NMAG436 - HW1

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1

By definition of L being AFF given by $f(\alpha, \beta) = 0$ we know that f is irreducible in $\mathbb{Q}$ and $L = \mathbb{Q}(\alpha, \beta)$.

(a) Using proposition 4.7 we know that α is transcendental over $\mathbb{Q}$ because $deg_y(f) = 2 > 0$ and also that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 2$. In other words, the basis A has 2 elements. Let $A = (1, \beta)$ (first element of basis A is 1 and the second is β. We can see that 1 and β are linearly independent:

$$c_1, c_2 \in \mathbb{Q}(\alpha) : 1c_1 + \beta c_2 = 0 \overset{?}{\iff} c_1 = 0 = c_2$$
$$1c_1 + \beta c_2 = 0 \iff \beta c_2 = -c_1 \overset{c_2 \neq 0}{\iff} \beta = \frac{-c_1}{c2} \in \mathbb{Q}(\alpha)$$
$$\implies \text{contradiction with } [\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 2 \implies c_1 = 0 = c_2$$

We know the basis has 2 elements and therefore A is a basis.

(b) Using the same reasoning (β is also transcendental over $\mathbb{Q}$) using symetry (we can just replace y with x) we get that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\beta)] = deg_x(f) = 3$. Therefore the basis B has 3 elements. Let $B = (1, \alpha, \alpha^2)$. Using the similar reasoning we can see that elements of B are linearly independent therefore they form basis.

(c) We will use equalities given by $f(\alpha, \beta) = 0 \implies \beta^2 = \alpha^3 + 2\alpha^2 + 1, \alpha^3 = \beta^2 - 2\alpha^2 - 1$.

$$[\alpha^3\beta^3]_A : \alpha^3\beta^3 = \alpha^3((\alpha^3 + 2\alpha^2 + 1)\beta) = \beta(\alpha^6 + \alpha^5 + \alpha^3), (\alpha^6 + \alpha^5 + \alpha^3) \in \mathbb{Q}(\alpha)$$
$$A = (1, \beta) \implies [\alpha^3\beta^3]_A = (0, \alpha^6 + \alpha^5 + \alpha^3)$$
$$[\alpha^3\beta^3]_A : \alpha^3\beta^3 = (\beta^2 - 2\alpha^2 - 1)\beta^3 = 1(\beta^5 - \beta^3) + \alpha^2(-2\beta^3)$$
$$B = (1, \alpha, \alpha^2) = (\beta^5 - \beta^3, 0, -2\beta^3)$$

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(a) Let $\gamma := (1, 2)^T \in \mathbb{Q}^2$. By calculating derivatives $\frac{\partial f}{\partial x}(\gamma) = -7, \frac{\partial f}{\partial y}(\gamma) = 4$ we get $t_\gamma(f) = -7(x - 1) + 4(y - 2) = -7x + 4y - 1$. Let $(a_1, a_2) := (-7, 4)$ from lemma 5.6. Using lemma 5.6 we know we are looking for $\sigma := \theta_A \tau_{-\gamma}$ where A is a regular rational matrix of the form $A = \begin{pmatrix} b_1 & b_2 \\ -7 & 4 \end{pmatrix}$. From definition of σ we get $\sigma(1, 2)^T = A(0, 0)^T = 0$.
Let $(b_1, b_2) = (1, 0)$ for example. We can see that A is regular. We now have to calculate the polynomial $\hat{f}$ following the proof of lemma 5.6. We will first calculate

$\tilde{f}(x,y)$.

$$
\begin{gathered}
p(x,y) \coloneqq \tilde{f}(x-1,y-2) = f(x,y) - t_\gamma(f) = y^2 - x^3 - 2x^2 + 7x - 4y \\
\tilde{f}(x,y) \overset{\text{substitution}}{=} p(x+1,y+2) = y^2 - x^3 - 5x^2 \\
A^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{7}{4} & \frac{1}{4} \end{pmatrix} \implies \theta^*_{A^{-1}}(p(x,y)) = p(x, \frac{7}{4}x + \frac{1}{4}y) \\
\hat{f}(x,y) = \theta^*_{A^{-1}}(\tilde{f}(x,y)) = \tilde{f}(x, \frac{7}{4}x + \frac{1}{4}y) = -x^3 - \frac{31x^2}{16} + \frac{7xy}{8} + \frac{y^2}{16} \\
\implies h(x) \coloneqq -x^3 - \frac{31x^2}{16}, g(x,y) \coloneqq \frac{7x}{8} + \frac{y}{16} \\
\sigma^*(p(x,y)) = p(x-1, -7x + 4y - 1)
\end{gathered}
$$

We can check $\sigma^*(\hat{f}+y) = \sigma^*(h(x)+yg(x,y)+y) = f$. The desired map is σ, where A is for example written above.

(b) From the lemma 5.6 we know σ has the form $\sigma \coloneqq \theta_A \tau_{-\gamma}$, where $A = \begin{pmatrix} b_1 & b_2 \\ -7 & 4 \end{pmatrix}$ is regular $\iff det(A) \neq 0$. We will now calculate $\sigma(0,0)^T$:

$$
\begin{gathered}
A = \begin{pmatrix} b_1 & b_2 \\ -7 & 4 \end{pmatrix}, \sigma(0,0)^T = A \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -1 \begin{pmatrix} b_1 \\ -7 \end{pmatrix} - 2 \begin{pmatrix} b_2 \\ 4 \end{pmatrix} = \begin{pmatrix} -b_1 - 2b_2 \\ -1 \end{pmatrix} \\
\text{we want } det(A) \neq 0 \iff 4b_1 + 7b_2 \neq 0 \iff b_1 \neq \frac{-7}{4} b_2 \\
\text{in other words } \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{Q}^2 \setminus Span_{\mathbb{Q}} \begin{pmatrix} -7 \\ 4 \end{pmatrix}
\end{gathered}
$$

To sum it up. There exists σ satisfying given conditions if and only if $\mathbf{a} = \begin{pmatrix} -b_1 - 2b_2 \\ -1 \end{pmatrix}$ where $(b_1, b_2)^T$ is not in the span of the vector $(-7, 4)$.