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Contents

1 Definitions

The geometric mean g of a 4-tuple $(a_1, a_2, a_3, a_4) \in (\mathbb{R}^+)^4$ is defined as [1]

$$g = \sqrt[4]{a_1 a_2 a_3 a_4}. \tag{1.1}$$

We say that a vector (a_1, a_2, a_3, a_4) contains its geometric mean if $g = a_i$ for at least one $i = 1, 2, 3, 4$.

We write "$1..n$ vectors" to mean vectors that contain integer components $1 \leq a_i \leq n$.

A "primitive vector" is a vector that contains its geometric mean, is not an integer multiple of a previously counted vector and has ordered elements from left to right. Any vector that contains its geometric mean and contains a 1 is immediately primitive.

2 Examples and non-examples

The vector $(1, 1, 1, 1)$ contains its geometric mean because

$$g = \sqrt[4]{1 \cdot 1 \cdot 1 \cdot 1} = 1 \tag{2.1}$$

and 1 is a member of $(1, 1, 1, 1)$. The vector $(1, 2, 2, 2)$ does not contain its geometric mean, because

$$g = \sqrt[4]{1 \cdot 2 \cdot 2 \cdot 2} = \sqrt[4]{8} \tag{2.2}$$

and $\sqrt[4]{8}$ is not a member of $(1, 2, 2, 2)$ (or even an integer).

The first non-trivial example is $(1, 2, 2, 4)$ which contains its geometric mean as

$$g = \sqrt[4]{1 \cdot 2 \cdot 2 \cdot 4} = \sqrt[4]{16} = 2. \tag{2.3}$$

3 Sequence

Let $a(n)$ denote the number of length four $1..n$ vectors that contain their geometric mean.

Then

$$\begin{aligned}
a(n) =&1, 2, 3, 16, 17, 18, 19, 56, 105, 106, 107, 144, 145, 146, 147, 208, 209, 306, 307, 320, && (3.1)\\
&321, 322, 323, 432, 529, 530, 723, 736, 737, 738, 739, 968, 969, 970, 971, 1176, 1177, && (3.2)\\
&1178, 1179, 1288, 1289, 1290, 1291, 1304, 1401, 1402, 1403, 1608, 1777, 2018, && (3.3)\\
&2019, 2032, 2033, 2418, 2419, 2528, 2529, 2530, 2531, 2616, 2617, 2618, 2715, && (3.4)\\
&3160, 3161, 3162, 3163, 3176, 3177, 3178, 3179, 3600, 3601, 3602, 3915, 3928, && (3.5)\\
&3929, 3930, 3931, 4136, 4545, 4546, 4547, 4608, 4609, 4610, 4611, 4648, 4649, && (3.6)\\
&4866, 4867, \cdots && (3.7)\\
& && (3.8)
\end{aligned}$$

The non-trivial first difference $\Delta a(n) = a(n) - a(n-1) - 1$ is

$$\begin{aligned}
\Delta a(n) =&0, 0, 0, 12, 0, 0, 0, 36, 48, 0, 0, 36, 0, 0, 0, 60, 0, 96, 0, 12, 0, 0, 0, 108, 96, 0, 192, 12, 0, 0, 0, 228, && (3.9)\\
&0, 0, 0, 204, 0, 0, 0, 108, 0, 0, 0, 12, 96, 0, 0, 204, 168, 240, 0, 12, 0, 384, 0, 108, 0, 0, 0, 84, 0, 0, 96, && (3.10)\\
&444, 0, 0, 0, 12, 0, 0, 0, 420, 0, 0, 312, 12, 0, 0, 0, 204, 408, 0, 0, 60, 0, 0, 0, 36, 0, 216, 0, 12, 0, 0, 0, 492, && (3.11)\\
&0, 360, 48, 564, \cdots && (3.12)
\end{aligned}$$

and dividing through by 12 we have

$$\begin{aligned}
(\Delta a(n))/12 =&0, 0, 0, 1, 0, 0, 0, 3, 4, 0, 0, 3, 0, 0, 0, 5, 0, 8, 0, 1, 0, 0, 0, 9, 8, && (3.13)\\
&0, 16, 1, 0, 0, 0, 19, 0, 0, 0, 17, 0, 0, 0, 9, 0, 0, 0, 1, 8, 0, 0, 17, && (3.14)\\
&14, 20, 0, 1, 0, 32, 0, 9, 0, 0, 0, 7, 0, 0, 8, 37, 0, 0, 0, 1, 0, 0, 0, && (3.15)\\
&35, 0, 0, 26, 1, 0, 0, 0, 17, 34, 0, 0, 5, 0, 0, 0, 3, 0, 18, 0, 1, 0, 0, 0, && (3.16)\\
&41, 0, 30, 4, 47, \cdots . && (3.17)
\end{aligned}$$

4 Modular Condition

4.1 Lemma

There exist no length four vectors that have only two distinct components that contain their geometric mean. Let $a, d \in \mathbb{Z}^+$, such that $a \neq d$. Let $v = (a, a, a, d)$. There are two cases, either the geometric mean is a, or d.

Mean is a: If the geometric mean is a, then $a^3 d = a^4$. Clearly we must have $a = d$, which is a contradiction.

Mean is d: If the geometric mean is d, then $a^3 d = d^4$, and it follows that $a^3 = d^3$, so $a = d$, another contradiction.

Two for two: Let $a, d \in \mathbb{Z}^+$, such that $a \neq d$. Let $v = (a, a, d, d)$. Then there is only one case, up to a permutation. Assume w.l.o.g that the mean is a. Then we have $a^2 d^2 = a^4$ which implies that $a = d$ which is yet another contradiction. This completes the proof of the lemma.

4.2 Proposition and proof

Proposition 1.

$$a(n) - a(n-1) \equiv 1 \pmod{12}. \tag{4.1}$$

Proof. Let $v = (a, b, c, d)$ be a solution to be counted in $a(n)$. Then there are five cases:

Case 1: If the elements are all the same, then the only possible new solution is (n, n, n, n), and this contribution can be written as $1 = 12 \cdot 0 + 1$.

Case 2: All elements are different, so (a, b, c, d) has 24 permutations, which is $24 = 12 \cdot 2$.

Case 3: Two elements are the same, rest are different. Without loss of generality, we can write this as (a, a, c, d), which has 12 permutations, which is $12 = 12 \cdot 1$.

Case 4: Two elements are the same, other two are the same.
No vectors of this form contain their geometric mean, by the Lemma.

Case 5: Three elements are the same, other is different.
No vectors of this form contain their geometric mean, by the Lemma. This completes the proof. □

Corollary If n is even, then $a(n)$ is even, and if n is odd, then $a(n)$ is odd.

5 Analysis

Write $v = (a_1, a_2, a_3, a_4)$. For v to be a new vector, it must contain n otherwise it would have been counted earlier. Trivially, we have one vector (n, n, n, n) and without loss of generality assume $a_4 = n$ and that the geometric mean is $a_2 = g$. We then have $v = (a_1, g, a_3, n)$

$$a_1 g a_3 n = g^4 \implies a_1 a_3 n = g^3. \tag{5.1}$$

The question now becomes how many ways can g^3 be written as $a_1 a_3 n$.
A simple set of bounds is

$$n \leq a(n) \leq n^4. \tag{5.2}$$

The lower bound comes from the fact that a trivial vector of the form (n, n, n, n) is obtained at every step, and the upper bound comes from there being n^4 possible length four $1..n$ vectors. By being the simplest, they are also the worst possible meaningful bounds.

Proposition 2.

$$x^3 \equiv 0 \pmod{n} \tag{5.3}$$

has only one solution if and only if

$$a(n) = a(n-1) + 1. \tag{5.4}$$

Proof. The forward proof was given by David A. Corneth on the OEIS [2], with minor changes for clarity.
Let (a, b, c, n) be such a tuple. Let without loss of generality c be the geometric mean of the tuple. Then $abcn = c^4$ and as c is not 0 we have $c^3 = abn$. So then $c^3 \equiv 0 \pmod{n}$. If $c^3 \equiv 0 \pmod{n}$ has only 1 solution then $c = n$. We can then write $n^3 = abn$, and this gives the tuple (n, n, n, n) which has 1 permutation. So giving $a(n) = a(n-1) + 1$.

Reverse: Suppose $a(n) = a(n-1) + 1$. Let $n, m \in \mathbb{Z}^+$ with $1 \leq m < n$, be distinct solutions to $x^3 \equiv 0 \pmod{n}$.
Let $v = (a, b, c, n)$ be a new solution to be counted, and w.l.o.g assume c is the geometric mean such that

$$c^4 = abcn \implies c^3 = abn \tag{5.5}$$

and so $c^3 \equiv 0 \pmod{n}$. Either $c = n$ or $c = m$. If $c = n$, then we gain the solution (n, n, n, n), and if $c = m$ then we gain another solution (a, b, m, n). By the modular lemma we know that a solution cannot have only two elements, so a and b are not equal to both m or n. This means that a contribution from this vector gives twelve or twenty-four permutations, but $a(n) = a(n-1) + 1$, so it must be the case that $(a, b, m, n) = (n, n, n, n)$, which means $m = n$. It follows that there is only one solution to $x^3 \equiv 0 \pmod{n}$.

□

Proposition 3. Given two vectors that contain their geometric mean, say $(a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)$ with (w.l.o.g) geometric mean a_3, b_3 respectively, then the elementwise product given by

$$(a_1, a_2, a_3, a_4) \odot (b_1, b_2, b_3, b_4) = (a_1b_1, a_2b_2, a_3b_3, a_4b_4) \tag{5.6}$$

also contains its geometric mean, given by a_3b_3.

Proof. Observe that $a_3 = \sqrt[4]{a_1a_2a_3a_4}, b_3 = \sqrt[4]{b_1b_2b_3b_4}$, and multiplying these together we have

$$a_3b_3 = \left(\sqrt[4]{a_1a_2a_3a_4}\right) \cdot \left(\sqrt[4]{b_1b_2b_3b_4}\right) = \sqrt[4]{(a_1b_1)(a_2b_2)(a_3b_3)(a_4b_4)}. \tag{5.7}$$

$\square$

Corollary Let $k \in \mathbb{Z}^+$. If $v = (a, b, g, n)$ contains its geometric mean g, then kv also contains its geometric mean equal to kg. Simply consider the elementwise product $(k, k, k, k) \odot (a, b, g, n) = (ka, kb, kg, kn)$.

Corollary Let $k \in \mathbb{Z}^+$. If $v = (a, b, g, n)$ does not contain its geometric mean, then kv also does not contain its geometric mean..

Corollary Given two primitive vectors of the form $(1, a_2, a_3, a_4), (1, b_2, b_3, b_4)$, then $(1, a_2b_2, a_3b_3, a_4b_4)$ is also primitive. E.g. $(1, 2, 2, 4) \odot (1, 3, 3, 9) = (1, 6, 6, 36)$.

5.1 Lower bound improvements

We can always improve the lower bound of future $a(n)$ by using proposition 3. For example, take the first two primitive vectors,

$$(1, 1, 1, 1), \quad (1, 2, 2, 4). \tag{5.8}$$

The $(1, 1, 1, 1)$ vector is responsible for the lower bound of $n \leq a(n)$. We can add an additional vector with every n that is a multiple of 4, as that is the largest value in the vector $(1, 2, 2, 4)$, which by proposition 3 implies that for say, $a(8)$, we know $(2, 4, 4, 8)$ is a solution and so we gain at least 12 solutions due to the permutations, so

$$n + 12\left\lfloor\frac{n}{4}\right\rfloor \leq a(n). \tag{5.9}$$

In fact, this is precisely equal to $a(n)$ for $n = 1, 2, \ldots, 7$. The next primitive vectors are $(1, 1, 2, 8)$ (12 permutations) and $(1, 4, 8, 8)$ (12 permutations), attained at $n = 8$. By proposition 3 we can write

$$n + 12\left\lfloor\frac{n}{4}\right\rfloor + 24\left\lfloor\frac{n}{8}\right\rfloor \leq a(n). \tag{5.10}$$

Taking this one step further, for $n = 9$ we have the vectors $(1, 3, 3, 9), (3, 6, 8, 9), (4, 6, 6, 9)$, which gives $12+24+12 = 48$ permutations, thus our new lower bound is

$$n + 12\left\lfloor\frac{n}{4}\right\rfloor + 24\left\lfloor\frac{n}{8}\right\rfloor + 48\left\lfloor\frac{n}{9}\right\rfloor \leq a(n). \tag{5.11}$$

This lower bound is exact for $n = 1, \ldots, 11$.

Proposition 4. I claim that there exist coefficients $\alpha_m \equiv 0 \pmod{12}$ such that for every $n \in \mathbb{Z}^+$,

$$a(n) = n + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n}{m} \right\rfloor . \tag{5.12}$$

Any truncation yields a lower bound for $a(n)$.

Proof. Base case: $n = 2$,

$$a(2) = 2 = 2 + \sum_{m=2}^{2} \alpha_m \left\lfloor \frac{2}{m} \right\rfloor = 2 + \alpha_2 \tag{5.13}$$

so clearly $\alpha_2 = 0$ thus $\alpha_2 \equiv 0 \pmod{12}$. Let $n \in \mathbb{Z}^+$, and suppose $a(n) = n + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n}{m} \right\rfloor$, with $\alpha_m \equiv 0 \ (\mathrm{mod}(n))$ for each $m = 2, ..., n$. Then

$$\begin{aligned}
a(n+1) &= (n+1) + \sum_{m=2}^{n+1} \alpha_m \left\lfloor \frac{n+1}{m} \right\rfloor && (5.14)\\
&= (n+1) + \alpha_{n+1} \left\lfloor \frac{n+1}{n+1} \right\rfloor + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n+1}{m} \right\rfloor && (5.15)\\
&= 1 + \alpha_{n+1} + n + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n+1}{m} \right\rfloor . && (5.16)
\end{aligned}$$

We note that because $2 \le m \le n$, we have $\left\lfloor \frac{n+1}{m} \right\rfloor, \left\lfloor \frac{n}{m} \right\rfloor \ge 1$. It follows that we can find some integer k such that

$$12k = \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n+1}{m} \right\rfloor - \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n}{m} \right\rfloor . \tag{5.17}$$

We can then write

$$\begin{aligned}
a(n+1) &= 1 + \alpha_{n+1} + 12k + n + \sum_{m=2}^{n} \alpha_m \left\lfloor \frac{n}{m} \right\rfloor && (5.18)\\
&= 1 + \alpha_{n+1} + 12k + a(n). && (5.19)
\end{aligned}$$

Using the fact that $a(n+1) - a(n) \equiv 1 \pmod{12}$, we must have $\alpha_{n+1} \equiv 0 \ (\mathrm{mod}(12))$. ☐

The sequence of α_m's, starting with $m = 2$, is

$$\alpha_m = 0, 0, 12, 0, 0, 0, 24, 48, 0, 0, 24, 0, 0, 0, 24, 0, 48, 0, 0, 0, 0, 0, 48, \cdots \tag{5.20}$$

We note that if $x^3 \equiv 0 \ (\mathrm{mod}(m))$ has only one solution then $\alpha_m = 0$, which follows from proposition 1. The converse is not true, e.g. $\alpha_{20} = 0$, but $x^3 \equiv 0 \ (\mathrm{mod}(20))$ has two solutions, $x = 10, 20$. We can then write

$$a(n) = n + 12 \left\lfloor \frac{n}{4} \right\rfloor + 24 \left\lfloor \frac{n}{8} \right\rfloor + 48 \left\lfloor \frac{n}{9} \right\rfloor + 24 \left\lfloor \frac{n}{12} \right\rfloor + 24 \left\lfloor \frac{n}{16} \right\rfloor + \cdots \tag{5.21}$$

and truncation yields

$$n + 12 \left\lfloor \frac{n}{4} \right\rfloor + 24 \left\lfloor \frac{n}{8} \right\rfloor + 48 \left\lfloor \frac{n}{9} \right\rfloor + 24 \left\lfloor \frac{n}{12} \right\rfloor + 24 \left\lfloor \frac{n}{16} \right\rfloor \le a(n). \tag{5.22}$$

5.2 Determining the coefficients

The coefficients of (5.12) can be determined iteratively. Suppose we know all the coefficients below some $m \in \mathbb{Z}^+$ i.e. up to some say, $p \in \mathbb{Z}^+$, and know the value of $a(m)$. Then to determine α_m, observe that

$$a(m) = m + 12\left\lfloor \frac{m}{4} \right\rfloor + 24\left\lfloor \frac{m}{8} \right\rfloor + \cdots + \alpha_m \underbrace{\left\lfloor \frac{m}{m} \right\rfloor}_{=1}. \tag{5.23}$$

Then

$$\alpha_m = a(m) - \left(m + 12\left\lfloor \frac{m}{4} \right\rfloor + 24\left\lfloor \frac{m}{8} \right\rfloor + \cdots \alpha_p \left\lfloor \frac{m}{p} \right\rfloor \right). \tag{5.24}$$

The coefficients can also be determined by computing the number of primitive vectors associated with m, and summing the permutations.

5.3 Upper bound improvements

Let $n \in \mathbb{Z}^+$, and $a \in \mathbb{Z}^+$ such that $a < n$. Noting that vectors of the form (a, a, a, n), (a, n, n, n) and (a, a, n, n) are not allowed forms, we can improve the upper bound by removing them from n^4. The vector (a, n, n, n) has $n - 1$ possible variations (since $a \neq n$), and by symmetry so does (a, a, a, n) and these both have four permutations. The vector (a, a, n, n) also has $n - 1$ variations with 6 permutations. Subtracting these off of our upper bound yields a new upper bound,

$$a(n) \leq n^4 - 14(n - 1). \tag{5.25}$$

6 Graphs

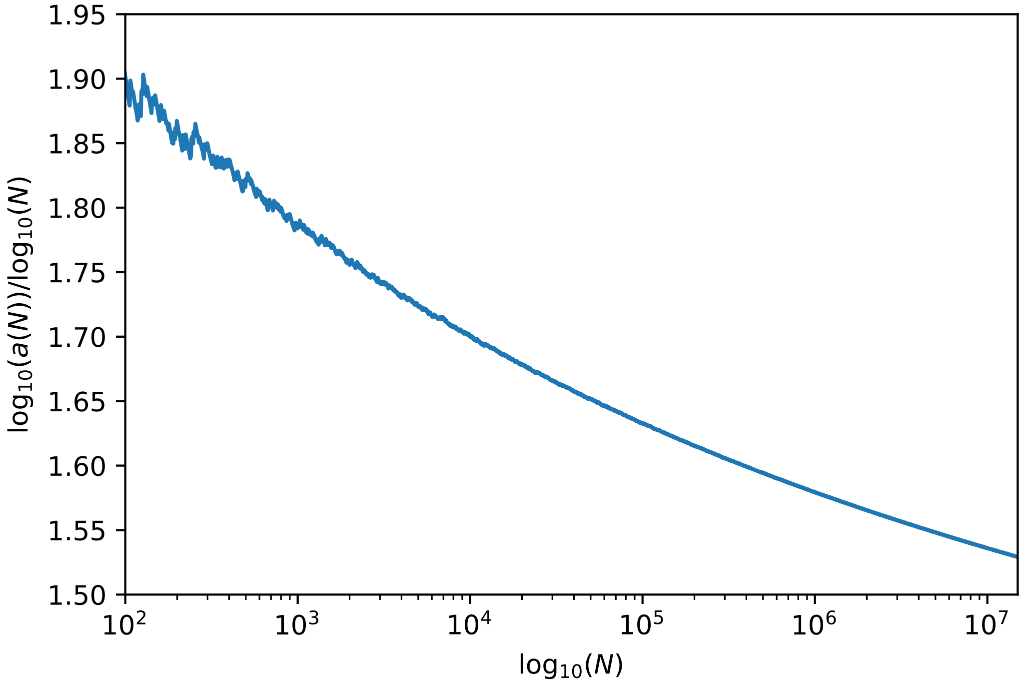

Figure 1. Log log plot of the terms from 10^2 to $1.5 \cdot 10^7$ of $a(n)$.

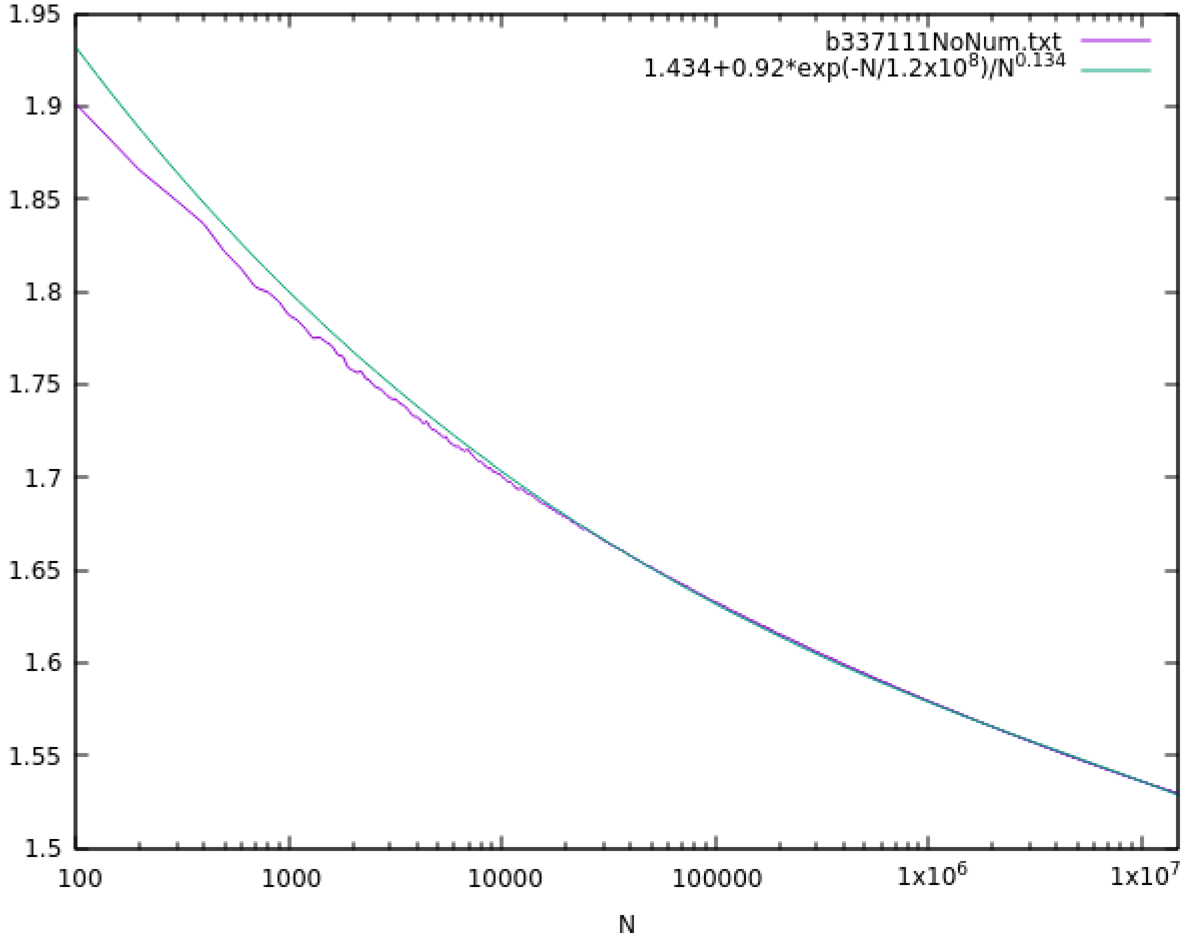

Figure 2. Fit to a product of power law and exponential, with an offset.

References

[1] Weisstein, Eric W. "Geometric Mean." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/GeometricMean.html Accessed Oct 2020.

[2] https://oeis.org/A337111 Accessed Oct 2020.

[3] https://oeis.org/A000189 Accessed Oct 2020.