(3) $L(U,V) = -\frac{1}{2c}U^TU - \frac{1}{2c}V^TV - \frac{1}{2\sigma^2}\sum_{(i,j)\in\Omega}\left(u_i^T v_j v_j^T u_i - 2u_i^T v_j E_q[\phi_{ij}]\right) + \text{const}$

$\nabla L(u_i) = 0 \quad \Rightarrow \quad u_i = \arg\max_{u_i} L(u_i)$

$\Rightarrow u_i = \left(\frac{1}{c}I + \sum_{j=1}^{M} v_j v_j^T/\sigma^2\right)^{-1}\left(\sum_{j=1}^{M} v_j E_q[\phi_{ij}]/\sigma^2\right)$

$\nabla L(v_j) = 0 \quad \Rightarrow \quad v_j = \arg\max_{v_j} L(v_j)$

$\Rightarrow v_j = \left(\frac{1}{c}I + \sum_{i=1}^{N} u_i u_i^T/\sigma^2\right)^{-1}\left(\sum_{i=1}^{N} u_i E_q[\phi_{ij}]/\sigma^2\right)$

$(U, V) = (\{u_i\}, \{v_j\})$.

(4)

1. Initialize $U_0$, $V_0$

2. For iteration $t = 1, \dots, T$.

(a) E-step: Calculate $\vec{E}_{q_t}[\phi] = \{E_{q_t}[\phi_{ij}]\}_{(i,j)\in\Omega}$

$$E_{q_t}[\phi_{ij}] = \begin{cases} u_{t-1}^T v_{t-1} + \sigma \times \dfrac{\Phi'(-u_{t-1}^T v_{t-1}/\sigma)}{1-\Phi(-u_{t-1}^T v_{t-1}/\sigma)} & \text{if } r_{ij} = 1 \\ u_{t-1}^T v_{t-1} + \sigma \times \dfrac{-\Phi'(-u_{t-1}^T v_{t-1}/\sigma)}{\Phi(-u_{t-1}^T v_{t-1}/\sigma)} & \text{if } r_{ij} = -1 \end{cases}$$

(b) M-step: update $U_t$ and $V_t$ using following equation:

$$\begin{cases} U_t = \left(\frac{1}{c}I + \sum_{j=1}^{M} v_j v_j^T/\sigma^2\right)^{-1}\left(\sum_{j=1}^{M} v_j E_q[\phi_{ij}]/\sigma^2\right) \\ V_t = \left(\frac{1}{c}I + \sum_{i=1}^{N} u_i u_i^T/\sigma^2\right)^{-1}\left(\sum_{i=1}^{N} u_i E_q[\phi_{ij}]/\sigma^2\right) \end{cases}$$

(c). Calculate $\ln p(R,U,V)$ using the following equation:

$\cancel{\ln P(R,U,V) = -\frac{c}{2}U^TU - \frac{c}{2}V^TV - \frac{1}{2\sigma^2}\sum_{(i,j)\in\Omega}(}$

$$\ln P(R,U,V) = \frac{d}{2}\ln\left(\frac{c}{2\pi}\right) - \frac{c}{2}U^TU - \frac{c}{2}V^TV + \sum_{(i,j)\in\Omega} r_{ij}\ln\Phi\left(\frac{u_i^Tv_j}{\sigma}\right) + \sum_{(i,j)\in\Omega}\left(\frac{1-r_{ij}}{2}\right)\ln\left(1-\Phi\left(\frac{u_i^Tv_j}{\sigma}\right)\right)$$

(3). $L(U,V) = -\frac{1}{2c}U^TU - \frac{1}{2c}V^TV - \frac{1}{2\sigma^2}\sum_{(i,j)\in\Omega}(u_i^T v_j v_j^T u_i - 2u_i^T v_j E_q(\phi_{ij})) + const$

$\nabla L(u_i) = 0 \Rightarrow u_i = \arg\max_{u_i} L(u_i)$

$\Rightarrow u_i = (\frac{1}{c}I + \sum_{j=1}^{M} v_j v_j^T/\sigma^2)^{-1}(\sum_{j=1}^{M} v_j E_q(\phi_{ij})/\sigma^2)$

$\nabla L(v_j) = 0 \Rightarrow v_j = \arg\max_{v_j} L(v_j)$

$\Rightarrow v_j = (\frac{1}{c}I + \sum_{i=1}^{N} u_i u_i^T/\sigma^2)^{-1}(\sum_{i=1}^{N} u_i E_q(\phi_{ij})/\sigma^2)$

$(U, V) = (\{u_i\}, \{v_j\})$.

(4)

1. Initialize $u_0$, $v_0$

2. For iteration $t = 1, \dots, T$.

(a). E-step: Calculate $\vec{E}_{q_t}[\phi] = \{ E_{q_t}[\phi_{ij}]\}_{(i,j)\in\Omega}$

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(b) M-step: update $u_t$ and $v_t$ using following equation:

$$\begin{cases} u_t = (\frac{1}{c}I + \sum_{j=1}^{M} v_j v_j^T/\sigma^2)^{-1}(\sum_{j=1}^{M} v_j E_q(\phi_{ij})/\sigma^2) \\ v_t = (\frac{1}{c}I + \sum_{i=1}^{N} u_i u_i^T/\sigma^2)^{-1}(\sum_{i=1}^{N} u_i E_q(\phi_{ij})/\sigma^2) \end{cases}$$

(c). Calculate $\ln p(R,U,V)$ using the following equation:

~~$\ln P(R,U,V) = -\frac{c}{2}U^TU - \frac{c}{2}V^TV - \frac{1}{2\sigma^2}\sum_{(i,j)\in\Omega} c$~~

$$\ln P(R,U,V) = \frac{d}{2}\ln(\frac{1}{2\pi}) - \frac{1}{2c}U^TU - \frac{1}{2c}V^TV + \sum_{(i,j)\in\Omega} r_{ij}\ln\Phi(\frac{u_i^Tv_j}{\sigma}) + \sum_{(i,j)\in\Omega}(-r_{ij})\ln(1-\Phi(\frac{u_i^Tv_j}{\sigma}))$$