NMAG436 - HW1

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1

Let $w := y^2 - f(x)$, where $f(x) := x^3 - 4x^2 - x + 4$. By definition, we see that w is a short WEP.

(a) We will use the proposition 3.12 and the definition of separable polynomials. We can see that 1 and -1 are roots of $f(x)$. We can get the 3rd root by polynomial division. We now know $f(x)$ has 3 roots: $\{-1, 1, 4\}$ therefore $f(x)$ has no multiple roots because it has degree 3 (we could check this without polynomial division by checking if -1 and 1 are roots of $f'(x)$ therefore multiple roots). This means $f(x)$ is separable and by proposition 3.12 3) w is smooth ($\mathbb{R}$ does not have characteristic 2).

(b) From a) $f(x) = (x-1)(x+1)(x-4) \overset{\mathbb{F}_5}{=} (x+4)(x+1)(x+1)$. We see 4 is a multiple root of $f(x) \implies f(x)$ is not separable. $\mathbb{F}_5$ does not have characteristic 2 so we again use proposition 3.12 3) and we get that w is not smooth $\iff w$ is singular.

2

$\mathbb{R}, \mathbb{F}_3$ don't have characteristic 2.

(a) Let $w := y^2 + y(2-2x) - (x^3 + x^2 + 3x - 1)$, w is a WEP but not a short WEP. To apply proposition 3.12, we need the short WEP of w. Lemma 3.1 gives the affine bijection needed to get the short WEP form of w. By appling lemma 3.1 ($a = -2, c = 2$) we have:

$$A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, b := \begin{pmatrix} 0 \\ -1 \end{pmatrix} \implies \sigma := \tau_{\theta_A(b)} \circ \theta_A \implies \sigma^*(f(x,y)) = f(x, x+y-1)$$

$\sigma^*(w) = y^2 - (x^3 + 2x^2 + x)$ is now a short WEP. Let $f(x) := (x^3 + 2x^2 + x)$, we can see -1 is a root of $f(x)$ and also a root of $f'(x)$ therefore $\sigma^*(w)$ is singular by 3.12 3) and we have only one singularity by 3.12 1) and the point is $s := (-1, 0)^T$. Using lemma 3.10 2) we get a singular point of w. The point is $\sigma(s) = (-1, -2)^T$. Combination of proposition 3.12 1) and lemma 3.10 2) says there is only one singular point of w.

(b) We will use the same steps and reasoning as in (a). Let $w := y^2 + y(2x+1) - (x^3 + 2x^2 + 2x)$. Applying lemma 3.1 ($a = 2, c = 1$):

$$A := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, b := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \sigma := \tau_{\theta_A(b)} \circ \theta_A \implies \sigma^*(f(x,y)) = f(x, 2x+y+1)$$

$\sigma^*(w) = y^2 - (x^3 + 1)$ is a short WEP. Let $f(x) := x^3 + 1$. We can guess 2 is a root of $f(x)$ and a root of $f'(x)$. Therefore $s := (2, 0)^T$ is the only singular point of $\sigma^*(w) \implies \sigma(s) = (2, 2)^T$ is the only singular point of w.

3

Let w, f be the same polynomials as in the 1st exercise (a). Then $W = (w)$. The roots of $f(x)$ are $\{-1, 1, 4\}$ therefore points $\alpha_1 := (-1, 0)^T, \alpha_2 := (1, 0)^T, \alpha_3 := (4, 0)^T \in V_w$. By lemma 4.1 $I_{\alpha_1} = (x + 1, y)$. For example $y \in I_{\alpha_1}$, y is irreducible in $\mathbb{R}[x, y]$ obviously $y \notin W$.

We can apply the same reasoning for the other 2 points:

$I_{\alpha_2} = (x - 1, y)$, for example y is irreducible and obviously $\notin W$.

$I_{\alpha_3} = (x - 4, y)$, for example y is irreducible and obviously $\notin W$.