Laplace Transform

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Two Famous Linear Transforms

Laplace transform

Pierre-Simon Laplace:
23 March 1749 – 5 March 1827) was a French scholar whose work was important to the development of engineering, mathematics, statistics, physics, astronomy, and philosophy.

Fourier transform

Jean-Baptiste Joseph Fourier:
21 March 1768 – 16 May 1830) was a French mathematician and physicist and best known for initiating the investigation of Fourier series, which eventually developed into Fourier analysis and harmonic analysis, and their applications to problems of heat transfer and vibrations.

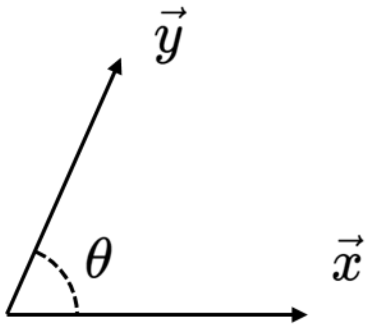
Inner Product

Inner Product

- For real valued vector

$$\vec{x} \cdot \vec{y} = \langle \vec{x}, \vec{y} \rangle = x^T y = \sum x_i y_i \quad \text{: similarity index}$$

- Orthogonal



$$x^T y = 0 \implies x \perp y$$

Inner Product

- Suppose $\hat{b}_i$ orthogonal basis
- Any $\vec{x}$ can be represented (decomposed) with linear combinations of $\hat{b}_i$

$$\vec{x} = c_1 \hat{b}_1 + c_2 \hat{b}_2 + \cdots + c_n \hat{b}_n$$

- Question: How to find c_i ?

$$\langle \vec{x}, \hat{b}_i \rangle = c_1 \langle \hat{b}_1, \hat{b}_i \rangle + \cdots + c_i \langle \hat{b}_i, \hat{b}_i \rangle + \cdots + c_n \langle \hat{b}_n, \hat{b}_i \rangle$$

$$\therefore c_i = \frac{\langle x, \hat{b}_i \rangle}{\langle \hat{b}_i, \hat{b}_i \rangle}$$

- Meaning: c_i indicates how much $\hat{b}_i$ information contains in $\vec{x}$

Hermitian Transpose

- Inner product for complex values

$$\langle \vec{x}, \vec{y} \rangle = y^H x \quad \text{where } y^H \text{ is complex conjugate transpose of } y$$

- Want to know how much $e^{j\omega t}$ component is contained in $x(t)$

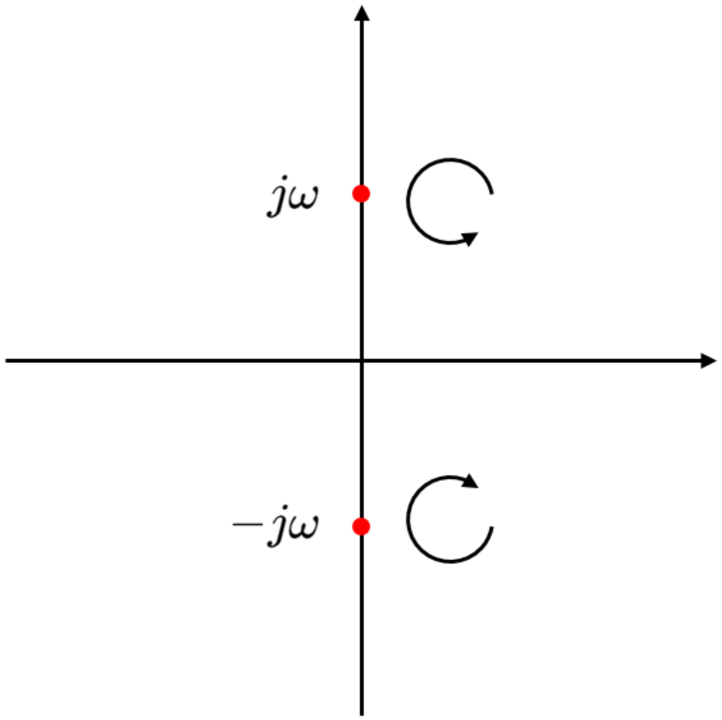
$$\begin{aligned} \langle x, e^{j\omega t} \rangle &= \int \left(e^{j\omega t}\right)^H x(t)\, dt \\ &= \int x(t) e^{-j\omega t}\, dt \\ &= X(j\omega) \end{aligned}$$

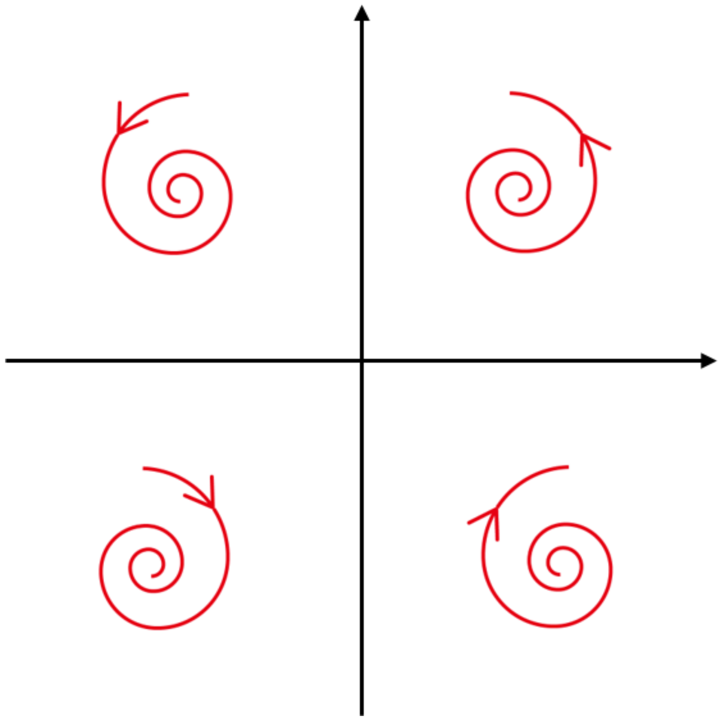
Two Linear Transformations

Fourier Transform $\xleftarrow{s\,=\,j\omega}$ Laplace Transform

$$X(j\omega) = \int x(t)e^{-j\omega t}\,dt$$

$$X(s) = \int x(t)e^{-st}dt$$





Laplace Transform

Laplace Transform

- Want to know how much $e^{-\sigma+j\omega t} = e^{-\sigma}e^{j\omega t}$ component is contained in $x(t)$

- Let $s = \sigma + j\omega t$

$$
\begin{aligned}
\left\langle x, e^{-\sigma+j\omega t}\right\rangle &= \int \left(e^{-\sigma+j\omega t}\right)^{H} x(t)\, dt \\
&= \int x(t) e^{-\sigma-j\omega t} dt \\
&= \int x(t) e^{-st} dt \\
&= X(s)
\end{aligned}
$$

Laplace Transform

- The system output is given by

$$\begin{aligned} y(t) &= H\{x(t)\} \\ &= h(t) * x(t) \\ &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \end{aligned}$$

- When $x(t) = e^{st}$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau \\ &= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \end{aligned}$$

- We define the <u>transfer function</u>

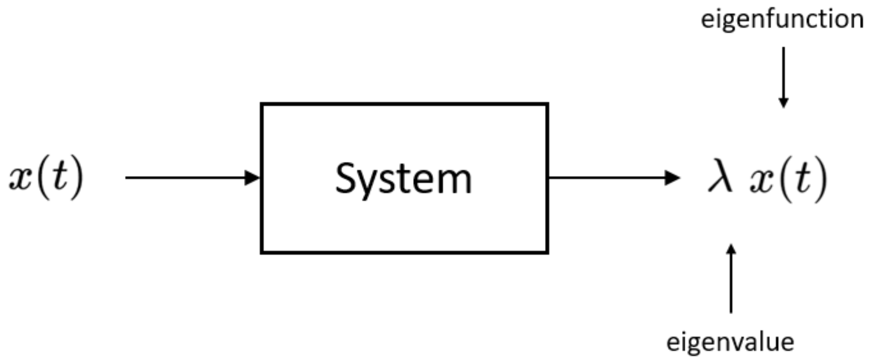
$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

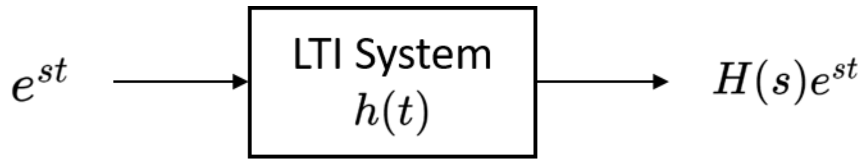
- We may write

$$y(t) = H\{e^{st}\} = H(s)e^{st}$$

Eigenfunction

- If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue





- The eigenvalue associated with eigenfunction e^{st} is $H(s)$

Definition: Laplace Transform

- Laplace transform maps a function of time t to a function of s (complex frequency)
  - The Laplace transform is similar to the Fourier transform.

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Example 1

$$x(t) = e^{-t}u(t) = \begin{cases} e^{-t} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{\infty} e^{-t}e^{-st}dt = \int_{0}^{\infty} e^{-(s+1)t}dt$$

$$= \frac{e^{-(s+1)t}}{-(s+1)}\Bigg|_{0}^{\infty} = \frac{1}{s+1}$$

- Provided $Re(s+1) > 0$ which implies that

$$\underbrace{\mathrm{Re}(s) > -1}_{\text{Regions of Convergence (ROC)}}$$

Example 2

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = \underbrace{\frac{3}{s+2}}_{\mathrm{Re}(s)>-2} - \underbrace{\frac{2}{s+1}}_{\mathrm{Re}(s)>-1} = \frac{3(s+1)-2(s+2)}{(s+2)(s+1)} = \underbrace{\frac{s-1}{s^2+3s+2}}_{\mathrm{Re}(s)>-1}$$

Example 3

- Laplace transform of the <u>impulse function</u>

$$x(t) = \delta(t)$$

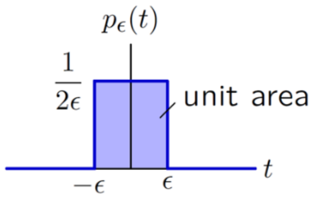
$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) e^{-st}\Big|_{t=0} dt \\ &= \int_{-\infty}^{\infty} \delta(t) \cdot 1 dt = 1 \end{aligned}$$

- Sifting property
  - Multiplying $f(t)$ by $\delta(t)$ and integrating over t sifts out $f(0)$

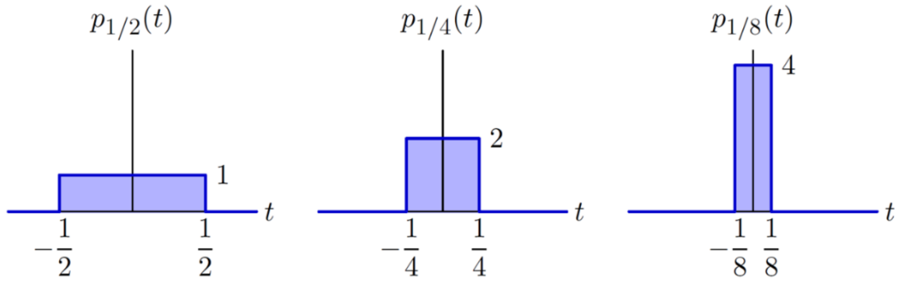
- Remember the Laplace transform of the impulse function is 1

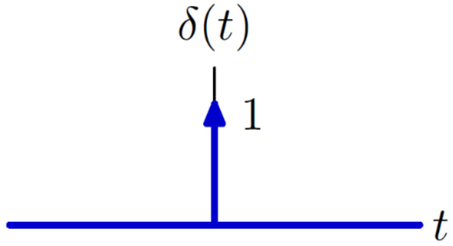
Unit Impulse Signal

- The unit-impulse signal acts as a pulse with unit area but zero width



$$\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)$$





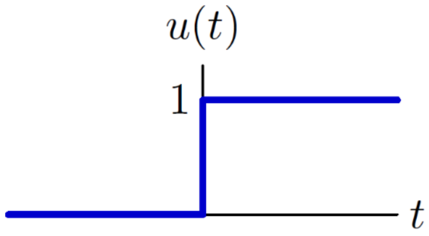
- The unit-impulse function is represented by an arrow with the number 1, which represents its area
- It has two seemingly contradictory properties :
  - It is nonzero only at $t = 0$ and
  - Its definite integral $(-\infty, \infty)$ is 1

Laplace Transform of Unit-Impulse Signal

- Laplace transform of unit-impulse function

$$D(s) = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = e^{-s0} = 1$$

Unit Step Signal



- The indefinite integral of the unit-impulse is the unit-step

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(t) \longrightarrow \boxed{\int} \longrightarrow u(t)$$

- Laplace transform of unit-step function

$$U(s) = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} 1 \cdot e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_{0}^{\infty} = \frac{1}{s}, \qquad \mathrm{Re}\{s\} > 0$$

Convolution Property of Laplace Transform

- Suppose

$$f \overset{\mathscr{L}}{\longleftrightarrow} F, \qquad F(s) = \int_{-\infty}^{\infty} f(u)e^{-su}du$$
$$g \overset{\mathscr{L}}{\longleftrightarrow} G, \qquad G(s) = \int_{-\infty}^{\infty} g(v)e^{-sv}dv$$

- Then

$$f * g \overset{\mathscr{L}}{\longleftrightarrow} F \cdot G$$

$$F(s) \cdot G(s) = \int_{-\infty}^{\infty} f(u)e^{-su}du \cdot \int_{-\infty}^{\infty} g(v)e^{-sv}dv$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)g(v)e^{-s(u+v)}dudv$$

- Convolution in time ↔ multiplication in s-domain

Convolution Property of Laplace Transform

- Change of variables

$$u + v = t$$

$$u = u$$
$$v = t - u$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{t} f(u)g(t-u)e^{-st}dudt = \int_{-\infty}^{\infty} e^{-st}\left[\int_{-\infty}^{t} f(u)g(t-u)du\right]dt$$

$$= \int_{-\infty}^{\infty} e^{-st}(f*g)(t)dt = \mathscr{L}\{f*g\}$$

Response to General Input

- Response to LTI system with impulse response $h(t)$

$$y(t) = h(t) * x(t)$$

$$Y(s) = H(s) \cdot X(s)$$

$$\therefore \; H(s) = \frac{Y(s)}{X(s)}$$

Laplace Transform of a Derivative

- Find the Laplace transform of $y(t) = \dot{x}(t)$, given $\mathscr{L}\{x(t)\} = X(s)$

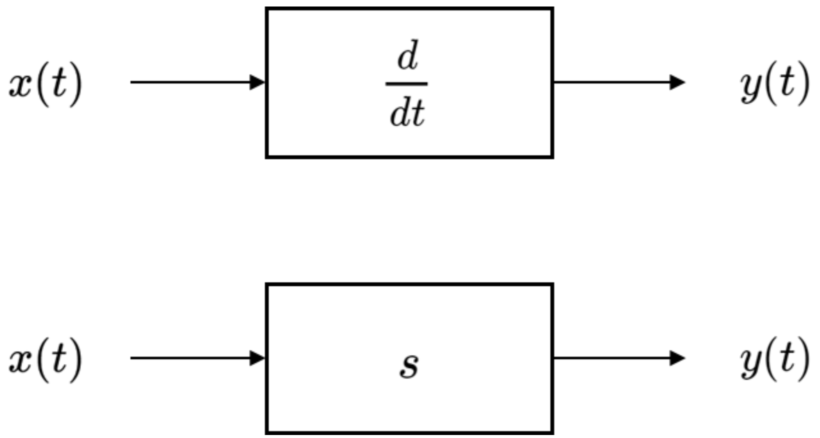
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$Y(s) = \int_{-\infty}^{\infty} y(t)e^{-st}dt = \int_{-\infty}^{\infty} \dot{x}(t)e^{-st}dt$$

$$= \underbrace{x(t)e^{-st}\Big|_{-\infty}^{\infty}}_{\text{must be zero since } X(s) \text{ converged}} - \int_{-\infty}^{\infty} x(t)(-se^{-st})dt, \qquad \text{using } \left(\int \dot{v}u = vu - \int v\dot{u}\right)$$

$$= s\int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$= sX(s)$$



Laplace Transform of a Derivative

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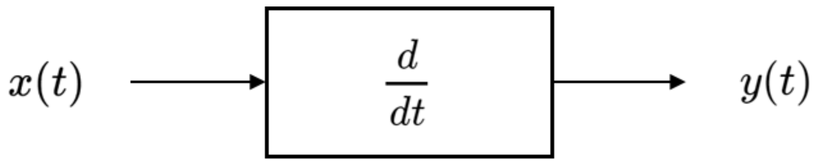
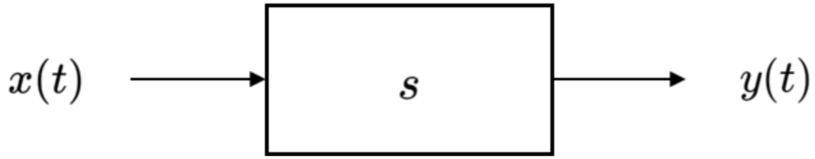




Solving Differential Equations with Laplace Transform

- Example of the first ODE

$$\dot{y}(t) + y(t) = \delta(t)$$

- Laplace transform

$$sY(s) + Y(s) = 1$$

- This is a simple algebraic expression
- Laplace transform converts a differential equation to an equivalent algebraic equation

$$Y(s) = \frac{1}{s+1}$$

$$y(t) = e^{-t}u(t)$$

Solving Differential Equations with Laplace Transform

- Example of the second ODE

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \delta(t)$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = 1$$

$$Y(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$= \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

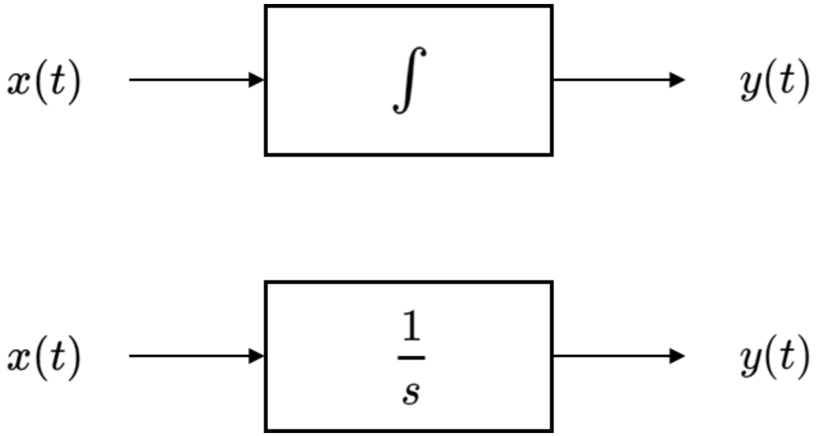
Laplace Transform of Integral

- Laplace transform of <u>integral</u>

$$\int_{-\infty}^{t} x(\tau)d\tau = u(t) * x(t)$$

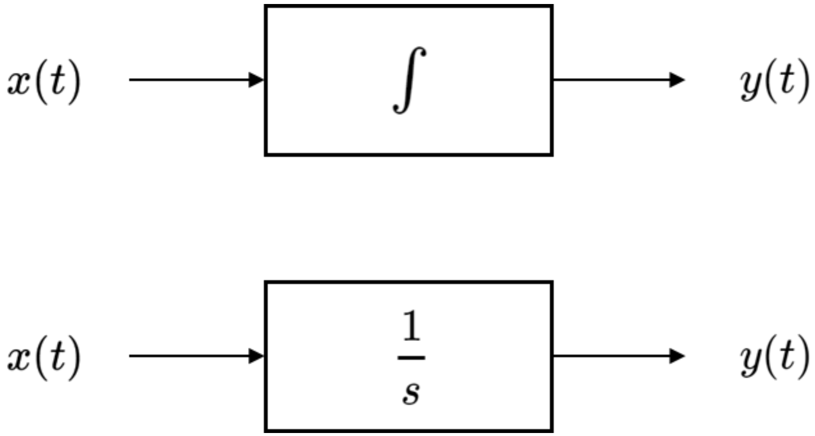
$$\downarrow \mathscr{L}$$

$$Y(s) = \frac{1}{s}X(s)$$



Laplace Transform of Integral

- Laplace transform of <u>integral</u>



Transfer Function

- Transfer function of an LTI system is defined as the Laplace transform of the impulse response.

$$y(t) = h(t) * x(t)$$

$$Y(s) = H(s) \cdot X(s)$$

$$\therefore \; H(s) = \frac{Y(s)}{X(s)}$$

- Describe input-output behavior of the system.

Rational Transfer Function

- Example

$$\ddot{y} + 3\dot{y} + 2y = 2\dot{x} - 3x, \qquad \text{where } \; y : \text{output and } x : \text{input}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s - 3}{s^2 + 3s + 2} = \frac{2(s - \frac{3}{2})}{(s+1)(s+2)}$$

Transfer Function

- Transfer function of differential equation system

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

$$\Downarrow \mathscr{L}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Poles and Zeros

- The poles and zeros of a rational transfer function offer much insight into LTI system characteristics

$$H(s) = \frac{\tilde{b}\prod_{k=1}^{M}(s-c_k)}{\prod_{k=1}^{N}(s-d_k)}, \qquad c_k \text{ zeros and } d_k \text{ poles}$$

- Determine the frequency response from poles and zeros (<u>will discuss later</u>)
  - Transfer function by substituting $j\omega$ for s
  - Evaluating the transfer function along the $j\omega$-axis in the s-plane
  - Graphical evaluation or Bode plot

$$H(j\omega) \quad \Longrightarrow \quad |H(j\omega)| \text{ and } \angle H(j\omega)$$