# Pouring a Fizzy Drink

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## 1 Pouring a Fizzy Drink

We will work entirely in terms of height, assuming the volume is made up of circular disks of infinitesimal height stacked upon each other. Let's suppose we have a cylindrical glass of height $H_g \in \mathbb{R}_{>0}$. Let the height of the drink be denoted by $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Let the foam that is formed by the pouring process be denoted by $b : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

Now suppose we wished to fill this glass up from empty to $H_0 \in (0, H_g)$ without the foam spilling out of the top of the glass, *assuming the foam is always on top of the drink*. Then we would require for all times $t \in \mathbb{R}_{\geq 0}$ that the height of the foam and drink are less than the height of the glass,

$$0 \leq h(t) + b(t) \leq H_g \tag{1.1}$$

along with the initial conditions

$$h(0) = b(0) = 0. \tag{1.2}$$

Whenever the foam would spill over the glass, we reduce the foam down to $b = H_g - h$ to simulate overflowing. A diagram of this is shown in Figure 1. We can choose to pour the drink into the glass however we wish, given by some pouring function $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ in the form

$$\frac{\mathrm{d}h}{\mathrm{d}t} = p(t). \tag{1.3}$$

The foam generated when the drink is poured is proportional to the amount being poured in, yielding a term

$$\text{Foam in} : \mu \frac{\mathrm{d}h}{\mathrm{d}t} = \mu p(t) \tag{1.4}$$

where $\mu \in \mathbb{R}^+$ is a dimensionless proportionality constant. The foam "evaporates", leaving the glass, according to some function $q : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$, so that $b$ satisfies the differential equation

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \mu p(t) - q(b). \tag{1.5}$$

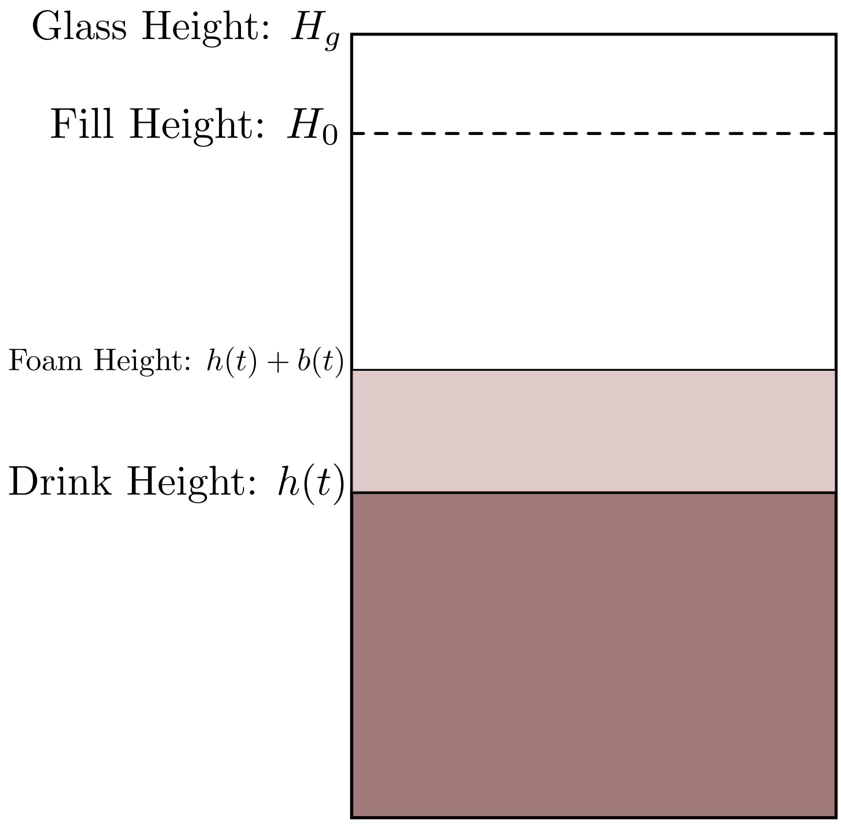



**Figure 1**: Diagram of the glass, partially filled with drink and foam.

### 1.1 Another example

We could try

$$p(t) = |\cos(\ln(1 + t/3))|, \tag{1.6}$$
$$q(t, b) = \sqrt{b} + b^{10}. \tag{1.7}$$

The resulting solution is shown in Figure 2.

Again, the $p, q$ graphs are shown in Figures 3a and 3b

### 1.2 An exact form

Let's suppose that

$$q(t, b) = kb(t) \tag{1.8}$$

for some constant $k > 0$.

Then we have

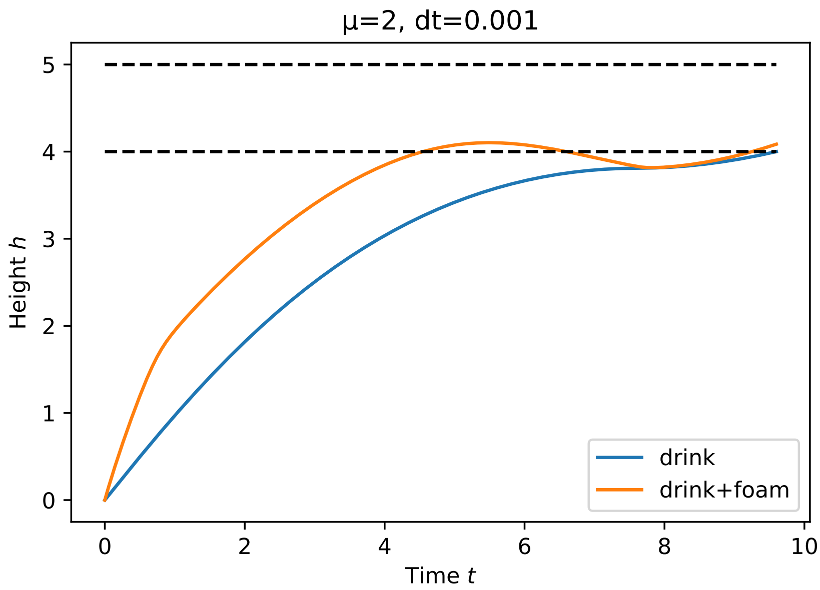
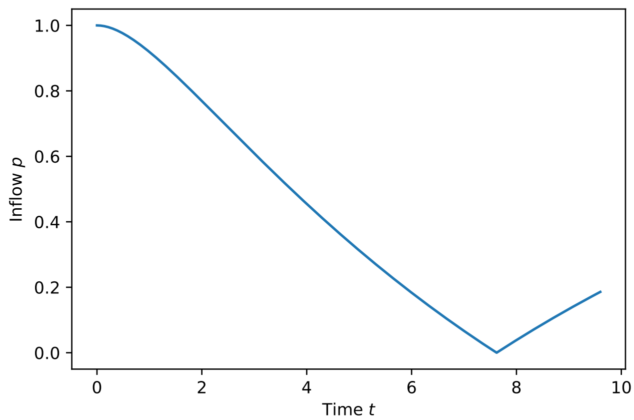
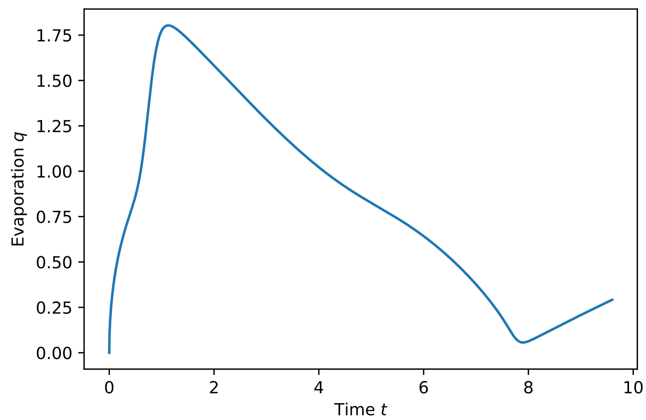



**Figure 2**: Drink and foam height as a function of time. The orange line is the foam height, on top of the drink shown in blue. The time taken to fill the drink to the required height is $\approx 9.599$ units.




(a) Inflow function $p(t)$.




(b) Evaporation function $q(t, b(t))$.

**Figure 3**: Derivative functions.

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \mu p(t) - kb(t). \tag{1.9}$$

Multiplying by an integrating factor $e^{kt}$, we rearrange to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{kt}b(t)\right) = \mu e^{kt}p(t). \tag{1.10}$$

Integrating from 0 to $t$ yields

$$e^{kt}b(t) - \underbrace{b(0)}_{=0} = \mu \int_0^t e^{ky} p(y)\,\mathrm{d}y \tag{1.11}$$

so that $b$ is given by

$$b(t) = \mu e^{-kt} \int_0^t e^{ky} p(y)\,\mathrm{d}y. \tag{1.12}$$

The spill condition (1.1) is then

$$0 \leq \int_0^t p(y)\,\mathrm{d}y + \mu e^{-kt} \int_0^t e^{ky} p(y)\,\mathrm{d}y \leq H_g. \tag{1.13}$$

Now, at the final time $t_\mathrm{f}$ we have

$$-H_0 \leq \mu e^{-kt_\mathrm{f}} \int_0^{t_\mathrm{f}} e^{ky} p(y)\,\mathrm{d}y \leq H_g - H_0. \tag{1.14}$$

The integrand is positive, so we can replace the lower bound with 0,

$$0 \leq \mu e^{-kt_\mathrm{f}} \int_0^{t_\mathrm{f}} e^{ky} p(y)\,\mathrm{d}y \leq H_g - H_0. \tag{1.15}$$

### 1.2.1 Specific example

Let $H_g = 1$ so that $H_0 \in (0, 1)$. We let

$$p(t) = |\cos(t)| \tag{1.16}$$

so that

$$h(t) = \sin(t) \tag{1.17}$$

omitting the "sgn(cos($t$)) factor", which means that the final time occurs at

$$t_\mathrm{f} = \sin^{-1}(H_0). \tag{1.18}$$

Then, using equation (1.12), $b$ is given by

$$b(t) = \frac{\mu(k\cos(t) + \sin(t))}{1+k^2} - \frac{\mu k e^{-kt}}{1+k^2}. \tag{1.19}$$

At the final time $t_\mathrm{f}$ we get

$$b(t_\mathrm{f}) = \frac{\mu\left(k\sqrt{1-H_0^2} + H_0\right)}{1+k^2} - \frac{\mu k e^{-k\sin^{-1}(H_0)}}{1+k^2}. \tag{1.20}$$

For this system to abide by the spill condition (due to $b$ and $h$ monotonically increasing on $(0, t_\mathrm{f})$), we require $\mu, k, H_0$ to satisfy

$$0 \leq H_0\left(1 + \mu + k^2\right) + \mu k\sqrt{1-H_0^2} - \mu k e^{-k\sin^{-1}(H_0)} \leq H_g\left(1+k^2\right) \tag{1.21}$$

## 2 Minimisation

For example, let $\mu = 2, k = 25, H_g = 1, H_0 = 0.8$.

We will once again use the foam function

$$q(b) = kb. \tag{2.1}$$

### 2.1 Constant input

Suppose we tried the pouring function

$$p(t) = c \tag{2.2}$$

for some $c > 0$. Then

$$h(t) = ct \tag{2.3}$$

and

$$b(t) = \mu e^{-kt} \int_0^t e^{ky} c \, \mathrm{d}y = \frac{\mu c}{k} \left(1 - e^{-kt}\right). \tag{2.4}$$

Now

$$(h + b)(t) = c \left[t + \frac{\mu}{k} \left(1 - e^{-kt}\right)\right]. \tag{2.5}$$

The finishing time is given by

$$t_\mathrm{f} = H_0/c. \tag{2.6}$$

The derivative of $h + b$ is given by

$$\frac{\mathrm{d}(h + b)}{\mathrm{d}t} = c\left(1 + \mu e^{-kt}\right) \geq 0 \tag{2.7}$$

so the only maximum that can occur is at time $t_\mathrm{f}$. That means that the drink doesn't spill if and only if

$$(h + b)(t_\mathrm{f}) = c \left[t_\mathrm{f} + \frac{\mu}{k} \left(1 - e^{-kt_\mathrm{f}}\right)\right] \leq H_g. \tag{2.8}$$

That is

$$(h + b)(t_\mathrm{f}) = \left[H_0 + \frac{\mu c}{k} \left(1 - e^{-kH_0/c}\right)\right] \leq H_g \tag{2.9}$$

which rearranges to

$$c\left(1 - e^{-kH_0/c}\right) \leq \frac{k}{\mu}(H_g - H_0). \tag{2.10}$$

This is an increasing function, and since we want to minimise $t_\mathrm{f} = H_0/c$ we need to choose the largest allowed $c$ value. We set

$$c\left(1 - e^{-kH_0/c}\right) = \frac{k}{\mu}(H_g - H_0). \tag{2.11}$$

Now, the above has an upper bound as $c \to \infty$, given by $kH_0$.

## 2.2 Harder

For an example of trying to find a minimum time, we will use a trial function

$$p(t) = \alpha t \tag{2.12}$$

for some constant $\alpha \in \mathbb{R}^+$.

Now,

$$h(t) = \int_0^t p(y)\,\mathrm{d}y = \int_0^t \alpha y\,\mathrm{d}t = \frac{1}{2}\alpha t^2. \tag{2.13}$$

Using (1.12) we have

$$b(t) = \mu e^{-kt} \int_0^t e^{ky}\,(\alpha y)\,\mathrm{d}y \tag{2.14}$$

$$= \alpha\gamma \left( \frac{e^{-kt}}{k} + t - \frac{1}{k} \right) \tag{2.15}$$

where $\gamma = \mu/k$.

This means that the height of the foam at time $t$ is given by

$$(h+b)(t) = \alpha \left[ \frac{1}{2}t^2 + \gamma \left( \frac{e^{-kt}}{k} + t - \frac{1}{k} \right) \right]. \tag{2.16}$$

This means we need to find $\alpha \in \mathbb{R}_0^+$ to minimise $t_\mathrm{f}$ where

$$h(t_\mathrm{f}) = \frac{1}{2}\alpha t_\mathrm{f}^2 = H_0 \tag{2.17}$$

subject to

$$(h+b)(t) = \alpha \left[ \frac{1}{2}t^2 + \gamma \left( \frac{e^{-kt}}{k} + t - \frac{1}{k} \right) \right] \le H_g \tag{2.18}$$

for all time $t \in (0, t_\mathrm{f})$. Furthermore,

$$t_\mathrm{f} = \sqrt{\frac{2H_0}{\alpha}}. \tag{2.19}$$

Now, seeking an upper bound on $(h+b)$, the derivative of $(h+b)$ is given by

$$\frac{\mathrm{d}(h+b)}{\mathrm{d}t} = \alpha t + \alpha(1 - e^{-kt}) = 0, \tag{2.20}$$

which reduces to

$$(t+1)e^{kt} = 1. \tag{2.21}$$

The solution to this is given by

$$\hat{t} = \frac{W(e^k k) - k}{k} \tag{2.22}$$

where $W$ is the Lambert $W$ function. By the extreme value theorem, $h+b$ takes its maximum, and we deduce that this at either $\hat{t}$ or $t_{\text{f}}$. For the foam to not spill, we need

$$\max\left((h+b)\left(\hat{t}\right),(h+b)(t_{\text{f}})\right) \leq H_g. \tag{2.23}$$

#### 2.2.1 Case 1

Suppose that

$$t_{\text{f}} = \sqrt{\frac{2H_0}{\alpha}} \leq \frac{W(e^k k) - k}{k} = \hat{t}. \tag{2.24}$$