# On A337111

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## Contents



## 1 Definitions

The geometric mean $g$ of a 4-tuple $(a_1, a_2, a_3, a_4) \in (\mathbb{R}^+)^4$ is defined as [1]

$$g = \sqrt[4]{a_1 a_2 a_3 a_4}. \tag{1.1}$$

We say that a vector $(a_1, a_2, a_3, a_4)$ contains its geometric mean if $g = a_i$ for at least one $i = 1, 2, 3, 4$.

We write "$1..n$ vectors" to mean vectors that contain integer components $1 \leq a_i \leq n$.

A "primitive vector" is a vector that contains its geometric mean, is not an integer multiple of a previously counted vector and has ordered elements from left to right. Any vector that contains its geometric mean and contains a 1 is immediately primitive.

## 2 Examples and non-examples

The vector $(1, 1, 1, 1)$ contains its geometric mean because

$$g = \sqrt[4]{1 \cdot 1 \cdot 1 \cdot 1} = 1 \tag{2.1}$$

and 1 is a member of $(1, 1, 1, 1)$. The vector $(1, 2, 2, 2)$ does not contain its geometric mean, because

$$g = \sqrt[4]{1 \cdot 2 \cdot 2 \cdot 2} = \sqrt[4]{8} \tag{2.2}$$

and $\sqrt[4]{8}$ is not a member of $(1, 2, 2, 2)$ (or even an integer).

The first non-trivial example is $(1, 2, 2, 4)$ which contains its geometric mean as

$$g = \sqrt[4]{1 \cdot 2 \cdot 2 \cdot 4} = \sqrt[4]{16} = 2. \tag{2.3}$$

## 3 Sequence

Let $a(n)$ denote the number of length four $1..n$ vectors that contain their geometric mean.

Then

$$\begin{aligned}a(n) =& 1, 2, 3, 16, 17, 18, 19, 56, 105, 106, 107, 144, 145, 146, 147, 208, 209, 306, 307, 320, &(3.1)\\ & 321, 322, 323, 432, 529, 530, 723, 736, 737, 738, 739, 968, 969, 970, 971, 1176, 1177, &(3.2)\\ & 1178, 1179, 1288, 1289, 1290, 1291, 1304, 1401, 1402, 1403, 1608, 1777, 2018, &(3.3)\\ & 2019, 2032, 2033, 2418, 2419, 2528, 2529, 2530, 2531, 2616, 2617, 2618, 2715, &(3.4)\\ & 3160, 3161, 3162, 3163, 3176, 3177, 3178, 3179, 3600, 3601, 3602, 3915, 3928, &(3.5)\\ & 3929, 3930, 3931, 4136, 4545, 4546, 4547, 4608, 4609, 4610, 4611, 4648, 4649, &(3.6)\\ & 4866, 4867, \cdots &(3.7)\\ & &(3.8)\end{aligned}$$

The non-trivial first difference $\Delta a(n) = a(n) - a(n-1) - 1$ is

$$\begin{aligned}\Delta a(n) =& 0, 0, 0, 12, 0, 0, 0, 36, 48, 0, 0, 36, 0, 0, 0, 60, 0, 96, 0, 12, 0, 0, 0, 108, 96, 0, 192, 12, 0, 0, 0, 228, &(3.9)\\ & 0, 0, 0, 204, 0, 0, 0, 108, 0, 0, 0, 12, 96, 0, 0, 204, 168, 240, 0, 12, 0, 384, 0, 108, 0, 0, 0, 84, 0, 0, 96, &(3.10)\\ & 444, 0, 0, 0, 12, 0, 0, 0, 420, 0, 0, 312, 12, 0, 0, 0, 204, 408, 0, 0, 60, 0, 0, 0, 36, 0, 216, 0, 12, 0, 0, 0, 492, &(3.11)\\ & 0, 360, 48, 564, \cdots &(3.12)\end{aligned}$$

and dividing through by 12 we have

$$\begin{aligned}(\Delta a(n))/12 =& 0, 0, 0, 1, 0, 0, 0, 3, 4, 0, 0, 3, 0, 0, 0, 5, 0, 8, 0, 1, 0, 0, 0, 9, 8, &(3.13)\\ & 0, 16, 1, 0, 0, 0, 19, 0, 0, 0, 17, 0, 0, 0, 9, 0, 0, 0, 1, 8, 0, 0, 17, &(3.14)\\ & 14, 20, 0, 1, 0, 32, 0, 9, 0, 0, 0, 7, 0, 0, 8, 37, 0, 0, 0, 1, 0, 0, 0, &(3.15)\\ & 35, 0, 0, 26, 1, 0, 0, 0, 17, 34, 0, 0, 5, 0, 0, 0, 3, 0, 18, 0, 1, 0, 0, 0, &(3.16)\\ & 41, 0, 30, 4, 47, \cdots . &(3.17)\end{aligned}$$

## 4 Modular Condition

### 4.1 Lemma

There exist no length four vectors that have only two distinct components that contain their geometric mean. Let $a, d \in \mathbb{Z}^+$, such that $a \neq d$. Let $v = (a, a, a, d)$. There are two cases, either the geometric mean is $a$, or $d$.

**Mean is $a$:** If the geometric mean is $a$, then $a^3 d = a^4$. Clearly we must have $a = d$, which is a contradiction.

**Mean is $d$:** If the geometric mean is $d$, then $a^3 d = d^4$, and it follows that $a^3 = d^3$, so $a = d$, another contradiction.

**Two for two:** Let $a, d \in \mathbb{Z}^+$, such that $a \neq d$. Let $v = (a, a, d, d)$. Then there is only one case, up to a permutation. Assume w.l.o.g that the mean is $a$. Then we have $a^2 d^2 = a^4$ which implies that $a = d$ which is yet another contradiction. This completes the proof of the lemma.

### 4.2 Proposition and proof

**Proposition 1.**

$$a(n) - a(n-1) \equiv 1 \pmod{12}. \tag{4.1}$$

*Proof.* Let $v = (a, b, c, d)$ be a solution to be counted in $a(n)$. Then there are five cases:

**Case 1:** If the elements are all the same, then the only possible new solution is $(n, n, n, n)$, and this contribution can be written as $1 = 12 \cdot 0 + 1$.

**Case 2:** All elements are different, so $(a, b, c, d)$ has 24 permutations, which is $24 = 12 \cdot 2$.

**Case 3:** Two elements are the same, rest are different. Without loss of generality, we can write this as $(a, a, c, d)$, which has 12 permutations, which is $12 = 12 \cdot 1$.

**Case 4:** Two elements are the same, other two are the same.
No vectors of this form contain their geometric mean, by the Lemma.

**Case 5:** Three elements are the same, other is different.
No vectors of this form contain their geometric mean, by the Lemma. This completes the proof. □

**Corollary** If $n$ is even, then $a(n)$ is even, and if $n$ is odd, then $a(n)$ is odd.

## 5 Analysis

Write $v = (a_1, a_2, a_3, a_4)$. For $v$ to be a new vector, it must contain $n$ otherwise it would have been counted earlier. Trivially, we have one vector $(n, n, n, n)$ and without loss of generality assume $a_4 = n$ and that the geometric mean is $a_2 = g$. We then have $v = (a_1, g, a_3, n)$

$$a_1 g a_3 n = g^4 \implies a_1 a_3 n = g^3. \tag{5.1}$$

The question now becomes how many ways can $g^3$ be written as $a_1 a_3 n$.
A simple set of bounds is

$$n \leq a(n) \leq n^4. \tag{5.2}$$

The lower bound comes from the fact that a trivial vector of the form $(n, n, n, n)$ is obtained at every step, and the upper bound comes from there being $n^4$ possible length four $1..n$ vectors. By being the simplest, they are also the worst possible meaningful bounds.

**Proposition 2.**

$$x^3 \equiv 0 \pmod{n} \tag{5.3}$$

has only one solution if and only if

$$a(n) = a(n-1) + 1. \tag{5.4}$$

*Proof.* The forward proof was given by David A. Corneth on the OEIS [2], with minor changes for clarity.
Let $(a, b, c, n)$ be such a tuple. Let without loss of generality $c$ be the geometric mean of the tuple. Then $abcn = c^4$ and as $c$ is not 0 we have $c^3 = abn$. So then $c^3 \equiv 0 \pmod{n}$. If $c^3 \equiv 0 \pmod{n}$ has only 1 solution then $c = n$. We can then write $n^3 = abn$, and this gives the tuple $(n, n, n, n)$ which has 1 permutation. So giving $a(n) = a(n-1) + 1$.

**Reverse:** Suppose $a(n) = a(n-1) + 1$. Let $n, m \in \mathbb{Z}^+$ with $1 \leq m < n$, be distinct solutions to $x^3 \equiv 0 \pmod{n}$.
Let $v = (a, b, c, n)$ be a new solution to be counted, and w.l.o.g assume $c$ is the geometric mean such that

$$c^4 = abcn \implies c^3 = abn \tag{5.5}$$

and so $c^3 \equiv 0 \pmod{n}$. Either $c = n$ or $c = m$. If $c = n$, then we gain the solution $(n, n, n, n)$, and if $c = m$ then we gain another solution $(a, b, m, n)$. By the modular lemma we know that a solution cannot have only two elements, so $a$ and $b$ are not equal to both $m$ or $n$. This means that a contribution from this vector gives twelve or twenty-four permutations, but $a(n) = a(n-1) + 1$, so it must be the case that $(a, b, m, n) = (n, n, n, n)$, which means $m = n$. It follows that there is only one solution to $x^3 \equiv 0 \pmod{n}$.

□

**Proposition 3.** Given two vectors that contain their geometric mean, say $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)$ with (w.l.o.g) geometric mean $a_3, b_3$ respectively, then the elementwise product given by

$$(a_1, a_2, a_3, a_4) \odot (b_1, b_2, b_3, b_4) = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4) \tag{5.6}$$

also contains its geometric mean, given by $a_3 b_3$.

*Proof.* Observe that $a_3 = \sqrt[4]{a_1 a_2 a_3 a_4}, b_3 = \sqrt[4]{b_1 b_2 b_3 b_4}$, and multiplying these together we have

$$a_3 b_3 = \left(\sqrt[4]{a_1 a_2 a_3 a_4}\right) \cdot \left(\sqrt[4]{b_1 b_2 b_3 b_4}\right) = \sqrt[4]{(a_1 b_1)(a_2 b_2)(a_3 b_3)(a_4 b_4)}. \tag{5.7}$$

$\square$

**Corollary** Let $k \in \mathbb{Z}^+$. If $v = (a, b, g, n)$ contains its geometric mean $g$, then $kv$ also contains its geometric mean equal to $kg$. Simply consider the elementwise product $(k, k, k, k) \odot (a, b, g, n) = (ka, kb, kg, kn)$.

**Corollary** Let $k \in \mathbb{Z}^+$. If $v = (a, b, g, n)$ does not contain its geometric mean, then $kv$ also does not contain its geometric mean..

**Corollary** Given two primitive vectors of the form $(1, a_2, a_3, a_4), (1, b_2, b_3, b_4)$, then $(1, a_2 b_2, a_3 b_3, a_4 b_4)$ is also primitive. E.g. $(1, 2, 2, 4) \odot (1, 3, 3, 9) = (1, 6, 6, 36)$.

### 5.1 Lower bound improvements

We can always improve the lower bound of future $a(n)$ by using proposition 3. For example, take the first two primitive vectors,

$$(1, 1, 1, 1), \quad (1, 2, 2, 4). \tag{5.8}$$

The $(1, 1, 1, 1)$ vector is responsible for the lower bound of $n \leq a(n)$. We can add an additional vector with every $n$ that is a multiple of 4, as that is the largest value in the vector $(1, 2, 2, 4)$, which by proposition 3 implies that for say, $a(8)$, we know $(2, 4, 4, 8)$ is a solution and so we gain at least 12 solutions due to the permutations, so

$$n + 12 \left\lfloor \frac{n}{4} \right\rfloor \leq a(n). \tag{5.9}$$

In fact, this is precisely equal to $a(n)$ for $n = 1, 2, \ldots, 7$. The next primitive vectors are $(1, 1, 2, 8)$ (12 permutations) and $(1, 4, 8, 8)$ (12 permutations), attained at $n = 8$. By proposition 3 we can write

$$n + 12 \left\lfloor \frac{n}{4} \right\rfloor + 24 \left\lfloor \frac{n}{8} \right\rfloor \leq a(n). \tag{5.10}$$

Taking this one step further, for $n = 9$ we have the vectors $(1, 3, 3, 9), (3, 6, 8, 9), (4, 6, 6, 9)$, which gives $12 + 24 + 12 = 48$ permutations, thus our new lower bound is

$$n + 12 \left\lfloor \frac{n}{4} \right\rfloor + 24 \left\lfloor \frac{n}{8} \right\rfloor + 48 \left\lfloor \frac{n}{9} \right\rfloor \leq a(n). \tag{5.11}$$

This lower bound is exact for $n = 1, \ldots, 11$.

**Proposition 4.** I claim that there exist coefficients $\alpha_m \equiv 0 \pmod{12}$ such that for every $n \in \mathbb{Z}^+$,

$$a(n) = n + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n}{m} \right\rfloor . \tag{5.12}$$

Any truncation yields a lower bound for $a(n)$.

*Proof.* Base case: $n = 2$,

$$a(2) = 2 = 2 + \sum_{m=2}^{2} \alpha_m \left\lfloor \frac{2}{m} \right\rfloor = 2 + \alpha_2 \tag{5.13}$$

so clearly $\alpha_2 = 0$ thus $\alpha_2 \equiv 0 \pmod{12}$. Let $n \in \mathbb{Z}^+$, and suppose $a(n) = n + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n}{m} \right\rfloor$, with $\alpha_m \equiv 0 \pmod{n}$ for each $m = 2, ..., n$. Then

$$\begin{aligned} a(n+1) &= (n+1) + \sum_{m=2}^{n+1} \alpha_m \left\lfloor \frac{n+1}{m} \right\rfloor & (5.14) \\ &= (n+1) + \alpha_{n+1} \left\lfloor \frac{n+1}{n+1} \right\rfloor + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n+1}{m} \right\rfloor & (5.15) \\ &= 1 + \alpha_{n+1} + n + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n+1}{m} \right\rfloor . & (5.16) \end{aligned}$$

We note that because $2 \le m \le n$, we have $\left\lfloor \frac{n+1}{m} \right\rfloor, \left\lfloor \frac{n}{m} \right\rfloor \ge 1$. It follows that we can find some integer $k$ such that

$$12k = \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n+1}{m} \right\rfloor - \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n}{m} \right\rfloor . \tag{5.17}$$

We can then write

$$\begin{aligned} a(n+1) &= 1 + \alpha_{n+1} + 12k + n + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n}{m} \right\rfloor & (5.18) \\ &= 1 + \alpha_{n+1} + 12k + a(n). & (5.19) \end{aligned}$$

Using the fact that $a(n+1) - a(n) \equiv 1 \pmod{12}$, we must have $\alpha_{n+1} \equiv 0 \pmod{12}$. $\square$

The sequence of $\alpha_m$'s, starting with $m = 2$, is

$$\alpha_m = 0, 0, 12, 0, 0, 0, 24, 48, 0, 0, 24, 0, 0, 0, 24, 0, 48, 0, 0, 0, 0, 0, 48, \cdots \tag{5.20}$$

We note that if $x^3 \equiv 0 \pmod{m}$ has only one solution then $\alpha_m = 0$, which follows from proposition 1. The converse is not true, e.g. $\alpha_{20} = 0$, but $x^3 \equiv 0 \pmod{20}$ has two solutions, $x = 10, 20$. We can then write

$$a(n) = n + 12 \left\lfloor \frac{n}{4} \right\rfloor + 24 \left\lfloor \frac{n}{8} \right\rfloor + 48 \left\lfloor \frac{n}{9} \right\rfloor + 24 \left\lfloor \frac{n}{12} \right\rfloor + 24 \left\lfloor \frac{n}{16} \right\rfloor + \cdots \tag{5.21}$$

and truncation yields

$$n + 12 \left\lfloor \frac{n}{4} \right\rfloor + 24 \left\lfloor \frac{n}{8} \right\rfloor + 48 \left\lfloor \frac{n}{9} \right\rfloor + 24 \left\lfloor \frac{n}{12} \right\rfloor + 24 \left\lfloor \frac{n}{16} \right\rfloor \le a(n). \tag{5.22}$$

## 5.2 Determining the coefficients

The coefficients of (5.12) can be determined iteratively. Suppose we know all the coefficients below some $m \in \mathbb{Z}^+$ i.e. up to some say, $p \in \mathbb{Z}^+$, and know the value of $a(m)$. Then to determine $\alpha_m$, observe that

$$a(m) = m + 12\left\lfloor \frac{m}{4} \right\rfloor + 24\left\lfloor \frac{m}{8} \right\rfloor + \cdots + \alpha_m \underbrace{\left\lfloor \frac{m}{m} \right\rfloor}_{=1}. \tag{5.23}$$

Then

$$\alpha_m = a(m) - \left( m + 12\left\lfloor \frac{m}{4} \right\rfloor + 24\left\lfloor \frac{m}{8} \right\rfloor + \cdots \alpha_p \left\lfloor \frac{m}{p} \right\rfloor \right). \tag{5.24}$$

The coefficients can also be determined by computing the number of primitive vectors associated with $m$, and summing the permutations.

## 5.3 Upper bound improvements

Let $n \in \mathbb{Z}^+$, and $a \in \mathbb{Z}^+$ such that $a < n$. Noting that vectors of the form $(a, a, a, n)$, $(a, n, n, n)$ and $(a, a, n, n)$ are not allowed forms, we can improve the upper bound by removing them from $n^4$. The vector $(a, n, n, n)$ has $n - 1$ possible variations (since $a \neq n$), and by symmetry so does $(a, a, a, n)$ and these both have four permutations. The vector $(a, a, n, n)$ also has $n - 1$ variations with 6 permutations. Subtracting these off of our upper bound yields a new upper bound,

$$a(n) \leq n^4 - 14(n - 1). \tag{5.25}$$

## 6 Graphs

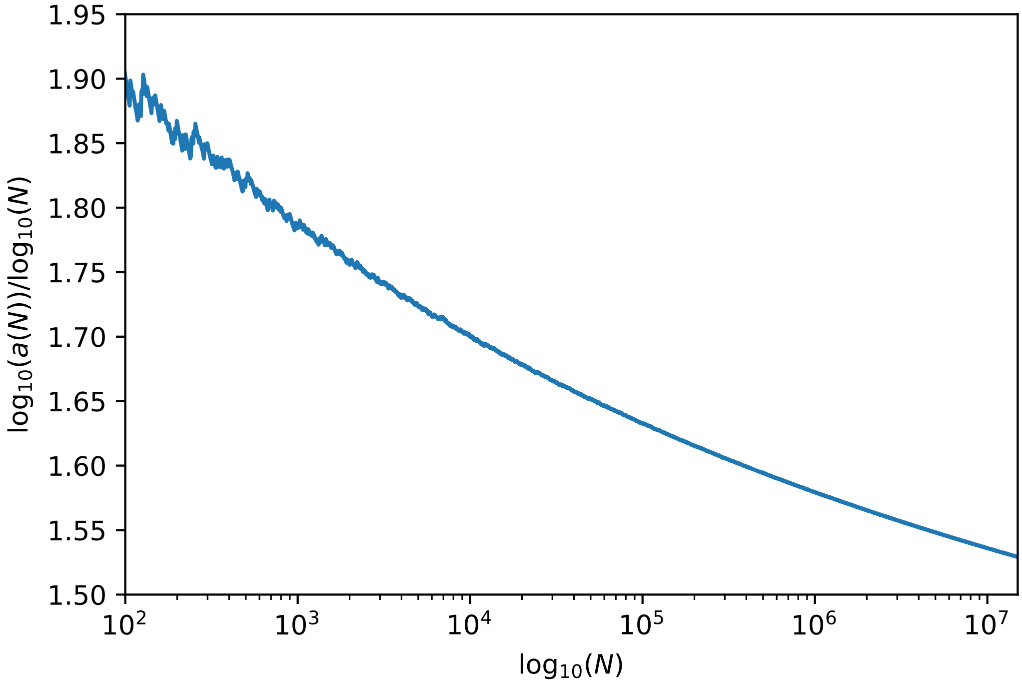


**Figure 1**. Log log plot of the terms from $10^2$ to $1.5 \cdot 10^7$ of $a(n)$.

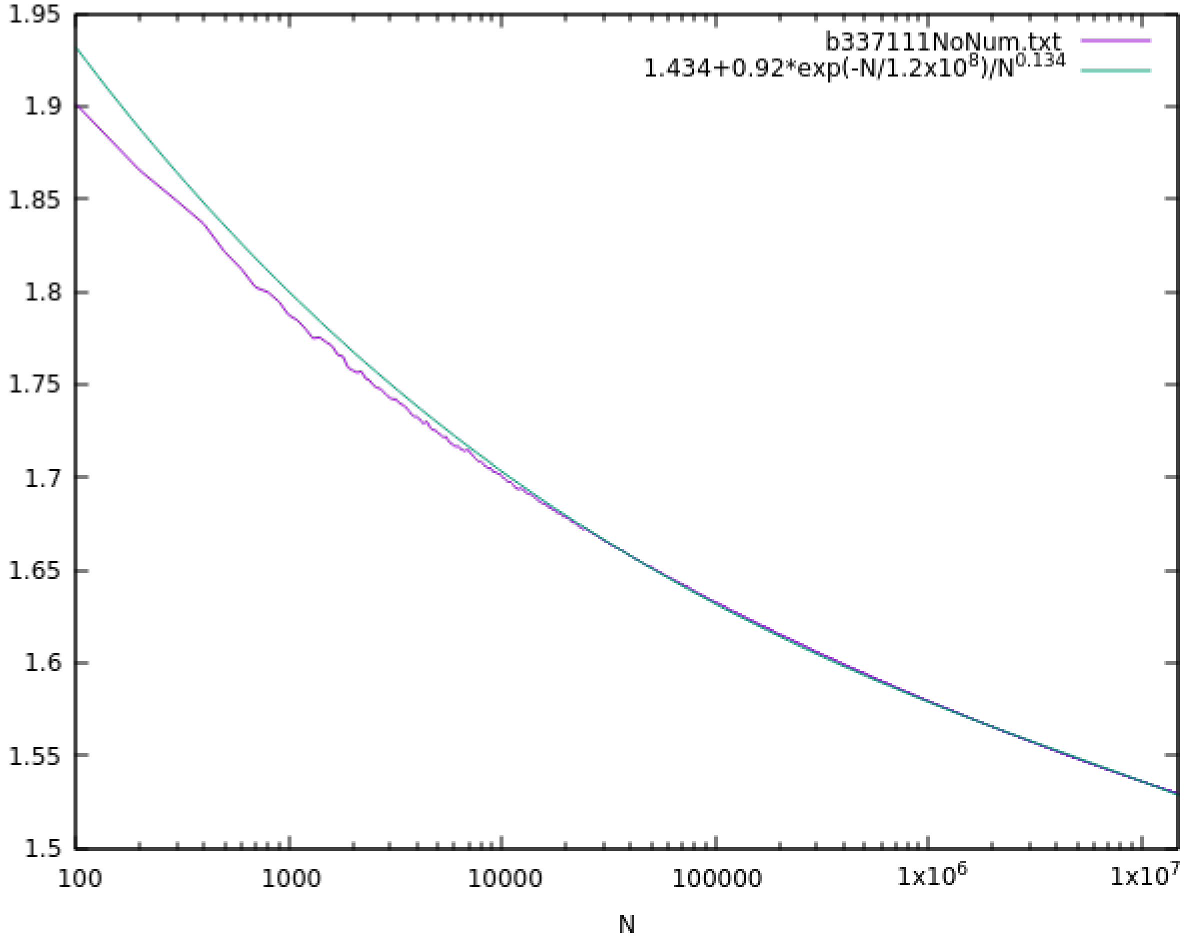


**Figure 2**. Fit to a product of power law and exponential, with an offset.

## References


[1] Weisstein, Eric W. "Geometric Mean." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/GeometricMean.html Accessed Oct 2020.

[2] https://oeis.org/A337111 Accessed Oct 2020.

[3] https://oeis.org/A000189 Accessed Oct 2020.