Differential Equation 1

True_hOREP

Contents

1 The Differential Equation

Let's try to solve the following initial value problem. Given that

$$y'(x) = y(x)\sin(xy)e^{-xy} \tag{1.1}$$
$$y(0) = y_0 \tag{1.2}$$

then find the behaviour of y as x approaches infinity. If $y \to L$, then find L.

1.1 Observations

1. If $y_0 = 0$, then $y \equiv 0$. It follows that $y \to 0$ as $x \to \infty$.
2. If $y_0 > 0$ then $y \to L$ for some $y_0 + \frac{1}{2} > L > y_0$ as $x \to \infty$.
3. As $y_0 \to \infty, L \to y_0 + \frac{1}{2}$.
4. If $y_0 < 0$ then $y \to 0$ as $x \to 0$.

1.2 Existence and Uniqueness

Let $y_0 > 0$. By the Peano existence theorem, since $f(x, y) = y\sin(xy)e^{-xy}$ is continuous we have that there exists a local solution around $(0, y_0)$. Now, for $x, y \geq 0$ we have

$$\left|\frac{\partial f}{\partial y}\right| = \left|e^{-xy}\sin(xy) - xye^{-xy}\sin(xy) + xye^{-xy}\cos(xy)\right| \tag{1.3}$$
$$\leq e^{-xy} + xye^{-xy} + xye^{-xy} \tag{1.4}$$
$$= (1 + 2xy)e^{-xy}. \tag{1.5}$$

Clearly if x or $y = 0$ then $(1 + 2xy)e^{-xy} = 1$. Seeking to maximise this on $x, y > 0$, we can find the stationary points as being on the curve $y = \frac{1}{2x}$. Then

$$\left|\frac{\partial f}{\partial y}\right|_{y=\frac{1}{2x}} = \left(1+2x\left(\frac{1}{2x}\right)\right)e^{-x\left(\frac{1}{2x}\right)} \tag{1.6}$$

$$= (1+1)\,e^{\left(-\frac{1}{2}\right)} \tag{1.7}$$

$$= \frac{2}{\sqrt{e}}. \tag{1.8}$$

It follows that $\sup_{x,y\geq 0}\left|\frac{\partial f}{\partial y}\right|$ is non-zero and bounded above, so exists and is positive in that

$$1 \leq \sup_{x,y\geq 0}\left|\frac{\partial f}{\partial y}\right| \leq \frac{2}{\sqrt{e}} \tag{1.9}$$

and from the Mean-Value theorem f has a Lipschitz condition on any non-negative rectangle and there exists a unique solution to the differential equation by Picard-Lindelof.

1.3 Geometric Intuition

Looking for the stationary points of y, where $y'(x) = 0$, we have

$$y(x)\sin(xy)e^{-xy} = 0 \tag{1.10}$$

and assuming that $y \neq 0$ we must have for $n \in \mathbb{Z}$

$$\sin(xy) = 0 \quad \implies xy = n\pi \tag{1.11}$$

and from that

$$y = \frac{n\pi}{x}. \tag{1.12}$$

Qualitatively, when $y_0 > 0$, y increases until it touches $y = \frac{\pi}{x}$, and then starts decreasing, until it touches $y = \frac{2\pi}{x}$, and then increases until $\frac{3\pi}{x}$ and so on. This quickly traps y since the distance between the n^{th} and $(n+1)^{\text{th}}$ pi curve quickly decreases for large x, and at large x values $y' \to 0$ due to the exponential term. When $y_0 < 0$, y increases very quickly due to the exponential term.

1.4 Calculated Values

These limits have been approximated using mpmath's odefun method, which uses a high order Taylor method.

$$L(y_0 = 1) = 1.4173018980313441078\dots < 1 + \frac{1}{2}. \tag{1.13}$$

```
import mpmath as mp
mp.mp.dps = 60
f = mp.odefun(lambda x, y: y*mp.sin(x*y)*mp.exp(-x*y), 0, 1)

print(mp.limit(f, mp.inf, verbose=True))
```

In table 2, looking at when $y_0 = 10^n$ for some $n \in \mathbb{Z}^+$, there is a pattern apparent in the decimal expansion, with $(n-1)$ 9's after the four, then 875, then $(n-2)$ zeroes, and then a sequence beginning with 71314. We can then make an approximate guess as

$$L(y_0 = 10^n) \approx 10^n + \frac{1}{2} - 125 \cdot 10^{-(n+3)} \tag{1.14}$$

$$= 10^n + \frac{1}{2} - \frac{1}{8}10^{-n}. \tag{1.15}$$

From this we can conjecture that

$$L(y_0) \approx y_0 + \frac{1}{2} - \frac{1}{8y_0}. \tag{1.16}$$

This makes sense for $y_0 > \frac{1}{4}$, since below this the approximation is in $(0, y_0)$, which we have observed to not be true.

y_0	L
0.01	0.10361281668075782384
0.02	0.15436636860328459555
0.03	0.19002796830301691716
0.04	0.21933982991761656316
0.05	0.24496968471867051292
0.06	0.26814995536882772512
0.07	0.28956530781142192081
0.08	0.3096402710930634373
0.09	0.32865829251129756917
0.1	0.34681903645663355187

Table 1. y_0 close to zero

y_0	L
$\ln(2)$	1.08799766343773913078037154317
G	1.32817726272330443105514325
1	1.4173018980313441078260787177
ϕ	2.05981190346327134245359710724
e	3.17995457030405286078497413654
π	3.60769768188523276002491652985
10	10.4881651704230857861505591043
100	100.498757080024119681228967152
1000	1000.49987507126297947462768742
10000	10000.4999875007130962746825725
100000	100000.4999987500071314296042
1000000	1000000.49999987500007131476294

Table 2. Special Cases

2 Asymptotic Behaviour

Assuming that $y \to L$ as $x \to \infty$, we can write for sufficiently large x

$$y' = y\sin(xL)e^{-xL} \tag{2.1}$$

and get

$$y(x) = ce^{\frac{-e^{-Lx}}{\sqrt{2}L}\sin\left(Lx+\frac{\pi}{4}\right)}. \tag{2.2}$$

Using the assumption that $y \to L$ as $x \to \infty$, we get $c = L$ and thus

$$y(x) = Le^{\frac{-e^{-Lx}}{\sqrt{2}L}\sin\left(Lx+\frac{\pi}{4}\right)}. \tag{2.3}$$

This copies much of the behaviour of the actual solution and is close to the actual $y(0) = y_0$ initial condition as

$$y_0 \approx L e^{\frac{-e^{-0}}{\sqrt{2}L}\sin\left(\frac{\pi}{4}\right)} \tag{2.4}$$
$$= L e^{\frac{-1}{2L}} \tag{2.5}$$
$$= L\left(1 - \frac{1}{2L} + \frac{1}{8L^2} - \frac{1}{48L^3} + O\left(L^{-4}\right)\right) \tag{2.6}$$
$$= L - \frac{1}{2} + \frac{1}{8L} - \frac{1}{48L^2} + O\left(L^{-3}\right). \tag{2.7}$$

and using our earlier approximation $L \approx y_0 + \frac{1}{2} - \frac{1}{8y_0}$ we get

$$L - \frac{1}{2} + \frac{1}{8L} - \frac{1}{48L^2} \approx y_0 + \frac{1}{2} - \frac{1}{8y_0} - \frac{1}{2} + \frac{1}{8\left(y_0 + \frac{1}{2} - \frac{1}{8y_0}\right)} \tag{2.8}$$
$$= y_0 - \frac{1}{8y_0} + \frac{1}{8y_0} - \frac{1}{16y_0^2} + O(y_0^{-3}) \tag{2.9}$$
$$= y_0 - \frac{1}{16y_0^2} + O(y_0^{-3}). \tag{2.10}$$

If assume that we have our function is close enough to y_0 at $x = 0$, then we can write

$$y_0 \approx L e^{\frac{-1}{2L}} \quad \implies \frac{1}{2L} e^{\frac{1}{2L}} \approx \frac{1}{2y_0} \tag{2.11}$$

and the inverse of this can be written with the Lambert W function as

$$L \approx \frac{1}{2W\left(\frac{1}{2y_0}\right)}. \tag{2.12}$$

For confirmation of our earlier approximation, we have for sufficiently large x [1][2],

$$\left[W\left(\frac{1}{x}\right)\right]^{-1} = x + 1 - \frac{1}{2x} + \frac{2}{3x^2} - \frac{9}{8x^3} + \frac{32}{15x^4} + O\left(x^{-5}\right) \tag{2.13}$$
$$= \sum_{n=-1}^{\infty} \frac{(-n)^n}{(n+1)!} x^{-n} \tag{2.14}$$

and thus

$$L \approx \frac{1}{2W\left(\frac{1}{2y_0}\right)} = y_0 + \frac{1}{2} - \frac{1}{8y_0} + \frac{1}{12y_0^2} - \frac{9}{128y_0^3} + \frac{1}{15y_0^4} + O\left(y_0^{-5}\right) \tag{2.15}$$
$$= \frac{1}{2}\sum_{n=-1}^{\infty} \frac{(-n)^n}{(n+1)!}(2y_0)^{-n}. \tag{2.16}$$

For a specific example:

$$L(2) = 2.4506956305254182795706 4\cdots \tag{2.17}$$
$$\frac{1}{2W\left(\frac{1}{4}\right)} = 2.4523225013522873571418 1\cdots. \tag{2.18}$$

This is about 0.07% error.

3 Sequence Approach

Define the sequence of functions for $n \in \mathbb{Z}^+$ by

$$\varphi_0(x;L) = L \tag{3.1}$$
$$\varphi'_{n+1}(x;L) = \varphi_{n+1}\sin(x\varphi_n)e^{-x\varphi_n} \tag{3.2}$$
$$\varphi_{n+1}(x;L) = L\exp\left(-\int_x^\infty \sin(t\varphi_n(t))e^{-t\varphi_n(t)}\,\mathrm{d}t\right). \tag{3.3}$$

Then $\varphi_1(x;L)$ is simply described by the derivation above, given by

$$\varphi_1(x;L) = Le^{\frac{-e^{-Lx}}{\sqrt{2}L}\sin\left(Lx+\frac{\pi}{4}\right)}. \tag{3.4}$$

Given that $\varphi_n(0;L) \approx y_0$, the hope would be to perform the integration analytically, and then either invert for L (unlikely except for φ_0, φ_1) or apply a method such as Newton's method to find L. We can find L as the value that satisfies

$$y_0 = L\exp\left(-\int_0^\infty \sin(t\varphi_n(t;L))e^{-t\varphi_n(t;L)}\,\mathrm{d}t\right). \tag{3.5}$$

This is probably only feasible to do for φ_1, so L is approximately given as the value that satisfies

$$y_0 = L\exp\left(-\int_0^\infty \sin\left(tLe^{\frac{-e^{-Lt}}{\sqrt{2}L}\sin\left(Lt+\frac{\pi}{4}\right)}\right)e^{-tLe^{\frac{-e^{-Lt}}{\sqrt{2}L}\sin\left(Lt+\frac{\pi}{4}\right)}}\,\mathrm{d}t\right). \tag{3.6}$$

I conjecture that the sequence $(\varphi_n)_{n=0}^\infty$ converges to the solution of our desired differential equation.

For the purposes of numerical calculation of L, I would recommend using the Secant method over Newton's method, as the latter requires a derivative. It *is* possible to write a derivative function of $y_0 - \varphi_n(0;L)$, however it requires the Leibniz rule, a huge amount of product rule and chain rule, and results in a hugely complicated function.

3.1 Numerical Analysis

Using the code

```
import mpmath as mp
from mpmath import mpf
mp.mp.dps = 20


def phi_0(x, L):
    return L


def phi_1(x, L):
    return (L
            * mp.exp(-mp.exp(-L*x)*mp.sin(L*x + mp.pi/4)
                     / (mp.sqrt(2)*L))
            )


def phi_2(x, L):
    integral = mp.quad(lambda t: mp.sin(t * phi_1(t, L))
                       * mp.exp(-t * phi_1(t, L)),
                       [x, mp.inf])
    return L * mp.exp(-integral)


def FindL(y_0):
    def f(L):
        return y_0 - phi_2(0, L)
    return mp.findroot(f, y_0)
```

The 2^{nd} order approximation for $L(1)$ can be computed. This is compared to the Lambert W approximation and the actual value below

$$\mathrm{FindL}(1) = 1.41736483928081901506 \tag{3.7}$$

$$L(1) = 1.41730189803134410782 62 \tag{3.8}$$

$$\frac{1}{2W\left(\frac{1}{2}\right)} = 1.42152993588311662685 18. \tag{3.9}$$

This integral method is clearly much more accurate to the actual value than just the Lambert W.

References

[1] https://www.wolframalpha.com/input/?i=1%2F%28W%281%2Fx%29%29 Accessed July 2020

[2] https://en.wikipedia.org/wiki/Lambert_W_function#Integer_and_complex_powers Accessed July 2020