Numerical Algorithms for Lyapunov Stability Analysis of Interpolative Control Structures

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***Abstract* - The complexity and specificity of stability analysis applied to interpolative-type control structures makes the study of such property a difficult, almost impossible task - at least in the analytical manner. Therefore, a better solution could be represented by the numerical approach.**

In this context, this paper starts with developing some methods and techniques with applicability for analysis of the interpolative-type controllers, based on Lyapunov method perspective. These methodological aspects are gathered together into a specific procedural algorithm. Based on this algorithm, a set of MATLAB-SIMULINK programs able to offer, in a flexible and user-interactive way, a possible solution to Lyapunov-stability analysis for a class of interpolative-type control systems with linear or non-linear processes of 2nd and 3rd order are developed. The solution based on the implemented software packages was finally validated through some practical examples.

***Index Terms*-numerical algorithms, Lyapunov-stability analysis, interpolative controllers, Shepard interpolation**

I. INTRODUCTION

The stability of interpolative-type control structures (as fuzzy, neural or purely interpolative control systems) – as a particular case of non-linear systems - is a challenging and actual problem, due to its multiple applicative directions. The stability analysis for fuzzy systems or purely interpolative ones represent a delicate problem and an issue difficult to approach, considering their complexity degree and their intrinsic non-linearity.

In the specialized literature there are many approaches, especially regarding the fuzzy systems stability [1]. Hence, many of the approaches related to the fuzzy systems present some limitations in the applicative area. One of the limitation is that, in the case of a system with fuzzy controllers, most of the approaches are based on Takagi-Sugeno models for the processes, as in [2]; consequently, for the other cases (numerous) when the process is based on classical models, a supplementary effort is necessary.

The purely interpolative control systems (systems containing interpolative blocks involved in the control process) developed usually for non-linear processes need some specific approaches on stability analysis issue. In the literature there are just a few studies related to their stability analysis. The existing studies are limited to the stability of the interpolation methods [3].

In this context, the paper has the aim to develop, at theoretical, procedural and applicative level, some methods and techniques with large applicability for analysis and synthesis of the interpolative-type controllers, through Lyapunov method perspective. The extended theory can be found in [4]. The starting point of the study is represented by the stability analysis method for a class of non-linear processes integrated in fuzzy control systems. In the next step, the study enlarges the applicability domain of this approach to the processes integrated in control systems with controller based on Shepard interpolation.

In section II, the theoretical results presented in [4] are concluded through a Unified Stability Theorem stated for both fuzzy and Shepard interpolative controllers, offering the possibility to study the stability of the interpolative-type control systems.

Section III presents some methodological aspects regarding the implementation algorithm and the basic structure of the implemented software package. The algorithms presented separately in [5] for fuzzy and Shepard interpolative structures are unified in the Procedural Algorithm, ready to use for the implementation of a software package dedicated to the stability analysis via Lyapunov method for a class of interpolative-type control systems. The software package developed further consists in a set of MATLAB-SIMULINK programs and control schemes build block-by-block in order to ensure maximum efficiency, transparency and accuracy of the results. In this context, the usage of pre-defined blocks from Fuzzy Toolbox was avoided and also was developed from scratch the Shepard interpolative controllers. In order to prove the efficiency of the developed numerical algorithms, a case study based on a second order non-linear system, together with some experimental results, is exposed in section IV. Finally, a few pertinent observations conclude the present paper in chapter V.

II. THEORETICAL BACKGROUND

The results produced in the following chapters are based on the previous work developed in [4] and [5] where the theoretical and procedural aspects of the numerical algorithms for Lyapunov stability analysis are treated in detail. In order to offer a consistent view and a background for the present research results, a short presentation of the basic theoretical aspects of the method and the vertebral spine of the software package is given in the following paragraphs.

The basic line of the approach is stated in terms of Lyapunov stability analysis for fuzzy systems as developed by Wong team in [6], referred as *Wong-Leung-Tam method.* The method treats the stability of the fuzzy systems for a class of non-linear processes, without a fuzzy model and without limitations on the form and distribution of the membership functions implicated on fuzzy controller inputs and outputs description.

In a second step, stability analysis is extended to interpolative structures based on controllers with Shepard interpolation. In Fig.1 a unified image of the fuzzy and interpolative control structure is depicted. The control structures consist on a fuzzy controller FC / interpolative controller IC and a non-linear process.

The controlled process in both structures described by Fig.1 is of SISO type, of n^{th} order, non-linear, given by (1):

$$\dot{x} = f(x) + b(x)\cdot g(u) + p\,, \qquad (1)$$

where $x = [x_1, x_2, \ldots, x_n]^T$ represents the state vector, u represents the command signal generated by the fuzzy controller, $p = [p_1, p_2, \ldots, p_n]^T$ is a vector describing some external disturbances, $f(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T$ and also $b(x) = [b_1(x), b_2(x), \ldots, b_n(x)]^T$ are the functional vectors describing the system dynamics and $g(u)$ is a scalar function depending on u.

The inner structure and the process of individual command u_i elaboration in both fuzzy and interpolative case are detailed in [4].

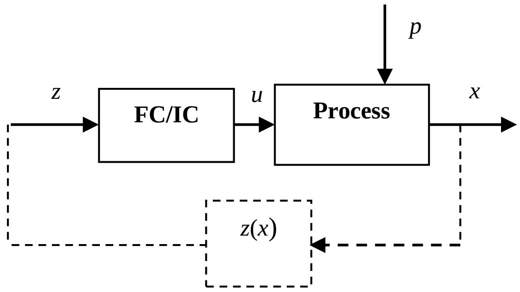

Fig.1. Unified scheme for the fuzzy and interpolative control system

The stability theorem enunciated by Wong team can be applied to linear and non-linear systems with controlled processes as in (1), with fuzzy controllers which use as a defuzzification method any weighted-sum-type method. This fact will guarantee the fulfillment of condition (2).

$$u_p \le u_i(x) \le u_q \qquad (2)$$

The calculation of command limits $u_p = \min(u_i)$ and $u_q = \max(u_i)$ is not necessary, their existence is sufficient. In order to develop a systematic approach, the Wong-Leung-Tam stability theorem is reformulated and then extended for the cases of some systems with interpolative controllers.

Shepard interpolative method for developing interpolative controllers presumes the support points obtained from the rules of the rule basis are provided directly; the linguistic variables associated to IC controller are described by some real intervals, as in Regelbasierte Interpolation method developed in [8]; the method used is the Shepard interpolation as in (3), where α represents a weight factor, x_i the locations of the support points for the rule basis and u_i the value of the command associated to the support points.

$$u = \frac{\sum_{i=1}^{r} w_i \cdot u_i}{\sum_{i=1}^{r} w_i}\;, \quad w_i = \frac{1}{\|x - x_i\|^{\alpha}} \qquad (3)$$

As the command elaboration method is different in the interpolative case [7], the stability theorem in a unified form for both fuzzy and Shepard interpolative structures can be formulated in terms of Unified Stability Theorem.

Unified Stability Theorem: If, for a structure with fuzzy or Shepard interpolative controller as in Fig.1 the following conditions are fulfilled:

1. there is a quadratic matrix, symmetric and positive definite and the candidate function

$$V = x^T \cdot P \cdot x \to \infty \text{ when } \|x\| \to \infty\,;$$

2. the first derivative is negative definite for each value of x;
3. the rule basis completely cover the inputs domains;
4. for the calculation of the fuzzy command u is applied a defuzzification method and for the calculation of the interpolative command u the Shepard interpolation method is applied - both respecting condition (2),

then the equilibrium point of the process, situated in the origin is asymptotically stable.

III. NUMERICAL ALGORITHM AND SOFTWARE PACKAGE DESCRIPTION

Having in mind the complexity of the calculus made in the situations in which the activation domains of the rules are overlapped, it's possible to consider only a numerical, computer assisted approach, for the stability analysis issue. Applying Unified Stability Theorem in order to analyze the stability of a fuzzy or a Shepard interpolative control system implies to follow the steps described in Procedural Algorithm.

Procedural Algorithm

i) The completion of the rule basis for the fuzzy controller is verified (in the interpolative case is not necessary to verify the completion condition, because by applying the Shepard interpolation a complete rule basis will be obtained).

ii) A Lyapunov candidate function for the given system/process is adopted, with respect of first condition in Unified Stability Theorem.

iii) For each x, the third condition in Unified Stability Theorem is verified, for the chosen Lyapunov function.

iv) For the process given by state-space model, the candidate Lyapunov function is developed, considering as parameters the elements of P matrix.

v) The first derivative $\dot{V}$ is calculated using the state-space model of the process and the $\dot{V} \leq 0$ condition is imposed. Consequently, some relations between the elements of P matrix will result.

vi) The values of elements in P matrix which respect the condition v) are determined.

In order to implement the theoretical and procedural steps traveled so far, a software package was developed. The software package consists in a set of configurable SIMULINK schemes for the control systems, a set of MATLAB programs for the calculation of Lyapunov function and a user interface, all presented in [9]. After the introducing of the process data, the fuzzy or interpolative controller is designed and the non-linearity is configured. In order to prepare the calculus background, the values of P matrix are set or, if the P matrix is unknown, an adequate matrix is adopted (using the Procedural Algorithm). The calculation of the Liapunov candidate function first derivative $\dot{V}$ for all the admissible values of x is delivered and the user can verify the fulfillment of the 2nd condition in the Unified Stability Theorem. The functionality of the environment is described in detail in [5].

Through the user interface the control structure can be configured by choosing the control algorithm (fuzzy or Shepard) and introducing process data (order, mathematical model, initial conditions).

Depending on the chosen control structure the user can configure the corresponding SIMULINK schema. Next, the stability calculation is made, depending on the Lyapunov candidate matrix adopted, following the steps described in the procedural algorithm. The procedure will operate off-line and is meant to verify the 3rd condition in Unified Stability Theorem. The result will be available for the user in numerical and graphical format.

The unified scheme for the fuzzy and interpolative open-loop control structures implemented in the development software tool is presented in Fig. 2. Based on this, the calculation and delivery – in numerical and graphical form – of the Lyapunov candidate function first derivative $\dot{V}$ for all the admissible values of x is possible.

The user can verify the fulfillment of the 3rd condition in the Unified Stability Theorem.

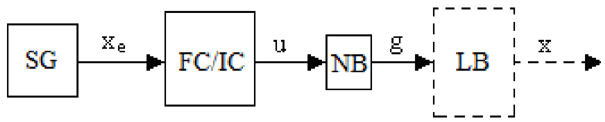

Fig.2. Unified scheme for the fuzzy and interpolative open-loop control structures implemented in the developed software tool

The signal generator SG was built for scanning through proper signals x_e the admissible variation domain of the state-vector x. The fuzzy controller FC and the interpolative controller IC was built block-by-block, following desiderates resulted from the Unified Stability Theorem and making them suitable for the implementation of the Procedural Algorithm. The structures presented in Fig.2 are the basis for the stability analysis of a class of processes described by state-space models in their linear parts LB and also by a non-linearity NB (when the case appears).

As a frame for applying the Unified Stability Theorem, a software development tool was constructed. The environment permits stability analysis applications based on Lyapunov direct method for interpolative-type control systems (with fuzzy and interpolative Shepard controllers) for linear and non-linear processes of 2nd and 3rd degree and can be implemented in MATLAB-SIMULINK. The processes can be given by state-space (for linear case) or by (4):

$$\dot{x} = A \cdot x(t) + B \cdot g(u)\,, \tag{4}$$

where g(u) represents a non-linear dependency.

IV. STABILITY ANALYSIS

In order to exemplify the numerical procedure described above, a process model and a fuzzy control system from [6] are considered as starting point. The process is given by the state-space equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot g(u) \qquad (5)$$

where $g(u)$ is the piecewise linear function represented in Fig.3. The function presents insensibility and saturation zones.

The process given in (5) is introduced into control structures with fuzzy and Shepard interpolative controllers, respectively. Both control structures has the aim to bring the state of the system to zero. For the stability analysis of the interpolative-type structures (fuzzy and Shepard ones) mentioned in the above chapters some functions from the software package were used.

These functions are dedicated to choose and calculate the derivative of the Lyapunov candidate function. For our given structures, using corresponding functions and following the procedural algorithm, from

$$\dot{V} \leq 0 , \qquad (6)$$

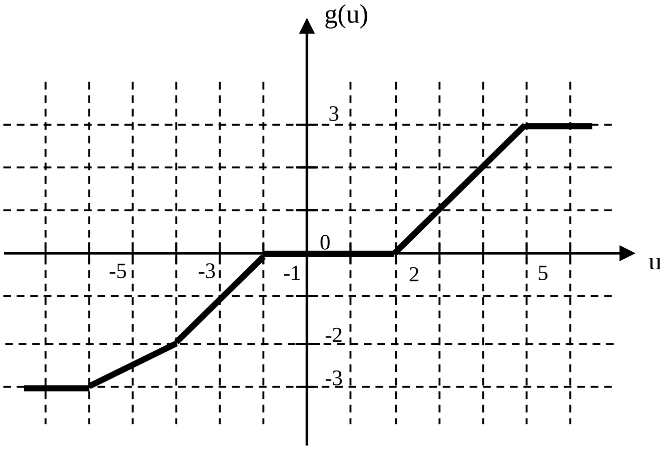

Fig.3. The non-linearity $g(u)$

for the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \qquad (7)$$

the final conditions result:

$$\begin{cases} p_{11} \geq p_{22} \\ p_{11} \leq 6p_{21} + 6p_{12} + 13p_{22} \\ p_{12} + p_{21} \leq 14p_{22} \end{cases} . \qquad (8)$$

Adopting the matrix (9) accordingly with relations (8):

$$P = \begin{bmatrix} 23 & 7 \\ 5 & 10 \end{bmatrix}, \qquad (9)$$

the Lyapunov candidate function is

$$V(x) = x^T \cdot P \cdot x = 23x_1^2 + 12x_1 \cdot x_2 + 10x_2^2 \qquad (10)$$

and its derivate is given as follows:

$$\dot{V}(x) = -24x_1^2 - 6x_1 \cdot x_2 - 8x_2^2 + g(u) \cdot (24x_1 + 40x_2) \qquad (11)$$

The calculus of the first derivative $\dot{V}(x)$ in every point of the state space $[-1,1] \times [-1,1]$ was made using different MATLAB functions implemented for the fuzzy structure and for the interpolative Shepard one.

The results are shown in Fig.5. In Fig.4 is given the signal provided by the signal generator SG, based on the principle mentioned above.

The measurement x_2 sweeps the admissible domain by a saw-tooth signal, while x_1 goes through the admissible domain by a step signal. This covering manner for the admissible time domain in which the state measurements varies from minimum to maximum explain the first derivative $\dot{V}(x)$ variation in Fig.5.

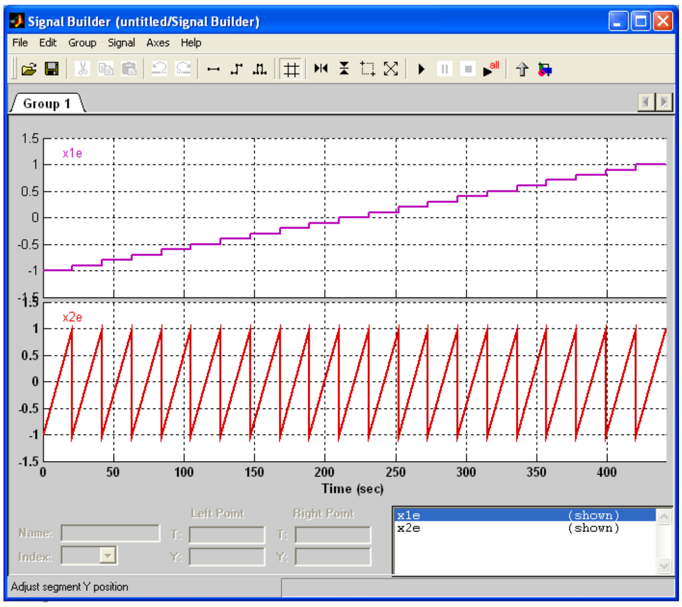

Fig.4. Measurements x_1 and x_2 given by Signal Generator SG.

The results available in Fig.5 and identical for both fuzzy and Shepard-interpolative structures.

The results illustrates also that the chosen candidate function is a Lyapunov function because it meets the given conditions and excepting the origin point, its derivative is negative. Implicitly, by this was demonstrated that the origin is an asymptotic stable equilibrium point for both fuzzy and Shepard interpolative studied systems, according the Unified Stability Theorem.

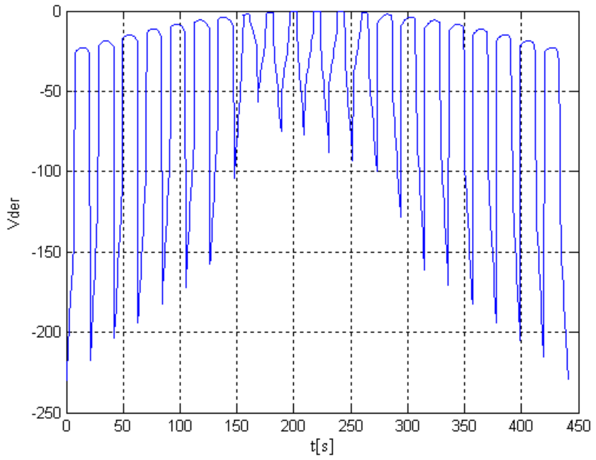

Fig.5. $\dot{V}$ representation for the case in which the states sweep their maximum admissible domains as in Fig. 4.

V. CONCLUSIONS

The paper describes a complete methodology for a Lyapunov stability analysis for interpolative-type control structures, with emphasis on the implementation and applicative results.

The foundation of the presented methodology is the enunciation of the Unified Stability Theorem for a class of fuzzy and Shepard interpolative control structures and the development of some methodological aspects gathered as a Procedural Algorithm. Based on this algorithm a package of software programs developed in MATLAB-SIMULINK was implemented. These programs are dedicated to the stability analysis of a class of non-linear systems, in both fuzzy and interpolative (Shepard) control implementation, through the direct method of Lyapunov.

The developed stability analysis software packages were tested on a non-linear study case. The obtained results are important because the stability analysis of an interpolative system - and even more, the possibility to prove the stability of an interpolative system represents a first attempt in this research field.

Choosing a Lyapunov candidate function can represent a challenge in most of the cases. The presented Procedural Algorithm did not always provide a solution (the conditions can be each other in conflict), and does not provide a unique solution, excepting the case when all the conditions are verified (through an analytical procedure this is impossible, they could be determined only the conditions for some chosen points from the state space).

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