



Deloitte  
Data Science  
Academy

## Lesson 4: **Optimization**

Optimization and its use cases, Linear Programming and Integer Linear Programming, Branch and Bound algorithm

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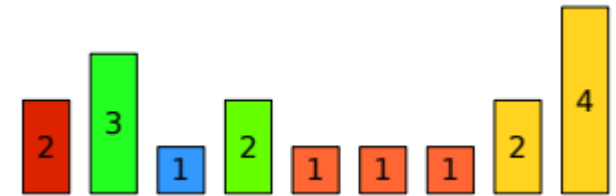


# Optimization

- **Optimization** is a term for the mathematical discipline that is concerned with the minimization/maximization of some **objective function** subject to **constraints** or decision that no solution exists.
- Combinatorics is the mathematics of discretely structured problems.
- Combinatorial optimization is an optimization that deals with discrete variables.
- Typical application areas:
  - Production (production speed up, cost reduction, efficient utilization of resources...)
  - Transportation (fuel saving, reduction of delivery time...)
  - Employees scheduling (reduction of human resources...)
  - Hardware design (acceleration of computations...)
  - Communication network design (end-to-end delay reduction)

# Problems

- **Bin packing problem:**
  - Objects of different sizes and containers of given capacity
- **Container loading:**
  - To store as much boxes as possible in a container
  - Constraints – size, capacity, loading process, stability, orientation,...
- **Assignments of shifts of employees, Project Scheduling:**
  - **Nurses scheduling**
    - Demand, holidays, regulation, personal preferences, fairness, ...
- **Project scheduling**
  - When should each activity of a project start, resources constraints, precedence, min-max time, etc.



# Problems

- **Route planning in automated warehouse:**
  - Can be formulated as assymmetric TSP
- **Computer graphics – coloring (Sýkora)**
  - Multiway cut problem
- **Supply network management:**
  - Given a prediction of customers demands, the aim is to plan production and transport of products in order to maximize the profit.
- **City planning**
  - New Fire department, EV charging station, etc..



# Why bother?

Typical goals of optimization:

- automation of the design/decision process
- increase the volume of the production (shorter production-line cycle)
- cost reduction (fuel saving, less machines)
- risk reduction (error elimination due to automated creation of production schedule)
- lean manufacturing (supply and stores reduction, outgrowths reduction when delay in supply)
- increase of the flexibility (faster reaction to structure or constraint change)
- user-friendly solutions (balanced schedule for all employees)

# Linear Programming (LP)

Let's state the problem of Linear Programming (LP).  
In the general form of LP, we are given

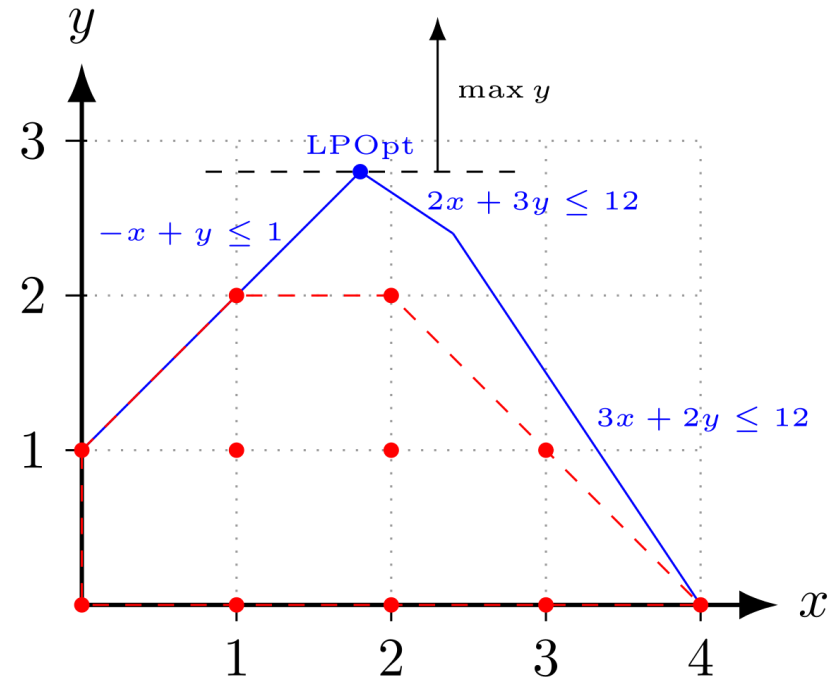
$$A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m, \mathbf{c} \in \mathbb{R}^n \text{ and } \mathbf{x} = [x_1, \dots, x_n] \in \mathbb{R}^n$$

And we have to minimize the objective function

$$\min \mathbf{c}^T \mathbf{x}$$

such that

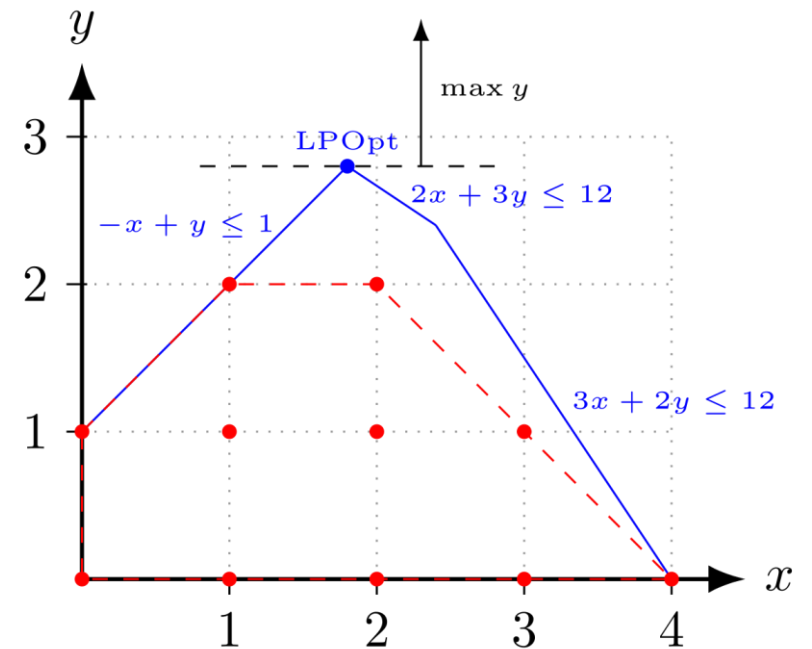
$$A\mathbf{x} \geq \mathbf{b}$$



# Integer Linear Programming (ILP/MIP/MILP)

Very similar to Linear Programming – only difference is that some or all of the  $x$  values can now be integers.

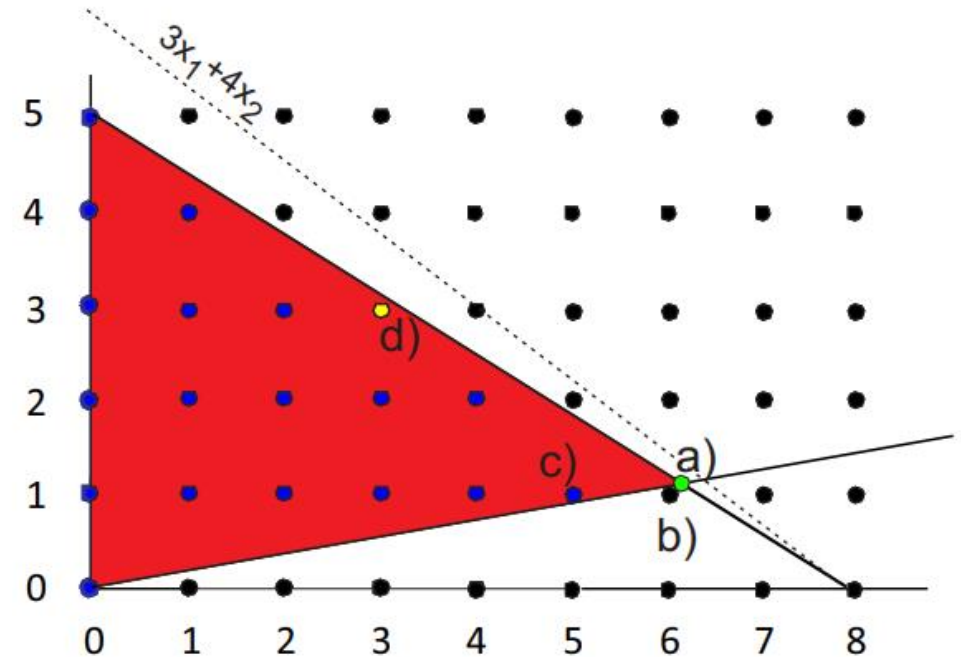
- It is not possible to just solve LP and round solution – we can arrive at suboptimal solution or not feasible one.
- ILP is not a convex set
- NP hard



# ILP vs LP caveats

$$\begin{aligned}\max z &= 3x_1 + 4x_2 \\ \text{s.t. } 5x_1 + 8x_2 &\leq 40 \\ x_1 - 5x_2 &\leq 0\end{aligned}$$

- a) LP solution  $z = 23.03$  for  $x_1 = 6.06, x_2 = 1.21$
- b) Rounding leads to infeasible solution  $x_1 = 6, x_2 = 1$
- c) Nearest feasible integer is not optimal  $z = 19$  for  $x_1 = 5, x_2 = 1$
- d) Optimal solution is  $z = 21$  for  $x_1 = 3, x_2 = 3$





# Knapsack problem

Given a set of items, each with a weight and a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible.

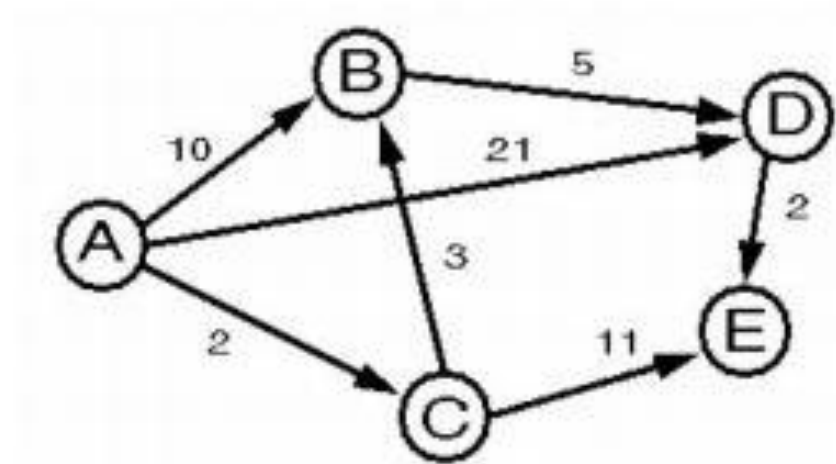
$$\begin{aligned} & \max \sum_i x_i \cdot v_i \\ & s. t. \sum_i x_i w_i \leq W \\ & x_i \in \{0,1\}, v_i \in R, w_i \in R \end{aligned}$$



# Shortest path in directed graph

Find shortest path from  $s$  to  $t$  or decide that  $t$  is unreachable from  $s$ .

$$\begin{aligned} & \max l_t \\ \text{subject to} \\ & l_s = 0 \\ & l_j \leq l_i + c_{i,j} \text{ for } i \in 1 \dots n, j \in 1 \dots n \\ & l_i \in R^{+,0}, c_{i,j} \in R^{+,0}, n \in Z^{+,0} \end{aligned}$$



# Traveling Salesman Problem

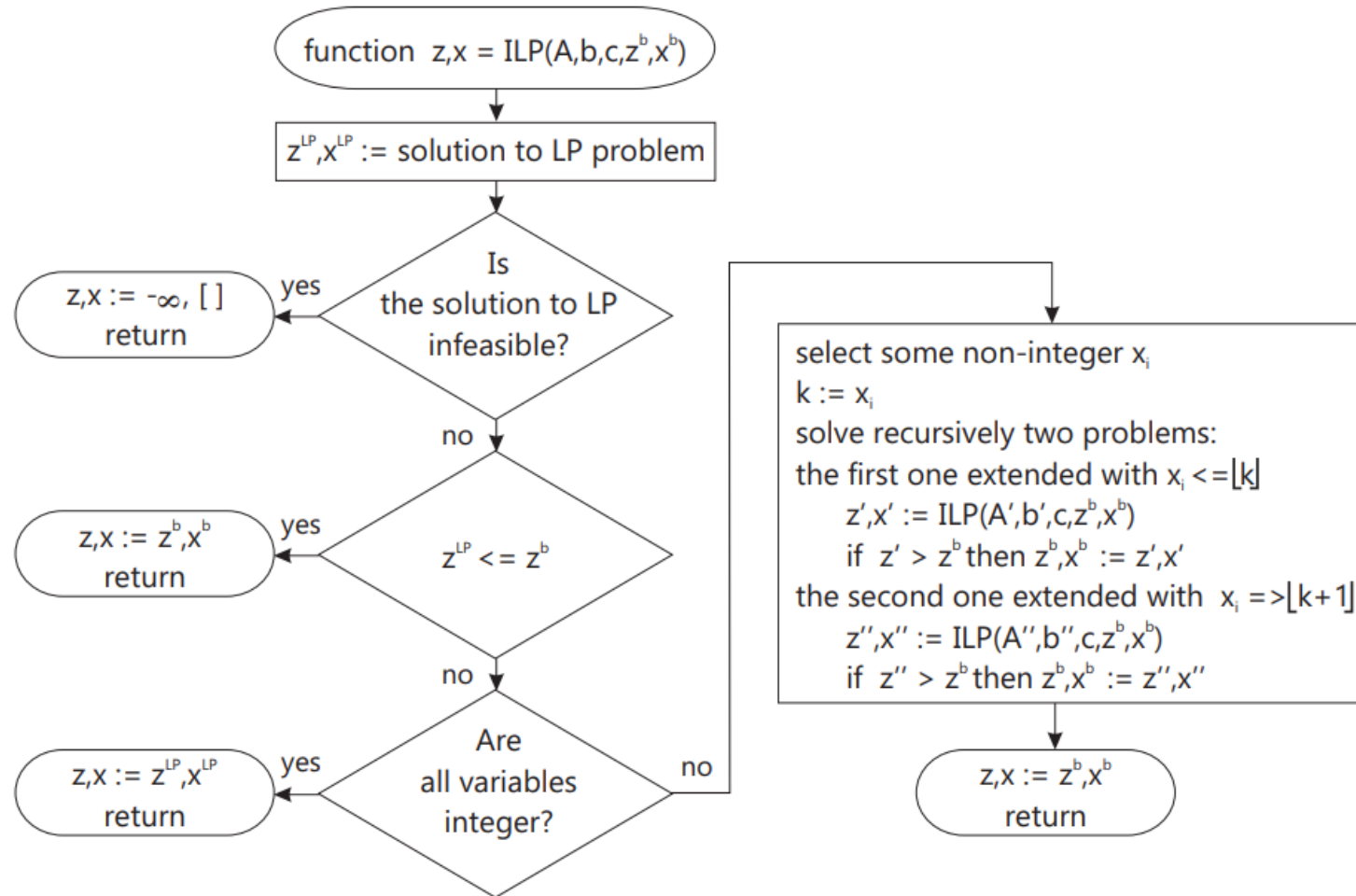
With complete directed graph and distance matrix, determine the order of visits of individual nodes such the overall price is minimized.

$$\begin{aligned} \min \quad & \sum_{i \in 1..n} \sum_{j \in 1..n} c_{i,j} * x_{i,j} \\ \text{subject to:} \quad & x_{i,i} = 0 \quad i \in 1..n \quad \text{avoid self-loop} \\ & \sum_{i \in 1..n} x_{i,j} = 1 \quad j \in 1..n \quad \text{enter once} \\ & \sum_{j \in 1..n} x_{i,j} = 1 \quad i \in 1..n \quad \text{leave once} \\ & s_i + c_{i,j} - (1 - x_{i,j}) * M \leq s_j \quad i \in 1..n, j \in 2..n \quad \text{cycle indivisibility} \\ \text{parameters:} \quad & M \in \mathbb{Z}_0^+, n \in \mathbb{Z}_0^+, c_{i \in 1..n, j \in 1..n} \in \mathbb{Q}^+ \\ \text{variables:} \quad & x_{i \in 1..n, j \in 1..n} \in \{0, 1\}, s_{i \in 1..n} \in \mathbb{R}_0^+ \end{aligned}$$

# Algorithm for solving ILP - Branch & Bound

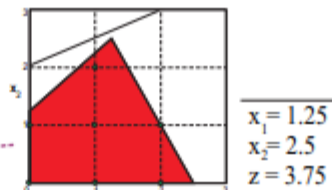
- The method is based on **splitting** the solution space into disjoint sets.
- It starts by relaxing on the integrality of the variables and **solves the LP problem**.
- If all variables  $x_i$  are integers, the computation ends. Otherwise one variable  $x_i \notin \mathbb{Z}$  is chosen and its value is assigned to  $k$ .
- Then the solution space is **divided into two sets** - in the first one we consider  $x_i \leq \lfloor k \rfloor$  and in the second one  $x_i \geq \lfloor k \rfloor + 1$ .
- The algorithm **recursively repeats** computation for the both new sets till feasible integer solution is found.

# Algorithm for solving ILP - Branch & Bound

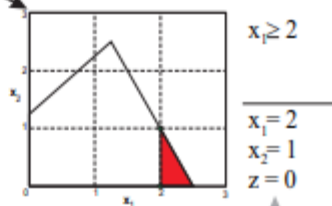
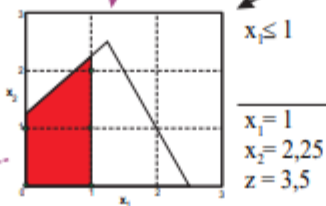


$$\begin{aligned} \max \quad & -x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 5 \\ & -4x_1 + 4x_2 \leq 5 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{Z} \end{aligned}$$

Search direction



$$x_1 \leq 1 \quad x_1 \geq 2$$

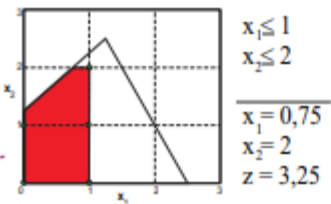


The third feasible solution

Infeasible solution

It is not needed to continue since  $z < z^*$ .

Since the search space has no other solution with  $z > z^*$ , algorithm terminates.

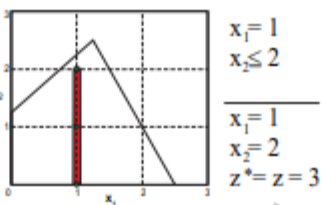
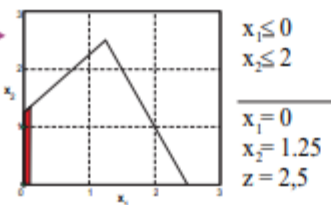


Infeasible solution

It is not needed to continue since  $z < z^*$ .

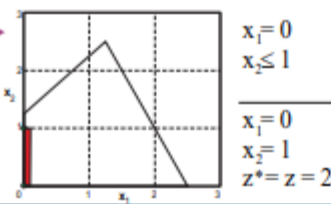
Since the search space has no other solution with  $z > z^*$ , algorithm terminates.

$$x_1 \leq 0 \quad x_1 \geq 1$$



The second feasible solution with a better value

$$x_2 \leq 1 \quad x_2 \geq 2$$



Infeasible solution

The first feasible solution

# Example for formulation

	T-shirt	Shirt	trousers	Capacity
Labor	3	2	6	150
Material	4	3	4	160
income	6	4	7	

- Goal: maximize the income (i.e. total income minus total expenses)
- Constraints: labor capacity is 150 person-hours material capacity is 160 meters
- **TODO: Formulate the problem with ILP**

# Example for formulation

	T-shirt	Shirt	trousers	Capacity
Labor	3	2	6	150
Material	4	3	4	160
income	6	4	7	

**Part 2:** Fixed cost has to be covered to rent the machine product. If at least 1 product is made, the rent has to be paid (will affect the objective function)

	T-shirt	Shirt	trousers
Machine Rent	200	150	100

- **TODO: Formulate the problem with ILP**



# Scheduling Problem – Simplified

You have set of engagements - E, weeks - W and people for a given level – P. Assign people to individual weeks of engagements based on their availability, experience and preference of the engagement managers.



# Optimization hands-on

MIP





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