

# NMAG436 - HW3

Jan Oupický

## 1

By calculating  $\frac{\partial f_1}{\partial x}(x, y) = 3x^2 + 1 = x^2 + 1$ ,  $\frac{\partial f_1}{\partial y}(x, y) = 2y = 0$  we see that  $f_1$  is singular at  $(1, 1)$  because  $\frac{\partial f_1}{\partial x}(1, 1) = 1 + 1 = 0$ ,  $\frac{\partial f_1}{\partial y}(1, 1) = 0$ . To be able to use lemma 5.11 we need to express  $L_1 = \mathbb{F}_2(\alpha, \beta)$  using a polynomial  $f'$  that is singular at  $(0, 0)$ . We define  $f' = \tau_{(1,1)}^*(f) = y^2 + x^3 + x^2$ . Now can say that  $L_1$  is AFF over  $\mathbb{F}_2$  given by  $f'(u, t) = 0$  where  $(u, t) = \tau_{(1,1)}(\alpha, \beta) = (\alpha + 1, \beta + 1)$ . Now we apply lemma 5.11 that tells us that  $\frac{u}{t} = \frac{\alpha+1}{\beta+1} \notin {}_{f'}\mathcal{O}_{(0,0)}$  and  $\frac{t}{u} = \frac{\beta+1}{\alpha+1} \notin {}_{f'}\mathcal{O}_{(0,0)}$ . We know that  ${}_f\mathcal{O}_{(0,0)} = {}_f\mathcal{O}_{(1,1)}$ . That means that  $a := \frac{\alpha+1}{\beta+1}$  is the element we are looking for.

## 2

Since we are in  $\mathbb{F}_2 \implies |V_{f_2}(\mathbb{F}_2)| \leq 2^2$ . So we can check all 4 possibilities for roots by substituting into  $f_2(x, y)$  and we get that truly  $V_{f_2}(\mathbb{F}_2) = \{(1, 0), (1, 1)\}$

- (a) We calculate  $\frac{\partial f_2}{\partial x}(x, y) = 3x^2 = x^2$ ,  $\frac{\partial f_2}{\partial y}(x, y) = 2y = 0$  and we see that  $\frac{\partial f_2}{\partial x}(1, 0) = 1$ ,  $\frac{\partial f_2}{\partial x}(1, 1) = 1$  so that means by definition that  $f_2$  is smooth at  $V_{f_2}(\mathbb{F}_2)$ .

Using theorem 5.8 (1) we get that there exists exactly one  $P \in \mathbb{P}_{L_2/\mathbb{F}_2}$  s.t.  $v_P(\alpha - 1) > 0$ ,  $v_P(\beta) > 0$ . Now we can use proposition 5.13 (2) that tells us  $P = P_{(1,0)}$ . This gives us  $P_{(1,0)} = P \in \mathbb{P}_{L_2/\mathbb{F}_2}$ . Using the same reasoning  $P_{(1,1)} \in \mathbb{P}_{L_2/\mathbb{F}_2}$ .

- (b) By definition of  $\mathcal{O}_{P_{(1,0)}}$  being a DVR, we know that there  $\exists p \in P_{(1,0)} : (p) = P_{(1,0)}$ . Let  $P := P_{(1,0)}$ . By definition of  $v_P$  we know that  $p$  is a generator of  $P \iff v_P(a) = 1$  because  $\forall a \in P = (p) : a = pb, b \in \mathcal{O}_P$ .

By following the proof of theorem 5.8 and using the knowledge that found  $P$  from 5.8 is  $P_{(1,0)}$  we'll find the element  $u : v_P(u) = 1$ . As in previous homeworks we calculate  $(a_1, a_2) = (1, 1)$  using partial derivatives. We choose  $(b_1, b_2) = (1, 0)$ , that gives us  $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  and  $\gamma = (1, 0)$ . We now have  $\sigma$  and we can calculate  $(u, t) = \sigma(\alpha, \beta) \implies (u, t) = (\alpha + 1, \alpha + \beta + 1)$ . We now have the generator we were looking for which is  $\alpha + 1$ .

## 3

- (a)  $\alpha^{-5} \in P \iff v_P(\alpha^{-5}) \geq 1 \iff -5v_P(\alpha) \geq 1 \iff v_P(\alpha) < 0 \xrightarrow{5.23} \exists! P \in \mathbb{P}_{L_1/\mathbb{F}_2} : v_P(\alpha) < 0$ . So the size of the set is 1. Assumptions of 5.23 are clearly satisfied and equivalences hold thanks to definitions and valuation properties.

- (b) First we will show that there does not exist a place  $P \in \mathbb{P}_{L_1/\mathbb{F}_2} : \alpha \in P \wedge \deg P = 1$ . For contradiction assume there is such  $P$ . Using corollary 5.17 (we have shown in 2) that we can use this) we get that either  $P = P_{(1,0)}$  or  $P = P_{(1,1)}$  or  $\alpha^{-1} \in P$ .

If  $\alpha \in P \implies v_P(\alpha) \geq 1$  and if  $\alpha^{-1} \in P \implies v_P(\alpha^{-1}) \geq 1 \iff v_P(\alpha) \leq -1$  which is clearly a contradiction.

So either  $P = P_{(1,0)}$  or  $P = P_{(1,1)}$ . In exercise 2) we have shown that  $P_{(1,0)}$  is generated by  $\alpha + 1$  which means  $P_{(1,0)} = \{(\alpha + 1)r(\alpha, \beta) | r \in R_{(1,0)}\}$ . So if  $\alpha \in P_{(1,0)} \implies \alpha + 1 | \alpha$  which is also a contradiction.

Using the same procedure (using the same  $A$  and  $\gamma = (1, 1)$ ) as in exercise 2) we find out that  $P_{(1,1)}$  is also generated by  $\alpha + 1$  ( $P_{(1,1)} = \{(\alpha + 1)r(\alpha, \beta) | r \in R_{(1,1)}\}$ ) and we find the same contradiction.

Therefore  $\alpha$  is not in a place of degree one. Using 4.6 we get  $|L : \mathbb{F}_2(\alpha)| = \deg_y(f_2) = 2$  and now we can use proposition 5.21 which says that if there are distinct  $P_1, \dots, P_n$  where  $n \in \mathbb{N}$  s.t.  $v_{P_i}(\alpha) \geq 1 \iff \alpha \in P_i, \forall i = 1, \dots, n$  this must hold  $2 \geq \sum_{i=1}^n v_{P_i}(\alpha) \deg P_i$  since  $v_{P_i} \geq 1$  and we have shown that  $\deg P_i \geq 1$  means there is at most one such  $P$ . Using observation after 5.17 and  $\alpha \in L \setminus \tilde{K}$  we know such place exists. Therefore the size of the set is 1.

- (c) In 2) and previous (b) we have shown that  $\alpha + 1 \in P_{(1,0)}, \alpha + 1 \in P_{(1,1)}$ . These places are distinct by definition. Also since  $|L : \mathbb{F}_2(\alpha)| = 2$  and  $\mathbb{F}_2(\alpha) = \mathbb{F}_2(\alpha + 1)$  we get  $|L : \mathbb{F}_2(\alpha + 1)| = 2$ . Now we use 5.21 again and we get  $2 \geq v_{P_{(1,0)}}(\alpha + 1) \deg P_{(1,0)} + v_{P_{(1,1)}}(\alpha + 1) \deg P_{(1,1)} + \dots$ . We know that  $v_{P_{(1,0)}}(\alpha + 1) = v_{P_{(1,1)}}(\alpha + 1) = 1$  and  $\deg P_i \geq 1$  since they are non trivial. Therefore there cannot be more places containing  $\alpha + 1 \implies$  the size of the set is 2.
- (d) Using c) we can see that  $\deg P_{(1,0)} = \deg P_{(1,1)} = 1$ . Since  $|V_{f_2}(\mathbb{F}_2)| = 2$  there can't be another place  $P$  of deg 1 s.t.  $P = P_\gamma$  for some  $\gamma \in V_{f_2}(\mathbb{F}_2)$ . The only other place of degree 1 left (according to 5.17) is  $P$  s.t.  $\alpha^{-1} \in P \iff -v_P(\alpha) \geq 1 \iff v_P(\alpha) \leq -1$ . Corollary 5.23 tells us there is exactly one such place. Therefore the size of the set is 3.