System Modeling:
Complex Number and Harmonic Motion

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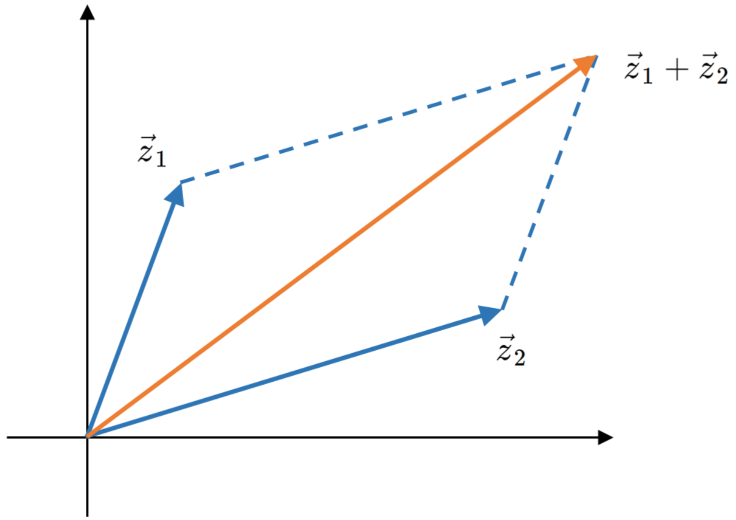
Complex Number

Complex Number

$$z_1 = a_1 + b_1 i, \quad \vec{z}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

$$z_2 = a_2 + b_2 i, \quad \vec{z}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$



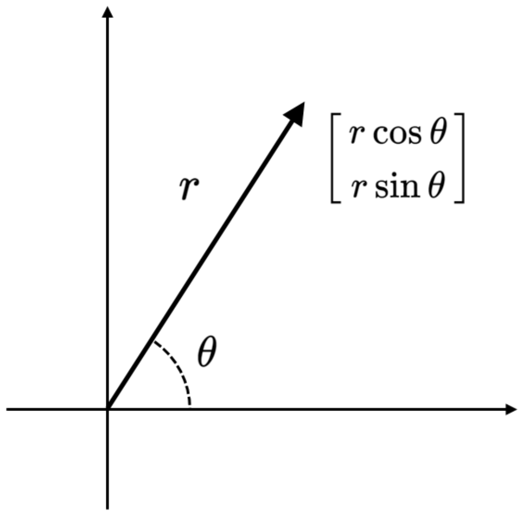
- Add

$$z = z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$$

$$\vec{z} = \vec{z}_1 + \vec{z}_2 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$



- Complex number in complex exponential

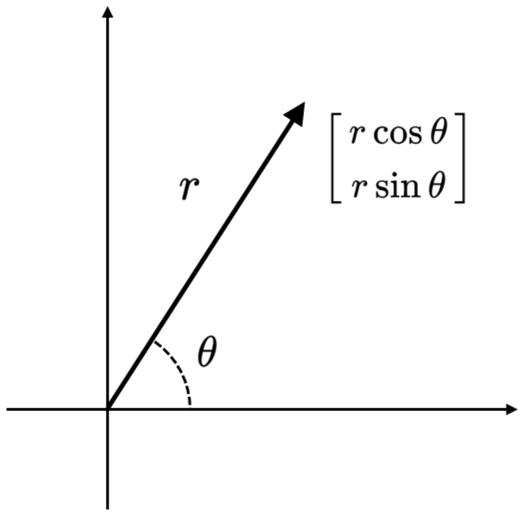
$$\begin{aligned}\vec{z} &= r\cos\theta + i\,r\sin\theta \\ &= r\left(\cos\theta + i\sin\theta\right) \\ &= re^{i\theta}\end{aligned}$$

r : magnitude (length)
θ : phase (angle)

Complex Number

- Multiply

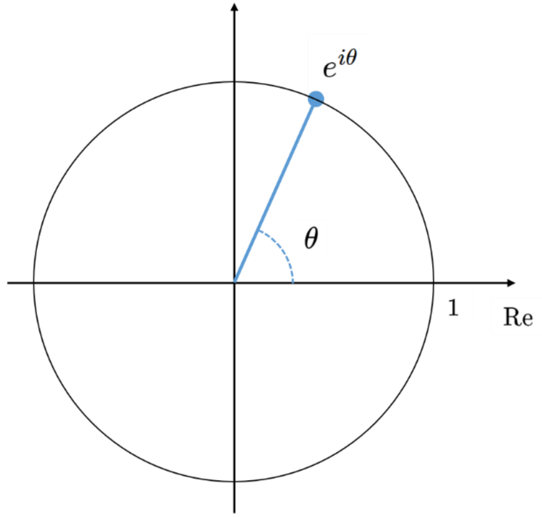
$$\begin{cases} z_1 = r_1 e^{i\theta_1} \\ z_2 = r_2 e^{i\theta_2} \end{cases} \implies \begin{cases} z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \\ \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \end{cases}$$



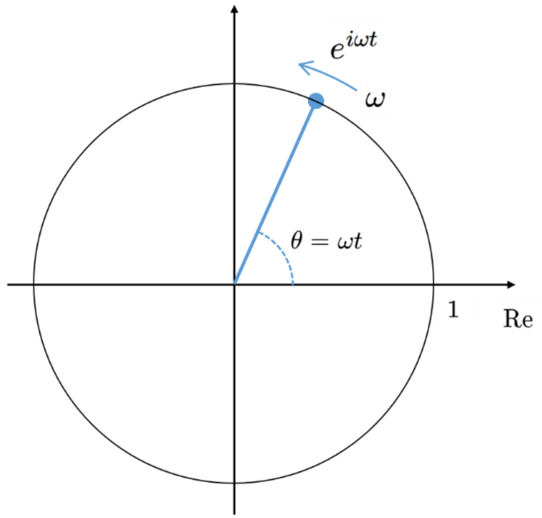
r : magnitude (length)
θ : phase (angle)

Geometrical Meaning of $e^{i\theta}$

- $e^{i\theta}$: point on the unit circle with angle of θ



- $\theta = \omega t$
- $e^{i\omega t}$: rotating on an unit circle with angular velocity of ω

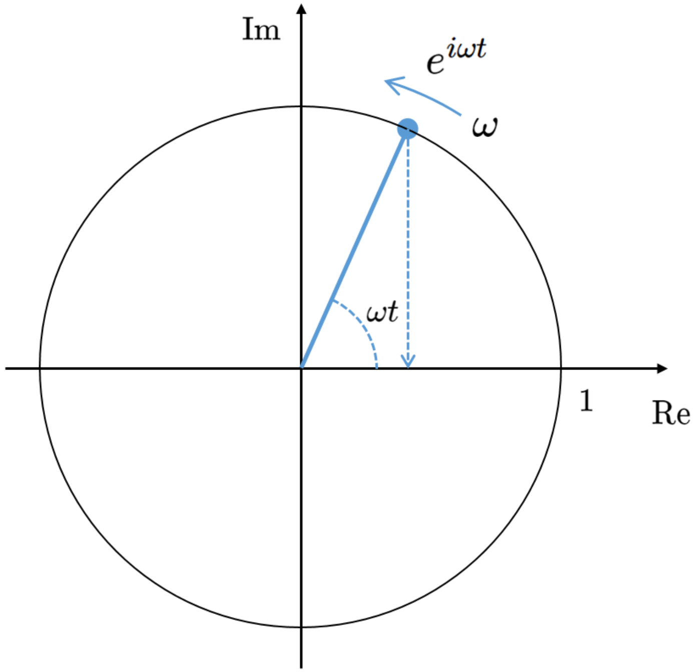


- Question: what is the physical meaning of $e^{-i\omega t}$?

Sinusoidal Functions from Circular Motions

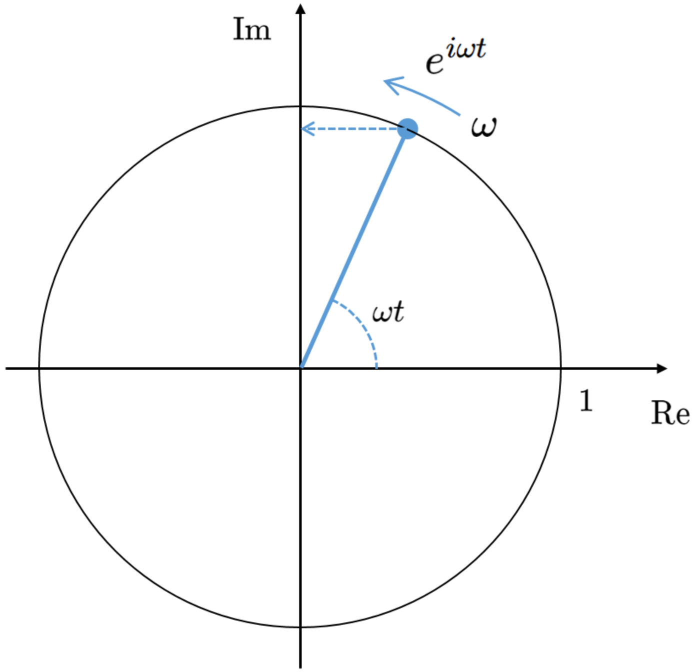
- Real part (cos term) is the projection onto the Re{} axis.
- Imaginary part (sin term) is the projection onto the Im{} axis.

projection of $e^{j\omega t}$ onto Re-axis



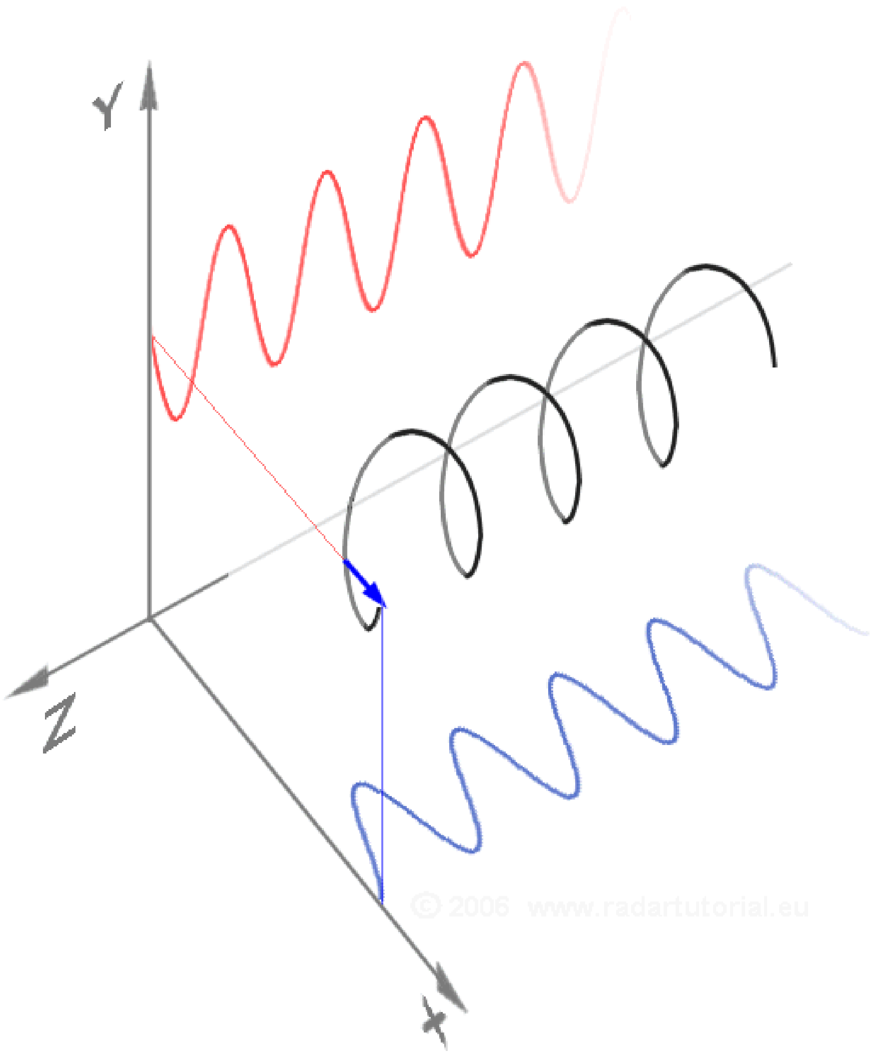
$$\mathrm{Re}(e^{i\omega t}) = \cos \omega t$$

projection of $e^{j\omega t}$ onto Im-axis



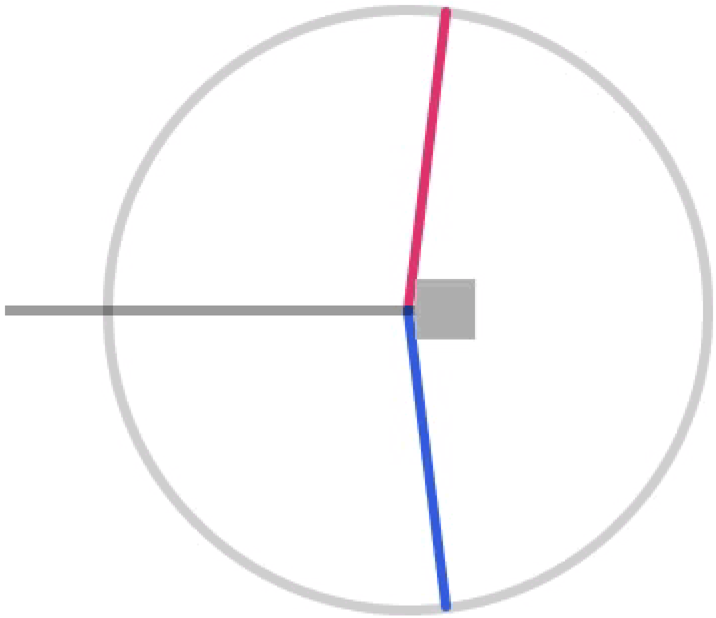
$$\mathrm{Im}(e^{i\omega t}) = \sin \omega t$$

Sinusoidal Functions from Circular Motions



Sinusoidal Functions from Circular Motions

$$\cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$



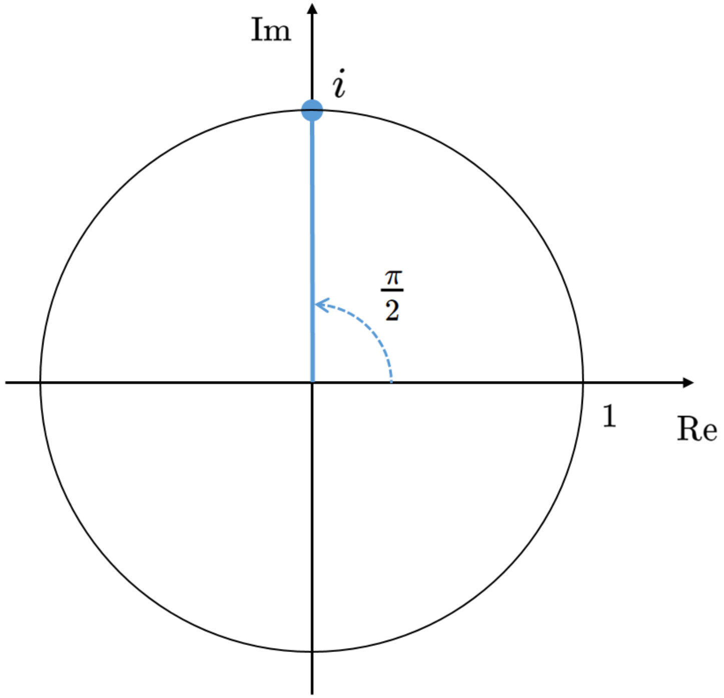
i Multiplying

- i multiplication $\Longrightarrow 90^o$ rotation forward

$$ie^{i\theta} = ?$$

$$z_1 = i = e^{i\frac{\pi}{2}}$$
$$z_2 = e^{i\theta}$$

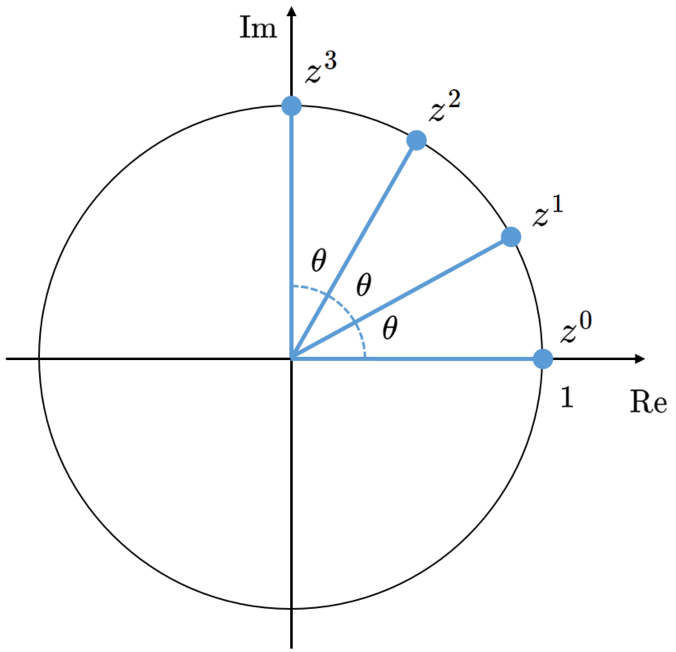
$$z_1 \cdot z_2 = e^{i\left(\frac{\pi}{2}+\theta\right)}$$



n-th Power of the Complex Exponential

$$z = e^{i\theta}$$

$$z^n = \left(e^{i\theta}\right)^n = e^{in\theta}$$



- Example
  - Find the solutions of $z^{12} = 1$

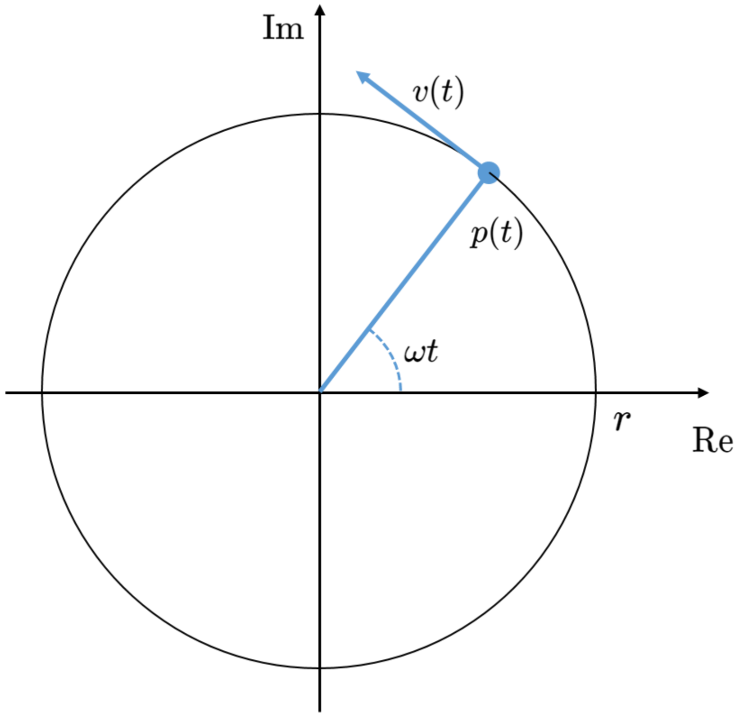
Circular Motion

Circular Motion

- Particle rotates on the circle with angular velocity of ω

$$p(t) = re^{i\omega t}$$

- Velocity in Circular Motion



$$v(t) = \frac{dp(t)}{dt} = r \cdot i\omega e^{i\omega t} = i\, r\omega\, e^{i\omega t}$$

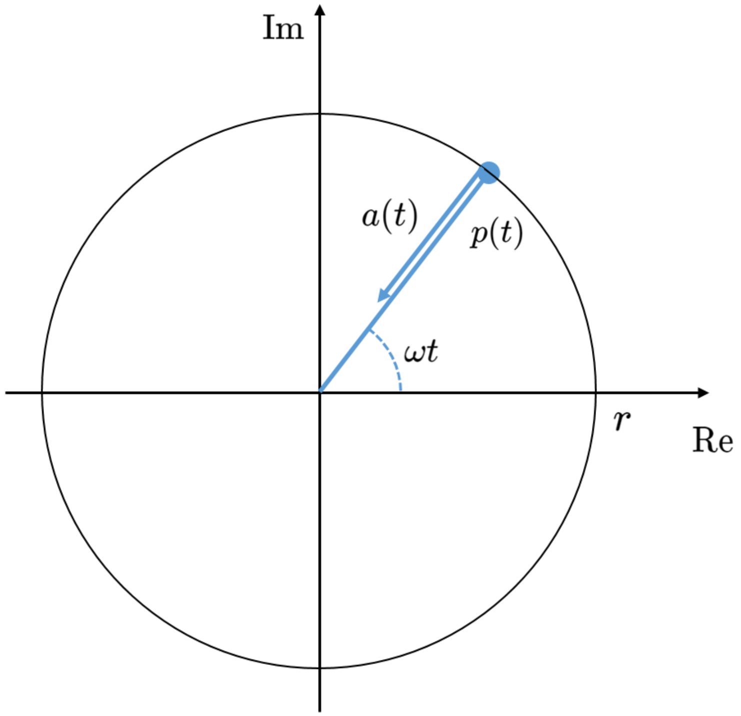
$$|v(t)| = r\omega$$
$$\angle v(t) = \omega t + \frac{\pi}{2}$$

Circular Motion

- Acceleration in Circular Motion

$$a(t) = \frac{dv(t)}{dt} = r\omega i \cdot i\omega e^{i\omega t} = -r\omega^2 e^{i\omega t}$$

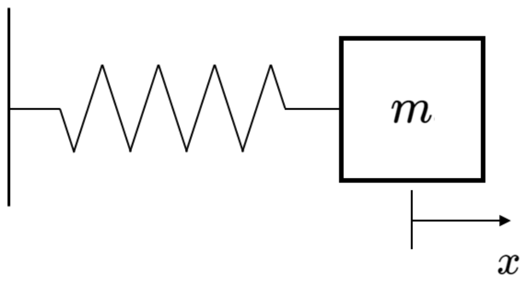
$$|a(t)| = r\omega^2$$
$$\angle a(t) = \omega t + \pi$$

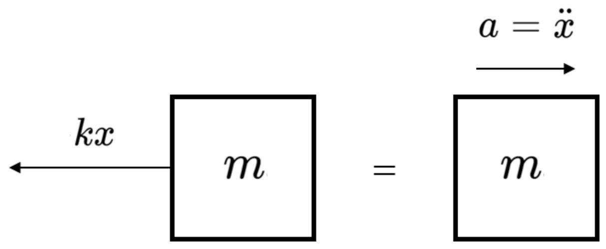


Harmonic Motion

Harmonic Motion

- Spring and Mass System





- Equations of motion

$$
\begin{aligned}
-kx &= m\ddot{x} \\
m\ddot{x} + kx &= 0 \\
\ddot{x} + \frac{k}{m}x &= 0
\end{aligned}
$$

$$\ddot{x} + \omega_n^2 x = 0, \;\; \omega_n = \sqrt{\frac{k}{m}}$$

Harmonic Motion

$$-kx = m\ddot{x}$$
$$m\ddot{x} + kx = 0$$
$$\ddot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \omega_n^2 x = 0,\ \ \omega_n = \sqrt{\frac{k}{m}}$$

- Differential Equation
  - 2nd order ODE (ordinary differential Equation)
  - No additional external force (suppose our system contains m, k)
  - spring force ($-kx$) is internal force
  - No input (= external) force
  - Two initial conditions determine the future motion $\begin{cases} x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$

- Solutions
  - Assume (or educated guess from Physics 1) that the solution is

$$x(t) = R\cos(\omega_n t + \phi)$$

  - Unknowns R and ∅ are determined by x_0, v_0

Seen as a Projection of a Circular Motion

- Sinusoidal can be seen as a projection of a circular motion

$$-kx = m\ddot{x}$$
$$m\ddot{x} + kx = 0$$
$$\ddot{x} + \frac{k}{m}x = 0$$

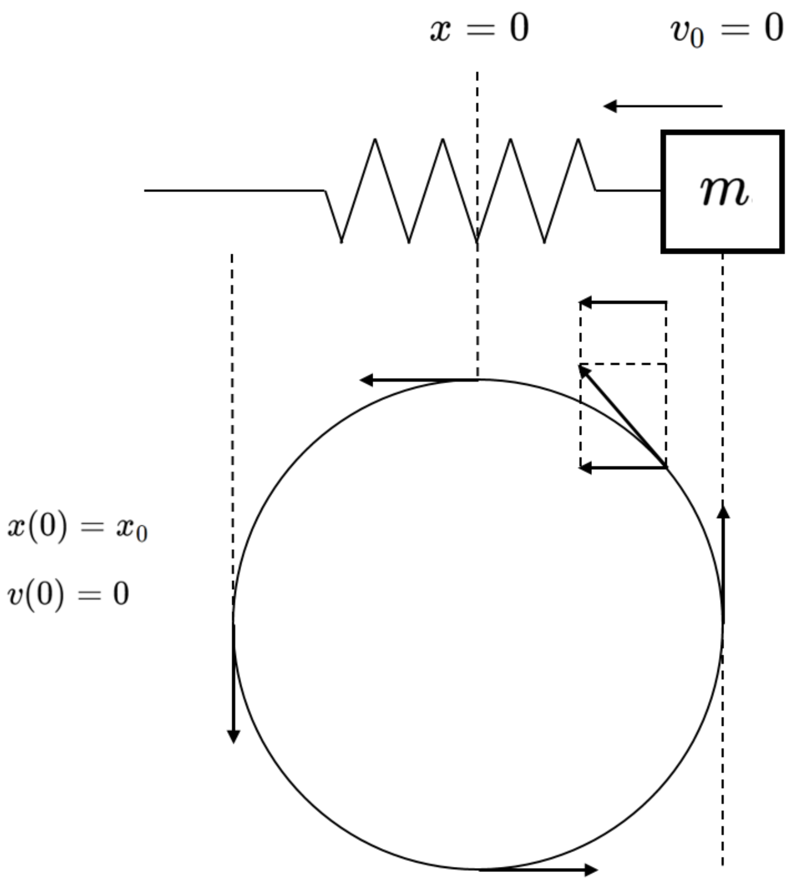
$$\ddot{x} + \omega_n^2 x = 0,\ \omega_n = \sqrt{\frac{k}{m}}$$

$$z(t) = Re^{j(\omega_n t+\phi)}$$
$$\dot{z}(t) = jR\omega_n e^{j(w_n t+\phi)} = j\omega_n z(t)$$
$$\ddot{z}(t) = -{\omega_n}^2 z(t)$$

$$\Rightarrow\ \ddot{z}(t) + {\omega_n}^2 z(t) = 0$$

Seen as a Projection of a Circular Motion

- We know that two initial conditions (x_0, v_0 at $t = 0$) will determine every motions.



Determine Unknown Coefficients

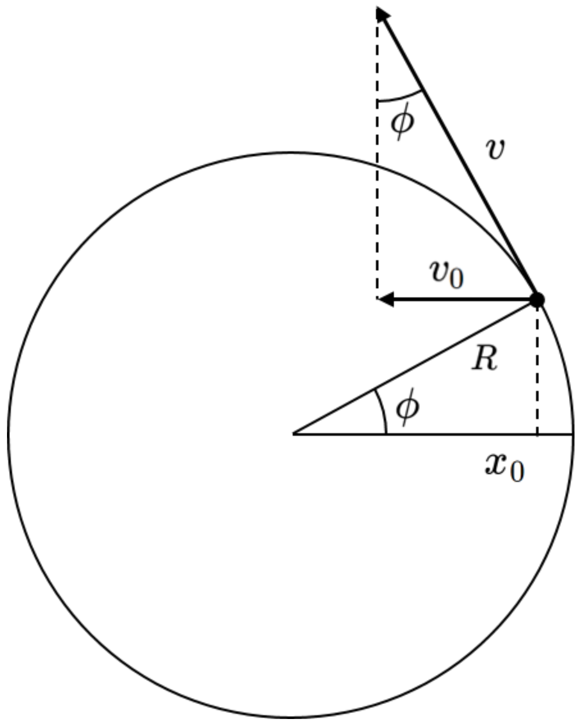
- How to obtain $A, \emptyset$ from x_0, v_0

$$x(t) = R\cos(\omega t + \phi) \qquad x(0) = R\cos\phi = x_0$$

$$\dot{x}(t) = -R\omega\sin(\omega t + \phi) \quad \dot{x}(0) = -R\omega\sin(\phi) = -v_0$$

- Determine Unknown Coefficients from circle

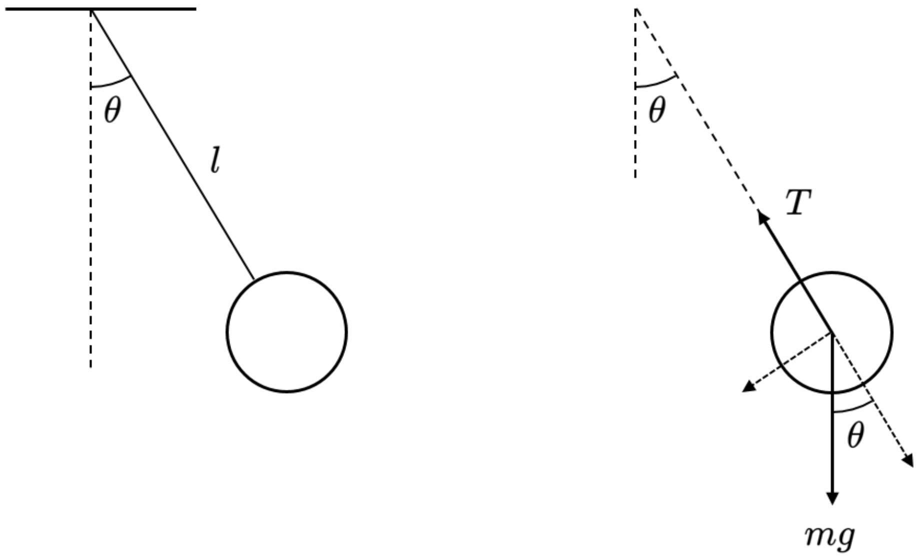
$$\begin{aligned} x_0 &= R\cos\phi \\ v_0 &= v\sin\phi = R\omega\sin\phi \end{aligned}$$



Pendulum

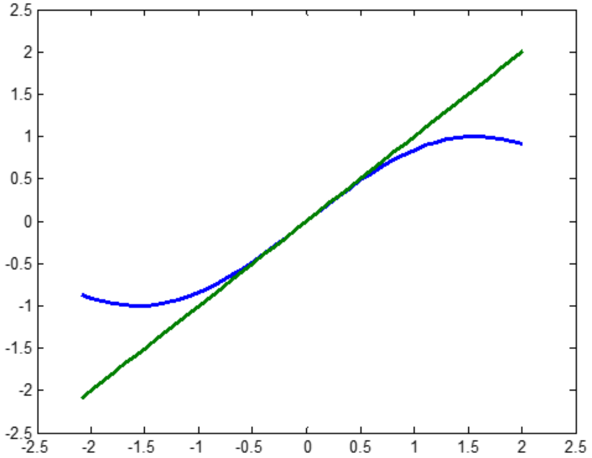
- Equations of motion

$$-T + mg\cos\theta = -ml\omega^2$$
$$-mg\sin\theta = ma = ml\ddot{\theta}$$



- From Nonlinear to Linear
  - Nonlinear system approximation possible?

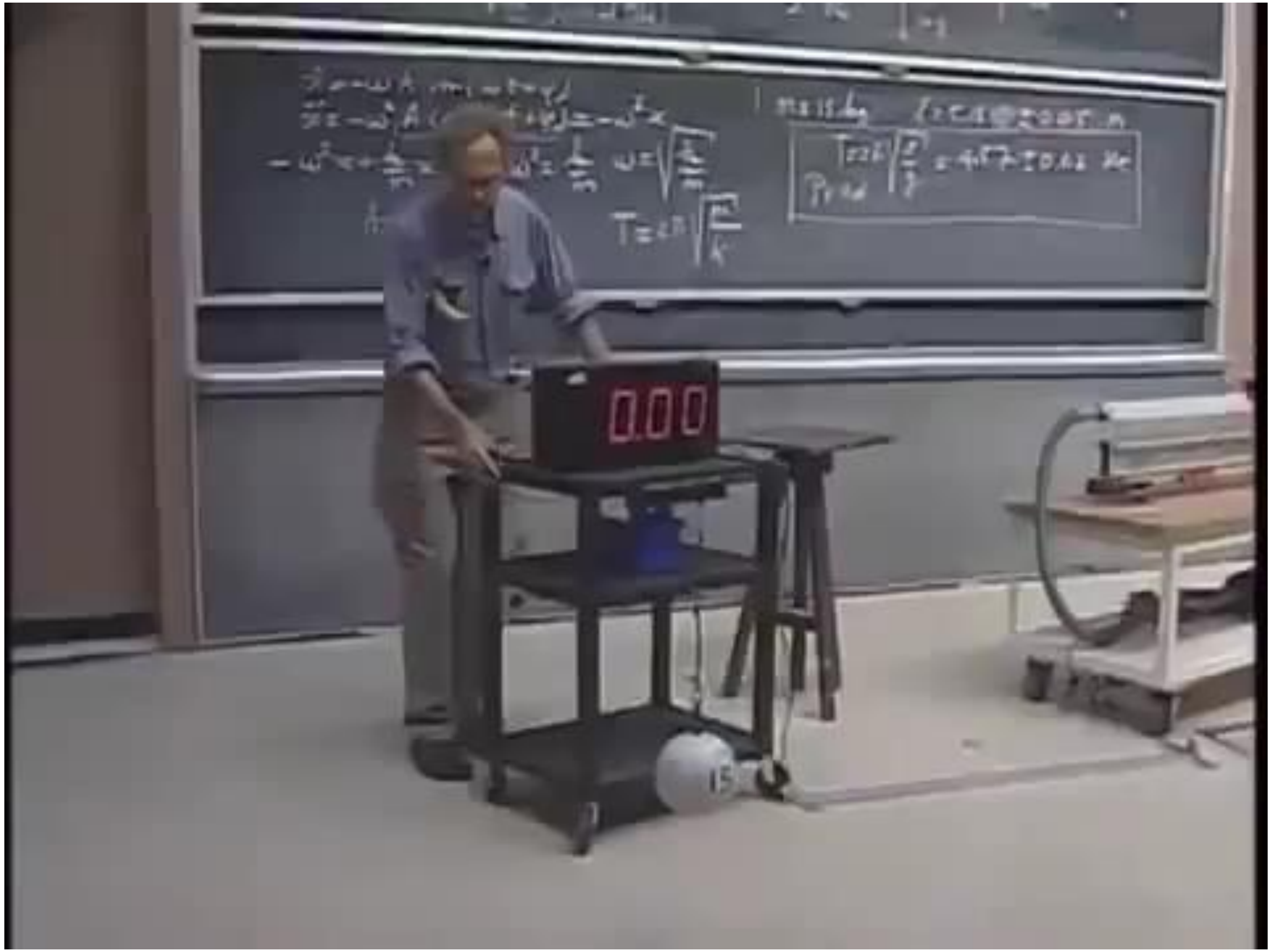
$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{l}\theta = 0$$



  - Period is independent of mass (non-intuitive)

$$\omega^2 = \frac{g}{l}$$

Period is Independent of Mass (non-intuitive)



From Physics (MIT 8.01) by Prof. Walter Lewin

Simulation of Free Vibration

$$z(t) = e^{j\omega t} = \cos \omega t + j \sin \omega t$$

- ω: angular velocity, [rad/sec]
- f: frequency, [rev/sec = Hz]

$$\omega = 2\pi f$$

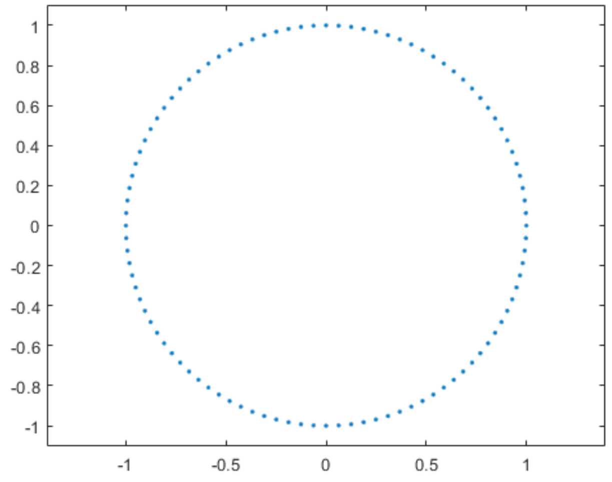
- One revolution per sec

$$\omega = 2\pi$$
$$f = 1$$

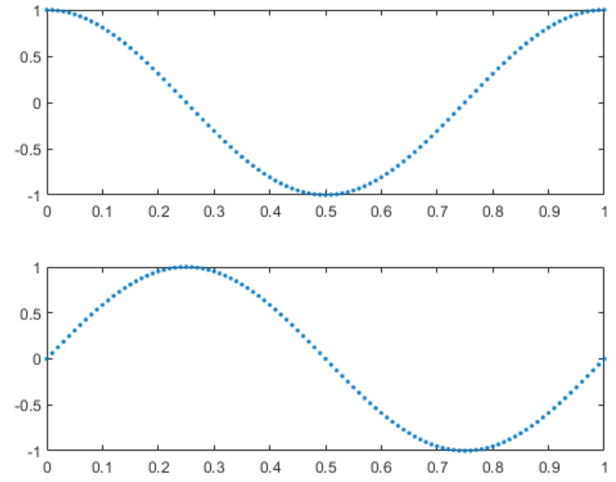
Simulation of Free Vibration

```
t = 0:0.01:1;
f = 1;
w = 2*pi*f;

z = exp(1j*w*t);
plot(real(z),imag(z),'.'), axis equal, ylim([-1.1 1.1])
```

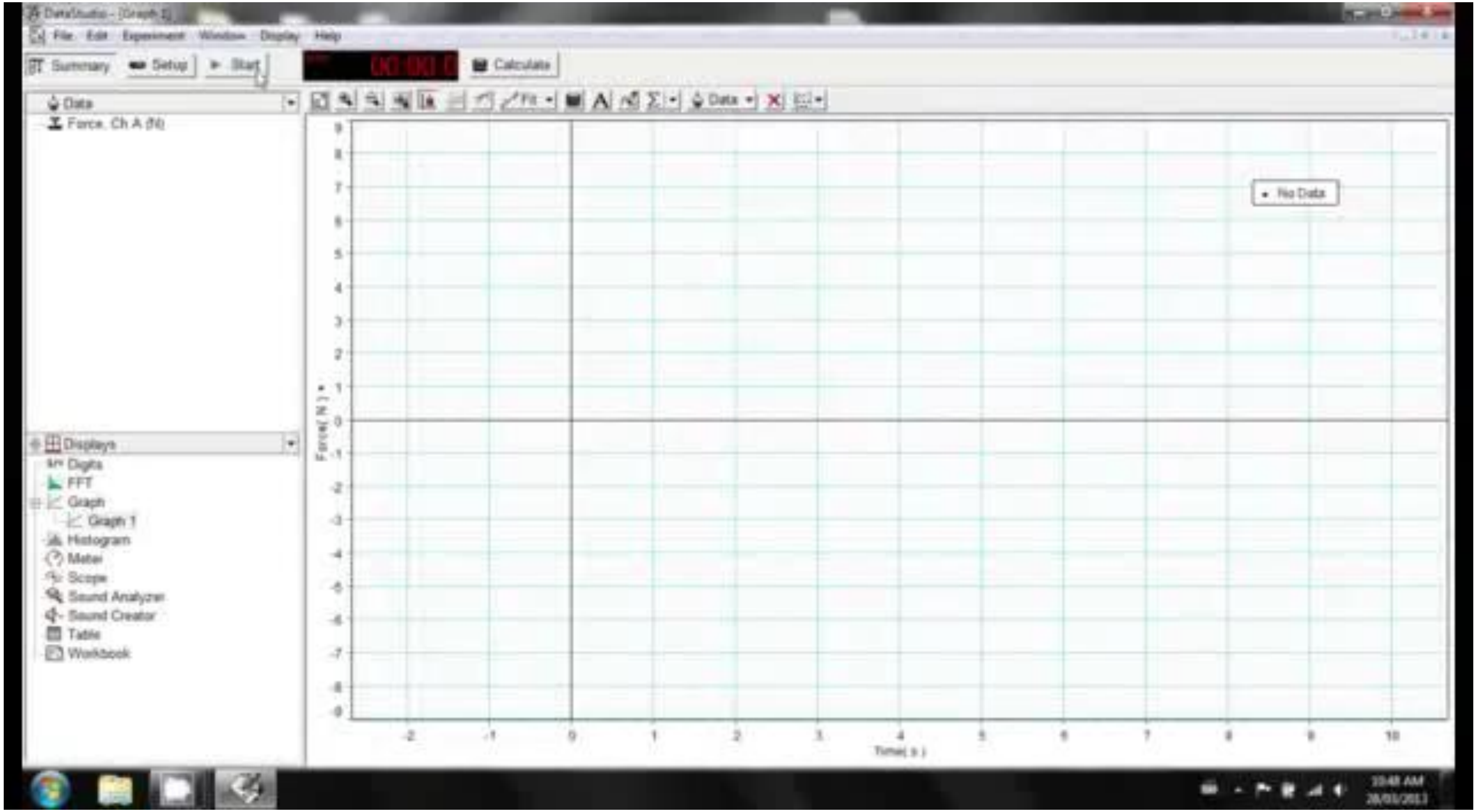



```
subplot(2,1,1), plot(t,real(z),'.')
subplot(2,1,2), plot(t,imag(z),'.')
```



Damped Free Vibration

Experiment First

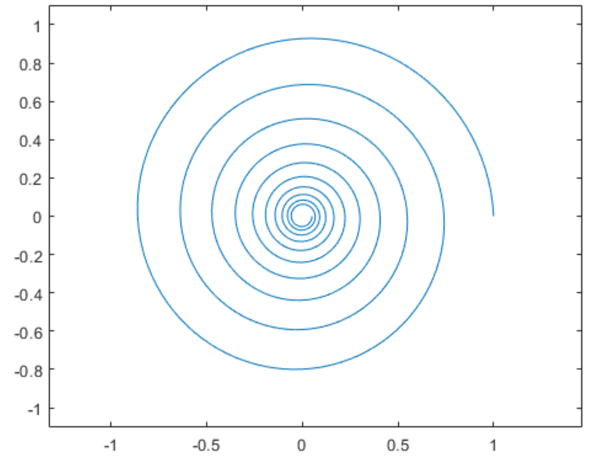


PHY245: Damped Mass On A Spring, https://www.youtube.com/watch?v=ZqedDWEAUN4

Damped Oscillating

- In a mathematical form (again from the educated guess)
- Exponentially decaying while oscillating

$$z(t) = e^{-\gamma t} e^{j\omega t}$$



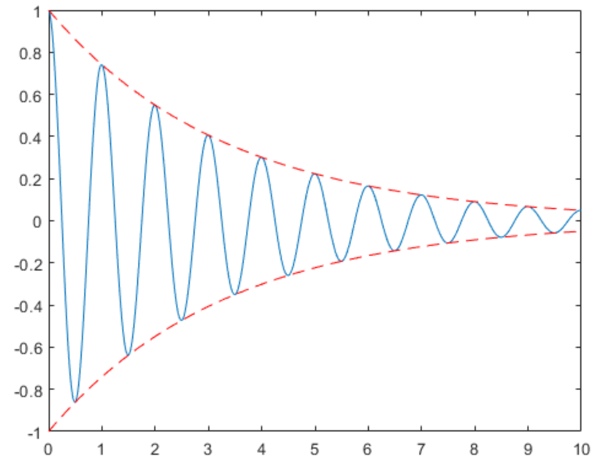

```
r = 0.3;

f = 1;
w = 2*pi*f;

t = 0:0.01:10;
z = exp(-1*r*t).*exp(1j*w*t);

plot(real(z),imag(z)), axis equal, ylim([-1.1,1.1])
```

```
plot(t,real(z),t,exp(-1*r*t),'r--',t,-exp(-1*r*t),'r--')
```



Damped Oscillating

- Assume damping causes exponential decay while oscillating

$$z(t) = e^{-\gamma t} e^{-j\omega t} = e^{-(\gamma + j\omega)t} \qquad \text{normalized for simplicity}$$

$$v(t) = \frac{dz(t)}{dt} = -(\gamma + j\omega) e^{-(\gamma + j\omega)t}$$

$$a(t) = \frac{d^2 z(t)}{dt^2} = (\gamma + j\omega)^2 e^{j\omega t} = (\gamma^2 - \omega^2 + j2\gamma\omega) e^{-(\gamma + j\omega)t}$$

$$\left\{(\gamma^2 - \omega^2 + j2\gamma\omega) e^{-(\gamma + j\omega)t}\right\} + 2\gamma \left\{\left(-(\gamma + j\omega) e^{-(\gamma + j\omega)t}\right)\right\} + \left(\gamma^2 + \omega^2\right) \left\{e^{-(\gamma + j\omega)t}\right\}$$

$$= a(t) + 2\gamma v(t) + \left(\gamma^2 + \omega^2\right) z(t) = \frac{d^2 z(t)}{dt^2} + 2\gamma \frac{dz(t)}{dt} + (\gamma^2 + \omega^2) z(t) = 0$$

- Show $z(t) = e^{-\gamma t} e^{j\omega t}$ also satisfies

$$\frac{d^2 z(t)}{dt^2} + 2\gamma \frac{dz(t)}{dt} + \left(\gamma^2 + \omega^2\right) z(t) = 0$$

Damped Oscillating

- Given the differential equation

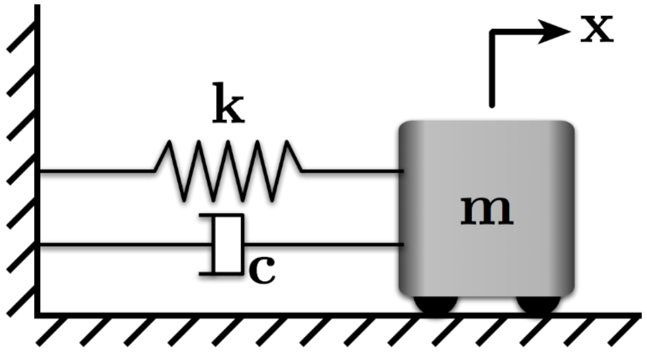
$$\ddot{z}(t) + 2\gamma\,\dot{z}(t) + \left(\gamma^2 + \omega^2\right) z(t) = 0$$

- Solution is a linear combination of

$$z(t) = e^{-\gamma t}\left(Ae^{j\omega t} + Be^{-j\omega t}\right)$$

- A, B are determined by initial conditions

Mass, Spring, and Damper System



$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \implies$$
$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0 \qquad \text{where } \omega_n^2 = \frac{k}{m} = \gamma^2 + \omega^2$$

- Parameters

$$\omega_n^2 = \frac{k}{m} = \gamma^2 + \omega^2 \qquad \text{: natural angular velocity}$$
$$\omega^2 = \omega_n^2 - \gamma^2 \qquad \text{: actual angular velocity}$$
$$\gamma = \zeta\omega_n \qquad \text{: decaying factor}$$
$$\omega^2 = \omega_n^2\left(1 - \zeta^2\right)$$
$$0 \le \zeta = \sqrt{1 - \left(\frac{\omega}{\omega_n}\right)^2} \le 1 \qquad \text{: damping ratio}$$

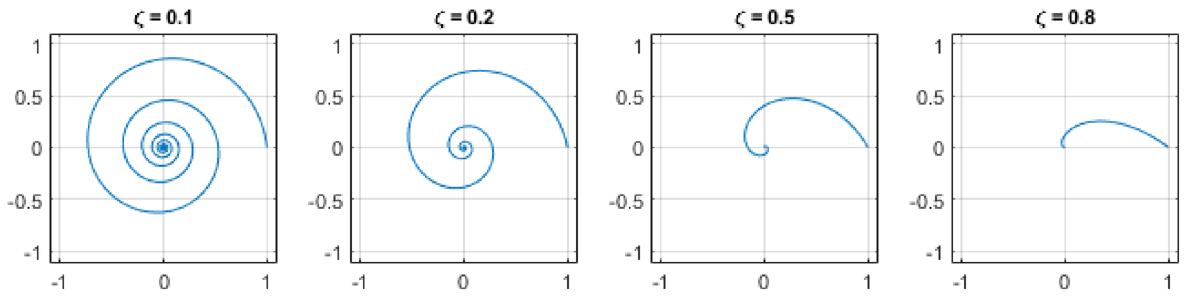
- Solution

$$z(t) = e^{-\zeta\omega_n t}\left(Ae^{j\omega_n\sqrt{1-\zeta^2}} + Be^{-j\omega_n\sqrt{1-\zeta^2}}\right)$$

Simulation of Damped Vibration

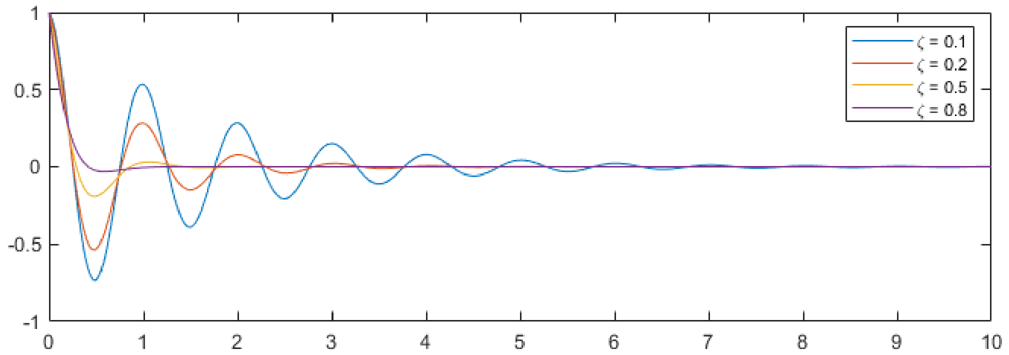
```
f = 1;
wn = 2*pi*f;
zeta = [0.1 0.2 0.5 0.8];
t = 0:0.01:10;

for i = 1:4
    r = zeta(i)*wn;
    w = wn*sqrt(1-zeta(i)^2);
    z = exp(-1*r*t).*exp(1j*w*t);
    subplot(1,4,i), plot(real(z),imag(z)), grid on
    axis equal, axis([-1.1,1.1 -1.1 1.1])
    title(['\zeta = ',num2str(zeta(i))],'fontsize',8)
end
```




```
for i = 1:4
    r = zeta(i)*wn;
    w = wn*sqrt(1-zeta(i)^2);
    z = exp(-1*r*t).*exp(1j*w*t);
    plot(t,real(z)), hold on
end

hold off
legend(['\zeta = ',num2str(zeta(1))],['\zeta = ',num2str(zeta(2))],...
['\zeta = ',num2str(zeta(3))],['\zeta = ',num2str(zeta(4))])
```



Example: Door Closer

Example: Torsional Pendulum

From Physics (MIT 8.01) by Prof. Walter Lewin

Example