

NMAG436 - HW3

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By calculating $\frac{\partial f_1}{\partial x}(x, y) = 3x^2 + 1 = x^2 + 1$, $\frac{\partial f_1}{\partial y}(x, y) = 2y = 0$ we see that f_1 is singular at $(1, 1)$ because $\frac{\partial f_1}{\partial x}(1, 1) = 1 + 1 = 0$, $\frac{\partial f_1}{\partial y}(1, 1) = 0$. To be able to use lemma 5.11 we need to express $L_1 = \mathbb{F}_2(\alpha, \beta)$ using a polynomial f' that is singular at $(0, 0)$. We define $f' = \tau_{(1,1)}^*(f) = y^2 + x^3 + x^2$. Now can say that L_1 is AFF over \mathbb{F}_2 given by $f'(u, t) = 0$ where $(u, t) = \tau_{(1,1)}(\alpha, \beta) = (\alpha + 1, \beta + 1)$. Now we apply lemma 5.11 that tells us that $\frac{u}{t} = \frac{\alpha+1}{\beta+1} \notin {}_{f'}\mathcal{O}_{(0,0)}$ and $\frac{t}{u} = \frac{\beta+1}{\alpha+1} \notin {}_{f'}\mathcal{O}_{(0,0)}$. We know that ${}_f\mathcal{O}_{(0,0)} = {}_f\mathcal{O}_{(1,1)}$. That means that $a := \frac{\alpha+1}{\beta+1}$ is the element we are looking for.

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Since we are in $\mathbb{F}_2 \implies |V_{f_2}(\mathbb{F}_2)| \leq 2^2$. So we can check all 4 possibilities for roots by substituting into $f_2(x, y)$ and we get that truly $V_{f_2}(\mathbb{F}_2) = \{(1, 0), (1, 1)\}$

- (a) We calculate $\frac{\partial f_2}{\partial x}(x, y) = 3x^2 = x^2$, $\frac{\partial f_2}{\partial y}(x, y) = 2y = 0$ and we see that $\frac{\partial f_2}{\partial x}(1, 0) = 1$, $\frac{\partial f_2}{\partial x}(1, 1) = 1$ so that means by definition that f_2 is smooth at $V_{f_2}(\mathbb{F}_2)$.

Using theorem 5.8 (1) we get that there exists exactly one $P \in \mathbb{P}_{L_2/\mathbb{F}_2}$ s.t. $v_P(\alpha - 1) > 0$, $v_P(\beta) > 0$. Now we can use proposition 5.13 (2) that tells us $P = P_{(1,0)}$. This gives us $P_{(1,0)} = P \in \mathbb{P}_{L_2/\mathbb{F}_2}$. Using the same reasoning $P_{(1,1)} \in \mathbb{P}_{L_2/\mathbb{F}_2}$.

- (b) By definition of $\mathcal{O}_{P_{(1,0)}}$ being a DVR, we know that there $\exists p \in P_{(1,0)} : (p) = P_{(1,0)}$. Let $P := P_{(1,0)}$. By definition of v_P we know that p is a generator of $P \iff v_P(a) = 1$ because $\forall a \in P = (p) : a = pb, b \in \mathcal{O}_P$.

By following the proof of theorem 5.8 and using the knowledge that found P from 5.8 is $P_{(1,0)}$ we'll find the element $u : v_P(u) = 1$. As in previous homeworks we calculate $(a_1, a_2) = (1, 1)$ using partial derivatives. We choose $(b_1, b_2) = (1, 0)$, that gives us $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\gamma = (1, 0)$. We now have σ and we can calculate $(u, t) = \sigma(\alpha, \beta) \implies (u, t) = (\alpha + 1, \alpha + \beta + 1)$. We now have the generator we were looking for which is $\alpha + 1$.

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- (a) $\alpha^{-5} \in P \iff v_P(\alpha^{-5}) \geq 1 \iff -5v_P(\alpha) \geq 1 \iff v_P(\alpha) < 0 \xrightarrow{5.23} \exists! P \in \mathbb{P}_{L_1/\mathbb{F}_2} : v_P(\alpha) < 0$. So the size of the set is 1. Assumptions of 5.23 are clearly satisfied and equivalences hold thanks to definitions and valuation properties.

- (b) First we will show that there does not exist a place $P \in \mathbb{P}_{L_1/\mathbb{F}_2} : \alpha \in P \wedge \deg P = 1$. For contradiction assume there is such P . Using corollary 5.17 (we have shown in 2) that we can use this) we get that either $P = P_{(1,0)}$ or $P = P_{(1,1)}$ or $\alpha^{-1} \in P$.

If $\alpha \in P \implies v_P(\alpha) \geq 1$ and if $\alpha^{-1} \in P \implies v_P(\alpha^{-1}) \geq 1 \iff v_P(\alpha) \geq -1$ which is clearly a contradiction.

So either $P = P_{(1,0)}$ or $P = P_{(1,1)}$. In exercise 2) we have shown that $P_{(1,0)}$ is generated by $\alpha + 1$ which means $P_{(1,0)} = \{(\alpha + 1)r(\alpha, \beta) | r \in R_{(1,0)}\}$. So if $\alpha \in P_{(1,0)} \implies \alpha | \alpha + 1$ which is also a contradiction.

Using the same procedure (using the same A and $\gamma = (1, 1)$) as in exercise 2) we find out that $P_{(1,1)}$ is also generated by $\alpha + 1$ ($P_{(1,1)} = \{(\alpha + 1)r(\alpha, \beta) | r \in R_{(1,1)}\}$) and we find the same contradiction.

Therefore α is not in a place of degree one. Using 4.6 we get $|L : \mathbb{F}_2(\alpha)| = \deg_y(f_2) = 2$ and now we can use proposition 5.21 which says that if there are distinct P_1, \dots, P_n where $n \in \mathbb{N}$ s.t. $v_{P_i} \geq 1 \iff \alpha \in P_i, \forall i = 1, \dots, n$ this must hold $2 \geq \sum_{i=1}^n v_{P_i} \deg P_i$ since $v_{P_i} \geq 1$ and we have shown that $\deg P_i \geq 1$ means there is at most one such P . Using observation after 5.17 and $\alpha \in L \setminus \bar{K}$ we know such place exists. Therefore the size of the set is 1.

- (c) In 2) and previous (b) we have shown that $\alpha + 1 \in P_{(1,0)}, \alpha + 1 \in P_{(1,1)}$. These places are distinct by definition. Also since $|L : \mathbb{F}_2(\alpha)| = 2$ and $\mathbb{F}_2(\alpha) = \mathbb{F}_2(\alpha + 1)$ we get $|L : \mathbb{F}_2(\alpha + 1)| = 2$. Now we use 5.21 again and we get $2 \geq v_{P_{(1,0)}}(\alpha + 1) \deg P_{(1,0)} + v_{P_{(1,1)}}(\alpha + 1) \deg P_{(1,1)} + \dots$. We know that $v_{P_{(1,0)}}(\alpha + 1) = v_{P_{(1,1)}}(\alpha + 1) = 1$ and $\deg P_i \geq 1$ since they are non trivial. Therefore there cannot be more places containing $\alpha + 1 \implies$ the size of the set is 2.
- (d) Using c) we can see that $\deg P_{(1,0)} = \deg P_{(1,1)} = 1$. Since $|V_{f_2}(\mathbb{F}_2)| = 2$ there can't be another place P of deg 1 s.t. $P = P_\gamma$ for some $\gamma \in V_{f_2}(\mathbb{F}_2)$. The only other place of degree 1 left (according to 5.17) is P s.t. $\alpha^{-1} \in P \iff -v_P(\alpha) \geq 1 \iff v_P(\alpha) \leq -1$. Corollary 5.23 tells us there is exactly one such place. Therefore the size of the set is 3.