Decision based on lattice order preference structure

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***Abstract*—Decision making alternatives are sometimes incomparable with each other and violate the completeness axiom of the rational behavior axiom of von Neumann–Morgenstern. Lattice order theory has therefore been used to deal with the incomparability issue among alternatives. This paper proves that the incomparable relation of the set of alternatives in same level under lattice order structure is equal to equivalence relation. The alternatives set with lattice ordered structure is classified as subsets of alternatives in terms of their corresponding location layers. The subsets make up a chain structure under the preference relations of equivalence classes, which realizes the chain-forming of preference structure, and transforms the lattice order structure to total order or asymmetric weak order structure. A new decision making method based on lattice-ordered preference structure is proposed and the proposed method is efficient to make decisions for the incomparable paired alternatives in the decision making alternatives set.**

Index Terms - Decision analysis; Lattice order; Preference structure; Equivalence relation; completeness axiom.

I. INTRODUCTION

The axioms of rational behavior initially proposed by von Neumann and Morgenstern [1] (in short VNM-rationality) pointed out that the preference relations of decision makers usually satisfy the following four characteristics such as completeness, transitivity, independence, continuity. However, the four axioms of VNM-rationality were challenged by scholars over the years. For instance, Allais [2-3] designed the famous Allais paradox, and pointed out that independence axiom could not always be satisfied. Flood [4], May [5], MacCrimmon and Larsson [6] and Fishburn [7] questioned the transitivity axiom because of the universal existence of preference cycle under multicriteria decision making environment. Georgescu [8], and Chipman [9] designed decision problems with possible infinitely many good alternatives or infinite bad alternatives to challenge the continuity axiom. Aumann [10] developed a utility theory that parallels the von Neumann-Morgenstern utility theory without completeness axiom.

Among the four axioms, the completeness axiom of VNM-rationality ensures each pair of the decision-making alternatives to be comparable. However, this axiom sometimes may be difficult to be satisfied due to the cognitive limitations of decision-makers, the incompleteness of information set, or the capacity constraints of decision-makers, which usually happens in the real world decision making, especially in the multicriteria decision making environment. That is, different criteria sometimes can cause the alternatives to be incomparable. For instance, a biography is incomparable with a novel since the criteria used are completely different. Generally, incomparability appears in such situations that one alternative has outstanding performance on some criteria and very bad performance on others [11]. Besides, there might exist a lot of incomparable alternatives when the scale of preferences is intangible and expressed by verbal variables. In such cases, preferences sometimes are hard to be quantified and ranked by worst or best, which makes it difficult for decision makers to judge the preference relations among the incomparable alternatives. Therefore, the completeness axiom has again been widely studied. For instance, Ok [12] studied the problem of representing an incomplete preference relation by means of a vector-valued utility functions. Maccheroni [13] investigated that how Yaari's dual theory of choice under risk naturally could extend to the case of incomplete preferences. Mandler [14] analyzed the link between completeness and transitivity and the assumption of rational self-interest, and concluded that if psychological preferences are incomplete, the revealed preferences can be intransitive without exposing agents to manipulations or violating outcome rationality. Mandler [15] used an agent's strict preferences to define indifference and incompleteness relations. Karni [16] studied the connections between continuity and completeness under alternative conceptions of preference relations. Dubra [17] discussed a potentially incomplete preference relation and continuity over lotteries under risk. Evren and Ok [18] developed the ordinal theory of (semi) continuous multi-utility representation for incomplete preference relations. Simply speaking, when decision makers are not able or refuse to express a preference or an indifference to two alternatives a and b due to lack of data or contradictory information, then incomparability relation between these two alternatives occurs. However, in practice, the related information set on the incomparable paired alternatives is usually not so poor that we even cannot determine their supremum and infimum. If we can determine the supremum and the infimum of the paired incomparable elements, the preference structure becomes a lattice that is a partially ordered set with unique least upper bounds and greatest lower bounds. All lattices will form a chain, a special form of the set with total order or asymmetric weak order. Therefore, Guo et al. [19] extended the total order in rational behavior axiom system of Von Neumann-Morgenstern to the lattice-ordered sequence, and built the behavior axiom system of lattice-ordered decision-making. Thus the lattice-ordered can reflect the preference structure of decision-makers more precisely than total-ordered. Since then, the lattice-ordered decision making method has been further studied [19-20].

The goal of this paper is to apply lattice theory [21] to deal with the incomparable elements in a partially ordered set by determining their unique least upper bounds (called supremum) and greatest lower bounds (called infimum) instead of considering the completeness axiom of preference relation. Specifically, a new lattice-ordered decision making method is proposed to handle the incomparable paired elements by considering the classification of equivalence classes of decision making alternatives that possess lattice-ordered preference structure. The steps of the proposed method include:

1) Establishing the equivalence classes of all paired elements;

2) Classifying the equivalence classes of incomparable elements;

3) Constructing the chain sequences using the preference relations of each equivalence class.

Through these steps, the chain-forming of the lattice-ordered preference structure can be obtained. Thus the preference structure can be transformed to total order or asymmetric weak order, and the lattice-ordered structure is transformed to total order or asymmetric weak order.

The rest of this paper is organized as follows. The next section briefly describes the preliminaries on lattice order theory. In Section 3, chain-forming of lattice-ordered preference structure is presented. An example on ranking eight alternatives is used to illustrate the proposed lattice-ordered decision making method in Section 4. Section 5 concludes the paper and discusses the future research directions.

II. PRELIMINARIES

Lattice order refers to a partially ordered set with unique least upper bounds (also called supremum) and greatest lower bounds (also called infimum) [21]. To understand the proposed method, the following definitions are firstly presented.

Definition 1: Suppose $(A,\leq)$ is a poset, $S\subseteq A$, $x\in A$. If $\forall s\in S$ and there is $s\leq x$, x is said to be an upper bound of S. If $\forall s\in S$ and $x\leq s$, x is defined as a lower bound of S. The set $S^u=\{x\in A|\forall s\in S, s\leq x\}$ of all upper bounds of S is called the upper bound of S; The set $S^l=\{x\in A|\forall s\in S, x\leq s\}$ of all lower bounds of S is called the lower bound of S. If x is the smallest element of S^u, x is defined as the supremum of set S, and written as: $\sup S=\bigvee_{s\in S} s$ .If y is the largest element of S^l, y is defined as the infimum of set S, and written as: $\inf S=\bigwedge_{s\in S} s$.

Definition 2: Let $(A,\leq)$ be a poset. If $\forall x$, $y\in A$, there exist the supremums and infimums of x and y, then A forms a lattice with respect to the the decision maker's preference relation "≤", and "≤" is defined as a lattice order of A.

Definition 3: Each ordered subset of the lattice is a chain of the lattice. If a chain of a lattice is not included in any other chains, the chain is defined as a maximum chain.

The above three definitions are applied to chain-forming of lattice-ordered preference structure.

III. CHAIN-FORMING OF LATTICE-ORDERED PREFERENCE STRUCTURE

As reviewed in the introductory part, many studies have shown that the completeness axiom is difficult to be satisfied due to the lack of information or indecisiveness of a single decision maker. Therefore, during the process of making decisions, there may exist some paired elements in a poset that are incomparable with each other. Although these elements are incomparable, in this paper, it is assumed that decision makers can determine their common upper bounds and lower bounds. Particularly, if the supremum and infimum of any pairs of elements is determined, a lattice is then formed by *Definition 2*, and a chain is finally formed by lattices in terms of *Definition 3*. Therefore, such preference relation is with the characteristic of the lattice-ordered or lattice-ordered preference relation, and the corresponding decision making model possesses the lattice-ordered preference structure.

A. Incomparability of the lattice elements in the same layer

Based on the concept of a lattice, the following remark is presented and proved.

Remark 1: *The lattice elements in the same layer are incomparable with each other.*

Proof by contradiction: Suppose a_1 and a_2 are two of the lattice elements in the same layer, there must exist a chain between a_1 and a_2 if they are comparable. Let $a_1\leq a_2$, the length of maximum chain from a_1 to the best element is larger than the length of maximum chain from a_2 to the best element, which contradicts to the condition that they are located at the same layer. Therefore, the lattice elements in the same layer are incomparable with each other.□

B. Determination of incomparable equivalence relation in the same layer

The equivalence relation is an important relation in mathematics. Once an equivalence relation is determined in a set A, the set is partitioned and decomposed into a union of all equivalence classes whose intersections are empty. Thus the study on certain questions can be attributed to the study on the representative elements of a subset whose elements are from each equivalence class set. In the context of multi-criteria or multi-attribute decision making, the decision makers need to judge the priority of decision alternatives with respect to different criteria or attributes. In this paper, we mainly focus

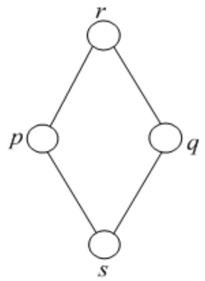


Fig. 1. The relation of two elements located at same layer and their bounders

on such set of alternatives that there exists some incomparable alternatives (at least two pairs of alternatives are incomparable), as Remark 1 stated that the lattice elements in the same layer are incomparable with each other. Therefore, in general, we assumed A is the set of decision-making alternatives, and it is important to analyze the equivalence relation in an incomparable alternative set.

Apply the equivalence relation to the theory of lattice-ordered structure, we can get the following theorem:

Theorem 1: *Suppose set A={$a_1, a_2, \ldots, a_n$} is a decision-making alternative set with lattice-ordered preference structure, and that single alternative itself satisfies the incomparable relation: $a_i \| a_i$ (i=1,…,n), where symbol "‖" denotes incomparable relation between two alternatives. According to the incomparability relation of the same layer, set A can be partitioned as m classes: A_i (i=1,2,..,m; m<n). The relation "‖" in the same layer is an equivalence relation, and A_i is an equivalence class with respect to set A.*

Proof: ①It is obvious from the assumption that the reflexivity holds. ②For any $a_i, a_j \in A$ in the same layer, if $a_i \| a_j$, then $a_j \| a_i$, thus the symmetry holds. ③For any $a_i, a_j, a_k \in A$ in the same layer, if $a_i \| a_j$ and $a_j \| a_k$, then $a_i \| a_k$, thus the transitivity holds.

In summary, A_i is an equivalence class of set A, the relation "‖" in the same layer is an equivalence relation of set A.

According to the equivalence relation, set *A* can be partitioned as *m* classes, and the following conditions: $A_1 \cup A_2 \cup \cdots \cup A_m = A$ and $A_1 \cap A_2 \cap \cdots \cap A_m = \phi$ hold.□

C. Chain-forming of lattice-ordered structure

In the axiom system of lattice-ordered decision making behavior, the axiom of completeness of the lattice is described as below:

Let P be a prospect space, for $\forall p, q \in P$, $\exists r, s \in P$, there exist $p \vee q = r$ and $p \wedge q = s$. If p and q locate at the same layer, the relation of the four elements is shown in Fig. 1.

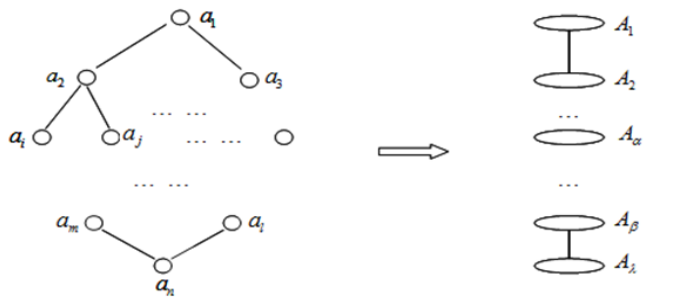
By *Theorem 1*, p and q can make up an equivalence class of A_1, that is, $A_1 = \{p, q\}$, and p is equivalent to q.

In the decision making alternatives set that has lattice-ordered preference structure, the elements in the same layer are satisfied with the incomparable equivalence relation. Therefore, the decision making alternatives can be classified in terms of the layers where they are located, and the elements in each layer are equivalent. According to the priority relations among all equivalence classes, the decision making alternatives set can be ranked as a chain sequence in order to transform the preference structure to the form of chain. The lattice-ordered can be transformed to a total order or asymmetric weak order to make the final decision. Therefore, the following theorem can be derived.

Theorem 2: Suppose the decision making alternatives set with lattice-ordered preference structure is $A = \left\{ \bigcup_{i=1}^{n} a_i \right\}$, where *n* is either finite or infinite numbers, then the elements in set *A* can be partitioned as finite or infinite λ classes: A_1, $A_2, \cdots, A_\lambda$, where λ denotes the number of layers, $A_1 \cup A_2 \cup \cdots \cup A_\lambda = A$, and $A_1 \cap A_2 \cap \cdots \cap A_\lambda = \phi$. The set $A = \{A_1, A_2, \cdots, A_\lambda\}$ is defined as the quotient of A by equivalence class. The elements in the same layer are equivalent to each other. The priority relations of all classes are ranked as chain sequence, and the corresponding preference relation is called total order or asymmetric weak order.

Proof: Since it is assumed that the decision making alternatives set $A = \{a_1, a_2, \cdots, a_n\}$ is with lattice-ordered preference structure, thus the set A must contain some incomparable alternatives, i.e., the incomparable relation $a_i \| a_j$ ($i,j \leq n$) will hold for certain pairs of alternatives in set A. By *Remark 1* and *Theorem 1*, the set A can be partitioned as λ classes: A_i *(i=1,2,.., λ; λ<n)*, and the class A_i is an equivalence class with respect to set A. By the property of equivalence class, we have $A_1 \cup A_2 \cup \cdots \cup A_\lambda = A$, and $A_1 \cap A_2 \cap \cdots \cap A_\lambda = \phi$, in which the elements in the same class A_i are equivalent, and a chain sequence can be formed by the different priority relations of all classes, and the set $A = \{A_1, A_2, \cdots, A_\lambda\}$ forms the quotient of A by equivalence class. □

To further demonstrate this theorem, an example with finite number of alternatives is introduced in the following. Assume there exist λ pairs of incomparable alternatives a_2 and a_3, a_i and a_j, $\cdots, a_m$ and a_l in a decision-making alternatives set $A = \{a_1, a_2, \cdots, a_n\}$. According to the equivalence relation of incomparability of the elements in the same layer, the elements in A can be partitioned as several finite classes： A_1, A_2, $\cdots$, A_α, $\cdots$, A_β, A_λ, in which $A_1 = \{a_1\}$, $A_2 = \{a_2, a_3\}, \ldots$, $A_\alpha = \{a_i, a_j\}, A_\beta = \{a_m, a_l\}$,



(a) Lattice-ordered preference structure (b) Chain structure of preference

Fig. 2. The transformation of preference structure

$A_\lambda = \{a_n\}$. Therefore, $A_1 \cup A_2 \cup \cdots \cup A_\alpha \cup \cdots \cup A_\beta \cup A_\lambda = A$ and $A_1 \cap A_2 \cap \cdots \cap A_\alpha \cap \cdots \cap A_\beta \cap A_\lambda = \phi$. Since the elements in the same layer are equivalent to each other, the priority relations among all classes can be ranked as the chain sequence , and the preference structure can be transformed to total order or asymmetric weak order.

The transformation of the preference structure is shown in Fig.2, where Fig. 2(a) denotes the lattice-ordered preference structure, and the preference structure in Fig. 2(b) represents chain structure. Assume the preferences in both Figures are ranked from the best (the upper layer) to the worst (the lower layer), after the chain-forming, the preference priority of each class is $A_1 \succ \cdots \succ A_\alpha \succ \cdots \succ A_\beta \succ A_\lambda$. Therefore, the best alternative set in set *A* is A_1 in the upper layer.

IV. ILLUSTRATIVE EXAMPLE

To illustrate the proposed method, an example on ranking eight alternatives with incomparable relations is introduced in this section. Assume a decision maker needs to rank eight alternatives in terms of the priorities of alternatives, denoted as $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ and a_8, respectively. The preference relations among the alternatives are further assumed as follows:

$$a_7 \succ a_3 \succ a_2 \succ a_1,\ a_7 \succ a_5 \succ a_2 \succ a_1,$$
$$a_7 \succ a_5 \succ a_4 \succ a_1,\ a_8 \succ a_5 \succ a_2 \succ a_1,$$
$$a_8 \succ a_5 \succ a_4 \succ a_1,\ a_8 \succ a_6 \succ a_4 \succ a_1$$
$$a_2 \| a_4, a_2 \| a_6,\ a_3 \| a_4,\ a_3 \| a_5,\ a_3 \| a_6,$$
$$a_3 \| a_8,\ a_5 \| a_6, a_6 \| a_7, a_7 \| a_8$$

where the symbol "‖" represents that the paired comparison elements are incomparable with each other, and "$\succ$" denotes "prefer to" (e.g. A$\succ$ B means A is preferred to B).

The related preference structure can be expressed as Hasse Graph, as shown in Fig.3.

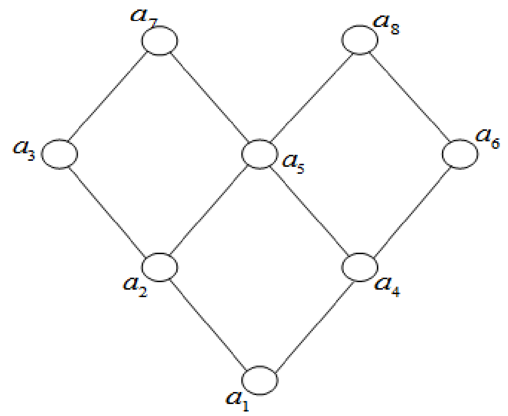


Fig. 3. Hasse Graph of preference structure

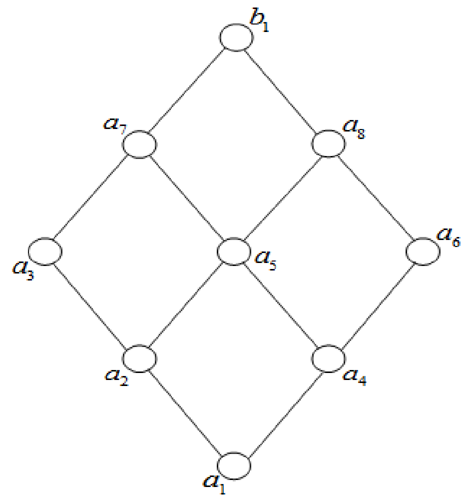


Fig. 4. The created full lattice of the expanded alternatives set

Fig.3 shows that the preference relation of alternatives set $A = \{a_1, a_2, \cdots, a_8\}$ is a partial order relation, and for any two alternatives a_i , a_j $(1 \le i \ne j \le 8)$, their corresponding infimum exists, thus if $a_i \succ a_j$, then $a_i \wedge a_j = a_j$. By this formula, the incomparable paired alternatives in Fig.3 and their infimums include:

$$a_2 \wedge a_4 = a_1,\ a_2 \wedge a_6 = a_1,\ a_3 \wedge a_4 = a_1,$$
$$a_3 \wedge a_5 = a_2,\ a_3 \wedge a_6 = a_1, a_3 \wedge a_8 = a_2,$$
$$a_5 \wedge a_6 = a_4,\ a_6 \wedge a_7 = a_4,\ a_7 \wedge a_8 = a_5$$

Therefore, the alternatives set *A* forms a "$\wedge -$" semilattice under the assumed preference relation.

To transform the semilattice structure to a lattice structure, a virtual alternative b_1 is added to the existing eight alternatives. The new alternatives set makes up an expansible alternatives set of the previous set, but the order relation of the original alternatives set is unchanged. Thus, the expanded alternatives set forms a full lattice, as shown in Fig. 4.

For any two alternatives, the supremum and infimum exist if they are comparable. For the incomparable alternatives, we can get the following results:

The infimums:

$$a_2 \wedge a_4 = a_1,\ a_2 \wedge a_6 = a_1,\ a_3 \wedge a_4 = a_1,$$
$$a_3 \wedge a_5 = a_2,\ a_3 \wedge a_6 = a_1,\ a_3 \wedge a_8 = a_2,$$
$$a_5 \wedge a_6 = a_4,\ a_6 \wedge a_7 = a_4,\ a_7 \wedge a_8 = a_5$$

The supremums:

$$a_2 \vee a_4 = a_5,\ a_2 \vee a_6 = a_8, a_3 \vee a_4 = b_2,$$
$$a_3 \vee a_5 = a_7,\ a_3 \vee a_6 = b_1,\ a_3 \vee a_8 = b_1,$$
$$a_5 \vee a_6 = a_8,\ a_6 \vee a_7 = b_1,\ a_7 \vee a_8 = b_1$$

According to the equivalence relation of the incomparability of the elements in the same layer, the expanded alternatives set can be classified layer-by-layer: $A_1 = \{b_1\}, A_2 = \{a_7, a_8\}, A_3 = \{a_3, a_5, a_6\}$, $A_4 = \{a_2, a_4\}$, $A_5 = \{a_1\}$, such that $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = A$ and $A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 = \phi$. In A_2, alternatives a_7 and a_8 are equivalent; in A_3, alternatives a_3, a_5 and a_6 are equivalent; while alternatives a_2 and a_4 is equivalent to each other in A_4. The ranking result of the alternatives set A, $A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$, forms a chain sequence and transforms lattice-ordered to total order or asymmetric weak-order. The order preference structure is shown in Fig.5.

Although we assumed that $a_2 \| a_4$, $a_2 \| a_6$, $a_3 \| a_4$, $a_3 \| a_5$, $a_3 \| a_6$, $a_3 \| a_8$, $a_5 \| a_6$, $a_6 \| a_7$, $a_7 \| a_8$, after deleting the virtual alternative set A_1, it can be concluded from the Fig.5 that $A_2 = \{a_7, a_8\} \succ A_3 = \{a_3, a_5, a_6\} \succ$ $A_4 = \{a_2, a_4\} \succ$ $A_5 = \{a_1\}$, indicating alternatives a_2 and a_4 is equivalent, alternative a_6 is preferred to a_2, alternative a_3 is preferred to a_4, alternative a_3 is equivalent to a_5 and a_6, alternative a_7 is preferred to a_6, while alternatives a_7 and a_8 are equivalent, therefore, these incomparable alternatives become comparable by the proposed lattice-ordered decision making.

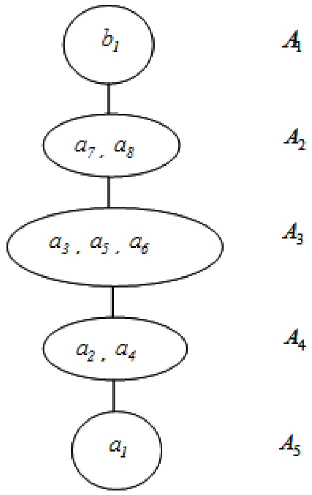


Fig. 5. The chain sequence of expanded alternatives set

V. CONCLUSIONS

The incomparable paired alternatives usually occur in the multicriteria decision making environment. In this paper, a new decision making method based on lattice-ordered preference structure is proposed to deal with the incomparable paired alternatives. Specifically, the decision making alternatives set that has lattice-ordered structure is classified as subsets layer-by-layer in terms of the incomparable equivalence relation, and the subsets (classes) is ranked as a chain sequence in terms of their preference relations. Then, the chain-forming of preference structure of lattice-ordered is formed. The preference structure of lattice-ordered is transformed to total-ordered or asymmetrical weak-ordered. Simply speaking, in this paper, the relation of incomparable elements in the same class is proved to be an equivalence relation. Based on this equivalence relation, we propose to group the decision alternatives into different classes, then all the decision alternatives can form a chain sequence and be ranked by the priorities of all the equivalence classes. Therefore, decision makers only need to select the decision making alternatives from the best alternatives set in the chain sequence. The proposed method is efficient to make decisions and simplifies the processes of decision making for the incomparable paired alternatives, which provides a new way to deal with the decision problem that contains incomparable decision alternatives.

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