An Optimization Model Integrating Different Preference Formats

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***Abstract*—These instructions give you guidelines for preparing papers for IEEE Transactions and Journals. Use this document as a template if you are using Microsoft *Word* 6.0 or later. Otherwise, use this document as an instruction set. The electronic file of your paper will be formatted further at IEEE. Paper titles should be written in uppercase and lowercase letters, not all uppercase. Avoid writing long formulas with subscripts in the title; short formulas that identify the elements are fine (e.g., "Nd–Fe–B"). Do not write "(Invited)" in the title. Full names of authors are preferred in the author field, but are not required. Put a space between authors' initials. Define all symbols used in the abstract. Do not cite references in the abstract. Do not delete the blank line immediately above the abstract; it sets the footnote at the bottom of this column.**

***Index Terms* - Group decision making; utility values; preference ordering; multiplicative preference relations; fuzzy preference relations; similarity measure; optimization approach.**

I. INTRODUCTION

Decision making based on different preference information formats is a natural issue in practice owing to differing experience, knowledge, personalities, skills, and viewpoints of different decision makers. Many substantial researches have been studied to devote to proposed problems which can be divided into two folds. The first class (Chiclana et al 1988, 2001; Delgado et al, 1998; Herrera et al, 2001; Herrera-viedma, 2002) transform all different preference structures into uniform formats, and then selection operator is implemented to rank alternatives. Other methods (Fan et al, 2006; Ma et al, 2006; Xu et al, 2011; Wang et al, 2007) employ optimization models to directly obtain collective priority vector instead of transform functions for obtaining uniform preference information. In this study, a new method is proposed to integrate different preference relations provided by different decision makers. We establish an optimization model based on a cosine similarity measure which can provide a new insight on integrating different formats of preference information.

The remainder of this study is organized as follows. Section 2 constructs the cosine similarity measure optimization model and solution. Section 3 shows illustrative example and numerical analyses of the model. Section 4 concludes the study.

II. Cosine Maximization Optimization Model

The cosine similarity index is the most widely used similarity measure. For instance, for two vectors $\vec{r}_i = (r_{i1}, r_{i2}, \ldots, r_{in})$ and $\vec{r}_j = (r_{j1}, r_{j2}, \ldots, r_{jn})$, the cosine similarity measure is a binary function defined as

$$\left\langle \vec{r}_i, \vec{r}_j \right\rangle = \frac{\vec{r}_i \bullet \vec{r}}{\|\vec{r}_i\| \|\vec{r}_j\|} = \frac{\sum_{k=1}^{n} r_{ik} r_{jk}}{\sqrt{\sum_{k=1}^{n} r_{ik}^2} \sqrt{\sum_{k=1}^{n} r_{jk}^2}} \in [0,1] \tag{1}$$

Kou and Lin (2014) first used the similarity measure method for derivation of priority vector in Analytic Hierarchy Process (AHP). Their basic idea was that the derived priority vector is to be most similar to each column of a Pair-wise Comparative Matrix(PCM) according to the cosine similarity measure.

Assume that $W = \{A_1, A_2, \ldots, A_n\}$ is a finite set containing n alternatives. $L = \{s_1, s_2, \ldots, s_K\}$ is a finite set containing the degrees of importance of decision makers. The collective group preference $w = \{w_1, w_2, \ldots, w_n\}$ indicates the degree of importance and best-to-worst selection order ranking the alternatives. $\{\Pi_1, \Pi_2, \Pi_3, \Pi_4\}$ is a finite set of utility value, preference ordering, multiplicative and fuzzy preference relations provided by decision makers respectively.

Let $(u_i^{(k)})_{1\times n}$, $u_i^{(k)} \in [0,1]$, $k \in \Pi_1$ be utility values provided by decision makers. We note that $\frac{w_i}{w_j} = \frac{u_i^{(k)}}{u_j^{(k)}}$ and it is clearly that

$$\left\langle \frac{u_i^{(k)}}{u_j^{(k)}}, w_i \right\rangle = \frac{\sum_{i=1}^{n} \frac{u_i^{(k)} w_i}{u_j^{(k)}}}{\sqrt{\sum_{i=1}^{n}\left(\frac{u_i^{(k)}}{u_j^{(k)}}\right)^2}\sqrt{\sum_{i=1}^{n} w_i^2}} = \frac{\sum_{i=1}^{n} u_i^{(k)} w_i}{\sqrt{\sum_{i=1}^{n}\left(u_i^{(k)}\right)^2}\sqrt{\sum_{i=1}^{n} w_i^2}} = 1 \tag{2}$$

Assume $(o_i^{(k)})_{1\times n}$, $o_i^{(k)} \in \{1, 2, ..., n\}$, $k \in \Pi_2$ is preference ordering, it is the ordering of alternatives. In this case, $\frac{w_i}{w_j} = \frac{n - o_i^{(k)}}{n - o_j^{(k)}}$ and

$$\left\langle \frac{n-o_i^{(k)}}{n-o_j^{(k)}}, w_i \right\rangle = \frac{\sum_{i=1}^{n} \frac{(n-o_i^{(k)})w_i}{n-o_j^{(k)}}}{\sqrt{\sum_{i=1}^{n}\left(\frac{n-o_i^{(k)}}{n-o_j^{(k)}}\right)^2}\sqrt{\sum_{i=1}^{n} w_i^2}} = \frac{\sum_{i=1}^{n}(n-o_i^{(k)})w_i}{\sqrt{\sum_{i=1}^{n}\left(n-o_i^{(k)}\right)^2 \sum_{i=1}^{n} w_i^2}} = 1 \tag{3}$$

Let $(a_{ij}^{(k)})_{n\times n}$, $k \in \Pi_3$ be multiplicative preference relations. Kou and Lin (2014) proved that the cosine similarity measure is equal to one if and only if the PCM is perfectly consistent. The higher the degree of consensus, the closer the cosine similarity measure is to 1.

Assume $(p_{ij}^{(k)})_{n\times n}$, $k \in \Pi_4$ be fuzzy preference relations. We can show that there are also existing similarity relations between a derived priority vector and each column of the PCM after the following transformation:

$$b_{ij}^{(k)} = \frac{p_{ij}^{(k)}}{1 - p_{ij}^{(k)}}. \tag{4}$$

When the PCM is perfectly consistent,

$$\langle \bar{b}_j, \bar{w} \rangle = \frac{\sum_i b_{ij} w_i}{\sqrt{\sum_i b_{ij}^2} \cdot \sqrt{\sum_i w_i^2}} = \frac{\sum_i \frac{p_{ij}}{1-p_{ij}} w_i}{\sqrt{\sum_i \left(\frac{p_{ij}}{1-p_{ij}}\right)^2} \cdot \sqrt{\sum_i w_i^2}} = \frac{\sum_i w_i^2}{\sqrt{\sum_i w_i^2} \cdot \sqrt{\sum_i w_i^2}} = 1. \tag{5}$$

In practice, we use $p_{ij}^{(k)} = 0.9999$ and $p_{ij}^{(k)} = 0.0001$ to replace $p_{ij}^{(k)} = 1$ or $p_{ij}^{(k)} = 0$ in fuzzy PCM.

Now, we transform the relation into mathematical language, the group priority vector must have the highest similarity measure between the derived collective priority and each column of decision makers' PCMs. We construct a cosine similarity measure maximization optimization model as follows:

$$\text{Maximize} \quad C = \sum_{k=1}^{K}\sum_{j=1}^{n} C_j^{(k)} = \sum_{k\in\Pi_1}\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_k \frac{w_i\left(\frac{u_i^{(k)}}{u_j^{(k)}}\right)}{\sqrt{\sum_{h=1}^{n} w_h^2}\sqrt{\sum_{i=1}^{n}\left(\frac{u_i^{(k)}}{u_j^{(k)}}\right)^2}} + \sum_{k\in\Pi_2}\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_k \frac{w_i\left(\frac{n-o_i^{(k)}}{n-o_j^{(k)}}\right)}{\sqrt{\sum_{h=1}^{n} w_h^2}\sqrt{\sum_{i=1}^{n}\left(\frac{n-o_i^{(k)}}{n-o_j^{(k)}}\right)^2}} +$$

$$\sum_{k\in\Pi_3}\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_k \frac{w_i a_{ij}^{(k)}}{\sqrt{\sum_{h=1}^{n} w_h^2}\sqrt{\sum_{h=1}^{n} a_{hj}^{(k)2}}} + \sum_{k\in\Pi_4}^{K_4}\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_k \frac{w_i b_{ij}^{(k)}}{\sqrt{\sum_{h=1}^{n} w_h^2}\sqrt{\sum_{h=1}^{n} b_{hj}^{(k)2}}}$$

$$\text{Subject to} \quad \begin{cases} \sum_{i=1}^{n} w_i = 1, \\ w_i > 0, i = 1, 2, ..., n \end{cases}$$

where C is the total similarity measure and $C^{(k)}$ is the similarity measure between the collective priority vector and the PCM of the k th decision maker.

To simplify the computing process, we set

$$\hat{w}_i = \frac{w_i}{\sqrt{\sum_{i=1}^{n} w_i^2}};$$

$$\bar{u}_{ij}^{(k)} = \frac{\frac{u_i^{(k)}}{u_j^{(k)}}}{\sqrt{\sum_{i=1}^{n}\left(\frac{u_i^{(k)}}{u_j^{(k)}}\right)^2}}; \; \bar{o}_{ij}^{(k)} = \frac{\frac{n-o_i^{(k)}}{n-o_j^{(k)}}}{\sqrt{\sum_{i=1}^{n}\left(\frac{n-o_i^{(k)}}{n-o_j^{(k)}}\right)^2}}; \tag{6}$$

$$\bar{a}_{ij}^{(k)} = \frac{a_{ij}^{(k)}}{\sqrt{\sum_{j=1}^{n} a_{ij}^{(k)2}}} \text{ and } \bar{b}_{ij}^{(k)} = \frac{b_{ij}^{(k)}}{\sqrt{\sum_{j=1}^{n} b_{ij}^{(k)2}}}$$

This transforms the optimization model into the following model:

$$\text{Max} \quad C = \sum_{k\in\Pi_1}\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_k \hat{w}_i \bar{u}_{ij}^{(k)} + \sum_{k\in\Pi_2}\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_k \hat{w}_i \bar{o}_{ij}^{(k)} + \sum_{k\in\Pi_3}\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_k \hat{w}_i \bar{a}_{ij}^{(k)} + \sum_{k\in\Pi_4}\sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_k \hat{w}_i \bar{b}_{ij}^{(k)}$$

$$S.t. \quad \begin{cases} \sum_{i=1}^{n} \hat{w}_i^2 = 1, \\ \hat{w}_i > 0, i = 1, 2, ..., n \end{cases}$$

We take the partial derivatives of the Lagrangian function of the objective function, and taking the partial derivatives to be equal to 0, it follows that

$$L(C,\lambda)=\sum_{k=1}^{K_1}\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_k\hat{w}_i\bar{u}_{ij}^{(k)}+\sum_{k=K_1+1}^{K_2}\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_k\hat{w}_i\bar{o}_{ij}^{(k)}+$$
$$\sum_{k=K_2+1}^{K_3}\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_k\hat{w}_i\bar{a}_{ij}^{(k)}+\sum_{k=k_3+1}^{K_4}\sum_{j=1}^{n}\sum_{i=1}^{n}\sigma_k\hat{w}_i\bar{b}_{ij}^{(k)}-\lambda(\sum_{i=1}^{n}\hat{w}_i-1) \tag{7}$$

$$\frac{\partial L(C,\lambda)}{\partial\hat{w}_i}=\sum_{k=1}^{K_1}\sum_{j=1}^{n}\sigma_k\hat{w}_i\bar{u}_{ij}^{(k)}+\sum_{k=K_1+1}^{K_2}\sum_{j=1}^{n}\sigma_k\hat{w}_i\bar{o}_{ij}^{(k)}+$$
$$\sum_{k=K_2+1}^{K_3}\sum_{j=1}^{n}\sigma_k\hat{w}_i\bar{a}_{ij}^{(k)}+\sum_{k=k_3+1}^{K_4}\sum_{j=1}^{n}\sigma_k\hat{w}_i\bar{b}_{ij}^{(k)}-2\lambda\hat{w}_i, \quad i=1,2,\ldots,n \tag{8}$$

By constraint conditions $\sum_{i=1}^{n}\hat{w}_i^2=1$, we can obtain

$$\hat{w}_i^G=\frac{\sum_{k=1}^{K_1}\sum_{j=1}^{n}\sigma_k\bar{u}_{ij}^{(k)}+\sum_{k=K_1+1}^{K_2}\sum_{j=1}^{n}\sigma_k\bar{o}_{ij}^{(k)}+\sum_{k=K_2+1}^{K_3}\sum_{j=1}^{n}\sigma_k\bar{a}_{ij}^{(k)}+\sum_{k=k_3+1}^{K_4}\sum_{j=1}^{n}\sigma_k\bar{b}_{ij}^{(k)}}{\sqrt{\sum_{i=1}^{n}\left(\sum_{k=1}^{K_1}\sum_{j=1}^{n}\sigma_k\bar{u}_{ij}^{(k)}+\sum_{k=K_1+1}^{K_2}\sum_{j=1}^{n}\sigma_k\bar{o}_{ij}^{(k)}+\sum_{k=K_2+1}^{K_3}\sum_{j=1}^{n}\sigma_k\bar{a}_{ij}^{(k)}+\sum_{k=k_3+1}^{K_4}\sum_{j=1}^{n}\sigma_k\bar{b}_{ij}^{(k)}\right)^2}},\quad i=1,2,\ldots,n \tag{9}$$

The solution is unique, since

$$w_i=\hat{w}_i^G\sqrt{\sum_{i=1}^{n}w_i^2},\quad i=1,2,\ldots,n \tag{10}$$

and

$$\sum_{i=1}^{n}w_i=\sum_{i=1}^{n}\hat{w}_i^G\sqrt{\sum_{i=1}^{n}w_i^2}=1 \tag{11}$$

It means

$$\sqrt{\sum_{i=1}^{n}w_i^2}=\frac{1}{\sum_{i=1}^{n}\hat{w}_i^G} \tag{12}$$

and following solution can be obtained:

$$w_i=\frac{\sum_{k=1}^{K_1}\sum_{j=1}^{n}\sigma_k\bar{u}_{ij}^{(k)}+\sum_{k=K_1+1}^{K_2}\sum_{j=1}^{n}\sigma_k\bar{o}_{ij}^{(k)}+\sum_{k=K_2+1}^{K_3}\sum_{j=1}^{n}\sigma_k\bar{a}_{ij}^{(k)}+\sum_{k=k_3+1}^{K_4}\sum_{j=1}^{n}\sigma_k\bar{b}_{ij}^{(k)}}{\sum_{i=1}^{n}\left(\sum_{k=1}^{K_1}\sum_{j=1}^{n}\sigma_k\bar{u}_{ij}^{(k)}+\sum_{k=K_1+1}^{K_2}\sum_{j=1}^{n}\sigma_k\bar{o}_{ij}^{(k)}+\sum_{k=K_2+1}^{K_3}\sum_{j=1}^{n}\sigma_k\bar{a}_{ij}^{(k)}+\sum_{k=k_3+1}^{K_4}\sum_{j=1}^{n}\sigma_k\bar{b}_{ij}^{(k)}\right)},\quad i=1,2,\ldots,n, \tag{13}$$

Since it is obviously that the supplementary angle (for examples, the angle $\pi-\arccos\sum_{i=1}^{n}w_i a_{ij}^{(k)}$ is the largest angle between priority vector and *j-th* column of *k-th* decision makers) corresponding to priority vector to each column vectors of the PCMs represents largest angle by means of inverse trigonometric function. That is, the $-w_i$, $i=1,2,\ldots,n$, is the minimum value of the proposed model. Therefore, The solution (13) represents the priority vector has the least angle to each columns of the PCMs and the lagrangian method assure the solution (13) is the needed uniqueness and maximality of the objective function.

III. Illustrative Example

The example includes four different preference formats, preference ordering, utility values, and multiplicative and fuzzy preference relations, which is studied by Chiclana et al. (2001), Ma et al (2006) and Xu et al(2011).

$$DM_1=\begin{pmatrix}1&1/7&1/3&1/5\\7&1&3&2\\3&1/3&1&1/2\\5&1/2&2&1\end{pmatrix}\quad DM_2=\begin{pmatrix}1&3&1/4&5\\1/3&1&2&1/3\\4&1/2&1&2\\1/5&3&1/2&1\end{pmatrix}$$

$$DM_3=\begin{pmatrix}0.5&0.1&0.6&0.7\\0.9&0.5&0.8&0.4\\0.4&0.2&0.5&0.9\\0.3&0.6&0.1&0.5\end{pmatrix}\quad DM_4=\begin{pmatrix}0.5&0.5&0.7&1\\0.5&0.5&0.8&0.6\\0.3&0.2&0.5&0.8\\0&0.4&0.2&0.5\end{pmatrix}$$

$$DM_5=\left\{u_i \middle| i=1,2,3,4\right\}=\left\{3,1,4,2\right\}$$
$$DM_6=\left\{u_i \middle| i=1,2,3,4\right\}=\left\{2,3,1,4\right\}$$
$$DM_7=\left\{o_i \middle| i=1,2,3,4\right\}=\left\{0.5,0.7,1.0,0.1\right\}$$
$$DM_8=\left\{o_i \middle| i=1,2,3,4\right\}=\left\{0.7,0.9,0.6,0.3\right\}$$

The priority vector obtained by our model is $(0.2303, 0.3588, 0.2563, 0.1547)^T$, and the ranking of alternatives is $A_2\succ A_3\succ A_1\succ A_4$ that is same as Chiclana et al. (2001), Ma et al.（2006）and Xu et al. (2011). However, our model’s result has highest cosine value, indicating the highest degree of consensus among decision makers.

TABLE I. COMPARATIVE RESULTS

	priority vector				***ranking***	***C***
	w_1	w_2	w_3	w_4		
Chiclana et al. (2001)	0.5651	0.7826	0.6619	0.4973	$A_2\succ A_3\succ A_1\succ A_4$	-
Ma et al.（2006）	0.2210	0.3426	0.2755	0.1159	$A_2\succ A_3\succ A_1\succ A_4$	26.9149
Xu et al. (2011)	0.2210	0.3426	0.2827	0.1537	$A_2\succ A_3\succ A_1\succ A_4$	26.8878
Our model	0.2303	0.3588	0.2563	0.1547	$A_2\succ A_3\succ A_1\succ A_4$	***27.0200***

IV. CONCLUSIONS

In this paper, an optimization model was developed on the basis of the cosine similarity measure for different preference formats. The basic idea is that the collective priority vector is most similar to each column of the PCMs provided by the decision makers. The model can be solved using a Lagrangian approach and the priority vector obtained by simple matrix operations. We use an example to compare the proposed method with existing approaches. The results show that the rankings generated by the different approaches are same and that the proposed model has highest degree of consensus among decision makers since greatest cosine values in all three examples.

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