

Reciprocal Gamma Integrals

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1 Introduction

Define the Gamma function [1] by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}. \quad (1.1)$$

The Fransén–Robinson constant F is defined as [2]

$$F = \int_0^{\infty} \frac{1}{\Gamma(x)} dx \approx 2.8077702420285 \dots \quad (1.2)$$

and is noticeably close to Euler’s number $e \approx 2.71828 \dots$. This is explained by

$$F = \int_0^{\infty} \frac{1}{\Gamma(x)} dx \approx \sum_{n=0}^{\infty} \frac{1}{n!} = e, \quad (1.3)$$

and the error can be written explicitly as

$$F - e = \int_0^{\infty} \frac{e^{-x}}{\pi^2 + (\ln x)^2} dx. \quad (1.4)$$

However, this is not the only extension of the factorial $n!$, as explained by Peter Luschny [3] on his website. We define the Digamma function ψ by

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \quad (1.5)$$

Then the Hadamard Gamma function H is defined by

$$H(x) = \frac{1}{\Gamma(1-x)} \frac{d}{dx} \ln \left(\frac{\Gamma(1/2 - x/2)}{\Gamma(1 - x/2)} \right) \quad (1.6)$$

or equivalently with ψ ,

$$H(x) = \frac{\psi(1 - x/2) - \psi(1/2 - x/2)}{2\Gamma(1 - x)}. \quad (1.7)$$

and

$$H(x) = \Gamma(x) \left[1 + \frac{\sin(\pi x)}{2\pi} \left\{ \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x+1}{2}\right) \right\} \right]. \quad (1.8)$$

Are there similar constants to F with other gamma functions? Do they satisfy similar properties?

2 Hadamard Reciprocal Integral

We are interested in the value of the integral

$$\hat{H} = \int_0^\infty \frac{1}{H(x)} dx, \quad (2.1)$$

that is,

$$\hat{H} = \int_0^\infty \frac{1}{\Gamma(x) \left[1 + \frac{\sin(\pi x)}{2\pi} \left\{ \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x+1}{2}\right) \right\} \right]} dx. \quad (2.2)$$

Using the mpmath PYTHON library mp.quad function over $(0, \infty)$, the integral has approximate value

$$\hat{H} = \int_0^\infty \frac{1}{H(x)} dx \approx 3.368202929607. \quad (2.3)$$

This value is very close to

$$3 + \frac{1}{e} = 3.367879441171442 \dots. \quad (2.4)$$

This suggests a similar identity to Equation 1.4. The value of this integral can be similarly approximated using the sum

$$\hat{H} = \int_0^\infty \frac{1}{H(x)} dx \approx \sum_{n=0}^\infty \frac{1}{H(n)}. \quad (2.5)$$

Now, $H(n) = n!$ for $n = 1, 2, 3, 4, \dots$ and

$$H(0) = \frac{\psi(1) - \psi(1/2)}{2\Gamma(1)} = \frac{-\gamma - (-\gamma - \ln(4))}{2} = \frac{\ln(4)}{2} = \ln(2). \quad (2.6)$$

It follows that

$$\sum_{n=0}^\infty \frac{1}{H(n)} = \frac{1}{\ln(2)} + \sum_{n=1}^\infty \frac{1}{n!} = \frac{1}{\ln(2)} + e - 1 \approx 3.16097686934800 \dots. \quad (2.7)$$

3 Luschny Reciprocal Integral

Define

$$g(x) = \frac{x}{2} \left[\psi \left(0, \frac{x+1}{2} \right) - \psi \left(0, \frac{x}{2} \right) \right] - 1/2, \quad (3.1)$$

and

$$P(x) = 1 - g(x) \frac{\sin(\pi x)}{\pi x}. \quad (3.2)$$

Then the Luschny Factorial L is defined as [3]

$$L(x) = \Gamma(x+1)P(x). \quad (3.3)$$

We now seek the value of

$$I = \int_0^\infty \frac{1}{L(x)} dx. \quad (3.4)$$

Applying the sum argument again, we have

$$\int_0^\infty \frac{1}{L(x)} dx \approx \sum_{n=0}^\infty \frac{1}{L(n)} \quad (3.5)$$

and we know that $L(n) = n!$ for $n = 1, 2, 3, \dots$, and $L(0) = 1/2$. We then get

$$\sum_{n=0}^\infty \frac{1}{L(n)} = 2 + e - 1 = e + 1 \approx 3.71, \quad (3.6)$$

however applying the mpmath quadrature we get

$$\int_0^\infty \frac{1}{L(x)} dx \approx 2.586705059786808227 \dots \quad (3.7)$$

This is a significant deviation! Clearly this simple approach will not work. We can instead apply the Abel-Plana formula [4] to reach a similar approximation. Assuming $1/L(x)$ is "nice enough" we have

$$\int_0^\infty \frac{1}{L(x)} dx = -\frac{1}{2L(0)} + \sum_{n=0}^\infty \frac{1}{L(n)} - i \int_0^\infty \frac{1/L(ix) - 1/L(-ix)}{e^{2\pi x} - 1} dx \quad (3.8)$$

$$= e - i \int_0^\infty \frac{1/L(ix) - 1/L(-ix)}{e^{2\pi x} - 1} dx \quad (3.9)$$

$$\approx e. \quad (3.10)$$

This is much closer, and the above relation can be "numerically verified to not be immediately false".

4 Generalised Luschny Reciprocal Integral

Define

$$g(x, \alpha) = \frac{x}{2} \left[\psi \left(0, \frac{x+1}{2} \right) - \psi \left(0, \frac{x}{2} \right) \right] - \alpha/2, \quad (4.1)$$

and

$$P(x, \alpha) = 1 - g(x, \alpha) \frac{\sin(\pi x)}{\pi x}. \quad (4.2)$$

Then the Generalised Luschny Factorial $L(x, \alpha)$ is defined as [3]

$$L(x, \alpha) = \Gamma(x+1)P(x, \alpha). \quad (4.3)$$

Then it easily shown that

$$L(x) = L(x, 1), \quad H(x+1) = L(x, 2). \quad (4.4)$$

Define

$$\mathcal{I}(\alpha) = \int_0^\infty \frac{1}{L(x, \alpha)} dx. \quad (4.5)$$

Graphing the parameter α against I yields Figure 1.

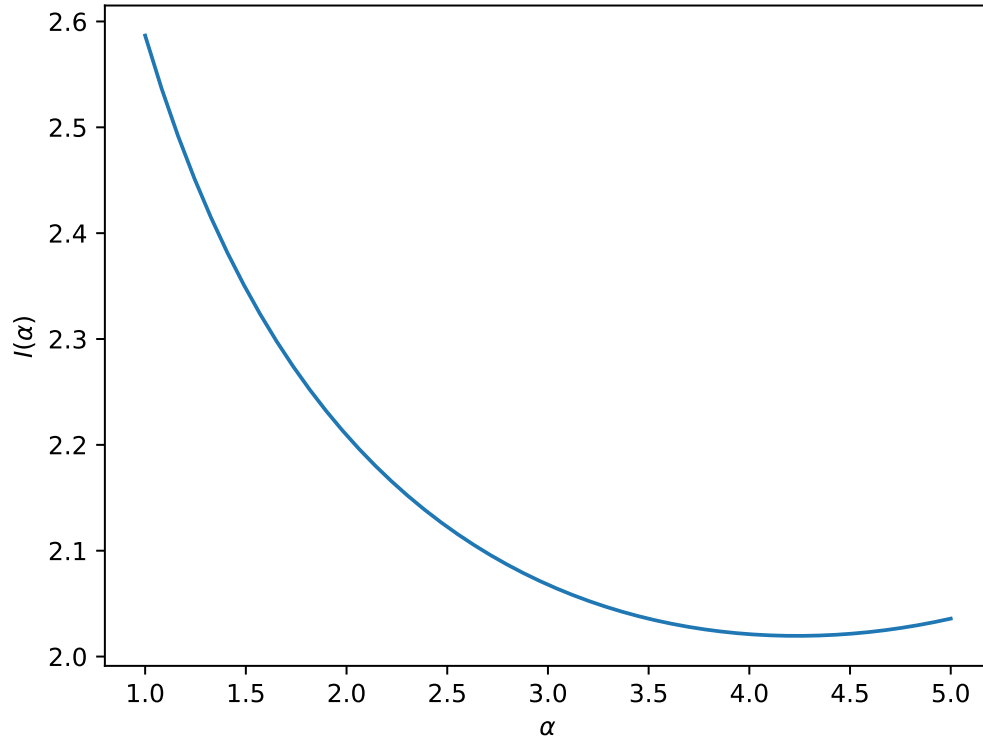


Figure 1. Relation between $\mathcal{I}(\alpha)$ and the input parameter α , over the interval $(1, 5)$. Note the appearance of a minimum between 4 and 4.5.

References

- [1] Weisstein, Eric W. "Gamma Function." From MathWorld—A Wolfram Web Resource.
<http://mathworld.wolfram.com/GammaFunction.html>
- [2] Weisstein, Eric W. "Fransén-Robinson Constant." From MathWorld—A Wolfram Web Resource.
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- [4] Anderson, David. "Abel-Plana Formula." From MathWorld—A Wolfram Web Resource, created by Eric W. Weisstein.
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