The Relativistic Pendulum

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Abstract

Following work by Cahit Erkal (2000), we derive the equation of motion for a relativistic pendulum under uniform gravity. We derive the time period of the relativistic pendulum, and give an upper bound in terms of the non-relativistic pendulum, verifying results by C. Erkal.

This document is based upon the work by [1].

The relativistic Lagrangian L for a pendulum with with point mass m is given by

$$L = -\frac{mc^2}{\gamma} - V(\vartheta) \tag{1}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2}$$

and V is the potential energy, which in our case is a gravitational potential and c is the speed of light. (We note that technically there should be a additional mc^2 term in the Lagrangian, but this is ignored as it is constant.)

1 Derivation of the potential

Ignoring the mass of the string, the potential energy of the pendulum at an angle ϑ with the vertical is given by

$$V(\vartheta) = \int_0^{\vartheta} \gamma mgl \sin(\vartheta)\, \mathrm{d}\vartheta. \tag{3}$$

By the principle of conservation of energy, and assuming that the pendulum is at rest at the initial angle ϑ_0 we have

$$V_1 + mc^2 = \gamma mc^2 + V(\vartheta) \tag{4}$$

where V_1 is the initial potential energy. We can rearrange for γ, which is

$$\frac{V_1 - V(\vartheta)}{mc^2} + 1 = \gamma. \tag{5}$$

We can substitute this into equation (3) to get

$$V(\vartheta) = \int_0^\vartheta \left(\frac{V_1 - V(\vartheta)}{mc^2} + 1 \right) mgl \sin(\vartheta) \, \mathrm{d}\vartheta. \tag{6}$$

Differentiating this, we have

$$\frac{\mathrm{d}V}{\mathrm{d}\vartheta} = \left(\frac{V_1 - V(\vartheta)}{mc^2} + 1 \right) mgl \sin(\vartheta) \tag{7}$$

and expanding we have

$$\frac{\mathrm{d}V}{\mathrm{d}\vartheta} + \frac{gl}{c^2} \sin(\vartheta) V(\vartheta) = \left(V_1 + mc^2\right) \frac{gl}{c^2} \sin(\vartheta). \tag{8}$$

Multiplying by an integrating factor given by

$$\exp\left(\int \frac{gl}{c^2} \sin(\vartheta) \, \mathrm{d}\vartheta \right) = \exp\left(-\frac{gl}{c^2} \cos(\vartheta) \right), \tag{9}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} \left(\exp\left(-\frac{gl}{c^2} \cos(\vartheta) \right) V(\vartheta) \right) = \left(V_1 + mc^2\right) \frac{gl}{c^2} \sin(\vartheta) \exp\left(-\frac{gl}{c^2} \cos(\vartheta) \right). \tag{10}$$

Integrating this, we se $u = -\frac{gl}{c^2} \cos(\vartheta)$, so

$$\exp\left(-\frac{gl}{c^2} \cos(\vartheta) \right) V(\vartheta) = \left(V_1 + mc^2\right) \int \frac{gl}{c^2} \sin(\vartheta) \exp\left(-\frac{gl}{c^2} \cos(\vartheta) \right) \mathrm{d}\vartheta \tag{11}$$

$$= \left(V_1 + mc^2\right) \int \exp(u) \, \mathrm{d}\vartheta \tag{12}$$

$$= \left(V_1 + mc^2\right) \exp\left(-\frac{gl}{c^2} \cos(\vartheta) \right) + C \tag{13}$$

for some constant $C \in \mathbb{R}$. Dividing through by the exponential term yields

$$V(\vartheta) = V_1 + mc^2 + C \exp\left(\frac{gl}{c^2} \cos(\vartheta) \right) \tag{14}$$

and we can find the constant using the initial condition $V(\vartheta_0) = V_1$, so

$$V_1 = V_1 + mc^2 + C \exp\left(\frac{gl}{c^2} \cos(\vartheta_0) \right) \tag{15}$$

and rearranging for C we have

$$C = -mc^2 \exp\left(-\frac{gl}{c^2}\cos(\vartheta_0)\right). \tag{16}$$

This gives the final potential as

$$V(\vartheta) = V_1 + mc^2\left[1 - \exp\left(\frac{gl}{c^2}(\cos(\vartheta) - \cos(\vartheta_0))\right)\right]. \tag{17}$$

2 Equations of Motion

Now, noting that the particle is constrained to the end of the rod, we have that

$$v = l\dot{\vartheta} \tag{18}$$

so the Lorentz factor can be written

$$\gamma = \frac{1}{\sqrt{1 - \left(l\dot{\vartheta}\right)^2/c^2}} \tag{19}$$

and thus write the Lagrangian as

$$L = -mc^2\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{1/2} - V(\vartheta) \tag{20}$$

which is written in the generalised coordinate ϑ. Applying the Euler-Lagrange equations [2] to this, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\vartheta}}\right) - \frac{\partial L}{\partial\vartheta} = 0. \tag{21}$$

Calculating the partial derivatives, we have

$$\frac{\partial L}{\partial\dot{\vartheta}} = \frac{\partial}{\partial\dot{\vartheta}}\left(-mc^2\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{1/2}\right) \tag{22}$$

$$= -mc^2\left(\frac{\partial}{\partial\dot{\vartheta}}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{1/2}\right) \tag{23}$$

$$= -mc^2\left(\frac{1}{2}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2}\right)\left(\frac{-2l^2\dot{\vartheta}}{c^2}\right) \tag{24}$$

$$= ml^2\dot{\vartheta}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2}, \tag{25}$$

and for the other partial derivative, we have

$$\frac{\partial L}{\partial \vartheta} = -\frac{\partial V}{\partial \vartheta} = -\frac{\mathrm{d}V}{\mathrm{d}\vartheta}. \tag{26}$$

Now, applying equation (7) and equation (17) we have

$$-\frac{\mathrm{d}V}{\mathrm{d}\vartheta} = -\left(\frac{V_1 - V_1 - mc^2\left[1 - \exp\left(\frac{gl}{c^2}(\cos(\vartheta) - \cos(\vartheta_0))\right)\right]}{mc^2} + 1\right) mgl\sin(\vartheta) \tag{27}$$

$$= -\left(-\left[1 - \exp\left(\frac{gl}{c^2}(\cos(\vartheta) - \cos(\vartheta_0))\right)\right] + 1\right) mgl\sin(\vartheta) \tag{28}$$

$$= -\left(-1 + \exp\left(\frac{gl}{c^2}(\cos(\vartheta) - \cos(\vartheta_0))\right) + 1\right) mgl\sin(\vartheta) \tag{29}$$

$$= -mgl\sin(\vartheta)\exp\left(\frac{gl}{c^2}(\cos(\vartheta) - \cos(\vartheta_0))\right). \tag{30}$$

We can now write the Euler-Lagrange equations (21), dividing through by ml^2, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vartheta}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2}\right) = -\frac{g}{l}\sin(\vartheta)\exp\left(\frac{gl}{c^2}(\cos(\vartheta) - \cos(\vartheta_0))\right). \tag{31}$$

Calculating the derivative on the left hand side, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vartheta}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2}\right) = \ddot{\vartheta}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2} + \dot{\vartheta}\left(-\frac{1}{2}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-3/2}\right)\cdot\left(\frac{-2l^2\dot{\vartheta}}{c^2}\right)\ddot{\vartheta} \tag{32}$$

$$= \ddot{\vartheta}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2} + \frac{l^2\dot{\vartheta}^2\ddot{\vartheta}}{c^2}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-3/2} \tag{33}$$

$$= \ddot{\vartheta}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2}\left[1 + \frac{l^2\dot{\vartheta}^2}{c^2}\left(1 - \frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1}\right]. \tag{34}$$

Now, noting that

$$1 + \frac{\frac{l^2\dot{\vartheta}^2}{c^2}}{1 - \frac{l^2\dot{\vartheta}^2}{c^2}} = \frac{1 - \frac{l^2\dot{\vartheta}^2}{c^2}}{1 - \frac{l^2\dot{\vartheta}^2}{c^2}} + \frac{\frac{l^2\dot{\vartheta}^2}{c^2}}{1 - \frac{l^2\dot{\vartheta}^2}{c^2}} = \frac{1}{1 - \frac{l^2\dot{\vartheta}^2}{c^2}} \tag{35}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\vartheta}\left(1-\frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2}\right)=\ddot{\vartheta}\left(1-\frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-3/2}. \tag{36}$$

Then (31) becomes

$$\ddot{\vartheta}=-\frac{g}{l}\left(1-\frac{l^2\dot{\vartheta}^2}{c^2}\right)^{3/2}\sin(\vartheta)\exp\left(\frac{gl}{c^2}(\cos(\vartheta)-\cos(\vartheta_0))\right). \tag{37}$$

Now, we can rearrange for γ in by inserting the potential (17) into (5), so

$$\begin{aligned}\gamma &= \frac{V_1-V_1-mc^2\left[1-\exp\left(\frac{gl}{c^2}(\cos(\vartheta)-\cos(\vartheta_0))\right)\right]}{mc^2}+1 && (38)\\ &= \exp\left(\frac{gl}{c^2}(\cos(\vartheta)-\cos(\vartheta_0))\right) && (39)\\ &= \left(1-\frac{l^2\dot{\vartheta}^2}{c^2}\right)^{-1/2}, && (40)\end{aligned}$$

and plugging this into (37) yields the equation of motion

$$\ddot{\vartheta}=-\frac{g}{l}\left(1-\frac{l^2\dot{\vartheta}^2}{c^2}\right)\sin(\vartheta). \tag{41}$$

Clearly in the limit $c\to\infty$ we obtain the Newtonian form of the pendulum.

3 Time Period

The time period for the non-relativistic pendulum started at rest from an angle ϑ_0 is given by

$$T_n(\vartheta_0) = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2(\phi)}} \, \mathrm{d}\phi \tag{42}$$

where $k = \sin(\vartheta_0/2)$.

Proposition 1. *Let the initial angle for the pendulum be ϑ_0, and let the pendulum begin from rest. Then the relativistic time period is given by*

$$T_r = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2(\phi)}\sqrt{u(\phi)}} \, \mathrm{d}\phi \tag{43}$$

with $k = \sin(\vartheta_0/2)$, where

$$u(\phi) = \frac{1 - \exp\left[-2\alpha k^2 \cos^2(\phi)\right]}{2\alpha k^2 \cos^2(\phi)}, \quad \alpha = \frac{2gl}{c^2}. \tag{44}$$

Proof. The velocity of the relativistic pendulum starting from rest at an angle $\vartheta_0 \neq 0$ can be determined as

$$v = \pm c\sqrt{1 - f(\vartheta)} \tag{45}$$

where

$$f(\vartheta) = \exp\left[\frac{2gl}{c^2}\left(\cos(\vartheta_0) - \cos(\vartheta)\right)\right]. \tag{46}$$

Thus via an abuse of notation,

$$\mathrm{d}\vartheta = \frac{v}{l}\,\mathrm{d}t = \pm\frac{c}{l}\sqrt{1 - f(\vartheta)}\,\mathrm{d}t. \tag{47}$$

For a full period, the pendulum swings from $\vartheta_0 \to 0$, then $0 \to -\vartheta_0$, and then back. This yields a factor of four, and we get the time period of the relativistic pendulum as

$$T_r(\vartheta_0) = \frac{4}{c} \int_0^{\vartheta_0} \frac{l}{\sqrt{1 - f(\vartheta)}} \, \mathrm{d}\vartheta. \tag{48}$$

Let $k = \sin(\vartheta_0/2)$, and $\sin(\phi) = \sin(\vartheta/2)/k$, and write $\alpha = 2gl/c^2$. Then the upper bound will be attained when $\sin(\phi) = 1$, so $\phi = \pi/2$, and the lower bound is at $\phi = 0$. We have

$$\cos(\phi)\,\mathrm{d}\phi = \frac{\cos(\vartheta/2)}{2k}\,\mathrm{d}\vartheta \tag{49}$$

thus

$$\mathrm{d}\vartheta = \frac{2k\cos(\phi)}{\cos(\vartheta/2)}\,\mathrm{d}\phi \tag{50}$$
$$= \frac{2k\cos(\phi)}{\sqrt{1-\sin^2(\vartheta/2)}}\,\mathrm{d}\phi \tag{51}$$
$$= \frac{2k\cos(\phi)}{\sqrt{1-k^2\sin^2(\phi)}}\,\mathrm{d}\phi. \tag{52}$$

Inserting, we have

$$T_r(\vartheta_0) = \frac{4}{c}\int_0^{\vartheta_0} \frac{2kl\cos(\phi)}{\sqrt{1-k^2\sin^2(\phi)}}\,\mathrm{d}\phi \frac{1}{\sqrt{1-\exp\left[\alpha\left(\cos(\vartheta_0)-\cos(\vartheta)\right)\right]}}. \tag{53}$$

Using the half angle formula $\cos(\vartheta) = 1 - 2\sin^2(\vartheta/2)$, we find that

$$\alpha\left(\cos(\vartheta_0)-\cos(\vartheta)\right) = -\alpha\left(-\cos(\vartheta_0)+\cos(\vartheta)\right) \tag{54}$$
$$= -\alpha\left(2\sin^2(\vartheta_0/2) - 2\sin^2(\vartheta/2)\right) \tag{55}$$
$$= -2\alpha\left(k^2 - \sin^2(\vartheta/2)\right) \tag{56}$$
$$= -2\alpha k^2\left(1-\sin^2(\phi)\right) \tag{57}$$
$$= -2\alpha k^2\cos^2(\phi). \tag{58}$$

This integral then becomes

$$T_r(\vartheta_0) = \frac{4}{c}\int_0^{\pi/2} \frac{2kl\cos(\phi)}{\sqrt{1-k^2\sin^2(\phi)}\sqrt{1-h(\phi)}}\,\mathrm{d}\phi, \tag{59}$$

with

$$h(\phi) = \exp\left[-2\alpha k^2\cos^2(\phi)\right]. \tag{60}$$

Define $u(\phi)$ by

$$2\alpha k^2\cos^2(\phi)u(\phi) = 1-h(\phi) = 2\alpha k^2\cos^2(\phi) - 2\alpha^2k^4\cos^4(\phi) + \frac{4}{3}\alpha^3k^6\cos^6(\phi) + \cdots \tag{61}$$

so that $u(\phi)$ itself is given by

$$u(\phi) = 1 - \alpha k^2\cos^2(\phi) + \frac{2}{3}\alpha^2k^4\cos^4(\phi) + \cdots \tag{62}$$

or explicitly,

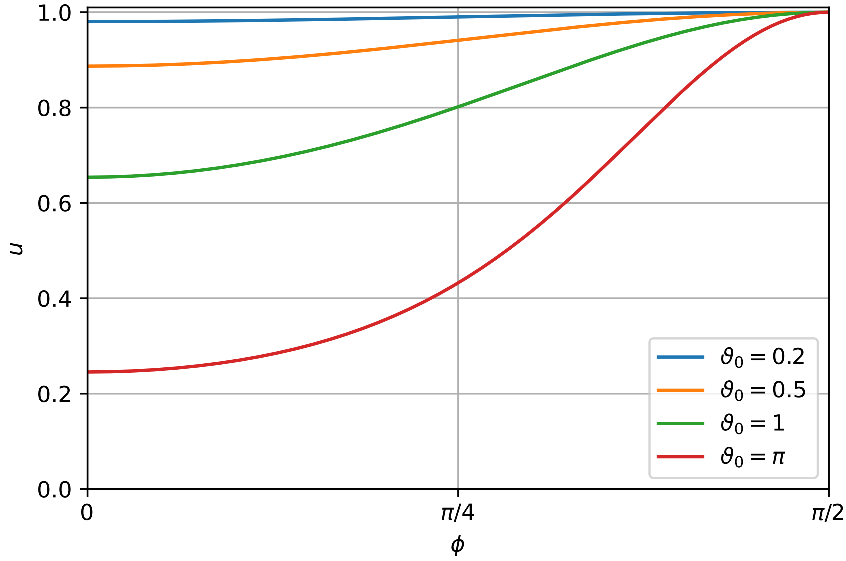


Figure 1: Graph of $u(\phi)$ over $\phi \in [0, \pi/2]$, for $\vartheta_0 = 0.2, 0.5, 1, \pi$. Here $\alpha = 2$.

$$u(\phi) = \frac{1 - \exp\left[-2\alpha k^2 \cos^2(\phi)\right]}{2\alpha k^2 \cos^2(\phi)}. \tag{63}$$

The graph of u is shown in Figure 1. Inserting this into (59) yields (noting that $\cos(\phi) > 0$ for $\phi \in (0, \pi/2)$ so $\sqrt{\cos^2(\phi)} = \cos(\phi)$)

$$T_r = \frac{4}{c} \int_0^{\pi/2} \frac{2kl\cos(\phi)}{\sqrt{1 - k^2 \sin^2(\phi)}\sqrt{1 - 1 + 2\alpha k^2 \cos^2(\phi) u(\phi)}} \, \mathrm{d}\phi \tag{64}$$

$$= \frac{4}{c} \int_0^{\pi/2} \frac{2l}{\sqrt{1 - k^2 \sin^2(\phi)}\sqrt{\frac{4gl}{c^2} u(\phi)}} \, \mathrm{d}\phi \tag{65}$$

$$= \frac{8l}{c} \sqrt{\frac{c^2}{4gl}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2(\phi)}\sqrt{u(\phi)}} \, \mathrm{d}\phi \tag{66}$$

$$= 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2(\phi)}\sqrt{u(\phi)}} \, \mathrm{d}\phi. \tag{67}$$

□

This is almost identical to (42), thus the time period difference $D(\vartheta_0) = T_r(\vartheta_0) - T_n(\vartheta_0)$ will solely depend upon how close $1/\sqrt{u(\phi)}$ is 1 over $(0, \pi/2)$. One can see from Figure 1 that smaller initial angles tend to result in u being closer to 1. We also see that u is bounded above and below, which we will exploit in the following proposition.

Proposition 2. *Let the initial angle for the pendulum be ϑ_0, and start the pendulum from rest. Let T_r, T_n denote the relativistic and non-relativistic time periods respectively. Then*

$$T_n \leq T_r \leq T_n \sqrt{\frac{2\alpha}{1 - \exp\left[-2\alpha\right]}}. \tag{68}$$

Proof. First, we show that u is bounded above. It is a fact that for $x < 0$ we have,

$$\frac{e^x - 1}{x} \leq 1. \tag{69}$$

It is clear that $2\alpha k^2 \cos^2(\phi) \geq 0$, and thus

$$|u(\phi)| = \left| \frac{\exp\left[-2\alpha k^2 \cos^2(\phi)\right] - 1}{-2\alpha k^2 \cos^2(\phi)} \right| \leq 1. \tag{70}$$

Dividing, square rooting and subtracting 1 then yields

$$\frac{1}{\sqrt{u(\phi)}} \geq 1. \tag{71}$$

It follows that $T_r \geq T_n$. We now show that u is bounded below. We let $\phi = 0$ (see Figure 1), and then we have

$$u(\phi = 0, \vartheta_0) = \frac{1 - \exp\left[-2\alpha \sin^2(\vartheta_0/2)\right]}{2\alpha \sin^2(\vartheta_0/2)}. \tag{72}$$

Differentiating,

$$\frac{\partial u}{\partial \vartheta_0}(\phi = 0, \vartheta_0) = \alpha \sin(\vartheta_0) \left(2\alpha \sin^2(\vartheta_0/2) e^{2\alpha \sin^2(\vartheta_0/2)} + e^{2\alpha \sin^2(\vartheta_0/2)} - 1 \right). \tag{73}$$

The minimum is $\vartheta_0 = \pi$. We deduce that a lower bound for u is

$$u(\phi) \geq u(\phi = 0, \vartheta_0 = \pi) = \frac{1 - \exp\left[-2\alpha\right]}{2\alpha}. \tag{74}$$

We then proceed as follows,

$$\frac{1-\exp[-2\alpha]}{2\alpha} \leq u(\phi) \tag{75}$$

$$\frac{1}{u(\phi)} \leq \frac{2\alpha}{1-\exp[-2\alpha]} \tag{76}$$

$$\frac{1}{\sqrt{u(\phi)}} \leq \sqrt{\frac{2\alpha}{1-\exp[-2\alpha]}}. \tag{77}$$

It follows that

$$T_n \leq T_r \leq T_n \sqrt{\frac{2\alpha}{1-\exp[-2\alpha]}}, \tag{78}$$

as required. □

It is then easily seen that as $\vartheta_0 \to 0$, we are forced to have $0 \leq T_r \leq 0$, so $T_r(0) = 0$. This can be rephrased as $D(\vartheta_0) \to 0$ as $\vartheta_0 \to 0$. Furthermore, if we let $c \to \infty$, then $\alpha \to 0$, so

$$\sqrt{\frac{2\alpha}{1-\exp[-2\alpha]}} \to 1. \tag{79}$$

Then we can deduce that $T_r \to T_n$ in the Newtonian limit.

3.1 Difference Plots

Plotting the difference function D along with the upper bounds for D yields Figure 2.

4 Angle as a function of time

To solve for $\vartheta(t)$, we start from an initial angle ϑ_0 at rest. We then define $y_1 = \vartheta, y_2 = \dot{\vartheta}$. Using equation (41), this is the set of first order equations

$$y_1' = y_2, \tag{80}$$

$$y_2' = -\frac{g}{l}\left(1 - \frac{l^2 y_2^2}{c^2}\right)\sin(y_1), \tag{81}$$

with initial conditions $y_0(0) = \vartheta_0, y_1(0) = 0$. In vector form we have

$$(y_1', y_2') = \mathbf{y}' = \mathbf{f}(t, \mathbf{y}) = \left(y_2, -\frac{g}{l}\left(1 - \frac{l^2 y_2^2}{c^2}\right)\sin(y_1)\right). \tag{82}$$

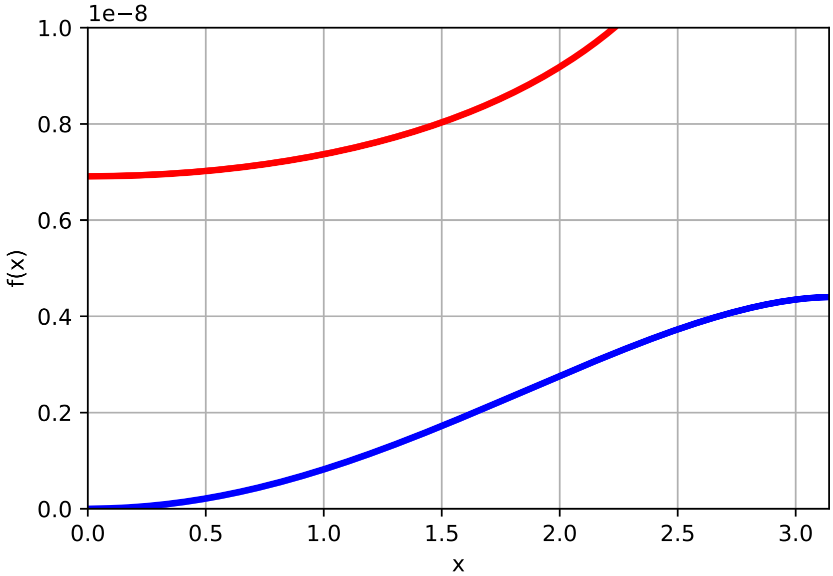
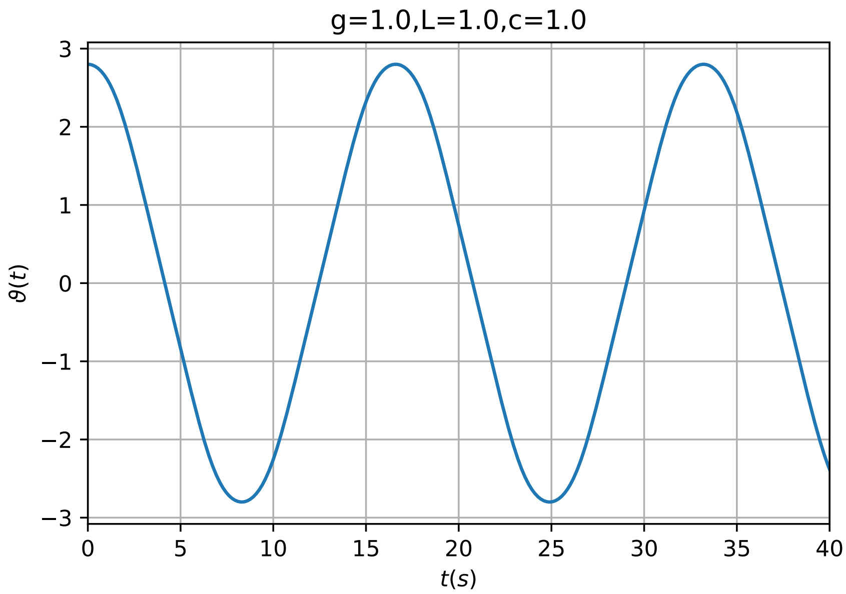


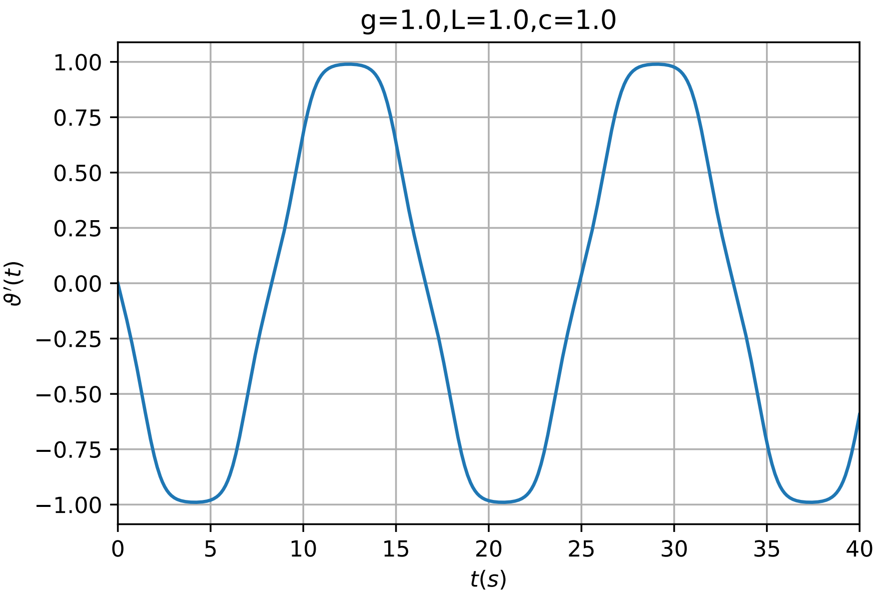
Figure 2: $D(\vartheta_0)$, with $g = 9.8, l = 100000, c = 3 \cdot 10^8$. The blue line is D, and the red line is the upper bound for D. Note the vertical scale is extremely small.

Now, $\mathbf{f}, \partial\mathbf{f}/\partial y_1, \partial\mathbf{f}/\partial y_2$ are continuous everywhere in $\mathbb{R}^2$, so there exists an interval $I \subset \mathbb{R}$ containing $t = 0$ on which a solution exists (see [3, Theorem 1.1]).

This can then be solved numerically using the Python `mpmath` library [4], specifically the `odefun` function. With constants $g = 1$, $l = 1$, $c = 1$ and initial condition $\vartheta_0 = 2.8$, this gives Figures 3a and 3b. The calculated time period is $T_r(2.8) = 16.6$, which can be seen to be accurate in the graph.



(a) $\vartheta(t)$.



(b) $\dot{\vartheta}(t)$.

Figure 3: $\vartheta, \dot{\vartheta}$ as functions of time, starting from rest at $\vartheta_0 = 2.8$.

References

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