State Space Representation

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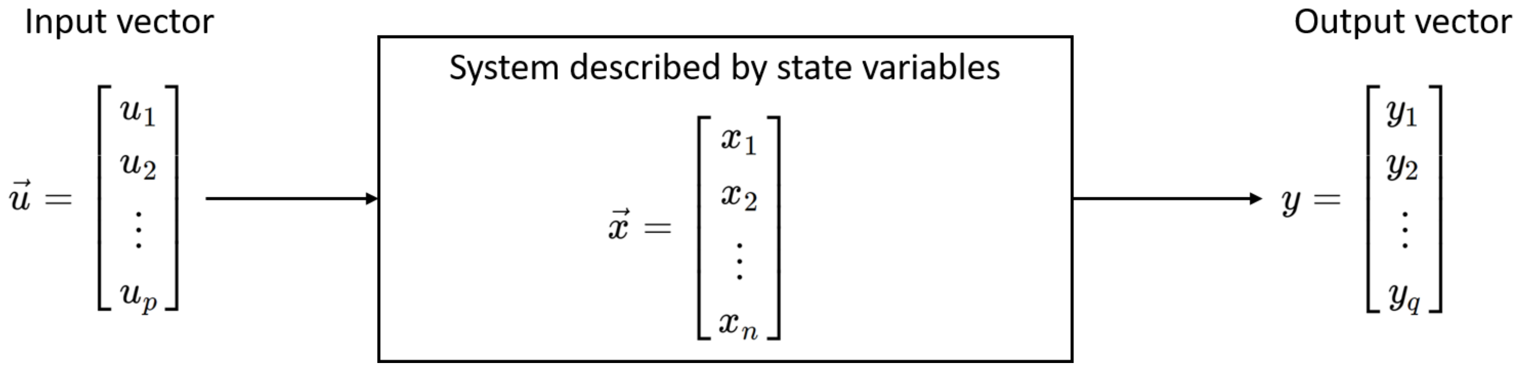
State of a Dynamic System

- A minimum set of variables, known as state variables, that fully describe the system and its response to any given set of inputs.



- The number of state variables, n, is equal to the number of independent "energy storage elements" in the system.
- The state equations

$$\begin{aligned} \dot{x_1} &= f_1(x, u, t) \\ \dot{x_2} &= f_2(x, u, t) \\ &\vdots \\ \dot{x_n} &= f_n(x, u, t) \end{aligned}$$

State Representation of LTI System

- We restrict attention primarily to linear and time-invariant (LTI) system. Then it becomes a set of n coupled first-order linear differential equations with constant coefficients.

$$
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + \cdots + a_{1n}x_n \ + \ b_{11}u_1 + \cdots + b_{1p}u_p \\
\dot{x}_2 &= a_{21}x_1 + \cdots + a_{2n}x_n \ + \ b_{21}u_1 + \cdots + b_{2p}u_p \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + \cdots + a_{nn}x_n \ + \ b_{n1}u_1 + \cdots + b_{np}u_p
\end{aligned}
$$

- Written compactly in a matrix form

$$
\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ & & \vdots & \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ & & \vdots & \\ & & \vdots & \\ & & & b_{np} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}
$$

$$
\dot{x} = Ax + Bu
$$

Output of LTI System

- Output equations

$$
\begin{aligned}
y_1 &= c_{11}x_1 + \cdots + c_{1n}x_n \; + \; d_{11}u_1 + \cdots + d_{1p}u_p \\
y_2 &= c_{21}x_1 + \cdots + c_{2n}x_n \; + \; d_{21}u_1 + \cdots + d_{2p}u_p \\
&\vdots \\
y_q &= c_{q1}x_1 + \cdots + c_{qn}x_n \; + \; d_{q1}u_1 + \cdots + d_{qp}u_p
\end{aligned}
$$

$$
\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ & & \vdots & \\ & & \vdots & \\ & & & c_{qn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1p} \\ & & \vdots & \\ & & \vdots & \\ & & & d_{qp} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}
$$

$y = Cx + Du$ or

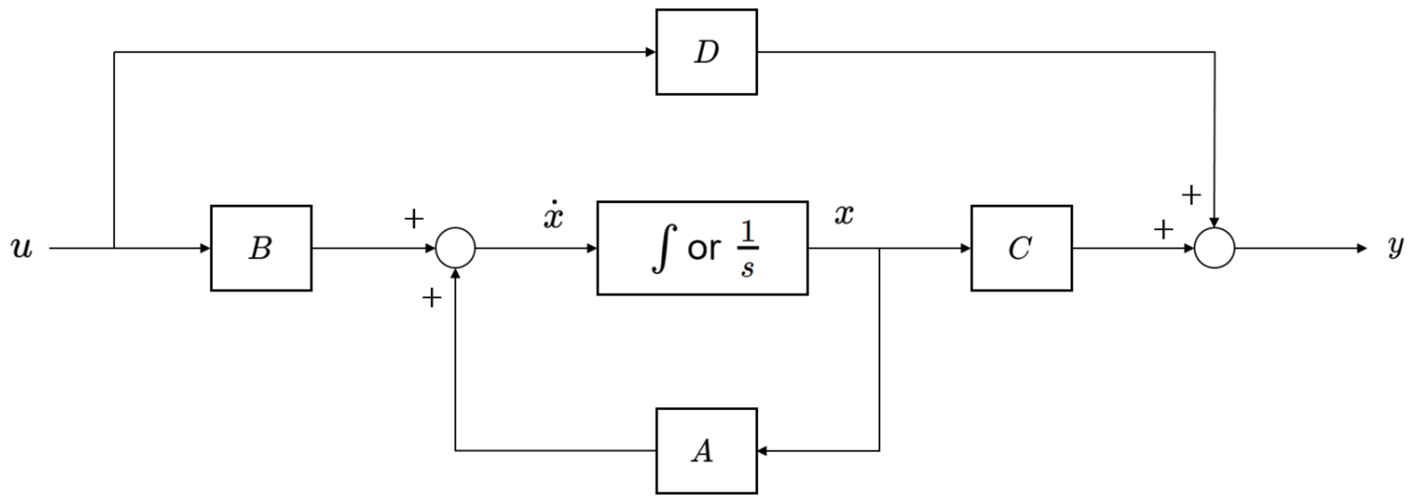
$y = Cx$ for many physical systems the matrix D is the null matrix

Block Diagram of LTI System

- The complete system model for LTI system in the standard state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Block diagram



Homogeneous State Response

Homogeneous State Response

- With zero input, $u(t) = 0$

$$\dot{x} = Ax$$

$$y(t) = e^{At} x\,(0)$$

- Let's figure out how such a system behaves
  - Start by ignoring the input term:

$$\begin{aligned} \dot{x} &= Ax \\ x(t_0) &= x_0 \end{aligned}$$

- What is the solution to this system?
  - If everything is scalar:

$$\dot{x} = ax, \quad x(t_0) = x_0 \implies x(t) = e^{at} x_0$$

  - How do we know?

$$\begin{aligned} x(0) &= e^0 x_0 = x_0 \\ \frac{dx(t)}{dt} &= a e^{at} x_0 = a x(t) \end{aligned}$$

Homogeneous State Response

- For higher-order systems, we just get a matrix version of this

$$\dot{x} = Ax, \quad x(0) = x_0 \implies x(t) = e^{At}x_0$$

- The definition is just like for scalar exponentials

$$e^{at} = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}, \qquad e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

- Derivative:

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\sum_{k=0}^{\infty} \frac{(At)^k}{k!} = 0 + \sum_{k=1}^{\infty} \frac{kA^k t^{k-1}}{k!} = A\sum_{k=1}^{\infty} \frac{(At)^{k-1}}{(k-1)!} = A\sum_{k=0}^{\infty} \frac{(At)^k}{k!} = Ae^{At}$$

- The matrix exponential plays such an important role that it has its own name: the state transition matrix, $\Phi(t)$

$$\dot{x} = Ax \implies \begin{aligned} x(t) &= \Phi(t)x(0) \\ e^{At} &= \Phi(t) \end{aligned}$$

Properties of State Transition Matrix

- Properties of state transition matrix

$$
\begin{aligned}
&\Phi(0) = I \\
&\Phi(-t) = \Phi^{-1}(t) \\
&\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)
\end{aligned}
$$

$$
\begin{aligned}
x(0) &= \Phi(-t_0)x(t_0) \\
x(t) = \Phi(t)x(0) &= \Phi(t)\Phi(-t_0)x(t_0) \\
&= \Phi(t - t_0)x(t_0)
\end{aligned}
$$

Example: State Transition Matrix

- Example

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix} x, \qquad x(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Phi(t) = e^{At} = S e^{\Lambda t} S^{-1}$$

$$\lambda_1 = -2 \quad \Longrightarrow \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \quad \vec{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

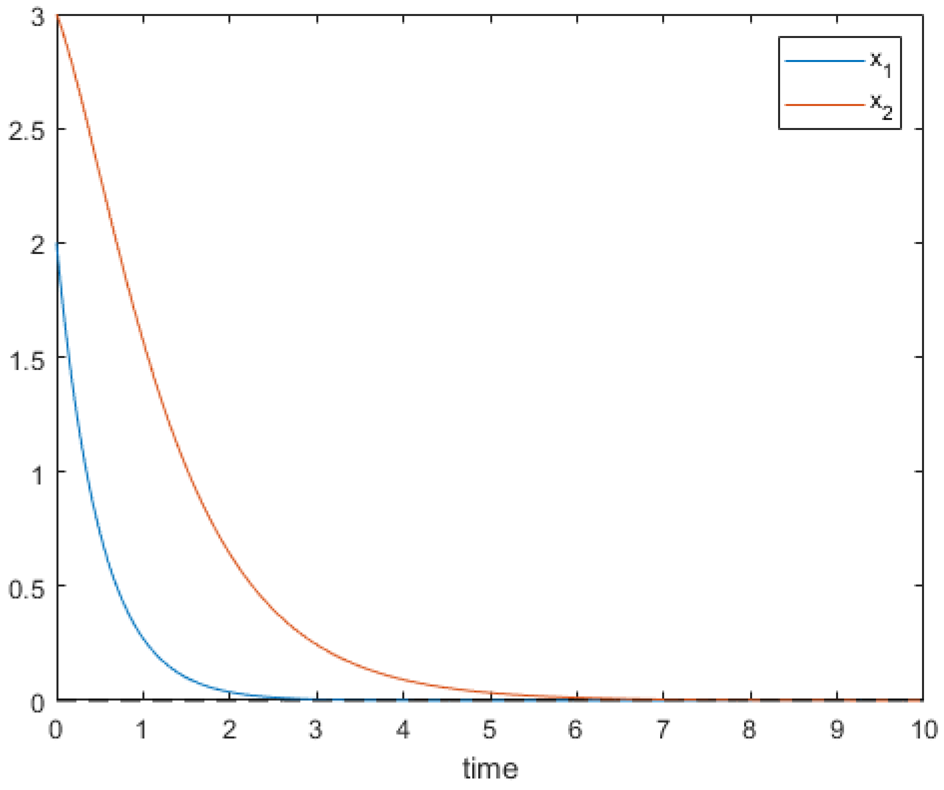
$$\lambda_2 = -1 \quad \Longrightarrow \quad \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \quad \vec{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad S^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$x(t) = S e^{\Lambda t} S^{-1} x(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2e^{-2t} \\ 5e^{-t} - 2e^{-2t} \end{bmatrix}$$

Example: State Transition Matrix

```
A = [-2, 0; 1, -1];
B = [0, 0]';
C = [1, 0;
     0, 1];
D = 0;

G = ss(A,B,C,D);

x0 = [2; 3];

t = linspace(0,10,500);
u = zeros(size(t));

[y, tout] = lsim(G,u,t,x0);

plot(tout,y,tout,zeros(size(tout)),'k--')
xlabel('time')
legend('x_1', 'x_2')
```



Forced State Response of LTI System

Forced State Response of LTI System

- But what if we have the controlled system:

$$\dot{x} = Ax + Bu$$

- Consider the complete response of a linear system to an input $u(t)$

$$\dot{x}(t) - ax(t) = bu(t)$$

- Derivation

$$e^{-at}\dot{x} - e^{-at}ax = \frac{d}{dt}(e^{-at}x(t)) = e^{-at}bu$$

$$\int_0^t \frac{d}{d\tau}(e^{-a\tau}x(\tau))d\tau = e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

- Complete solution

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

For Higher Order Systems

$$\dot{x} = Ax + Bu$$

- Complete solution

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- The output

$$y = Cx + Du$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- The first is a term similar to the system homogeneous response $x_h(t) = e^{At}x(0)$ that is dependent only on the system initial conditions $x(0)$

- The second term is in the form of a convolution integral, and it is the particular solution for the input $u(t)$ with zero initial conditions

For Higher Order Systems

- Note

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

$$x(0) = \Phi(0)x(0) + \int_0^0 \Phi(0-\tau)Bu(\tau)d\tau = x(0)$$

$$\frac{d}{dt}\int_{t_0}^t f(t,\tau)d\tau = f(t,t) + \int_{t_0}^t \frac{\partial f(t,\tau)}{\partial t}d\tau \quad \longleftarrow \quad \text{(Leibniz Integral Rule)}$$

$$\implies \Phi(0)Bu(t) + \int_{t_0}^t A\Phi(t-\tau)Bu(\tau)d\tau$$

$$\frac{d}{dt}x(t) = A\left(\Phi(t)x(0) + \int_{t_0}^t \Phi(t-\tau)Bu(\tau)d\tau\right) + Bu(t)$$

$$\frac{d}{dt}x(t) = Ax + Bu$$

Note: Leibniz Integral Rule

If $\mathbf{F}(t) = \int_{a(t)}^{b(t)} f(t,\tau)d\tau$, then $\dfrac{d\mathbf{F}}{dt} = \displaystyle\int_{a(t)}^{b(t)} \frac{\partial f(t,\tau)}{\partial t} d\tau + f(t,b(t))\frac{db}{dt} - f(t,a(t))\frac{da}{dt}$

Suppose f is a univariate function

$$F(t) = \int_{t_0}^{t} f(\tau)d\tau \quad \Longrightarrow \quad \frac{dF(t)}{dt} = f(t)$$

$$F(t) = \int_{t}^{t_0} f(\tau)d\tau \quad \Longrightarrow \quad \frac{dF(t)}{dt} = -f(t)$$

Now suppose f is a function of two variables (a and b are constant)

$$F(t) = \int_{a}^{b} f(t,\tau)d\tau \quad \Longrightarrow \quad \frac{dF(t)}{dt} = \int_{a}^{b} \frac{\partial f(t,\tau)}{\partial t} d\tau$$

Now suppose the limits of integration a and b themselves depend on t,

$$\mathbf{F}(t) = F(t, a(t), b(t)) = \int_{a(t)}^{b(t)} f(t,\tau)d\tau$$

$$\frac{d\mathbf{F}(t)}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial a}\frac{da}{dt} + \frac{\partial F}{\partial b}\frac{db}{dt}$$

$$\frac{d\mathbf{F}(t)}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial a}\frac{da}{dt} + \frac{\partial F}{\partial b}\frac{db}{dt}$$

$$= \int_{a(t)}^{b(t)} \frac{\partial f(t,\tau)}{\partial t} d\tau + f(t,b(t))\frac{db}{dt} - f(t,a(t))\frac{da}{dt}$$

$$\frac{d}{dt}\int_{t_0}^{t} f(t,\tau)d\tau = \int_{t_0}^{t} \frac{\partial f(t,\tau)}{\partial t} d\tau + f(t,t)$$

Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u, \qquad u = 2u(t) \ \text{ where } u(t) \text{ is a step function,} \qquad x(0) = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\Phi(t) = \begin{bmatrix}e^{-2t} & 0\\ e^{-t} - e^{-2t} & e^{-t}\end{bmatrix} = e^{At}$$

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau = \Phi(t)\int_0^t \Phi(-\tau)B1d\tau$$

$$= \begin{bmatrix}e^{-2t} & 0\\ e^{-t} - e^{-2t} & e^{-t}\end{bmatrix}\int_0^t \begin{bmatrix}e^{2\tau} & 0\\ e^{\tau} - e^{2\tau} & e^{\tau}\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix}d\tau$$

$$= \begin{bmatrix}e^{-2t} & 0\\ e^{-t} - e^{-2t} & e^{-t}\end{bmatrix}\int_0^t \begin{bmatrix}2e^{2\tau}\\ 2e^{\tau} - 2e^{2\tau}\end{bmatrix}d\tau$$

$$= \begin{bmatrix}e^{-2t} & 0\\ e^{-t} - e^{-2t} & e^{-t}\end{bmatrix}\begin{bmatrix}e^{2\tau}\big|_0^t\\ (2e^{\tau} - e^{2\tau})\big|_0^t\end{bmatrix}$$

$$= \begin{bmatrix}e^{-2t} & 0\\ e^{-t} - e^{-2t} & e^{-t}\end{bmatrix}\begin{bmatrix}e^{2t} - 1\\ 2e^{t} - e^{2t} - 1\end{bmatrix}$$

$$= \begin{bmatrix}1 - e^{-2t}\\ (e^{-t} - e^{-2t})(e^{2t} - 1) + e^{-t}(2e^{t} - e^{2t} - 1)\end{bmatrix}$$

$$= \begin{bmatrix}1 - e^{-2t}\\ 1 + e^{-2t}\end{bmatrix}$$

The Response of LTI to the Singularity Input Functions

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

- Impulse response

$$u(t) = K\delta(t) = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_p \end{bmatrix} \delta(t)$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}BK\delta(\tau)d\tau$$

$$= e^{At}x(0) + e^{At}BK$$

$$= e^{At}(x(0) + BK)$$

- The effect of impulse inputs on the state response is similar to changing a set of initial conditions $x(0) \rightarrow x(0) + BK$

The Response of LTI to the Singularity Input Functions

- Step response

$$u(t) = Ku_{\text{step}} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_p \end{bmatrix} u_{\text{step}}(t)$$

$$\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
&= e^{At}x(0) + \int_0^t e^{A(t-\tau)}BKd\tau \\
&= e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}d\tau BK \\
&= e^{At}x(0) + e^{At}[-A^{-1}e^{-A\tau}|_0^t]BK \\
&= e^{At}x(0) + e^{At}[-A^{-1}e^{-At} + A^{-1}]BK \\
&= e^{At}x(0) + e^{At}A^{-1}[I - e^{-At}]BK \\
&= e^{At}x(0) + A^{-1}[e^{At} - I]BK \qquad (Ae^{At} = e^{At}A)
\end{aligned}$$

$$x(\infty) = -A^{-1}BK \qquad \text{if the system is stable}$$

The Response of LTI to the Singularity Input Functions

- Step response

$$\ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y = g\,u_{\text{step}}(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \end{bmatrix} \cdot 1 \cdot u_{\text{step}}(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x(\infty) = -A^{-1}BK = -\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ g \end{bmatrix} \cdot 1 \cdot = \begin{bmatrix} \frac{mg}{k} \\ 0 \end{bmatrix}$$