Gaussian Arctan Integral

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1 Definition

Define $G : \mathbb{R} \to \mathbb{R}$

$$G(n,t;x) = \int_0^x \frac{e^{-ty^n}}{1+y^n}\,\mathrm{d}y \tag{1.1}$$

where $t, n \in \mathbb{R}$.

2 Properties

2.1 Limit to infinity

Proposition 1. *If $n > 0, t > 0$ then*

$$\lim_{x\to\infty} (G(n,t;x)) = e^t\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{n-1}{n}, t\right) \tag{2.1}$$

where [1][2]

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} \,\mathrm{d}x \tag{2.2}$$

$$\Gamma(s,z) = \int_z^\infty x^{s-1} e^{-x} \,\mathrm{d}x. \tag{2.3}$$

2.1.1 Pre-work

Lemma 1. *For $n, t \in \mathbb{R}^+$*

$$J = \int_0^\infty e^{-tx^n} \,\mathrm{d}x = \frac{\Gamma\left(\frac{n+1}{n}\right)}{\sqrt[n]{t}} \tag{2.4}$$

Proof. Introducing the substitution $v = \sqrt[n]{t}x$, we get

$$J = \frac{1}{\sqrt[n]{t}} \int_0^\infty e^{-v^n} \,\mathrm{d}v \tag{2.5}$$

This deals with the t, and now for the rest of the integral! Letting $u = v^n$, $\mathrm{d}u = nv^{n-1}\,\mathrm{d}v = nu^{1-\frac{1}{n}}\,\mathrm{d}v$ we then transform our integral to

$$J = \frac{1}{\sqrt[n]{t}} \frac{1}{n} \underbrace{\int_0^\infty u^{\frac{1}{n}-1} e^{-u} \,\mathrm{d}u}_{\Gamma\left(\frac{1}{n}\right)} \tag{2.6}$$

Using the identity

$$\Gamma(1+z) = z\Gamma(z) \tag{2.7}$$

completes the proof

$$J = \int_0^\infty e^{-tx^n} \,\mathrm{d}x = \frac{\Gamma\left(\frac{n+1}{n}\right)}{\sqrt[n]{t}}.$$

□

Lemma 2.

We can see that

$$0 < \int_0^\infty \frac{e^{-tx^n}}{1+x^n} \,\mathrm{d}x < \int_0^\infty e^{-tx^n} \,\mathrm{d}x \tag{2.8}$$

As we saw above, since the Gamma function we derived converges on $n \in (0, \infty)$ for $t > 0$, this integral also converges.

Also, since

$$\int_0^\infty e^{-tx^n} \,\mathrm{d}x = \frac{\Gamma\left(\frac{n+1}{n}\right)}{\sqrt[n]{t}}. \tag{2.9}$$

we can deduce

$$\lim_{t\to\infty}\int_0^\infty \frac{e^{-tx^n}}{1+x^n}\,\mathrm{d}x = 0. \tag{2.10}$$

Since we will want to include $t=0$ later, we also have for $n\in(1,\infty)$ and $t\geq 0$

$$0 < \int_0^\infty \frac{e^{-tx^n}}{1+x^n}\,\mathrm{d}x \leq \int_0^\infty \frac{1}{1+x^n}\,\mathrm{d}x \tag{2.11}$$

This means for $t=0$ we require $n>1$.

2.1.2 Proof of limit

Proof. Define the integral function of t, where n is a constant as

$$\mathcal{L}(t) = \int_0^\infty \frac{e^{-tx^n}}{1+x^n}\,\mathrm{d}x. \tag{2.12}$$

We can observe that $\frac{e^{-tx^n}}{1+x^n}$ is a continuous function in $[0,\infty)$, and its partial derivative

$$\frac{\partial}{\partial t}\left(\frac{e^{-tx^n}}{1+x^n}\right) = \frac{-x^n e^{-tx^n}}{1+x^n} \tag{2.13}$$

is also continuous on this interval. Since this function is dominated by the integrable function e^{-tx^n} and $\frac{1}{1+x^n}$ we can apply the Leibniz rule for differentiation under the integral sign [3]. Applying this yields

$$\begin{aligned}
\mathcal{L}'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \left(\frac{e^{-tx^n}}{1+x^n}\right)\mathrm{d}x \\
&= \int_0^\infty \frac{\partial}{\partial t}\left(\frac{e^{-tx^n}}{1+x^n}\right)\mathrm{d}x \\
&= -\int_0^\infty \frac{x^n e^{-tx^n}}{1+x^n}\,\mathrm{d}x
\end{aligned} \tag{2.14}$$

and then adding and subtracting a 1 inside the integral we get

$$\mathcal{L}'(t) = -\int_0^\infty \frac{(1+x^n)e^{-tx^n}}{1+x^n} - \frac{e^{-tx^n}}{1+x^n}\,\mathrm{d}x \tag{2.15}$$

and after simplification we have

$$\mathcal{L}'(t) = -\int_0^\infty e^{-tx^n}\,\mathrm{d}x + \mathcal{L}(t). \tag{2.16}$$

Applying Lemma 1 (2.4) we now have

$$\mathcal{L}'(t) - \mathcal{L}(t) = -\frac{\Gamma\left(\frac{n+1}{n}\right)}{\sqrt[n]{t}}. \tag{2.17}$$

This is a linear first order differential equation, and can be solved with an integrating factor e^{-t}. Multiplying throughout by this we get

$$\frac{d}{\mathrm{d}t}\left(e^{-t}\mathcal{L}(t)\right) = -\Gamma\left(\frac{n+1}{n}\right)e^{-t}t^{\frac{-1}{n}}. \tag{2.18}$$

Integrating this from t to ∞ we get

$$\lim_{t\to\infty}(e^{-t}\mathcal{L}(t)) - (e^{-t}\mathcal{L}(t)) = -\Gamma\left(\frac{n+1}{n}\right)\underbrace{\int_t^\infty e^{-t}t^{\frac{n-1}{n}-1}\,\mathrm{d}t}_{\Gamma\left(\frac{n-1}{n},t\right)}. \tag{2.19}$$

Applying Lemma 2 (2.10) we know that $\lim_{t\to\infty}(e^{-t}I(t)) = 0$ so we are left with

$$-e^{-t}\mathcal{L}(t) = -\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{n-1}{n},t\right) \tag{2.20}$$

and multiplying by e^t we complete the proof.

$$\mathcal{L}(t) = \int_0^\infty \frac{e^{-tx^n}}{1+x^n}\,\mathrm{d}x = e^t\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{n-1}{n},t\right) \tag{2.21}$$

□

We now denote this limit as $\mathcal{L}(n,t)$.

Proposition 2. *For every n,t where $\mathcal{L}(n,t)$ exists*

$$|G(n,t;x)| < \mathcal{L}(n,t). \tag{2.22}$$

Proof. The integrand of G is always positive

$$0 < \frac{e^{-ty^n}}{1+y^n} \tag{2.23}$$

which means that the integral is always increasing, but since $G(n,t;x) \to \mathcal{L}(n,t)$ as $x\to\infty$, we must have

$$|G(n,t;x)| < \mathcal{L}(n,t). \tag{2.24}$$

□

2.2 Corollaries

Corollary 1. *As $\mathcal{L}(n,t) < J$ we have*

$$e^t\Gamma\left(\frac{n-1}{n},t\right) < t^{\frac{-1}{n}} \tag{2.25}$$

and after some manipulation we have

$$\Gamma(s,z) < t^{s-1}e^{-z}. \tag{2.26}$$

Corollary 2. *If we let $t \to 0$ we have*

$$\int_0^\infty \frac{1}{1+x^n}\,\mathrm{d}x = \Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{n-1}{n}\right) \tag{2.27}$$

and using Euler's reflection formula [4]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.28}$$

it is simple to derive

$$\int_0^\infty \frac{1}{1+x^n}\,\mathrm{d}x = \frac{\pi}{n\sin\left(\frac{\pi}{n}\right)}. \tag{2.29}$$

Corollary 3. *For $a \neq 0$*

$$\int_0^\infty \frac{e^{-tx^n}}{a^n+x^n}\,\mathrm{d}x = a^{1-n}e^{ta^n}\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{n-1}{n}, ta^n\right). \tag{2.30}$$

3 Series around 0

Proposition 3. *For $|x| < 1$ and $n, t \in \mathbb{R}$*

$$G(n,t;x) = x\sum_{k=0}^{\infty}\frac{(-1)^k x^{kn}}{(1+kn)}e_k(t). \tag{3.1}$$

where $e_k(t)$ is the exponential sum function, defined

$$e_k(t) = \sum_{m=0}^{k}\frac{t^m}{m!} \tag{3.2}$$

Proof. This is a similar method to earlier, but we have a variable upper bound which complicates things slightly. Again, we consider x, n constant. Write

$$G(t) = \int_0^x \frac{e^{-ty^n}}{1+y^n}\,\mathrm{d}y. \tag{3.3}$$

We have already shown differentiation under the integral sign is valid, so continuing as previous we come to the differential equation

$$G'(t) = -\int_0^x e^{-ty^n}\,\mathrm{d}y + G(t). \tag{3.4}$$

Substituting $ty^n = v^n$ to the integral we get

$$G'(t) = -\frac{1}{n\sqrt[n]{t}}\int_0^{tx^n} u^{\frac{1}{n}-1}e^{-u}\,\mathrm{d}u + G(t). \tag{3.5}$$

This integral is a form of the lower incomplete gamma function, so we actually have

$$G'(t) = -\frac{1}{n\sqrt[n]{t}}\gamma\left(\frac{1}{n}, tx^n\right) + G(t). \tag{3.6}$$

Introducing the same integrating factor, e^{-t} we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}G(t)\right) = -e^{-t}\frac{1}{n\sqrt[n]{t}}\gamma\left(\frac{1}{n}, tx^n\right). \tag{3.7}$$

At this point, it will get somewhat confusing to have t's floating around, so we change the variable of integration from t to p to make it clearer. Integrating from t to ∞ we lose the upper limit term on the left due to (2.10), and then multiplying by e^t we finally have

$$G(t) = \frac{e^t}{n}\int_t^\infty e^{-p}p^{\frac{-1}{n}}\gamma\left(\frac{1}{n}, px^n\right)\mathrm{d}p. \tag{3.8}$$

This form still converges everywhere. Introducing the series for the lower incomplete gamma function

$$\gamma(s, z) = z^s\sum_{k=0}^\infty \frac{(-1)^k z^k}{k!(s+k)}. \tag{3.9}$$

which is uniformly convergent on $z \in (-1, 1)$ and any non-negative integer s , as can be shown with the Weierstrass M test with a geometric series. In our case we have

$$\gamma\left(\frac{1}{n}, px^n\right) = p^{\frac{1}{n}}x^{(n\cdot\frac{1}{n})}\sum_{k=0}^\infty \frac{(-1)^k p^k x^{nk}}{k!(s+k)} \tag{3.10}$$

$$= p^{\frac{1}{n}}x\sum_{k=0}^\infty \frac{(-1)^k p^k x^{nk}}{k!\left(\frac{1}{n}+k\right)}. \tag{3.11}$$

Inserting this into our integral we have

$$G(t) = \frac{e^t}{n}\int_t^\infty e^{-p}\underbrace{\left(p^{\frac{-1}{n}}\cdot p^{\frac{1}{n}}\right)}_{=1}x\sum_{k=0}^\infty \frac{(-1)^k p^k x^{nk}}{k!\left(\frac{1}{n}+k\right)}\mathrm{d}p \tag{3.12}$$

$$= e^t\int_t^\infty e^{-p}x\sum_{k=0}^\infty \frac{(-1)^k p^k x^{nk}}{k!\,(1+kn)}\mathrm{d}p. \tag{3.13}$$

Due to the uniform convergence, we are free to interchange integration and summation and we have

$$G(t) = xe^t\sum_{k=0}^\infty \frac{(-1)^k x^{kn}}{k!\,(1+nk)}\underbrace{\int_t^\infty e^{-p}p^{(k+1)-1}\,\mathrm{d}p}_{\Gamma(k+1,t)}. \tag{3.14}$$

There are no more integrals in this now, and our series is

$$G(n,t;x) = xe^t\sum_{k=0}^\infty \frac{(-1)^k\Gamma(k+1,t)}{k!\,(1+kn)}x^{kn}. \tag{3.15}$$

Because $k+1$ is an integer, we can use the identity

$$\frac{\Gamma(k+1,t)}{k!} = \frac{e_k(t)}{e^t}$$

to get our final form required

$$G(n,t;x) = x\sum_{k=0}^{\infty} \frac{(-1)^k x^{kn}}{(1+kn)} e_k(t). \tag{3.16}$$

Written out this looks like

$$G(n,t;x) = x\left(1 - \frac{x^n}{1+n}e_1(t) + \frac{x^{2n}}{1+2n}e_2(t) - \frac{x^{3n}}{1+3n}e_3(t) + \frac{x^{4n}}{1+4n}e_4(t) + \cdots\right). \tag{3.17}$$

We already know the radius of convergence, as our original integrand has poles at $\pm i$, and this has magnitude 1, so our radius of convergence is 1. □

3.1 Inverse Series

It is possible to apply series reversion [5] applying the general formula given in the citation. This is slightly difficult because the powers of x jump in increments of n, instead of 1. As a special case, for $n = 2$ we have

$$G(n,t;x) = x - \frac{x^3}{3}e_1(t) + \frac{x^5}{5}e_2(t) - \frac{x^7}{7}e_3(t) + \frac{x^9}{7}e_4(t) + \cdots. \tag{3.18}$$

so

$$a_1 = 1 \tag{3.19}$$
$$a_2 = a_4 = \cdots = 0 \tag{3.20}$$
$$a_3 = -\frac{e_1(t)}{3} \tag{3.21}$$
$$a_5 = \frac{e_2(t)}{5} \tag{3.22}$$
$$a_7 = -\frac{e_3(t)}{7} \tag{3.23}$$
$$a_9 = \frac{e_4(t)}{9} \tag{3.24}$$
$$\vdots \tag{3.25}$$
$$a_{2k+1} = \frac{(-1)^k}{(2k+1)}e_k(t) \tag{3.26}$$

and our inverted series has coefficients

$$A_1 = a_1^{-1} = 1 \tag{3.27}$$
$$A_2 = -a_1^{-3} a_2 = 0 \tag{3.28}$$
$$A_3 = a_1^{-5}\left(2a_2 - a_1 a_3\right) = -a_3 = \frac{e_1(t)}{3} \tag{3.29}$$
$$A_4 = 0 \tag{3.30}$$
$$A_5 = a_1^{-9}\left(6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 + 14a_2^4 - a_1^3 a_5 - 21 a_1 a_2^2 a_3\right) \tag{3.31}$$
$$= 3a_3^2 - a_5 = 3\left(-\frac{e_1(t)}{3}\right)^2 - \frac{e_2(t)}{5} == \frac{e_1(t)^2}{3} - \frac{e_2(t)}{5} \tag{3.32}$$
$$\vdots \tag{3.33}$$

It follows that

$$G^{-1}(2,t;x) = x + \frac{e_1(t)}{3}x^3 + \left(\frac{e_1(t)^2}{3} - \frac{e_2(t)}{5}\right)x^5 + \cdots . \tag{3.34}$$

4 Asymptotic Series

Proposition 4. *For* $x \in (1,\infty)$

$$G(n,t;x) = \mathcal{L}(n,t) - \sum_{k=0}^{\infty}(-1)^k \mathcal{R}_k \tag{4.1}$$

where

$$\mathcal{R}_k = \frac{1}{n} t^{\left(\frac{n(k+1)-1}{n}\right)} \Gamma\left(\frac{1-n(k+1)}{n}, tx^n\right) \tag{4.2}$$

Proof. We can see from the definition of G

$$G(n,t;x) = \int_0^x \frac{e^{-ty^n}}{1+y^n}\,\mathrm{d}y \tag{4.3}$$

that we could rewrite it is

$$G(n,t;x) = \int_0^\infty \frac{e^{-ty^n}}{1+y^n}\,\mathrm{d}y - \int_x^\infty \frac{e^{-ty^n}}{1+y^n}\,\mathrm{d}y \tag{4.4}$$

where this first integral is just $\mathcal{L}(n,t)$. This is

$$G(n,t;x) = \mathcal{L}(n,t) - \int_x^\infty \frac{e^{-ty^n}}{1+y^n}\,\mathrm{d}y. \tag{4.5}$$

Looking at this integral on its own, and calling it G_c

$$G_c(n,t,x) = \int_x^\infty \frac{e^{-ty^n}}{1+y^n}\,\mathrm{d}y \tag{4.6}$$

we prepare it to be expressed as a geometric series which is valid for $1 < |y^n|$, which restricts x to $(1,\infty)$.

$$G_c(n,t,x) = \int_x^\infty \frac{e^{-ty^n}}{y^n\left(1+\frac{1}{y^n}\right)}\,\mathrm{d}y \tag{4.7}$$

$$= \int_x^\infty e^{-ty^n}\sum_{k=0}^\infty \frac{(-1)^k}{y^{n(k+1)}}\,\mathrm{d}y \tag{4.8}$$

This geometric series is also uniformly convergent on $(1+\varepsilon,\infty)$, so we can interchange integration and summation.

$$G_c(n,t,x) = \sum_{k=0}^\infty (-1)^k \int_x^\infty e^{-ty^n} y^{-n(k+1)}\,\mathrm{d}y \tag{4.9}$$

$$= \sum_{k=0}^\infty (-1)^k \mathcal{R}_k. \tag{4.10}$$

If we solve for the $\mathcal{R}_k$ terms in this series we have

$$\mathcal{R}_k = \int_x^\infty e^{-ty^n} y^{-n(k+1)}\,\mathrm{d}y. \tag{4.11}$$

Making the substitution $u = ty^n$, $\frac{1}{n}t^{\frac{-1}{n}}u^{\frac{1}{n}-1}\,\mathrm{d}u = \mathrm{d}y$, $\frac{1}{y^{n(k+1)}} = t^{k+1}u^{-(k+1)}$ yields

$$\mathcal{R}_k = \frac{1}{n}t^{\left(k+1-\frac{1}{n}\right)}\int_{tx^n}^\infty e^{-u}u^{\left(\frac{1}{n}-k-1\right)-1}\,\mathrm{d}u. \tag{4.12}$$

This last integral is the upper gamma function again. We now have

$$\mathcal{R}_k = \frac{1}{n}t^{\left(k+1-\frac{1}{n}\right)}\Gamma\left(\frac{1}{n}-k-1, tx^n\right) \tag{4.13}$$

and slotting these into the summation gives

$$G_c(n,t,x) = \sum_{k=0}^\infty (-1)^k \frac{1}{n} t^{\left(\frac{n(k+1)-1}{n}\right)}\Gamma\left(\frac{1-n(k+1)}{n}, tx^n\right) \tag{4.14}$$

$$= \frac{t^{\frac{n-1}{n}}}{n}\sum_{k=0}^\infty (-1)^k t^k \Gamma\left(\frac{1-n(k+1)}{n}, tx^n\right). \tag{4.15}$$

□

4.1 Approximation

We can just use the first term of the asymptotic series if we want, which is

$$\mathcal{R}_0 = \frac{t^{\frac{n-1}{n}}}{n}\Gamma\left(\frac{1-n}{n}, tx^n\right) \tag{4.16}$$

$$= \int_x^\infty \frac{e^{-ty^n}}{y^n}\,\mathrm{d}y. \tag{4.17}$$

Proposition 5. *For every x, n, t for which $\mathcal{L}(n,t)$ exists*

$$\mathcal{L}(n,t) - \mathcal{R}_0(n,t,x) \backsim G(n,t;x). \tag{4.18}$$

Proof. It is obvious that for $y > 0$

$$0 < \frac{e^{-ty^n}}{1+y^n} < \frac{e^{-ty^n}}{y^n}. \tag{4.19}$$

It follows that for $x > 0$

$$\int_x^\infty \frac{e^{-ty^n}}{1+y^n}\,\mathrm{d}y < \int_x^\infty \frac{e^{-ty^n}}{y^n}\,\mathrm{d}y \tag{4.20}$$

and this term on the left is $\mathcal{L} - G$, so we have

$$\mathcal{L}(n,t) - G(n,t;x) < \int_x^\infty \frac{e^{-ty^n}}{y^n}\,\mathrm{d}y \tag{4.21}$$

and rearranging we get

$$\mathcal{L}(n,t) - \mathcal{R}_0(n,t,x) < G(n,t;x). \tag{4.22}$$

Since it is obvious that $\mathcal{R}_0 \to 0$ monotonically as $x \to \infty$, we deduce that $\mathcal{L} - \mathcal{R}_0$ asymptotically approaches G. □

We can get a weaker form of this using the asymptotic for the incomplete gamma function, namely

$$\Gamma(s,z) \backsim z^{s-1}e^{-z} \tag{4.23}$$

which yields

$$G(n,t;x) \backsim \mathcal{L}(n,t) - \frac{1}{nt}x^{1-\frac{2}{n}}e^{-tx^n}. \tag{4.24}$$

We can also find a condition that bounds $\mathcal{R}_0$ as follows: If $n \geq 1$ then

$$\begin{aligned} \mathcal{R}_0 = \int_x^\infty \frac{e^{-ty^n}}{y^n}\,\mathrm{d}y &\leq \int_x^\infty e^{-ty^n}\,\mathrm{d}y && (4.25)\\ &\leq \int_x^\infty e^{-ty}\,\mathrm{d}y && (4.26)\\ &= \frac{e^{-tx}}{t}. && (4.27) \end{aligned}$$

Suppose R_0 is a good approximation when $R_0 < \varepsilon$, then we require

$$x > -\frac{1}{t}\ln(\varepsilon t). \tag{4.28}$$

There is another condition that can be used if $n \geq 1$:

$$\mathcal{R}_0 = \int_x^\infty \frac{e^{-ty^n}}{y^n}\,\mathrm{d}y \leq \int_x^\infty y^{-n}\,\mathrm{d}y \tag{4.29}$$

$$= \frac{x^{1-n}}{n-1} \tag{4.30}$$

$$R_0 < \varepsilon \implies x > (\varepsilon(n-1))^{\frac{1}{n-1}}. \tag{4.31}$$

Then the approximation is valid for $x > \min\left\{(\varepsilon(n-1))^{\frac{1}{n-1}}, -\frac{1}{t}\ln(\varepsilon t)\right\}$.

Corollary 4. *When G is bounded, the bounds are given by*

$$\mathcal{L}(n,t) - \mathcal{R}_0(n,t,x) < G(n,t;x) < \mathcal{L}(n,t). \tag{4.32}$$

5 Relationships to other functions

Similar to earlier, we can write (via simple substitution)

$$\int_0^x \frac{e^{-ty^n}}{a^n+y^n}\,\mathrm{d}y = a^{1-n}G\left(n, ta^n, \frac{x}{a}\right). \tag{5.1}$$

We also have for every $x \in \mathbb{R}^+$

$$G(1,0,x) = \ln(1+x) \tag{5.2}$$

$$G(2,0,x) = \arctan(x) \tag{5.3}$$

$$\vdots \tag{5.4}$$

$$G(n,0,x) = {}_2F_1\left(1, \frac{1}{n}; 1+\frac{1}{n}; -x^n\right)x \tag{5.5}$$

where ${}_2F_1$ is the hypergeometric function [6],

$${}_2F_1\left(a,b;c;z\right) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}. \tag{5.6}$$

It can also be shown via a similar method that

$$\mathcal{L}(2,t) = \frac{\pi e}{2}\,\mathrm{erfc}\left(\sqrt{t}\right) \tag{5.7}$$

where $\mathrm{erfc}(z)$ is the complementary error function, from which

$$\Gamma\left(\frac{1}{2}, t\right) = \sqrt{\pi}\,\mathrm{erfc}\left(\sqrt{t}\right) \tag{5.8}$$

can be derived. Similarly,

$$\mathcal{L}(1,t) = -e^t\mathrm{Ei}(-t) \tag{5.9}$$

where $\mathrm{Ei}(z)$ is the exponential integral

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t}\,\mathrm{d}t. \tag{5.10}$$

6 Pointwise limit

Proposition 6. *$G(n,t;x) \to \mathcal{G}_\infty(x)$ as $n \to \infty$ pointwise, where $\mathcal{G}_\infty(x)$ is defined as the piecewise function*

$$\mathcal{G}_\infty(x) = \begin{cases} x, |x| < 1, \\ 1, x \geq 1. \end{cases} \tag{6.1}$$

Proof. For the first part, $|x| < 1$ we have a series that we can utilise.

$$\begin{aligned}
|G(n,t;x) - \mathcal{G}_\infty(x)| &= \left| x \sum_{k=0}^{\infty} \frac{(-1)^k x^{kn}}{(1+kn)} e_k(t) - x \right| && (6.2)\\
&= |x| \left| \sum_{k=0}^{\infty} \frac{(-1)^k x^{kn}}{(1+kn)} e_k(t) - 1 \right| && (6.3)\\
&= |x| \left| \sum_{k=1}^{\infty} \frac{(-1)^k x^{kn}}{(1+kn)} e_k(t) \right| && (6.4)\\
&\leq |x| \sum_{k=1}^{\infty} \frac{|x|^{kn}}{(1+kn)} |e_k(t)| && (6.5)\\
&\leq |x| e^t \sum_{k=1}^{\infty} \frac{|x|^{kn}}{(1+kn)} && (6.6)\\
&\leq |x| e^t \sum_{k=1}^{\infty} (|x|^n)^k && (6.7)\\
&= \frac{e^t |x|^{n+1}}{1 - |x|^n} && (6.8)
\end{aligned}$$

Now, seeking a N such that

$$\frac{e^t |x|^{N+1}}{1 - |x|^N} < \varepsilon \tag{6.9}$$

we find that choosing

$$N > \frac{\ln(\varepsilon) - \ln(e^t |x| + \varepsilon)}{\ln |x|} \tag{6.10}$$

satisfies for $n > N$

$$|G(n,t;x) - \mathcal{G}_\infty(x)| < \frac{e^t |x|^{n+1}}{1 - |x|^n} < \varepsilon. \tag{6.11}$$

For the other part, we consider the asymptotic form. However, first we prove that

$$\lim_{n\to\infty} \mathcal{L}(n,t) = 1. \tag{6.12}$$

Calculating

$$\lim_{n\to\infty} \mathcal{L}(n,t) = e^t \lim_{n\to\infty} \Gamma\left(\frac{n+1}{n}\right) \Gamma\left(\frac{n-1}{n}, t\right) \tag{6.13}$$

The gamma function, and upper gamma function are continuous functions, so we can pass the limit through.

$$\lim_{n\to\infty} \mathcal{L}(n,t) = e^t \Gamma\left(\lim_{n\to\infty} \frac{n+1}{n}\right) \Gamma\left(\lim_{n\to\infty} \frac{n-1}{n}, t\right) \tag{6.14}$$

$$= e^t \underbrace{\Gamma(1)}_{=1} \Gamma(1,t) \tag{6.15}$$

However,

$$\Gamma(1,t) = \int_t^\infty e^{-p} \, \mathrm{d}p \tag{6.16}$$

$$= e^{-t}. \tag{6.17}$$

Therefore

$$\lim_{n\to\infty} \mathcal{L}(n,t) = e^t e^{-t} \tag{6.18}$$

$$= 1. \tag{6.19}$$

Now because on $(1,\infty)$ we have

$$G(n,t;x) = \mathcal{L}(n,t) - \int_x^\infty \frac{e^{-ty^n}}{1+y^n} \, \mathrm{d}y. \tag{6.20}$$

and by similar justification to earlier, $\frac{e^{-ty^n}}{1+y^n}$ is dominated by e^{-ty^n}, we can interchange the integral and limit so that we get

$$\lim_{n\to\infty} G(n,t;x) = \lim_{n\to\infty} \mathcal{L}(n,t) - \lim_{n\to\infty} \int_x^\infty \frac{e^{-ty^n}}{1+y^n} \, \mathrm{d}y \tag{6.21}$$

$$= 1 - \int_x^\infty \lim_{n\to\infty} \left(\frac{e^{-ty^n}}{1+y^n}\right) \mathrm{d}y \tag{6.22}$$

$$= 1. \tag{6.23}$$

$\square$

7 Further Identities

Consider the integral for $k \geq 0$

$$\int_0^\infty y^{kn} e^{-ty^n} \, \mathrm{d}y. \tag{7.1}$$

The substitution $u = ty^n$ solves this, and we get

$$\int_0^\infty y^{kn} e^{-ty^n} \, \mathrm{d}y = \frac{1}{n} t^{-(1+kn)/n} \Gamma\left(\frac{1}{n} + k\right). \tag{7.2}$$

Now, multiplying by $\frac{1+y^n}{1+y^n}$ inside and separating out we get

$$\int_0^\infty \frac{y^{(k+1)n} e^{-ty^n}}{1+y^n} \, \mathrm{d}y + \int_0^\infty \frac{y^{kn} e^{-ty^n}}{1+y^n} \, \mathrm{d}y = \frac{1}{n} t^{-(1+kn)/n} \Gamma\left(\frac{1}{n} + k\right) \tag{7.3}$$

which is equivalent to

$$\int_0^\infty \frac{y^{(k+1)n} e^{-ty^n}}{1+y^n} \, \mathrm{d}y = \frac{1}{n} t^{-(1+kn)/n} \Gamma\left(\frac{1}{n} + k\right) - \int_0^\infty \frac{y^{kn} e^{-ty^n}}{1+y^n} \, \mathrm{d}y. \tag{7.4}$$

This allows us to calculate integrals for arbitrary k by recurrence relation. Now if we fix $k \in \mathbb{N}$, this is equivalent to the statement (via differentiating equation 2.17)

$$\frac{\partial^{k+1} \mathcal{L}}{\partial t^{k+1}} = \frac{1}{n} t^{-(1+kn)/n} \Gamma\left(\frac{1}{n} + k\right) - \frac{\partial^k \mathcal{L}}{\partial t^k}. \tag{7.5}$$

We also have the initial value

$$\int_0^\infty \frac{y^{0n} e^{-ty^n}}{1+y^n} \, \mathrm{d}y = \mathcal{L}(n, t). \tag{7.6}$$

7.1 Exact form

Let for $k \in \mathbb{N}$

$$p_k = t^{-k} \Gamma\left(\frac{1}{n} + k\right). \tag{7.7}$$

Then, it can be shown from the above linear recurrence relation that

$$\frac{\partial^k \mathcal{L}}{\partial t^k} = \begin{cases} \mathcal{L} + \frac{1}{n} t^{-\frac{1}{n}} (p_{k-1} - p_{k-2} + \cdots + p_1 - p_0), & \text{k even}, \\ -\mathcal{L} + \frac{1}{n} t^{-\frac{1}{n}} (p_{k-1} - p_{k-2} + \cdots - p_1 + p_0), & \text{k odd}. \end{cases} \tag{7.8}$$

Written in summation notation this is

$$\frac{\partial^k \mathcal{L}}{\partial t^k} = (-1)^k \mathcal{L} - \frac{1}{n} t^{-\frac{1}{n}} \sum_{i=0}^{k-1} (-1)^{i+k} t^{-i} \Gamma\left(\frac{1}{n} + i\right). \tag{7.9}$$

7.1.1 Example

For example, choose $k = 3$, an odd number. Then

$$\int_0^\infty \frac{y^{3n}e^{-ty^n}}{1+y^n}\,\mathrm{d}y = \frac{\partial^3 \mathcal{L}}{\partial t^3} \tag{7.10}$$
$$= -\mathcal{L} + \frac{1}{n}t^{-\frac{1}{n}}(p_2 - p_1 + p_0) \tag{7.11}$$
$$= -\mathcal{L} + \frac{1}{n}t^{-\frac{1}{n}}\left(t^{-2}\Gamma\left(\frac{1}{n}+2\right) - t^{-1}\Gamma\left(\frac{1}{n}+1\right) + \Gamma\left(\frac{1}{n}\right)\right). \tag{7.12}$$

If we also let $n = 2, t = 1$ then

$$\int_0^\infty \frac{y^6 e^{-y^2}}{1+y^2}\,\mathrm{d}y = -\mathcal{L}(2,1) + \frac{1}{2}\left(\Gamma\left(\frac{1}{2}+2\right) - \Gamma\left(\frac{1}{2}+1\right) + \Gamma\left(\frac{1}{2}\right)\right) \tag{7.13}$$
$$= -e\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2},1\right) + \frac{1}{2}\left(\frac{3\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2} + \sqrt{\pi}\right) \tag{7.14}$$
$$= \frac{5\sqrt{\pi}}{8} - \frac{\pi e}{2}\,\mathrm{erfc}(1) \tag{7.15}$$
$$= 0.43613694599\cdots. \tag{7.16}$$

8 Trigonometric Generalisation

Suppose that we let $t \in \mathbb{C}$, and parameterised this as $t = a + ib$ for some $a, b \in \mathbb{R}$ with the restriction that $n \in (1,\infty), a \geq 0$ or if $n \in (0,1], a > 0$. Then we can write

$$G(n, a+ib, x) = \int_0^x \frac{e^{-(a+ib)y^n}}{1+y^n}\,\mathrm{d}y. \tag{8.1}$$

Using Euler's formula we get

$$G(n, a+ib, x) = \int_0^x \frac{e^{-(a+ib)y^n}}{1+y^n}\,\mathrm{d}y \tag{8.2}$$
$$= \int_0^x \frac{e^{-ay^n}(\cos(by^n) - i\sin(by^n))}{1+y^n}\,\mathrm{d}y \tag{8.3}$$
$$= \int_0^x \frac{e^{-ay^n}\cos(by^n)}{1+y^n}\,\mathrm{d}y - i\int_0^x \frac{e^{-ay^n}\sin(by^n)}{1+y^n}\,\mathrm{d}y. \tag{8.4}$$

From this we can see

$$\int_0^x \frac{e^{-ay^n}\sin(by^n)}{1+y^n}\,\mathrm{d}y = -\mathrm{Im}(G(n, a+ib, x)) \tag{8.5}$$
$$\int_0^x \frac{e^{-ay^n}\cos(by^n)}{1+y^n}\,\mathrm{d}y = \mathrm{Re}(G(n, a+ib, x)). \tag{8.6}$$

All the other derived formulae also apply here, such as

$$\int_0^\infty \frac{e^{-ay^n}\sin(by^n)}{1+y^n}\,\mathrm{d}y = -\mathrm{Im}\left(e^{(a+ib)}\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{n-1}{n}, a+ib\right)\right) \tag{8.7}$$

$$\int_0^\infty \frac{e^{-ay^n}\cos(by^n)}{1+y^n}\,\mathrm{d}y = \mathrm{Re}\left(e^{(a+ib)}\Gamma\left(\frac{n+1}{n}\right)\Gamma\left(\frac{n-1}{n}, a+ib\right)\right). \tag{8.8}$$

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