On A337560

True_hOREP

Contents

1 Definition

Let $a(n)$ denote the number of length four $1..n$ vectors $v = (a, b, c, d)$ that contain their harmonic mean

$$h = \frac{4}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}}. \tag{1.1}$$

It can be shown that h must satisfy

$$h(abc + abd + acd + bcd) = 4abcd. \tag{1.2}$$

2 Generation

Seeking new vectors in $a(n)$ and not $a(n-1)$, the vector must contain n otherwise it would have already been counted, so write $v = (a, b, c, n)$. If $h = n$, then $v = (n, n, n, n)$. Suppose w.l.o.g. that $h = c$, so that

$$abc + abn + acn + bcn = 4abn \tag{2.1}$$

$$\implies c(ab + an + bn) = 3abn \tag{2.2}$$

$$\implies c = \frac{3abn}{ab + an + bn}. \tag{2.3}$$

Then for $1 \le a \le b \le n$, whenever $3abn = 0 \pmod{ab + an + bn}$, the vector

$$v = \left(a, b, \frac{3abn}{ab + an + bn}, n\right) \tag{2.4}$$

is a solution.

3 Modular

$$a(n) - a(n-1) = 1 \pmod{12} \tag{3.1}$$

Vectors of the form (a, a, a, d) and (a, a, d, d) cannot contain their harmonic mean, and all the other forms have 12 permutations (counting themselves) other than the trivial solution (n, n, n, n).

Proof. We show that (a, a, a, d) does not contain its harmonic mean if $a \neq d$. We have

$$h(a^3 + a^2d + a^2d + a^2d) = 4a^3d. \tag{3.2}$$

Cancelling away the a^2, yields

$$h = \frac{4ad}{a + 3d}. \tag{3.3}$$

Now let $h = a$. Then

$$a = \frac{4ad}{a + 3d} \implies 1 = \frac{4d}{a + 3d} \tag{3.4}$$

that is

$$a + 3d = 4d, \tag{3.5}$$

thus $a = d$. Now suppose that $h = d$. We get

$$d = \frac{4ad}{a + 3d} \tag{3.6}$$

which is equivalent to

$$a + 3d = 4a \tag{3.7}$$

which is precisely $d = a$. It follows that these vectors cannot contain their harmonic mean. The case for (a, a, d, d), $a \neq d$, is similar. We have

$$h = \frac{4a^2d^2}{2a^2d + 2ad^2} = \frac{2ad}{a + d}. \tag{3.8}$$

W.l.o.g assume that $h = a$. Then this becomes

$$a = \frac{2ad}{a + d} \implies a + d = 2d. \tag{3.9}$$

It follows that $a = d$, a contradiction.

□