# On A337110

**True_hOREP**

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# 1 Definition

Let $a(n)$ denote the number of length three $1..n$ vectors $(a_0, a_1, a_2)$ that contain their own geometric mean

$$\sqrt[3]{a_0 a_1 a_2} \in (a_0, a_1, a_2). \tag{1.1}$$

# 2 First difference

We note that at each step $n$, there is a trivial vector gained; $(n, n, n)$ trivially contains its geometric mean, as $\sqrt[3]{n^3} = n \in (n, n, n)$. Next, for the first difference $a(n) - a(n-1)$ to increase it must contain a vector that has $n$ as an element, or it would have already been counted. This means new vectors are of the form $(n, a_1, a_2)$. Since we have already counted the trivial case, we cannot have both $a_1 = a_2 = n$, so the geometric mean will be less than $n$ and thus the mean must equal either $a_1$ or $a_2$.

**Proposition 1.**

$$a(n) = a(n-1) + 1 + 6 \cdot A057918(n). \tag{2.1}$$

*Proof.* We first show that the only valid solutions $v = (a, b, c)$ either have all the same terms or all different terms. Suppose that two terms are the same, w.l.o.g $v = (a, a, c)$. Then we have two cases, either the geometric mean is $a$ and then $a^2 c = a^3$, which implies that $c = a$, which is a contradiction, or the geometric mean is $c$, so $a^2 c = c^3$ which implies that $c = a$, which is a contradiction. It follows that there are no vectors in the form $v = (a, a, c)$.

Now, consider new vectors $v = (a, b, c)$ counted by $a(n)$ that have not appeared before. For $v$ to be a new vector, one of the terms must be $n$, otherwise it would have already been counted. We observe that $(n, n, n)$ is the only possible new vector with all the elements the same. We now consider $v = (a, b, n)$ with $a \leq b \leq n$. We now demonstrate that $(a, b, n)$ is in a geometric progession.

We must have

$$an = b^2 \tag{2.2}$$

and because $a, n, b, an, b^2 \in \mathbb{Z}$ we know that $an$ must be a square, so

$$n = ar^2 \tag{2.3}$$

for some $r \in \mathbb{Z}$. It follows that $b = ar$, and we have established our geometric progession. The number of pairs of numbers (a,b) each less than n where (a,b,n) is in geometric progression is counted precisely by A057918, and since any permutation of a solution is a solution we add this to the trivial solution and the previously counted solutions to get

$$a(n) = a(n-1) + 1 + 6A057918(n). \tag{2.4}$$

□