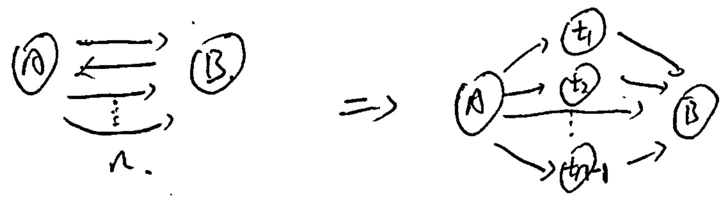
1. Answer:

If there are multiple ~~nodes~~ edges between pairs of nodes, we will add some middle node here. Assume there are a pair of nodes, and the connected edges are n:



(we add $n-1$ middle nodes).

The middle nodes $\{t_1, t_2, \cdots, t_{n-1}\}$ will connect to both node A and node B.

For edge $A-t_i$: we set the capacity to be original capacity, weight to be 0.

For edge t_i-B: we set the capacity to be original capacity, weight to be original weight.

Then the problem can be solved by max-flow as standard problem.

2. Answer:

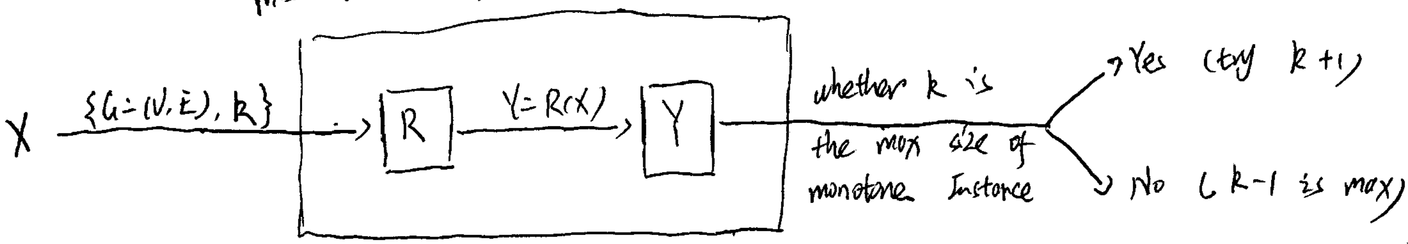
Assume: X: vertex cover $\quad Y$: Monotone Instance Minimum Number.

R: for vertice v_i in G $\Rightarrow$ variable x_i in Y.

for edge (v_i, v_j) in G $\Rightarrow$ clause $C_{ij} = (x_i \vee v_j)$ in Y.

(Note: if $C = (x_1 \vee x_2 \vee x_3)$, we can divide into $((x_1 \vee x_2) \vee (x_2 \vee x_3) \vee (x_1 \vee x_3))$ three edges.

This will correspond to $(v_1, v_2), (v_1, v_3), (v_2, v_3)$.



We have $C_1, C_2, \cdots, C_m$ clauses for each edge in $G = (V, E)$

We will judge whether they can be satisfied by setting maximum vertex cover.

① if $S \in G$ and $|S| = k$.

so each edge is covered by a vertice in S.

so each edge is covered by a vertice in S will have at least one corresponding variable equal to 1

$\Rightarrow$ each clause will include at least one corresponding variable equal to 1

so all clauses are satisfied.

② If $X \in Y$ and k variables are "1".

$\Rightarrow$ each edge will one corresponding node.

$\Rightarrow$ k nodes will cover the whole graph, the size of vertex cover is k,

And the reduction from X to Y's time is polynomial $O(n+m)$

So $X \le_p Y$.

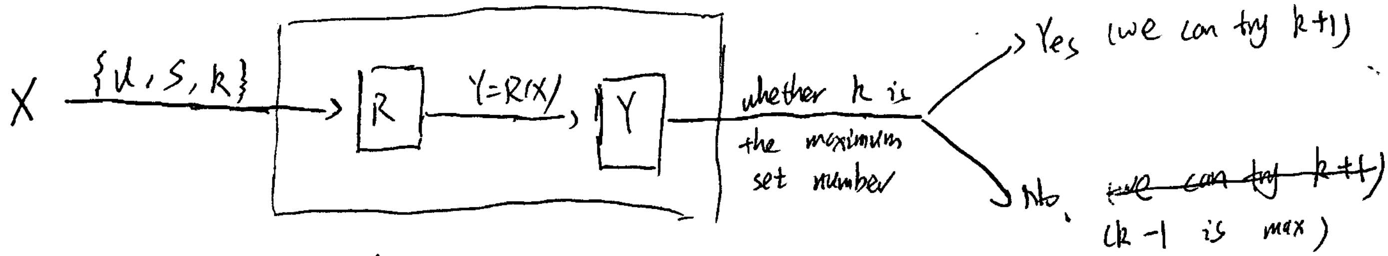
X is NP-complete problem, so Y is also NP-complete problem.

3. Answer:

Assume X: Set Packing Y: Maximum subset of orthogonal customers.

$U = \{U_1, U_2, \cdots, U_n\} \Rightarrow$ products n.

$S = \{S_1, S_2, \cdots, S_m\} \Rightarrow$ customers m.



each set will correspond to each row in $m \times n$ matrix A.

For each set $S_i \in S$ and $1 \le j \le n$, $A[i][j] = 1$ if $U_j \in S_i$.

① If k sets in X are not interactive,
the corresponding k rows in A have no common "1" in the same column.
because the k sets are not interactive.
$\Rightarrow$ We can select k rows from A that are orthogonal.

② If k customers are orthogonal,
the corresponding sets will not sharing the same element
so there will be k sets that are not interactive.

So the X problem can reduce to Y problem.
And the reduction is polynomial $O(mn)$, so $X \le_p Y$.

For X is NP-complete problem
$\Rightarrow Y$ is a NP-complete problem

4. Answer:

① We regard patients and hospitals as vertices in a graph;

② We connect each possible edge between patients and hospitals, and make the edge's capacity to be 1; (possible mean's the distance is within half-hour driving)

③ We add a source s and a sink t. The s will connect to every patient, and the capacity is 1. The t will connect to every hospital, and the capacity is $\frac{n}{k}$.

④ Use Ford-Fulkerson algorithm to find max-flow of the constructed network.

If the value of the maximum flow is n, then there is a solution that all patients can be sent evenly to the hospital.

Complexity:

Building graph: $O(nk + n + k) = O(nk)$.

Max flow: $\cancel{O(mC)} = O[(n+k)n^2k^2]$.

So the complexity is $O((n+k)n^2k^2)$

5- Answer:

(1) maximize $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} S_{ij}$

$$x_{ij} = \begin{cases} 1 & \text{employee } i \text{ is assigned to position } j; \\ 0 & \text{otherwise} \end{cases}$$

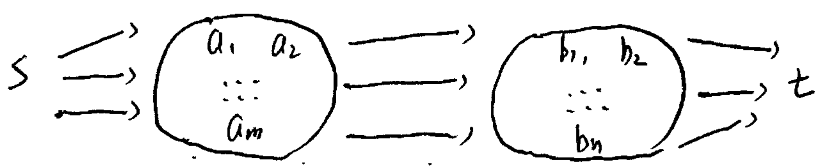
subject to $\sum_{i=1}^{m} x_{ij} \le 1$ for $i = 1, 2, \cdots, n$

$\sum_{j=1}^{n} x_{ij} = 1$ for $i = 1, 2, \cdots, m$

$x_{ij} \in \{0; 1\}$

(2) We can rewrite $w_{ij} = a - S_{ij}$ (a is a number which is larger than $\max\{S_{ij}\}$)

Then we assume there are nodes $\{a_1, a_2, \cdots, a_m\}$, $\{b_1, b_2, \cdots, b_n\}$, source s, sink t.



For each node a_i, the demand is 1; for ~~each~~ source s, the demand is m.

For each edge $a_i - b_j$: the capacity is 1, the weight is w_{ij}.

For $s \to a_i$, $b_j \to t$: the capacity is 1, the weight is 0.

The problem is:

minize $\sum f_{ij} w_{ij}$

subject to $\sum f(s, w) = \sum f(w, t) = d$ (w is the node)

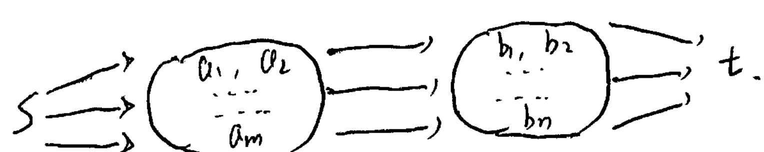
$\sum f(a, b) = 1$ for a_i in $\{a_1, \cdots, a_m\}$, b_j in $\{b_1, \cdots, b_n\}$

this is a minimum-cost flow problem.

If we get f_{ij} set to 1, the j position is assigned to employee i.

(3). maximize $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}$

The same as above:



If $S_{ij} = 1$, then we will connect a_i with b_j, and the capacity is 1

Then this is a max-flow problem.

we will try to get as may employees as possible ~~to~~ the jobs.

(4)

① $\max \quad x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23}$

subject to

$$x_{11} + x_{12} + x_{13} = 1$$
$$x_{21} + x_{22} + x_{23} = 1$$
$$x_{11} + x_{21} \le 1$$
$$x_{12} + x_{22} \le 1$$
$$x_{13} + x_{23} \le 1$$
$$x_{ij} \in \{0, 1\}$$

$$\max_{a_1, a_2, \cdots a_5} \min_{x_{11}, \cdots, x_{23}} -x_{11} - x_{12} - x_{13} - x_{21} - x_{22} - x_{23}$$
$$+ a_1 (x_{11} + x_{12} + x_{13} - 1)$$
$$+ a_2 (x_{21} + x_{22} + x_{23} - 1)$$
$$+ a_3 (x_{11} + x_{21} - 1)$$
$$+ a_4 (x_{12} + x_{22} - 1)$$
$$+ a_5 (x_{13} + x_{23} - 1)$$

$a_1, a_2, a_3 \ge 0, a_4 \ge 0, a_5 \ge 0$

$$\therefore \max_{a_1, a_2, a_3 \ge 0, a_4 \ge 0, a_5 \ge 0} \min_{x_{ij}} -(a_1 + a_2 + a_3 + a_4 + a_5)$$
$$+ x_{11} (a_1 + a_3 - 1)$$
$$+ x_{12} (a_1 + a_4 - 1)$$
$$+ x_{13} (a_1 + a_5 - 1)$$
$$+ x_{21} (a_2 + a_3 - 1)$$
$$+ x_{22} (a_2 + a_4 - 1)$$
$$+ x_{23} (a_2 + a_5 - 1)$$

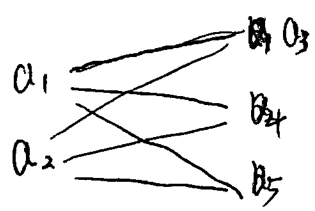
~~max~~ $\min_{a_1, a_2, a_3 \ge 0, a_4 \ge 0, a_5 \ge 0} \quad a_1 + a_2 + a_3 + a_4 + a_5$

subject to

$$a_1 + a_3 \le 1$$
$$a_1 + a_4 \le 1$$
$$a_1 + a_5 \le 1$$
$$a_2 + a_3 \le 1$$
$$a_2 + a_4 \le 1$$
$$a_2 + a_5 \le 1$$

② We can regard the problem as a ~~minimum cut problem~~ vertex cover problem.

~~min Σ~~



$\min \sum a_i$

subject to $a_i + a_j = 1$

The a_i is like node in Graph, a_1, a_2 connect to a_3, a_4, a_5.

Then we will ask how much node we need to cover the graph.

So it's a ~~vc~~ minimum vertex cover problem