Random Series

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1 Introduction

In the website [1] they consider the convergence of a random harmonic series,

$$R = \sum_{n=1}^{\infty} \frac{\sigma_n}{n} \tag{1.1}$$

where $\sigma_n \in \{1, -1\}$ is chosen randomly with probability $1/2$. The Kolmogorov three-series theorem yields that R converges almost surely. A good source for this and similar is [2].

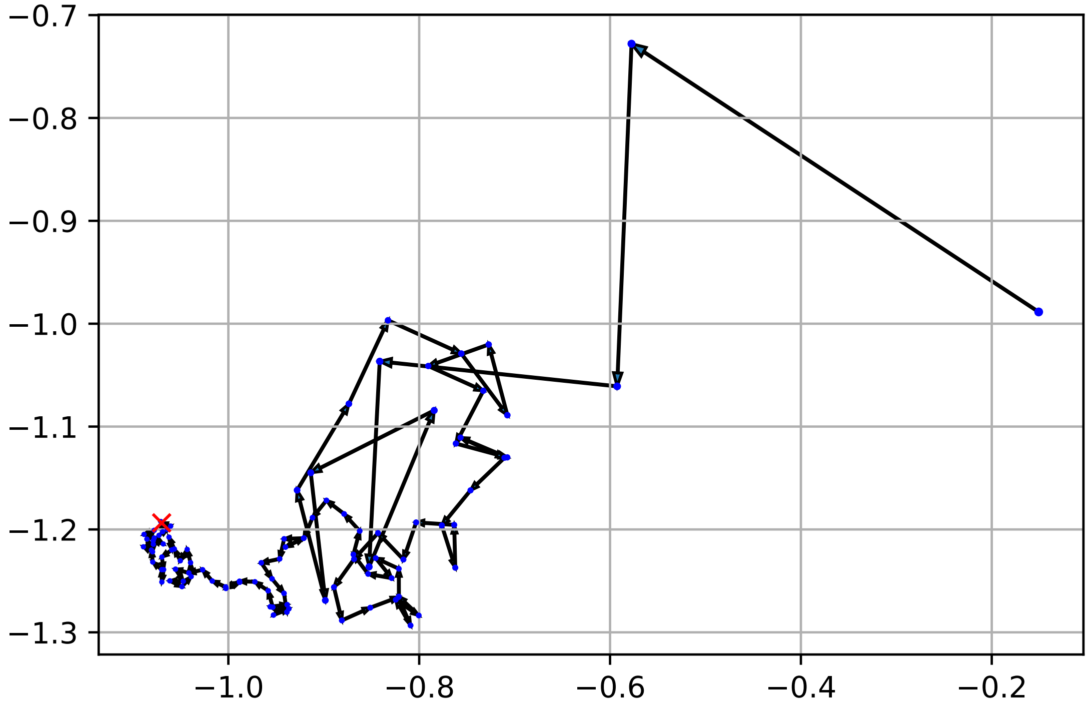


Figure 1. Complex Random Harmonic Series, with 100 terms. The end value is roughly $-1.1 - 1.2i$.

2 Complex Numbers

A natural question to ask is "can we extend this to the complex numbers?". Let $X_n \in [0, 1]$ be randomly chosen with uniform probability. Then consider the random harmonic series

$$\mathcal{R} = \sum_{n=1}^{\infty} \frac{e^{2\pi i X_n}}{n}. \tag{2.1}$$

An example of this is shown in Figure 1.

This is akin to starting from 0, drawing a circle with radius 1, and then picking a random direction and placing a new point there. Then we draw a circle with radius 1/2 and pick another random direction, and place a new point on the circle. We continue this with radius $1/n$ at each step n. Doing this for 1000 terms, we get Figure 3

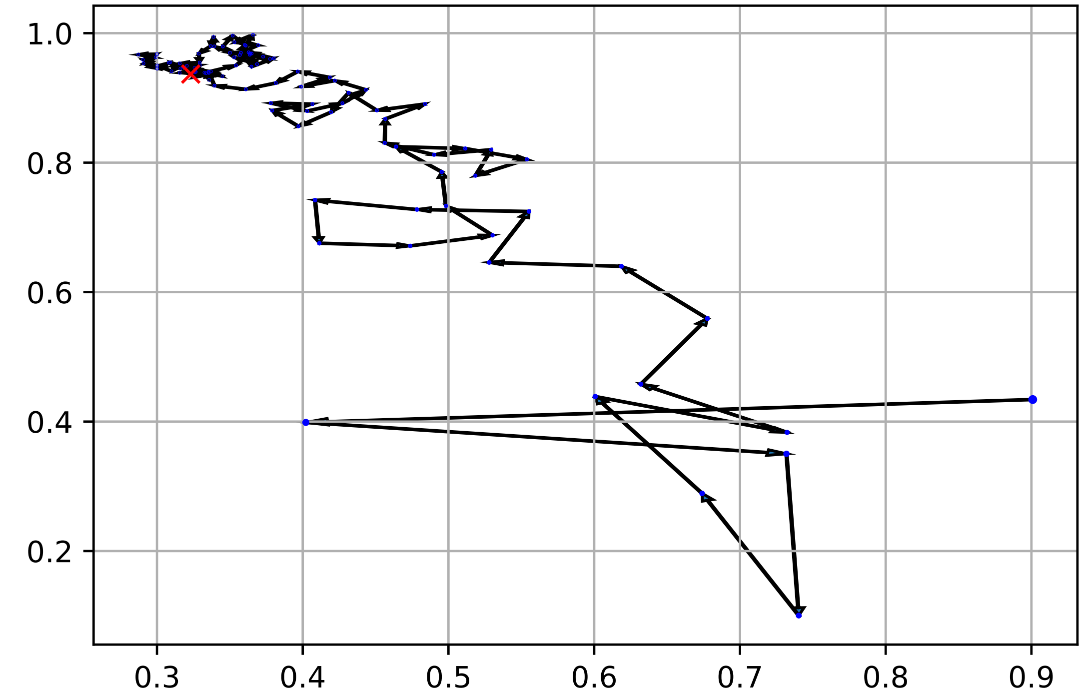


Figure 2. Complex Random Harmonic Series, with 100 terms. The end value is roughly $1 + 0.3i$.

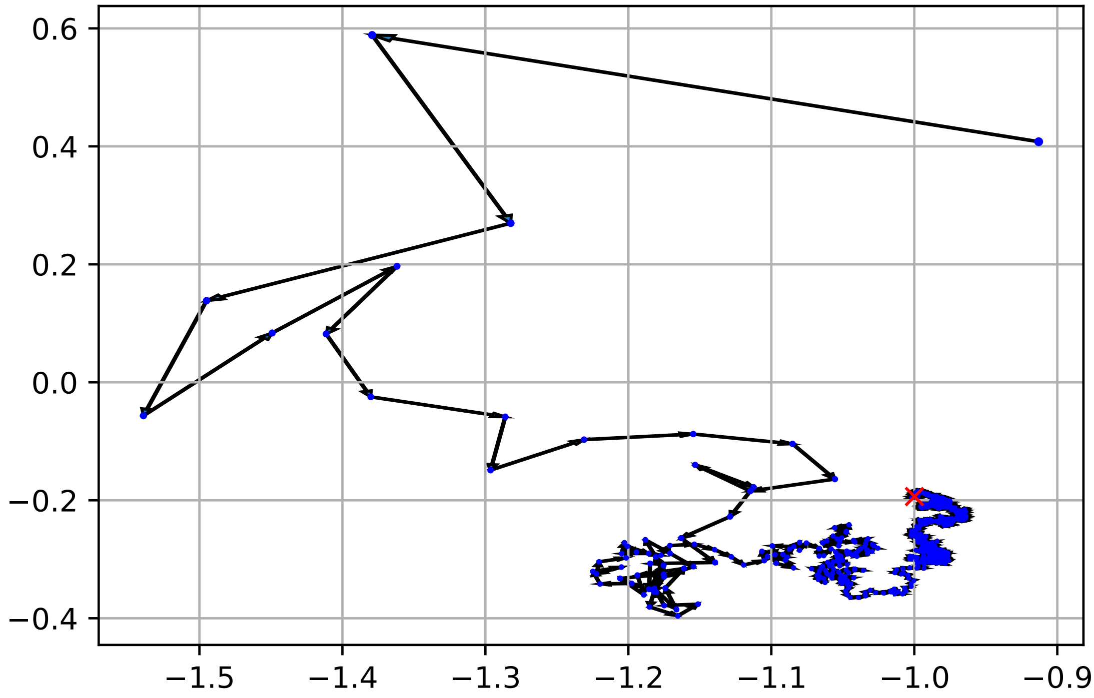


Figure 3. Complex Random Harmonic Series, with 1000 terms. The end value is roughly $-1 - 0.2i$.

3 Distribution

4 Python Code

```
import numpy as np
import matplotlib.pyplot as plt
fig, ax = plt.subplots()
# Number of terms to add
n = 100
# Define the random variables t_n
t_n = np.random.uniform(0, 1, n)
# Put the random variables into the exponential
weights = np.exp(2*np.pi*1j*t_n)
# Produce list (1,2,3,...,n)
integer_list = np.arange(1, n+1)
# Produce list (1,1/2,1/3,...,1/n)
recip = 1/integer_list
# Multiply each 1/n by exp(2 pi i t_n)
terms = weights*recip
sum_seq = []
# Add up terms of the sequence
for m in range(1, n+1):
    sum_seq.append(np.sum(terms[0:m]))
# Define an iterator i
i = 1
# Plot the sum sequence points, scaled down further down the sequence
for z in sum_seq:
    ax.plot(np.real(z), np.imag(z), 'bo', markersize=2*i**(-1/5))
    i += 1
# Split the sum sequence into real and imaginary parts
real_val = np.real(sum_seq)
imag_val = np.imag(sum_seq)
# ax.plot(real_val, imag_val)
# Plot "limit"
ax.plot(real_val[n-1], imag_val[n-1], 'rx')
# Draw arrows between points
for m in range(0, n-1):
    ax.arrow(real_val[m], imag_val[m],
             real_val[m+1] - real_val[m],
             imag_val[m+1] - imag_val[m],
             head_width=0.01*(m+1)**(-1/5),
             length_includes_head=True)
plt.grid()
fig.savefig("Complex_Random_Harmonic_Series.pdf")
```

5 Generalisation / Guaranteed Convergence

We can consider for $s > 1$, and $X_n \in [0, 1]$ chosen with uniform probability, the random Riemann Zeta series

$$\mathcal{R}_s = \sum_{n=1}^{\infty} \frac{e^{2\pi i X_n}}{n^s}. \tag{5.1}$$

It is obvious that $E(\mathcal{R}_s) = 0$ due to the circular symmetry. Clearly

$$|\mathcal{R}_s| = \left| \sum_{n=1}^{\infty} \frac{e^{2\pi i X_n}}{n^s} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) < \infty \tag{5.2}$$

so $|\mathcal{R}_s|$ is bounded. Furthermore, we can find a lower bound by trying to return to 0 with every term after the first. Without loss of generality, assume that $X_1 = 0$. Then assume that $X_n = 1/2$ for every other n so that

$$\begin{aligned} \mathcal{R}_s &= 1 - \sum_{n=2}^{\infty} \frac{1}{n^s} && (5.3) \\ &= 1 + 1 - \sum_{n=1}^{\infty} \frac{1}{n^s} && (5.4) \\ &= 2 - \zeta(s). && (5.5) \end{aligned}$$

We deduce that in general

$$|\mathcal{R}_s| \geq 2 - \zeta(s). \tag{5.6}$$

Combining the above, for each $s > 1$ we have, noting that absolute values are non-negative,

$$\max(0, 2 - \zeta(s)) \leq |\mathcal{R}_s| \leq \zeta(s). \tag{5.7}$$

In particular, for $s = 2$ we have $2 - \pi^2/6 \leq |\mathcal{R}_2| \leq \pi^2/6$. Letting $s \to \infty$ we have $\zeta(s) \to 1$ and it follows by the squeeze theorem that

$$\lim_{s \to \infty} |\mathcal{R}_s| = 1. \tag{5.8}$$

Plotting the distribution for $n = 2$, sampled over 10^4 terms summing to $n = 1000$, we get Figure 4. Because $\zeta(s) \leq 2$, zero is not reachable.

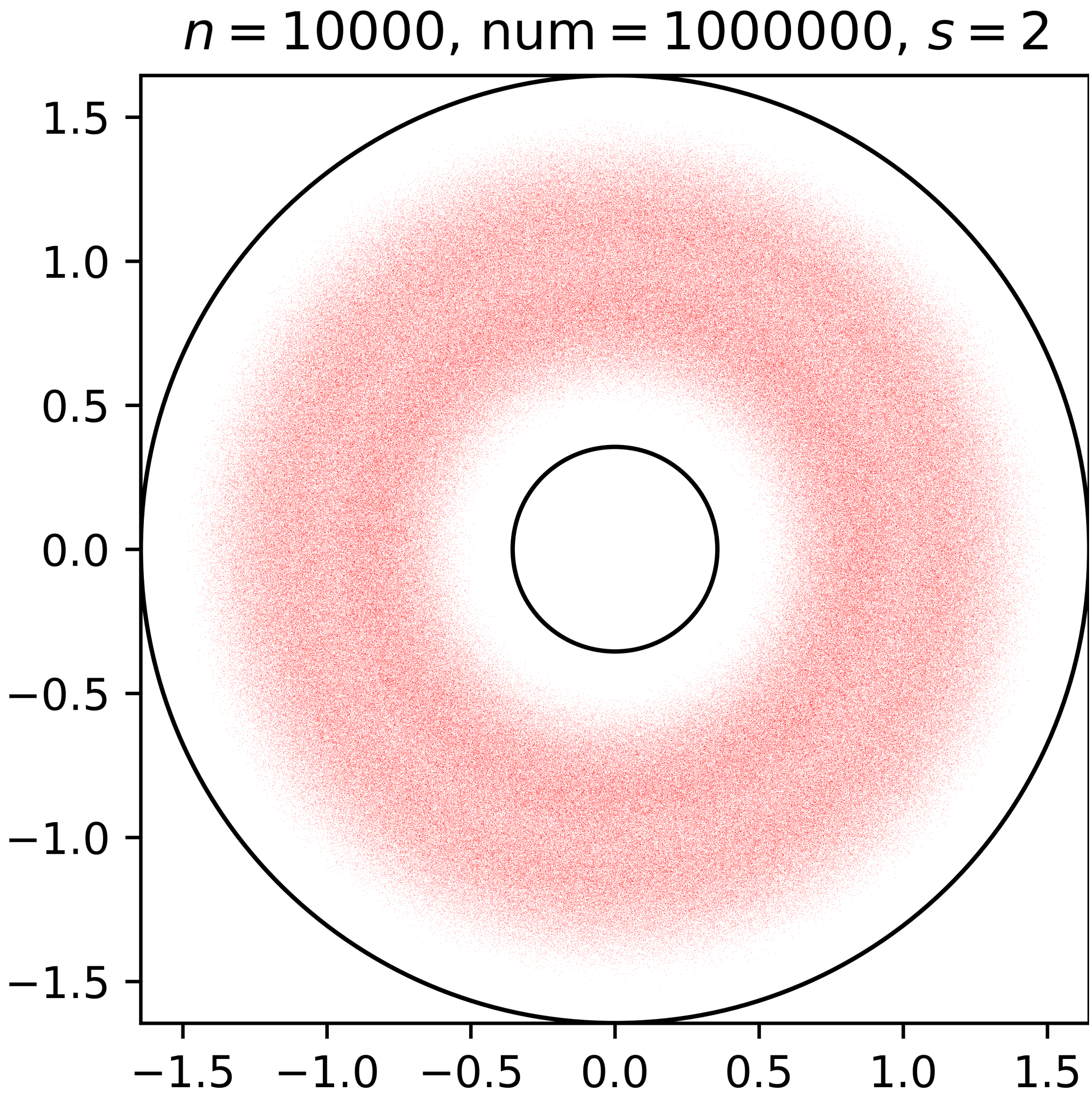


Figure 4. Distribution for random zeta series, with 10^6 points. Each point is the sum of 10^4 terms as defined in Equation 5.1. Notice that in the middle of the band, there are less points.

5.1 Radial Distribution

Given the angular symmetry of the problem, we can instead just consider the distribution in terms of the radius to investigate the two bands that form in the annulus shape. Calculating the magnitude of $\mathcal{R}_s$ explicitly with Euler's formula,

$$|\mathcal{R}_s|^2 = \left|\sum_{n=1}^{\infty} \frac{e^{2\pi i X_n}}{n^s}\right|^2 \tag{5.9}$$

$$= \left|\sum_{n=1}^{\infty} \frac{\cos(2\pi X_n) + i\sin(2\pi X_n)}{n^s}\right|^2 \tag{5.10}$$

$$= \left|\sum_{n=1}^{\infty} \frac{\cos(2\pi X_n)}{n^s} + i\sum_{n=1}^{\infty} \frac{\sin(2\pi X_n)}{n^s}\right|^2 \tag{5.11}$$

$$= \left(\sum_{n=1}^{\infty} \frac{\cos(2\pi X_n)}{n^s}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{\sin(2\pi X_n)}{n^s}\right)^2, \tag{5.12}$$

$$\implies |\mathcal{R}_s| = \sqrt{\left(\sum_{n=1}^{\infty} \frac{\cos(2\pi X_n)}{n^s}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{\sin(2\pi X_n)}{n^s}\right)^2}. \tag{5.13}$$

Doing this numerically for 10^7 points summed over 10^4 terms, we have Figure 6. Note the dip near $r = 1$. A qualitative explanation for this is that at the first step the sum is at radius 1. The next steps will tend to take outwards from radius 1, as most directions point away from the unit circle. The reason the right most bump is larger is that there are "more" directions outwards from the circle than inwards.

We can quantify this more precisely by splitting the allowed region into a pair of annuli, shown in Figure 5.

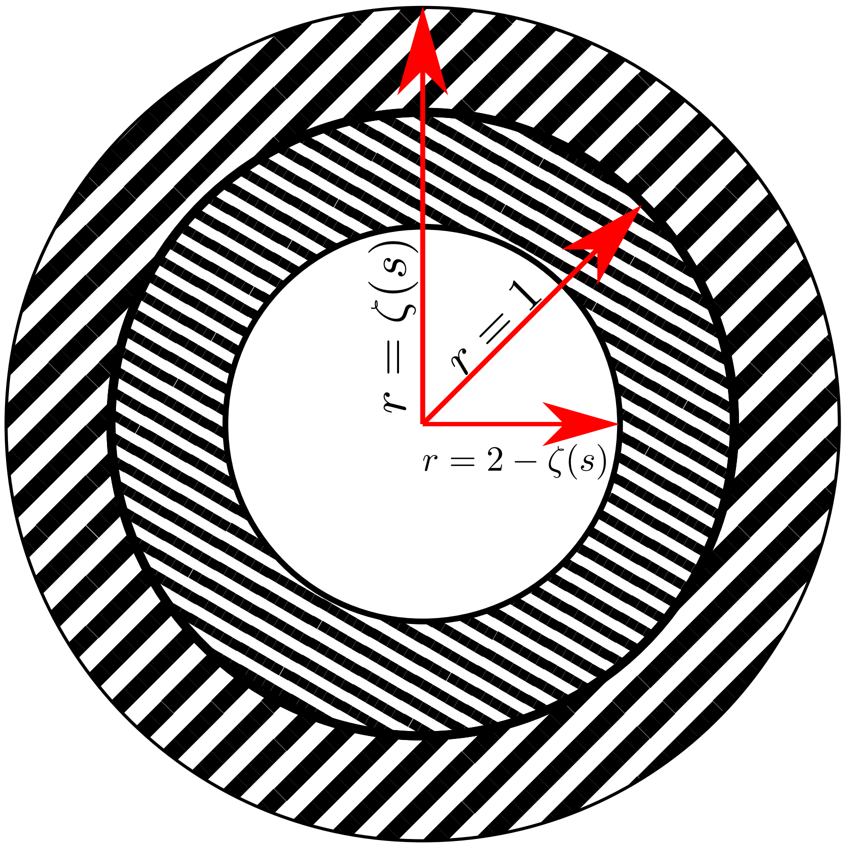


Figure 5. Caption

Let s be such that $2 \geq \zeta(s)$. Then

$$2 - \zeta(s) \leq |\mathcal{R}_s| \leq \zeta(s). \tag{5.14}$$

We let the inner circle have area A_1, the inner annulus have area A_2, and the outer annulus have area A_3. The area of the circle with radius $\zeta(s)$ is $\pi\zeta(s)^2$. The area of the circle with radius 1 is obviously π, and the area of the non-allowed circle is $A_1 = \pi(2 - \zeta(s))^2$. We then have

$$A_1 + A_2 + A_3 = \pi\zeta(s)^2, \tag{5.15}$$

$$A_1 + A_2 = \pi, \tag{5.16}$$

$$A_1 = \pi(2 - \zeta(s))^2. \tag{5.17}$$

We can solve for A_2, A_3, yielding

$$A_2 = 4\pi\zeta(s) - 3\pi - \pi\zeta(s)^2, \tag{5.18}$$

$$A_3 = \pi\left(\zeta(s)^2 - 1\right). \tag{5.19}$$

Then the relative sizes of A_2, A_3 are the areas divided by the total area

$$A_2 + A_3 = \pi\zeta(s)^2 - \pi(2-\zeta(s))^2 \tag{5.20}$$

$$= \pi\left(\zeta(s)^2 - (2-\zeta(s))^2\right) \tag{5.21}$$

$$= \pi(2\zeta(s) - 2)(\zeta(s) + 2 - \zeta(s)) \tag{5.22}$$

$$= 4\pi(\zeta(s) - 1). \tag{5.23}$$

In the case $s = 2$, this is

$$\frac{A_2}{A_2 + A_3} = \frac{4\zeta(2) - 3 - \zeta(2)^2}{4(\zeta(2) - 1)} \approx 0.34, \tag{5.24}$$

$$\frac{A_3}{A_2 + A_3} = \frac{\left(\zeta(s)^2 - 1\right)}{4(\zeta(2) - 1)} \approx 0.66. \tag{5.25}$$

This explains the small bias towards the second bump.

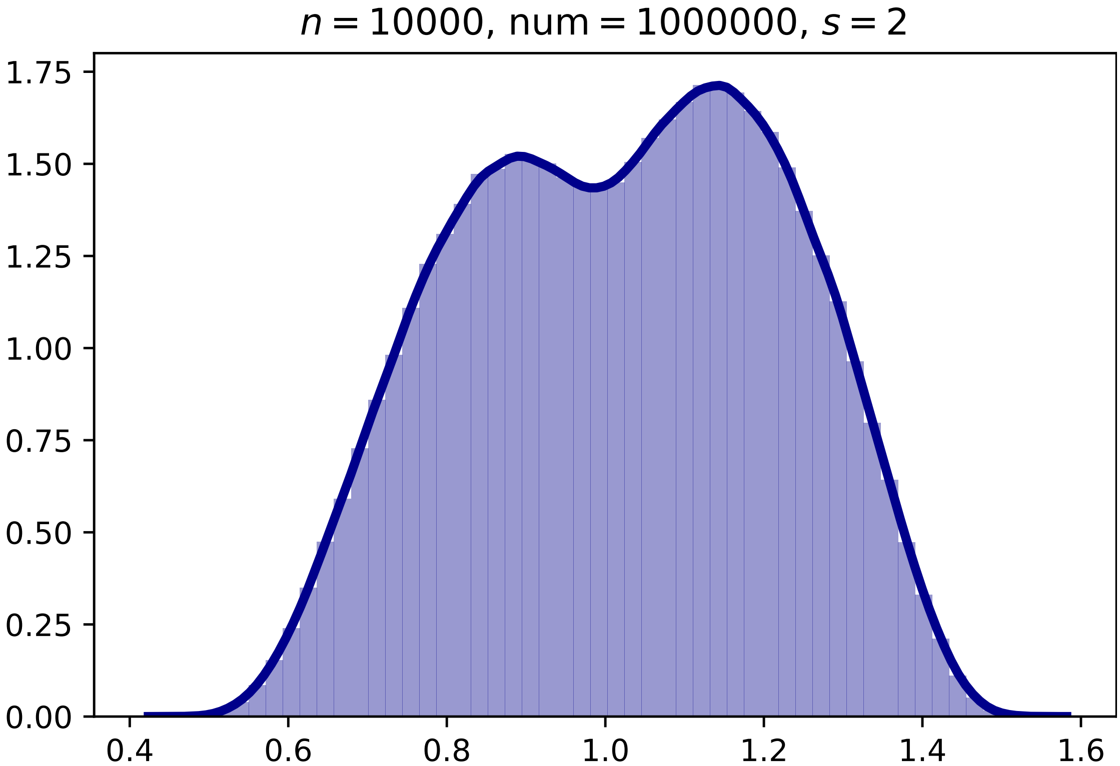
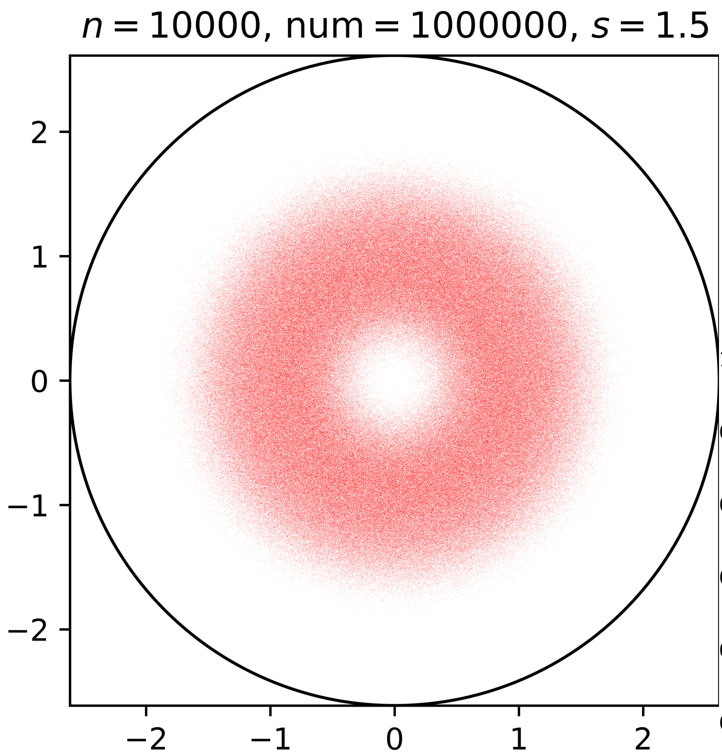


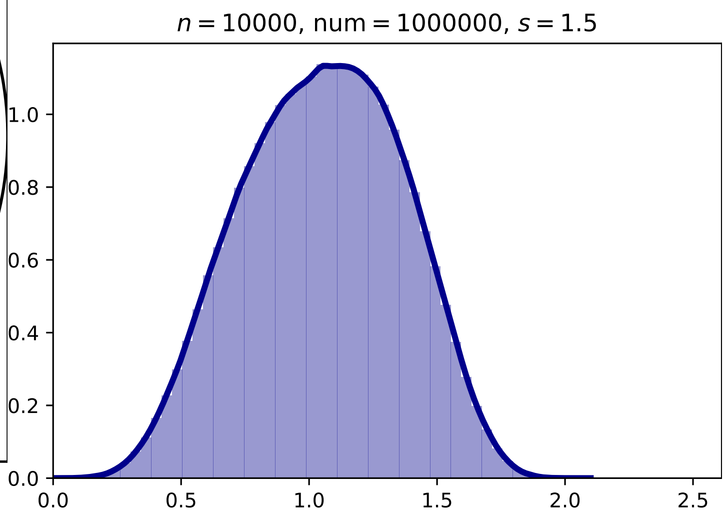
Figure 6. Histogram of the radial dependence

5.2 Distribution Change

Now if we let $2 \geq \zeta(s)$, say $s = 3/2$ ($\zeta(3/2) \approx 2.61$) then we get an alternative graph, shown in Figure 7a and associated radial graph shown in Figure 8a. Notice the lack of two bumps.



(a) Distribution for $s = 3/2$.



(b) Radial distribution for $s = 3/2$.

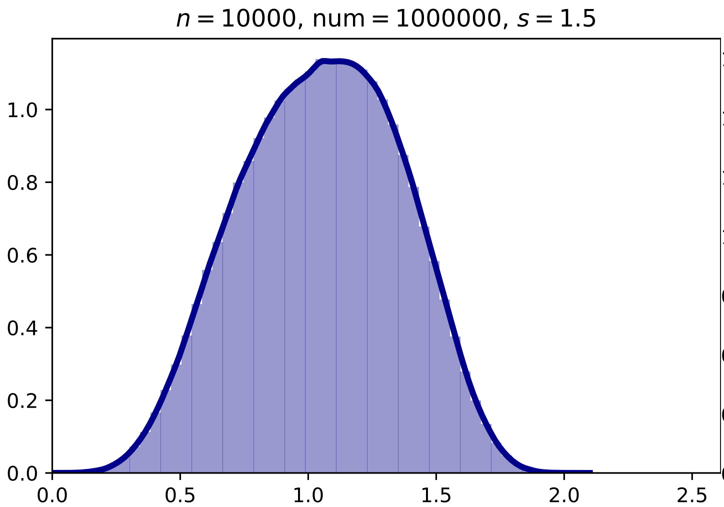
Figure 7.

More distributions are shown in the Figure below.

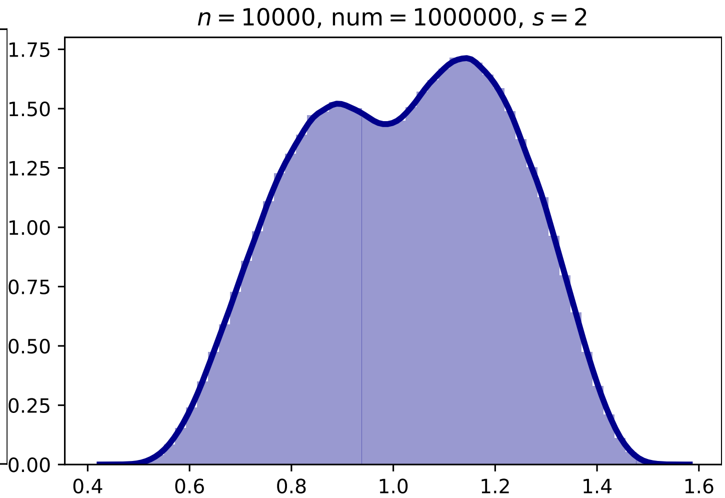
References

[1] https://thatsmaths.com/2016/07/28/random-harmonic-series/, Accessed Jan 2022.

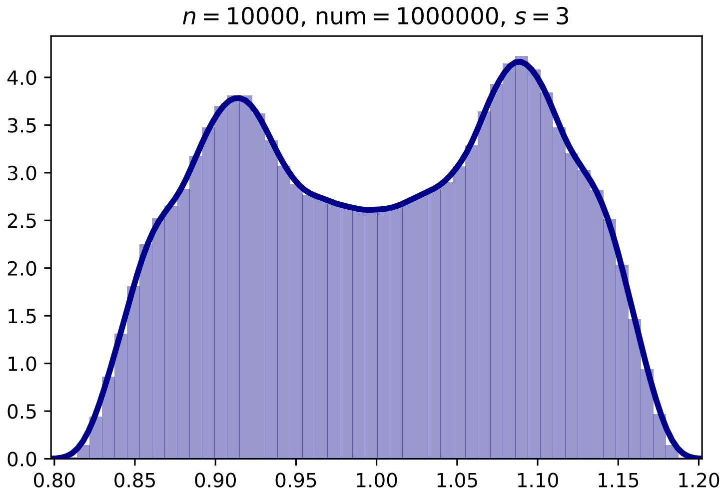
[2] Byron Schmuland. Random harmonic series. *The American Mathematical Monthly*, 110(5):407–416, 2003.



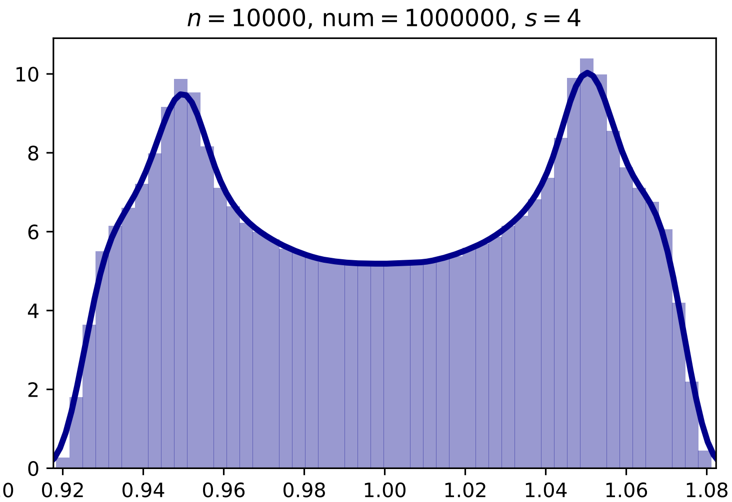
(a) Radial distribution for $s = 3/2$.



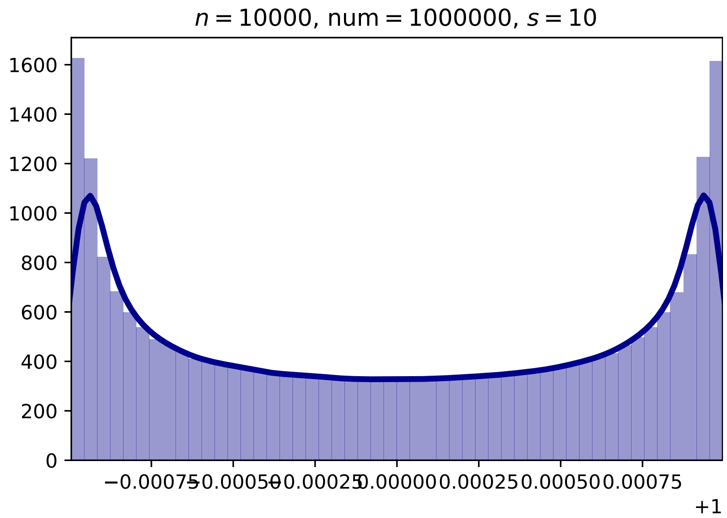
(b) Distribution for $s = 2$.



(c) Distribution for $s = 3$.



(d) Distribution for $s = 4$.



(e) Distribution for $s = 10$.

Figure 8. Radial distributions for $s = 3/2, 2, 4, 10$.