Fourier Transform

BEP
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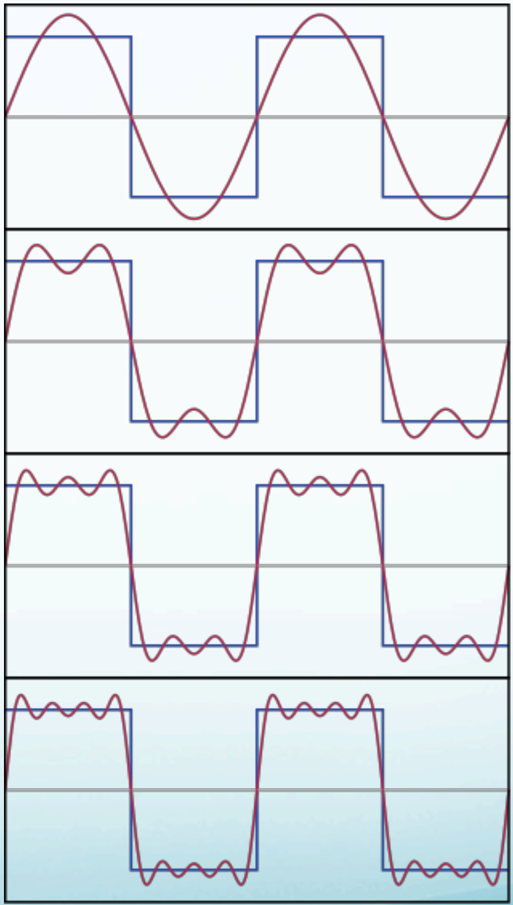
Fourier Series

- Used only for periodic functions
- “A way to represent a wave-like function as the sum of simple sine waves”
- Example: rect function



Fourier Series

- Euler's formula: $e^{jx} = \cos(x) + j \sin(x)$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}$$

- It can be derived, through a series of mathematical manipulations, that

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jnw_0 t}\, dt$$

Fourier Transform

- Used for periodic and non-periodic functions
- This is done by assuming that $T \rightarrow \infty$

$$\omega_0 = \frac{2\pi}{T} \rightarrow d\omega \qquad \lim_{T\rightarrow\infty} \frac{1}{T} = \lim_{T\rightarrow\infty} \frac{\omega_0}{2\pi} = \frac{d\omega}{2\pi}$$

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jnw_0 t}\, dt$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}$$

$$c_n = \frac{d\omega}{2\pi} \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \lim_{T\rightarrow\infty} \left(\sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t} \right)$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier Transform

$$c_n = \frac{d\omega}{2\pi}\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) = \lim_{T\to\infty}\left(\sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}\right)$$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}\,dt e^{j\omega t}\,d\omega$$

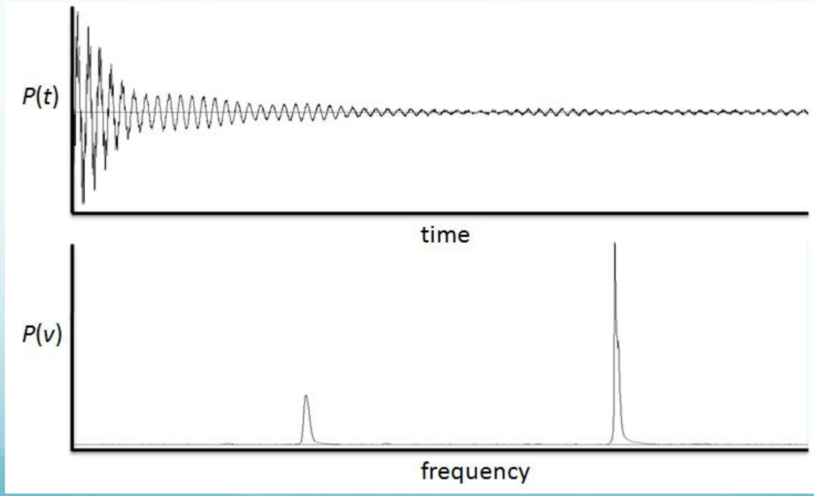
$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

DFT

- (Discrete) Fourier Transform
- We cannot assume that there are infinite number of frequencies in a signal if we wish to compute it
- Obtained by decomposing a sequence of values into components of different frequencies



DFT

DFT

FT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Sampling

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$$

DFT

$$X_s(j\omega) = \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)e^{-j\omega t}\,dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T} = \sum_{n=-\infty}^{\infty} x_n e^{-jn\omega T}$$

DFT

- Using Euler's Formula

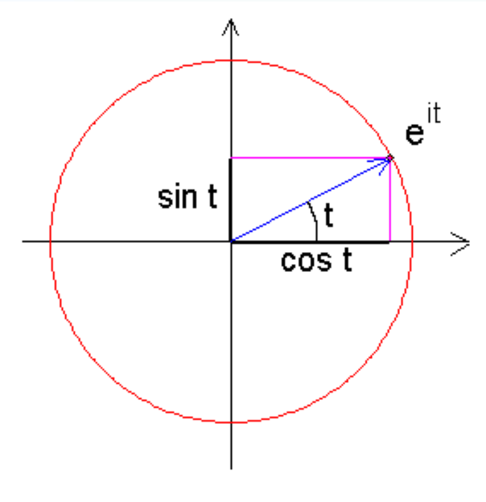
$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x_n \cos(n\omega T) - j\sum_{n=-\infty}^{\infty} x_n \sin(n\omega T)$$

- An angle θ is the same as $\theta + 2\pi$

$$X_s(j\omega) = X_s(j\omega + jk\frac{2\pi}{T})$$

- So the signal repeats itself every N samples

$$X_s(j\omega) = \sum_{n=0}^{N-1} x_n e^{-jn\omega T}$$



DFT

- Considering

$$\omega \rightarrow \omega_m = \frac{2\pi m}{NT} \qquad m = 0,1,\ldots N-1$$

- We have that

$$X_m = X_s(j2\pi\omega / NT) = \sum_{n=0}^{N-1} x_n e^{-j2\pi mn/N}$$

- Resulting in

$$X_m = \sum_{n=0}^{N-1} x_n e^{-j2\pi mn/N}, \quad m = 0,1,\ldots N-1$$

$$x_n = \frac{1}{N}\sum_{n=0}^{N-1} X_m e^{j2\pi mn/N}, \quad n = 0,1,\ldots N-1$$

2-D DFT

- DFT

$$F(m) = \frac{1}{N}\sum_{n=0}^{N-1} f(n)e^{-j2\pi mn/N}, \quad m = 0,1,\ldots N-1$$

$$f(n) = \sum_{n=0}^{N-1} F(m)e^{j2\pi mn/N}, \quad n = 0,1,\ldots N-1$$

- 2-D DFT

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

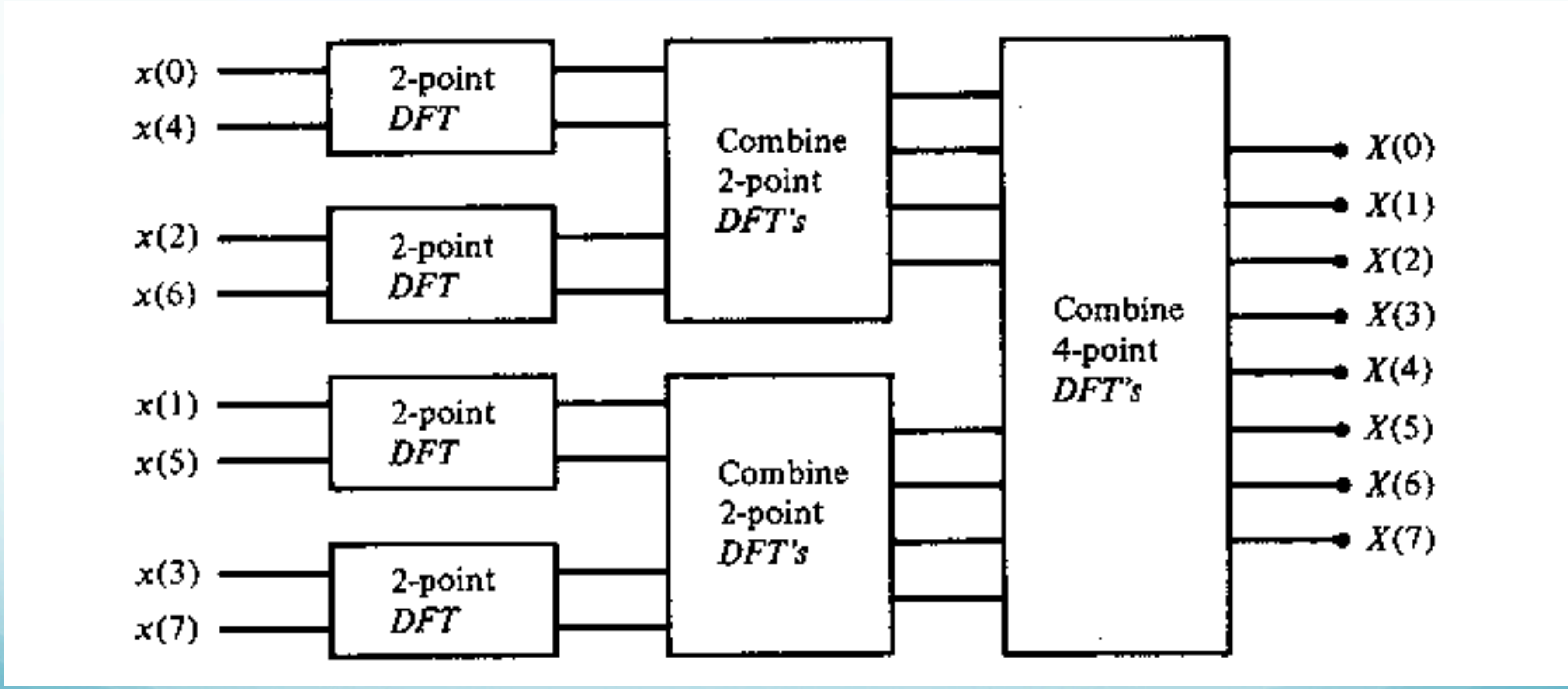
$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

FFT

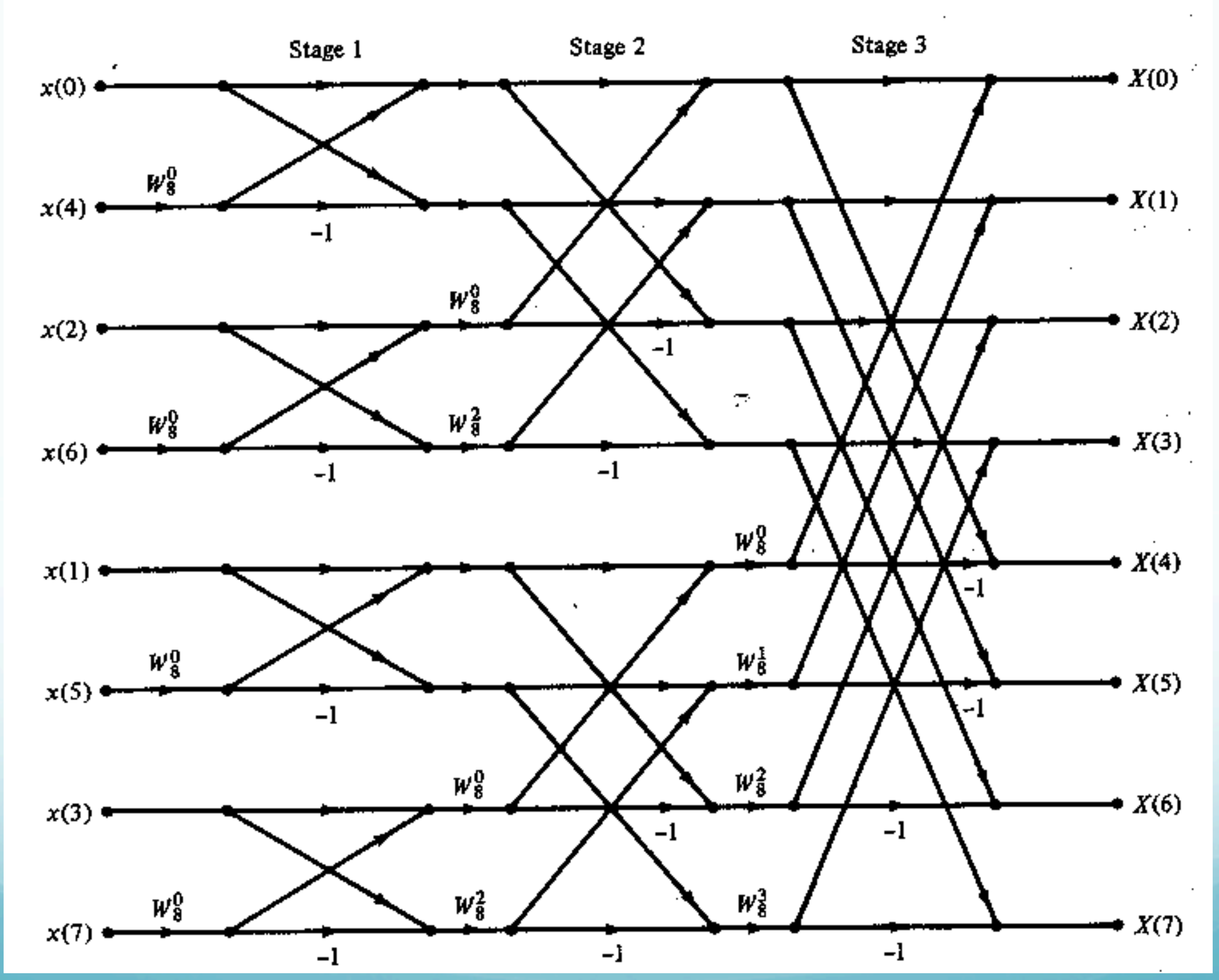
- Fast Fourier Transform
- Algorithm to compute the DFT of a sequence
- Exploits the symmetry and periodicity properties of the phase factor $W_N = e^{-j2\pi/N}$
- That's why it's efficient
- Symmetry property: $W_N^{k+N/2} = -W_N^k$
- Periodicity Property: $W_N^{k+N} = W_N^k$

Radix-2 FFT Algorithm

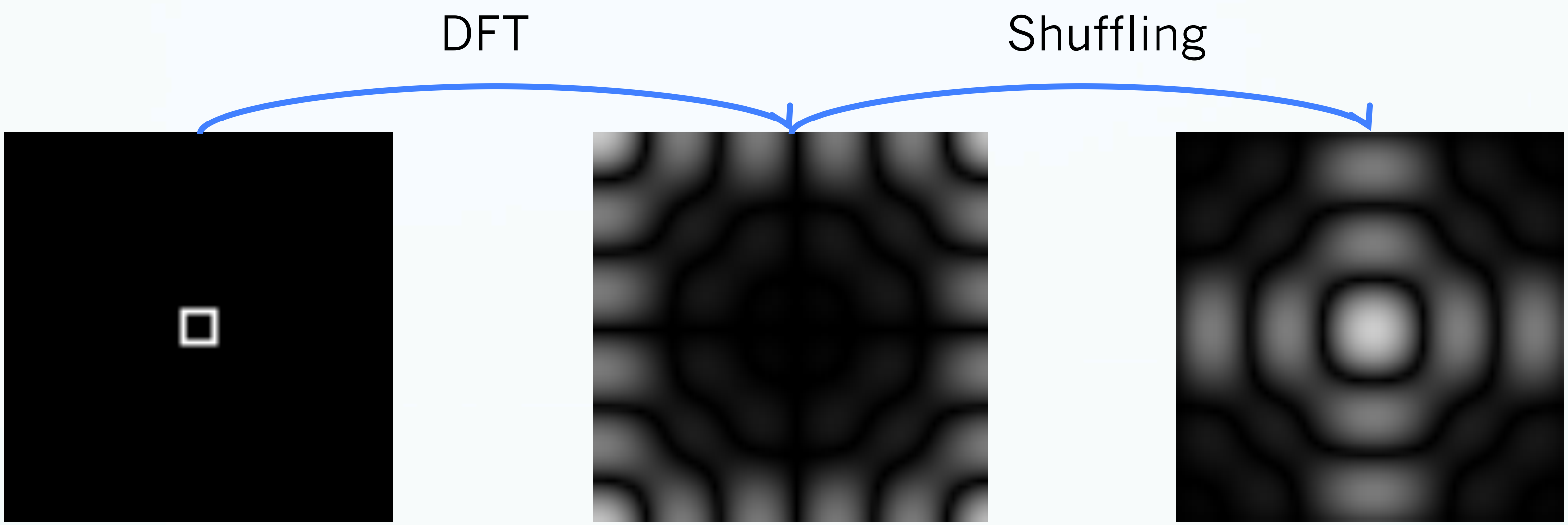
- Example of an N = 8-point DFT

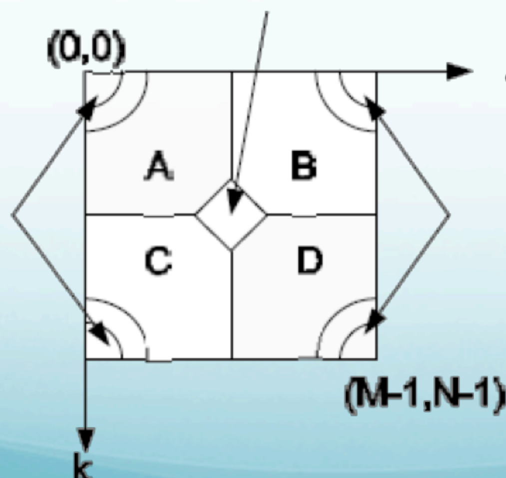


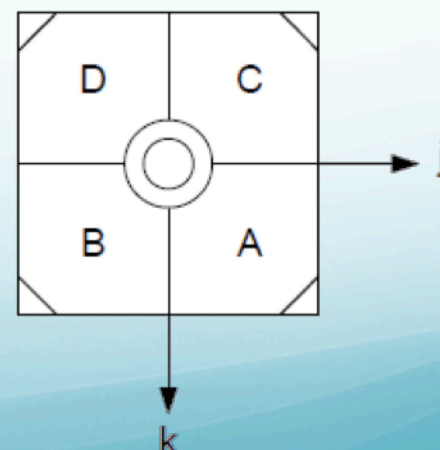
Radix-2 FFT Algorithm



Shuffling





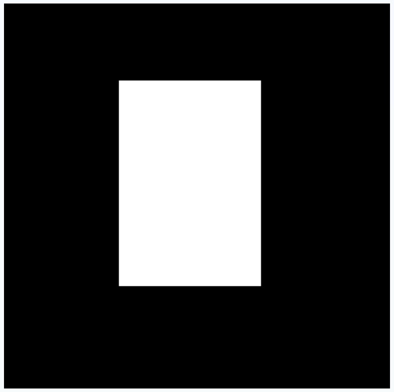


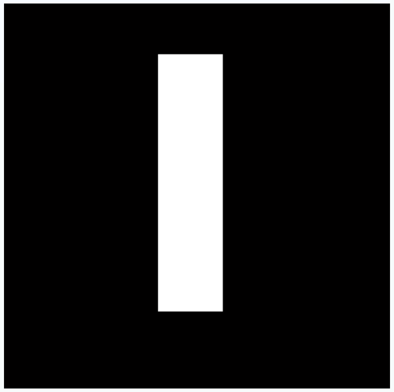
Homework

- Implement 2-D DFT
- Implement 2-D Inverse DFT

Example

Input





Output