NMMB538 - Zkouška

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Lemma Q.1. *Proof:*

Denote $h = x_2^2 - f(x_1)$ and assume $h = u \cdot v$ where $u, v \in \bar{K}[x_1, x_2]$.

First assume $u, v \in \bar{K}[x_1, x_2] \backslash \bar{K}[x_1]$ i.e. $\deg_{x_2}(u) > 0, \deg_{x_2}(v) > 0$. Because $\deg_{x_2}(u) + \deg_{x_2}(v) = \deg_{x_2}(h) = 2 \implies \deg_{x_2}(u) = 1 = \deg_{x_2}(v)$. W.l.o.g assume $lc_{x_2}(u) = 1 = lc_{x_2}(v)$, we can do that since $lc_{x_2}(h) = 1$. Therefore we can write $u = x_2 - s_1$ and $v = x_2 - s_2$ where $s_1, s_2 \in \bar{K}[x_1]$. This gives us

$$x_2^2 - f(x_1) = h = (x_2 - s_1)(x_2 - s_2) = x_2^2 - (s_1 + s_2)x_2 + s_1 s_2$$

So it must hold that $s_1 = -s_2$ and then $h = x_2^2 + s_1(-s_1) \implies f(x_1) = s_1^2$.

Now assume w.l.o.g $u \in \bar{K}[x_1]$.We compare the leading coefficients.

$$1 = lc_{x_2}(h) = lc_{x_2}(u) \cdot lc_{x_2}(v) = u \cdot lc_{x_2}(v)$$

This shows that u must be invertible in $\bar{K}[x_1, x_2] \implies u \in \bar{K}^*$. In other words h is absolutely irreducible.

$\square$

Sublemma Q.3.5 *Let F/K be an algebraic function field, char$(K) \neq 2$, that is given by $y^2 = f(x)$, f being a quartic polynomial that is absolutely irreducible. Let $P \in \mathbb{P}_{F/K}$. If $x \notin P$ or $y \notin P$, then $x, y \notin P$ and $2v_P(x) = v_P(y)$.*

Proof:

Lemma Q.4. *Proof:*

By sublemma Q.3.5 we know, that if $P \in \mathbb{P}_{F/K} : x^{-1} \in P \implies y^{-1} \in P$ and $2v_P(x) = v_P(y)$. Therefore $2 | v_P(y^{-1}) \implies v_P(y^{-1}) \geq 2$.

$$(y)_- = \sum_{y^{-1} \in P} v_P(y^{-1}) P \implies$$

$$\deg((y)_-) = \sum_{y^{-1} \in P} v_P(y^{-1}) \deg(P) = [F : K(y^{-1})] = [F : K(y)] = \deg(f) = 4$$

We know $v_P(y^{-1}) \geq 2$, this means there are 3 possibilities.

1. There is only one place P s.t. $\deg(P) = 1$ and $v_P(y^{-1}) = 4$.
2. There is only one place P s.t. $\deg(P) = 2$ and $v_P(y^{-1}) = 2$.
3. There are 2 distinct places P_1, P_2 s.t. $\deg(P_1) = 1 = \deg(P_2)$ and $v_{P_1}(y^{-1}) = 2 = v_{P_2}(y^{-1})$.

For every possibility it holds that $(y)_- = 2(x)_-$. And also that $\deg((x)_-) = 2$.

When determining the genus we can assume $K = \bar{K}$. Denote the genus of F/K as $g > 0$. By the Riemann theorem we get that for $D \in \text{Div}(F/K) : \deg(D) - l(D) < g$. Since $\deg_{\bar{K}}(D) = [F : K(x)] = 2 \implies 2 - l(D) < g$. $D \geq 0 \implies l(D) \geq 1 \implies l(D) \geq 2 - g$. Therefore $g = 1$ or $g = 0$.