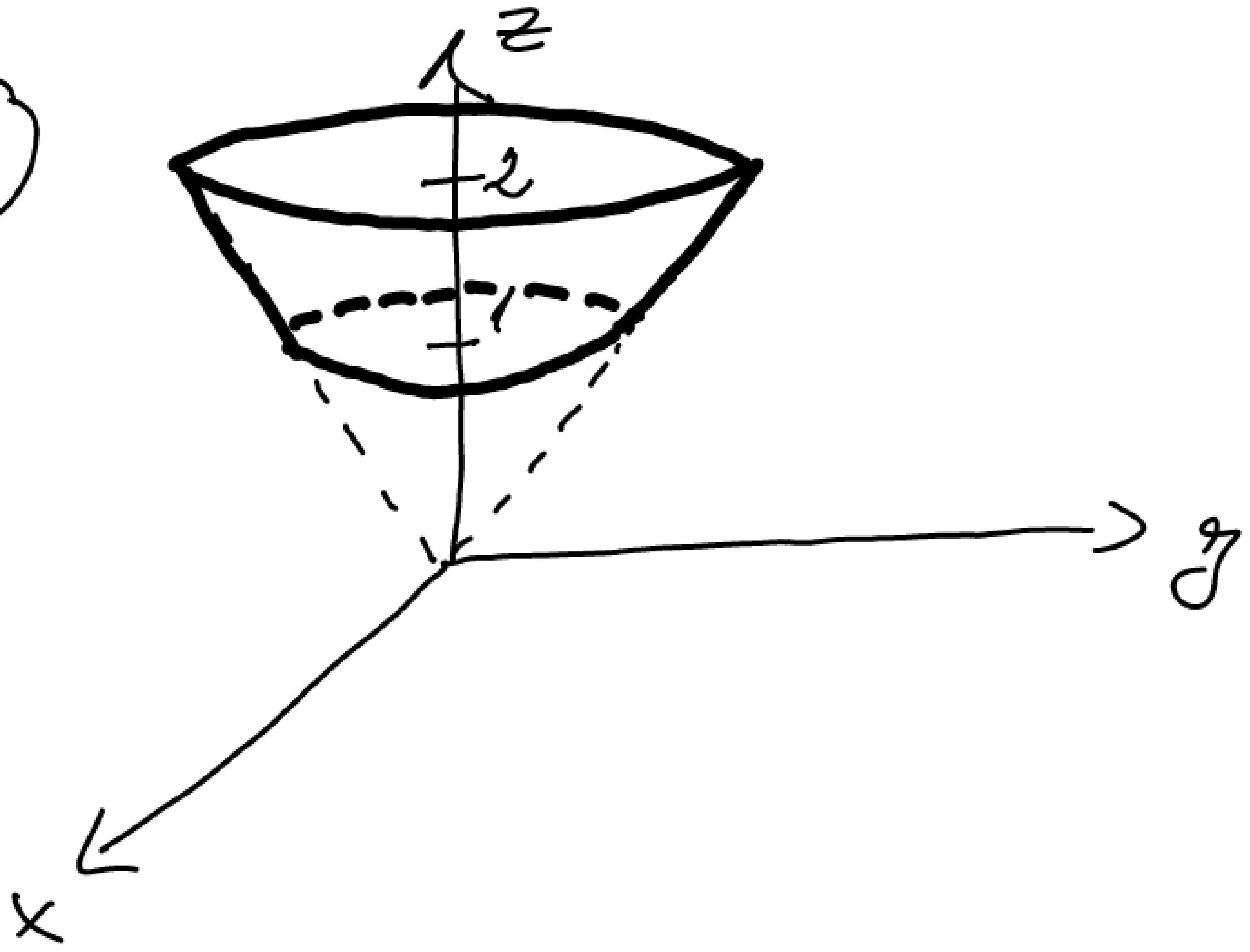


$$\int_M y = \int_{-1}^{1}\int_{x^2+1}^{3-x^2} y \, dy \, dx = \int_{-1}^{1} \left[\frac{y^2}{2} \right]_{x^2+1}^{3-x^2} dx$$

$$= \frac{1}{2}\int_{-1}^{1} \underbrace{(3-x^2)^2 - (x^2+1)^2}_{9-6x^2+x^4-(x^4+2x^2+1)=8(1-x^2)} dx$$

$$= 8\int_{0}^{1} 1-x^2 \, dx = 8\left(1-\frac{1}{3}\right) = \frac{16}{3} .$$

④



$$\nabla \cdot F = y^2 + x^2$$

válcové souřadnice

$$\Rightarrow \int_S F = \int_M x^2 + y^2 \, d\lambda_3(x,y,z) = \int_1^2 \int_0^{2\pi} \int_0^z r^2 r \, dr \, d\varphi \, dz$$

$$= \int_1^2 2\pi \frac{z^4}{4} \, dz = \frac{\pi}{2} \left[\frac{z^5}{5} \right]_1^2 = \frac{\pi}{10}(32-1)$$

$$= \frac{31\pi}{10}.$$

⑤ a)

integrujeme lichou funkci přes interval symetrický kolem 0.

$$a_\ell = \frac{1}{\pi}\int_{-\pi}^{\pi} -t\cos(\ell t)\,dt \overset{\vee}{=} 0 \quad \text{pro } \ell \in \mathbb{N}_0 .$$

per-partes

$$b_\ell = \frac{1}{\pi}\int_{-\pi}^{\pi} -t\sin(\ell t)\,dt = \frac{2}{\pi}\int_{0}^{\pi} \underbrace{-t\sin(\ell t)}_{t\left(\frac{\cos(\ell t)}{\ell}\right)'}\,dt \overset{\vee}{=}$$

$$= \frac{2}{\pi}\left\{\left[t\,\frac{\cos(\ell t)}{\ell}\right]_0^{\pi} - \int_0^{\pi} \frac{\cos(\ell t)}{\ell}\,dt\right\}$$

$$= \frac{2(-1)^\ell}{\ell} - \frac{2}{\pi\ell^2}\left[\sin(\ell t)\right]_0^{\pi} = \frac{2(-1)^\ell}{\ell} \quad \text{pro } \ell \in \mathbb{N}.$$

Fourierova řada funkce f je

$$\sum_{\ell=1}^{\infty} \frac{2(-1)^\ell}{\ell}\sin(\ell t)$$

b) Součet $\mathcal{F}_f(t)$ Fourierovy řady funkce f na intervalu $[5\pi, 7\pi)$ je

$$\mathcal{F}_f(t) = \mathcal{F}_f(\underbrace{t-6\pi}_{\in[-\pi,\pi)}) = \begin{cases} -t+6\pi, & t\in(5\pi,7\pi), \\ 0, & t=5\pi. \end{cases}$$