1 Answer:

(1) $\phi_{ij} \sim \text{Normal}(u_i^T v_j, \sigma^2)$. $r_{ij} \sim \text{sign}(\phi_{ij})$

$$q(\phi) = p(\phi_{ij}|R,u,v) = \frac{p(r_{ij}|\phi_{ij})\cdot p(\phi_{ij}|u_i,v_j)}{\int p(r_{ij}|\phi_{ij})\cdot p(\phi_{ij}|u_i,v_j)\,d\phi_{ij}}$$

$$\propto \text{sign}(\phi_{ij})\cdot \text{Normal}(u_i^T v_j, \sigma^2)$$

$$\underset{\phi_{ij}}{\cancel{q(\phi_{ij})}} \sim TN(u_i^T v_j, \sigma^2) \quad \begin{cases} \text{when } r_{ij}=1, & \overset{\phi_{ij}}{\cancel{q(\phi_{ij})}} \sim \text{TNormal}(u_i^T v_j, \sigma^2) \text{ on } R^+ \\ \text{when } r_{ij}=-1, & \overset{\phi_{ij}}{\cancel{q(\phi_{ij})}} \sim \text{TNormal}(u_i^T v_j, \sigma^2) \text{ on } R^- \end{cases}$$

(2) $$\overset{p}{\cancel{q}}(\phi|\vec{r},u,v) = \frac{\prod_{(i,j)\in\Omega} p(r_{ij}|u_i,v_j,\phi_{ij})\cdot p(\phi_{ij}|u_i,v_j)}{\prod_{(i,j)\in\Omega}\int p(r_{ij}|u_i,v_j,\phi_{ij})\cdot p(\phi_{ij}|u_i,v_j)\,d\phi_{ij}}$$

$$= \prod_{(i,j)\in\Omega} \frac{p(r_{ij}|u_i,v_j,\phi_{ij})\cdot p(\phi_{ij}|u_i,v_j)}{\int p(r_{ij}|u_i,v_j,\phi_{ij})\cdot p(\phi_{ij}|u_i,v_j)\,d\phi_{ij}}$$

$$= \prod_{(i,j)\in\Omega} q(\phi_{ij}).$$

(2) $$P(R,u,v,\phi) = p(u)\,p(v)\,P(R,\phi)$$
$$= p(u)\,p(v)\prod_{(i,j)\in\Omega} p(r_{ij}|\phi_{ij})\cdot p(\phi_{ij}|u_i,v_j)$$

$$L(u,v) = \ln p(u) + \ln p(v) + \sum_{(i,j)\in\Omega} E_q[\ln \text{sign}(\phi_{ij})] - \frac{1}{2\sigma^2}E_q[(\phi_{ij} - u_i^T v_j)^2] + \text{const}$$

$$= -\sum_{i=1}^{N}\frac{1}{2}u_i^T u_i - \sum_{j=1}^{M}\frac{1}{2}v_j^T v_j - \sum_{(i,j)\in\Omega}\frac{1}{2\sigma^2}\left(u_i^T v_j v_j^T u_i - 2u^T v\, E_q(\phi_{ij})\right) + \text{const}.$$

$$E_q(\phi_{ij}) = \begin{cases} u_i^T v_j + \sigma \times \dfrac{\Phi'(-u_i^T v_j/\sigma)}{1-\Phi(-u_i^T v_j/\sigma)} & \text{if } r_{ij} = 1 \\ u_i^T v_j + \sigma \times \dfrac{-\Phi'(-u_i^T v_j/\sigma)}{\Phi(-u_i^T v_j/\sigma)} & \text{if } r_{ij} = -1 \end{cases}$$

$\Phi'(s)$ is the probability density function of a Normal(0,1).

$\Phi(s)$ is the CDF of Normal(0,1).

(3). $L(U,V) = -\frac{1}{2c}U^TU - \frac{1}{2c}V^TV - \frac{1}{2\sigma^2}\sum_{(i,j)\in\Omega}(u_i^T v_j v_j^T u_i - 2u_i^T v_j E_q(\phi_{ij})) + \text{const}$

$\nabla L(u_i) = 0 \Rightarrow u_i = \arg\max_{u_i} L(u_i)$

$\Rightarrow u_i = (\frac{1}{c}I + \sum_{j=1}^{M} v_j v_j^T/\sigma^2)^{-1}(\sum_{j=1}^{M} v_j E_q(\phi_{ij})/\sigma^2)$

$\nabla L(v_j) = 0 \Rightarrow v_j = \arg\max_{v_j} L(v_j)$

$\Rightarrow v_j = (\frac{1}{c}I + \sum_{i=1}^{N} u_i u_i^T/\sigma^2)^{-1}(\sum_{i=1}^{N} u_i E_q(\phi_{ij})/\sigma^2)$

$(U, V) = (\{u_i\}, \{v_j\})$.

(4)

1. Initialize u_0, v_0

2. For iteration $t = 1, \dots, T$

(a). E-step: Calculate $\overrightarrow{E_{q_t}}[\phi] = \{E_{q_t}[\phi_{ij}]\}_{(i,j)\in\Omega}$

$$E_{q_t}[\phi_{ij}] = \begin{cases} u_{t-1}^T v_{t-1} + \sigma \times \dfrac{\Phi'(-u_{t-1}^T v_{t-1}/\sigma)}{1-\Phi(-u_{t-1}^T v_{t-1}/\sigma)} & \text{if } r_{ij} = 1 \\ u_{t-1}^T v_{t-1} + \sigma \times \dfrac{-\Phi'(-u_{t-1}^T v_{t-1}/\sigma)}{\Phi(-u_{t-1}^T v_{t-1}/\sigma)} & \text{if } r_{ij} = -1 \end{cases}$$

(b) M-step: update u_t and v_t using following equation:

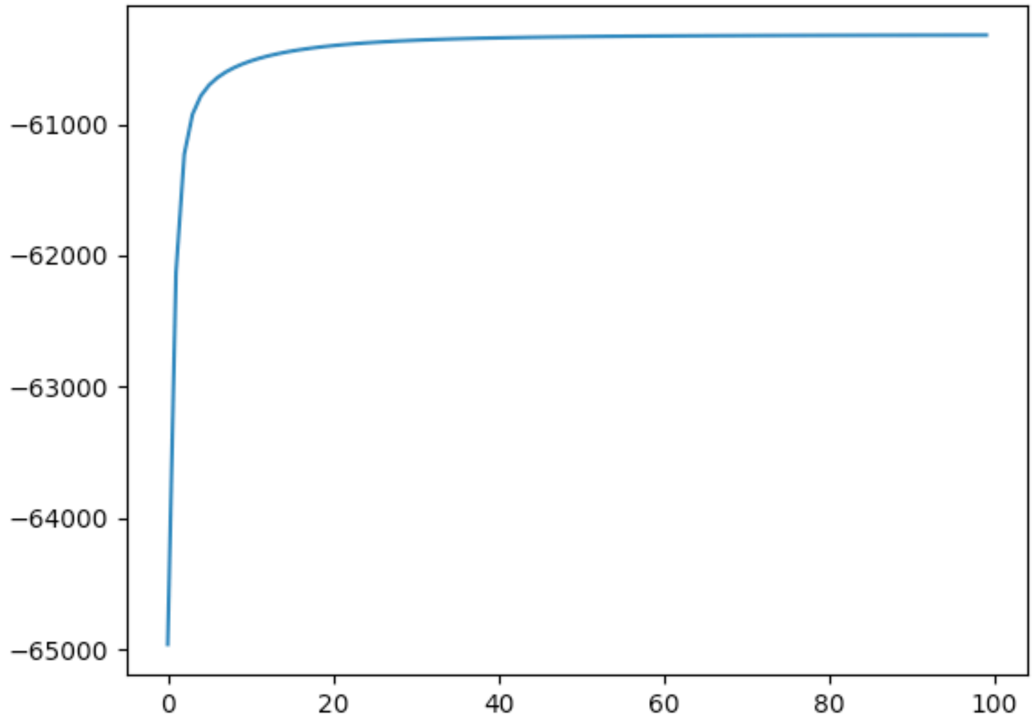
$$\begin{cases} u_t = (\frac{1}{c}I + \sum_{j=1}^{M} v_j v_j^T/\sigma^2)^{-1}(\sum_{j=1}^{M} v_j E_q(\phi_{ij})/\sigma^2) \\ v_t = (\frac{1}{c}I + \sum_{i=1}^{N} u_i u_i^T/\sigma^2)^{-1}(\sum_{i=1}^{N} u_i E_q(\phi_{ij})/\sigma^2) \end{cases}$$

(c). Calculate $\ln p(R,U,V)$ using ~~the~~ following equation:

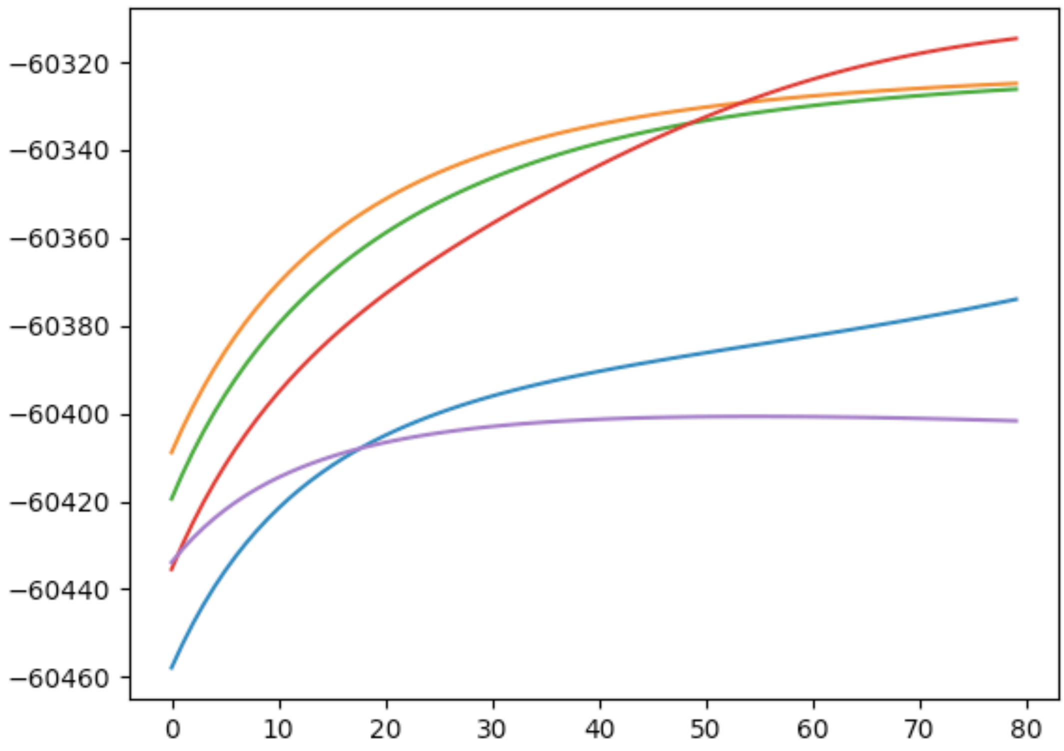
$\cancel{\ln P(R,U,V) = -\frac{c}{2}U^TU - \frac{c}{2}V^TV \ \frac{1}{2\sigma^2}\sum_{(i,j)\in\Omega}}$

$$\ln P(R,U,V) = \frac{d}{2}\ln(\frac{1}{2\pi}) - \frac{1}{2c}U^TU - \frac{1}{2c}V^TV + \sum_{(i,j)\in\Omega} r_{ij}\ln\Phi(\frac{u_i^T v_j}{\sigma}) + \sum_{(i,j)\in\Omega}(\frac{-r_{ij}}{\cancel{2}})\ln(1-\Phi(\frac{u_i^T v_j}{\sigma}))$$

1. Run your algorithm for 100 iterations and plot ln p(R;U;V) for iterations 2 through 100.



2. Plot 5 different objective functions for iterations 20 through 100



3. Predict the values given in the test set Like Dislike

	Test as 1	Test as -1
Predict as 1	2150	988
Predict as -1	584	1278

Accuracy = 68.56