Hyperbolic functions

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1 Basics

Definition

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\coth(x) = \frac{1}{\tanh(x)}$$

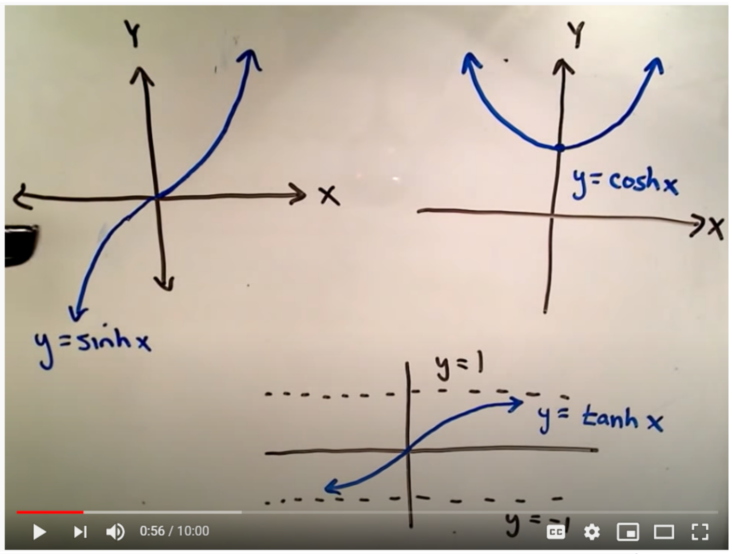
Examples

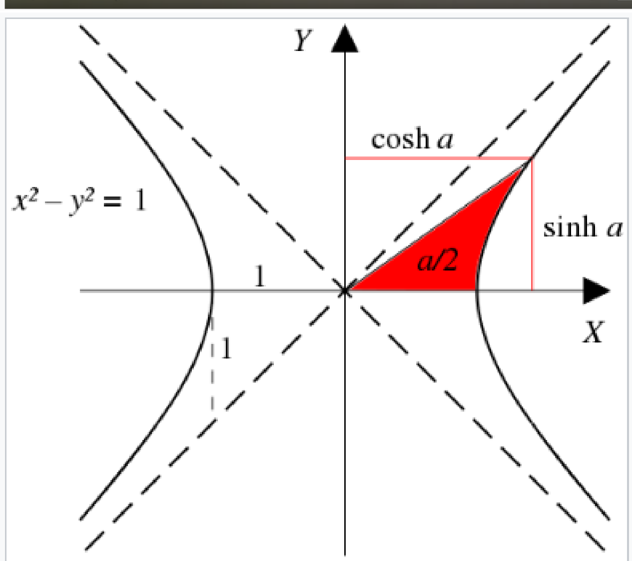
$$\sinh(0) = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$\cosh(0) = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$$

$$\tanh(1) = \frac{\sinh(1)}{\cosh(1)} = \frac{e^1 - e^{-1}}{e^1 + e^{-1}} = \frac{e^2 - 1}{e^2 + 1}$$

Graph





Functions

$$\sinh(2x) = 2 \cdot \sinh(x) \cdot \cosh(x)$$

$$\frac{e^{2x} - e^{-2x}}{2} = 2 \cdot (\frac{e^{2x} - e^{-2x}}{4}) = 2 \cdot (\frac{e^{x} - e^{-x}}{2} \cdot \frac{e^{x} + e^{-x}}{2}) = 2 \cdot \sinh(x) \cdot \cosh(x)$$

2 Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos -\theta + i\sin -\theta = \cos\theta - i\sin\theta$$
$$\cos\theta = \tfrac{e^{i\theta}+e^{-i\theta}}{2}$$
$$\sin\theta = \tfrac{e^{i\theta}-e^{-i\theta}}{2i}$$

Let

$$\theta = it$$

then,

$$\cos(it) = \tfrac{e^{t}+e^{-t}}{2}$$

and

$$\sin(it) = \tfrac{e^{-t}-e^{t}}{2i}, -i\sin(it) = \tfrac{e^{t}-e^{-t}}{2}$$

Set

$$x = \tfrac{e^{t}+e^{-t}}{2}, y = \tfrac{e^{t}-e^{-t}}{2}$$

Then, we get

$$x^2 - y^2 = 1$$

We can say that

$$\cosh t = \frac{e^t + e^t}{2} = \cos(it)$$
$$\sinh t = \frac{e^t - e^{-t}}{2} = \sin(it)$$

3 Derivatives

$$\frac{d}{dx}\sinh(x) = \cosh(x)$$
$$\frac{d}{dx}\cosh(x) = \sinh(x)$$
$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$
$$\frac{d}{dx}\operatorname{csch}(x) = -\operatorname{csch}(x)\coth(x)$$
$$\frac{d}{dx}\operatorname{sech}(x) = -\operatorname{sech}(x)\tanh(x)$$
$$\frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$$

Example Proofs

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

Example Problems

$$f(x) = \tanh(4x)$$
$$f'(x) = 4(\text{sech}^2(4x))$$

slightly harder

$$f(x) = \ln(\sinh(x))$$
$$f'(x) = \frac{\cosh(x)}{\sinh(x)} = \coth(x)$$

another one

$$f(x) = \sinh(x)\tanh(x)$$
$$f'(x) = (\cosh(x))(\tanh(x)) + \sinh(x) \cdot \text{sech}^2(x) = \cosh(x) \cdot \frac{\sinh(x)}{\cosh(x)} + \sinh(x) \cdot \text{sech}^2(x)$$
$$= \sinh(x) + \sinh(x) \cdot \text{sech}^2(x) = \sinh(x)(1 + \text{sech}^2(x))$$

4 Integrals

$$\int \sinh(ax)dx = a^{-1}\cosh(ax) + C$$
$$\int \cosh(ax)dx = a^{-1}\sinh(ax) + C$$
$$\int \tanh(ax)dx = a^{-1}ln(\cosh(ax)) + C$$
$$\int \coth(ax)dx = a^{-1}ln(\sinh(ax)) + C$$
$$\int \text{sech}(ax)dx = a^{-1}\arctan(\sinh(ax)) + C$$
$$\int \text{csch}(ax)dx = a^{-1}ln(\tanh(\frac{ax}{2})) + C = a^{-1}ln|\text{csch}(ax) - \coth(ax)| + C$$
$$\int \frac{1}{\sqrt{a^2+u^2}}du = \text{arsinh}(\frac{u}{a}) + C$$
$$\int \frac{1}{\sqrt{a^2+u^2}}du = \text{arcosh}(\frac{u}{a}) + C$$
$$\int \frac{1}{a^2-u^2}du = a^{-1}artanh(\frac{u}{a}) + C \qquad u^a 2 < a^2$$
$$\int \frac{1}{u^2-a^2}du = a^{-1}arcoth(\frac{u}{a}) + C \qquad u^a 2 > a^2$$
$$\int \frac{1}{u\sqrt{a^2-u^2}}du = -a^{-1}arsech(\frac{u}{a}) + C$$
$$\int \frac{1}{u\sqrt{a^2+u^2}}du = -a^{-1}\text{arcsch}(\frac{u}{a}) + C$$

5 Taylor series expressions

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdot = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdot = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

6 Inverse Hyperbolic Functions

$$\operatorname{arsinh}(x) = \ln(x + \sqrt{x^2+1})$$

$$\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2-1}) \qquad x \geq 1$$

$$\operatorname{artanh}(x) = \frac{1}{2}\ln(\frac{1+x}{1-x}) \qquad |x| < 1$$

$$\operatorname{arcoth}(x) = \frac{1}{2}\ln(\frac{x+1}{x-1}) \qquad |x| > 1$$

$$\operatorname{arsech}(x) = \ln(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}) \qquad 0 < x \leq 1$$

$$\operatorname{arcsch}(x) = \ln(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}) \qquad x \neq 0$$

Derivatives

$$\frac{d}{dx}\operatorname{arsinh}(x) = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\operatorname{arcosh}(x) = \frac{1}{\sqrt{x^2-1}} \qquad 1 < x$$

$$\frac{d}{dx}\operatorname{artanh}(x) = \frac{1}{1-x^2} \qquad |x| < 1$$

$$\frac{d}{dx}\operatorname{arcoth}(x) = \frac{1}{x^2-1} \qquad |x| > 1$$

$$\frac{d}{dx}\operatorname{arsech}(x) = -\frac{1}{x\sqrt{1-x^2}} \qquad 0 < x < 1$$

$$\frac{d}{dx}\operatorname{arcsch}(x) = -\frac{1}{|x|\sqrt{1-x^2}} \qquad x \neq 0$$

Example Problems

$$f(x) = \operatorname{artanh}(\sqrt{x})$$

$$f'(x) = \frac{1}{2\sqrt{x}} \frac{1 - (\sqrt{x})^2}{=} \frac{1}{2\sqrt{x}(1-x)}$$

another problem

$$f(x) = x^2 \cdot \operatorname{arsinh}(2x)$$

$$f'(x) = 2x \operatorname{arsinh}(2x) + x^2 \cdot 2 \cdot \frac{1}{(\sqrt{(2x)^2 + 1})} = 2x(\operatorname{arsinh}(2x) + \frac{x}{\sqrt{1 + 4x^2}})$$