Stability

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POSTECH

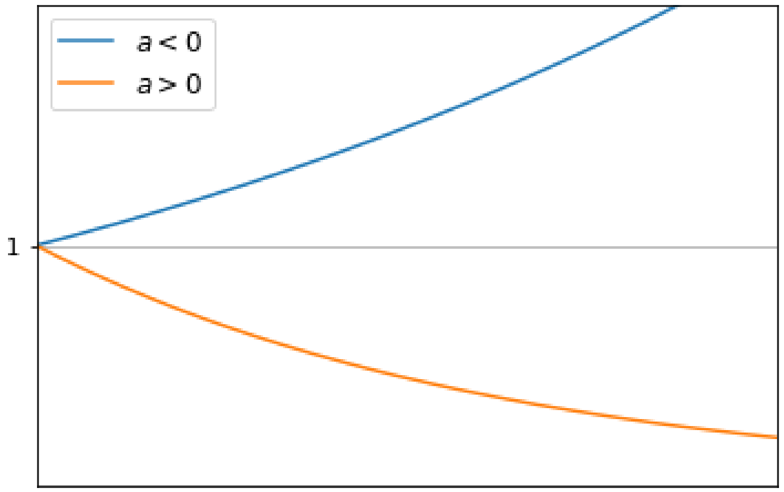
Most Slides from the Routh-Hurwitz Criterion by Brian Douglas and Control by Prof. Richard Hill

Stability of Open Loop System

- In order for a system $G(s) = \frac{N(s)}{D(s)}$ to be stable all of the roots of the characteristic polynomial need to lie in the left-half plane (LHP).
  - The characteristic equation is the denominator of the transfer function.
  - The roots of the characteristic equation are the exact same as the poles of the transfer function.
  - The eigenvalues of matrix A in the equivalent state space representation are the same as the roots of the characteristic polynomial.
  - In order to have a stable system, roots of $G(s)$ must be in LHP.

$$G(s) = \frac{1}{s+a}$$

$$\mathscr{L}^{-1}(G(s)) = e^{-at}u(t)$$



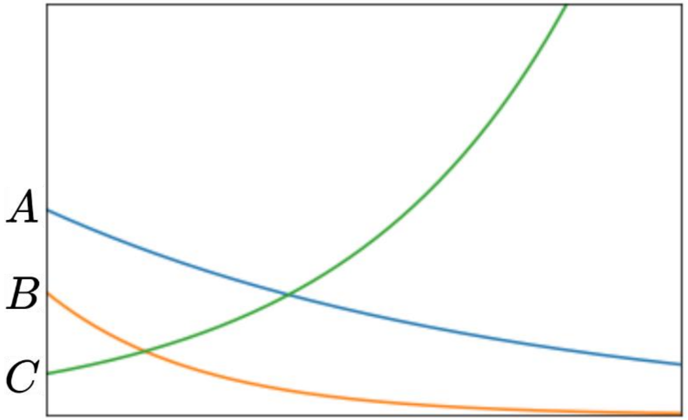
Stability of Open Loop System

- When a pole is negative
  - This root exists in the left half plane
  - Transfer function will ultimately die out
  - The system will eventually be at rest (stable)
- When a pole is positive
  - This root exists in the right half plane
  - Transfer function will blow up into infinity
  - The system is unstable
- Transfer function of multiple poles
  - The last one blows up to infinity to make the whole transfer function unstable
  - Conclusion: a single root in the right half plane makes the whole system unstable

$$G(s) = \frac{1}{s+1} \cdot \frac{1}{s+3} \cdot \frac{1}{s-2}$$

$$G(s) = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$\mathscr{L}^{-1}(G(s)) = Ae^{-t} + Be^{-3t} + Ce^{2t}$$



Routh-Hurwitz Criterion

Routh-Hurwitz Criterion

- Calculating the roots of the system for larger than the second-order polynomial becomes time-consuming and possibly even impossible in a closed-form

- How can we determine the stability of a higher order polynomial without solving for the roots directly?
  - The great thing about the Routh-Hurwitz criterion is that you do not have to solve for the roots of the characteristic equation

$$G(s) = \frac{1}{s^4 + 3s^3 - 5s^2 + s + 2}$$

  - If all of the signs are not the same, the system is unstable
  - If you build up a transfer function with a series of poles, then the only way to get a negative coefficient is to have at least one pole exists in right-half plane

Routh-Hurwitz Criterion

- However, we cannot claim that all positive coefficients are still either stable or unstable

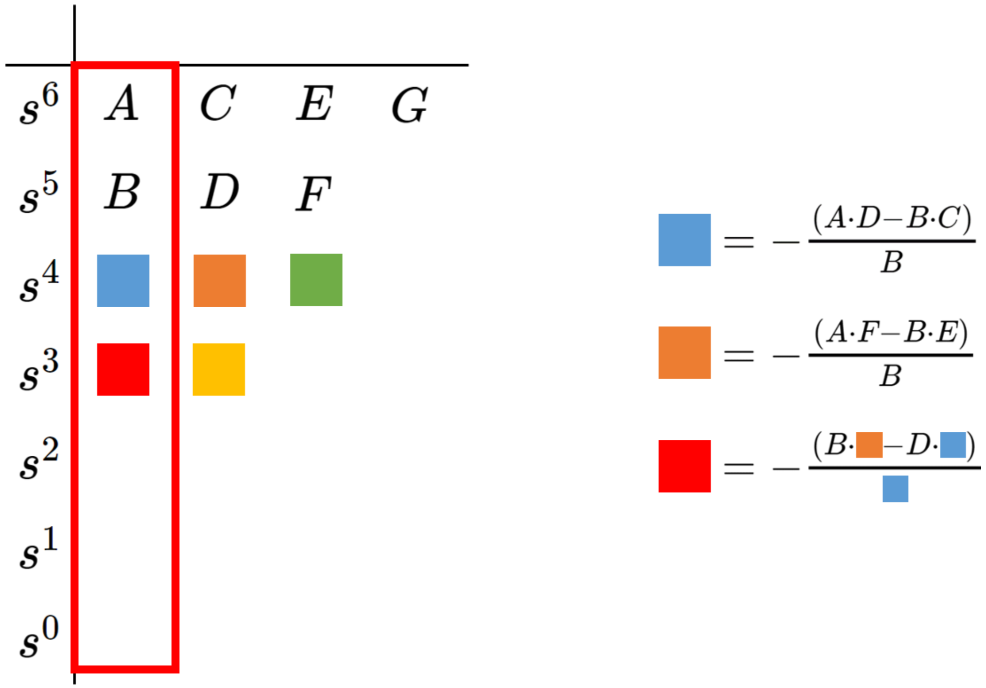
$$G(s) = \frac{1}{s^4 + 2s^3 + 3s^2 + 10s + 8}$$

$$= \frac{1}{s^2 - s + 4} \cdot \frac{1}{s + 2} \cdot \frac{1}{s + 1}$$

Normal Case (1/2)

- Routh array is a table that can be populated with the coefficients of the polynomial with a few simple rules
  - The number of RHP roots of $D(s)$ is equal to the number of sign changes in the left column of the Routh array

$$As^6 + Bs^5 + Cs^4 + Ds^3 + Es^2 + Fs + G$$



Normal Case (2/2)

- Determine the number of roots in RHP by counting the number of sign changes
  - We can determine the number of roots in the right-half plane by looking at this first column
  - It changes sign twice which means that there are two roots in the right half plane

$$G(s) = \frac{1}{s^4 + 2s^3 + 3s^2 + 10s + 8}$$

s^4	1	3	8
s^3	2	10	0
s^2	-2	8	0
s^1	18	0	
s^0	8		

Special Case 1 (1/2)

- A zero in a row with at least one non-zero appearing later in that same row
  - If you are attempting to access stability of the system, you do not need to complete the rest of the table at this point
  - The system is always unstable because completing Routh array will always result in a sign change of the first column

1	0	4
2	3	

1	2	5
2	4	
0	5	

Unstable

Special Case 1 (2/2)

- If you are interested in the number of roots located in the right half plane, you can complete the table like below
  - You replace that zero with the Greek symbol epsilon $\epsilon > 0$
  - When you finish completing the table, you can take the limit as epsilon ϵ goes to zero
  - You can see that we still have two unstable roots or two roots in the right half plane

s^4	1	2	5
s^3	2	4	
s^2	$\cancel{0}\ \epsilon$	5	
s^1	$\frac{4\epsilon-10}{\epsilon}$		
s^0	5		

s^4	1	2	5
s^3	2	4	
s^2	0^+	5	
s^1	$-\infty$		
s^0	5		

Special Case 2 (1/2)

- The second special case is when there is an entire row of zeros, not just a single zero in the row

$$D(s) = s^5 + 2s^4 + 6s^3 + 10s^2 + 8s + 12$$

s^5	1	6	8
s^4	2	10	12
s^3	1	2	0
s^2	6	12	0
s^1	0	0	0
s^0			

- Auxiliary polynomial $P(s)$: the row directly above the row of zeros

$$6s^2 + 12s^0 \implies P(s) = s^2 + 2$$

s^5	1	6	8
s^4	2	10	12
s^3	1	2	0
s^2	6	12	0
s^1	0	0	0
s^0			

Auxiliary polynomial

Special Case 2 (2/2)

- Then $P(s)$ is a factor of the original polynomial $D(s)$

$$D(s) = P(s) \cdot R(s)$$

$$s^5 + 2s^4 + 6s^3 + 10s^2 + 8s + 12 = \underbrace{(s^2 + 2)}_{\text{marginally stable}} \underbrace{(s^3 + 2s^2 + 4s + 6)}_{\text{stable}}$$

- Apply the Routh-Hurwitz criterion again to $R(s)$

s^3	1	4
s^2	2	6
s^1	1	0
s^0	6	

Stability with State Space Representation

Stability with State Space Representation

- It is useful to start with scalar systems to get some intuition about what is going on

$$\dot{x} = ax \qquad \Longrightarrow \quad x(t) = e^{at}x(0)$$

$$\begin{cases} a > 0 : \text{unstable} \\ a < 0 : \text{asymptotically stable} \\ a = 0 : \text{marginally stable} \end{cases}$$

- From scalars to matrices?

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

- We cannot say that $A > 0$, but we can do the next best thing - eigenvalues !

Stability with State Space Representation

- The eigenvalues tell us how the matrix A 'acts' in different directions (eigenvectors)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$$Av = \lambda v$$

$$\begin{cases} \mathrm{Re}\,(\lambda) > 0 : \text{unstable} \\ \mathrm{Re}\,(\lambda) < 0 : \text{asymptotically stable} \\ \mathrm{Re}\,(\lambda) \leq 0 : \text{critically stable} \end{cases}$$

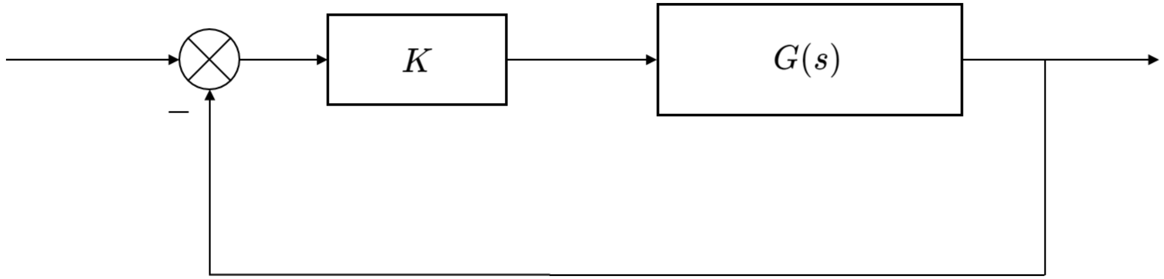
Stability of Closed Loop System

Root Locus (Stability in Time)

- We are interested in the stability of a closed loop system from an open loop system.

- The closed-loop system is

$$H(s) = \frac{KG}{1 + KG}$$



- A pole exists when the characteristic polynomial in the denominator becomes zero.

$$1 + KG(s) = 0 \implies KG(s) = -1 = 1\angle(2k+1)\pi, \quad k = 0, \pm 1, \pm 2, \cdots$$

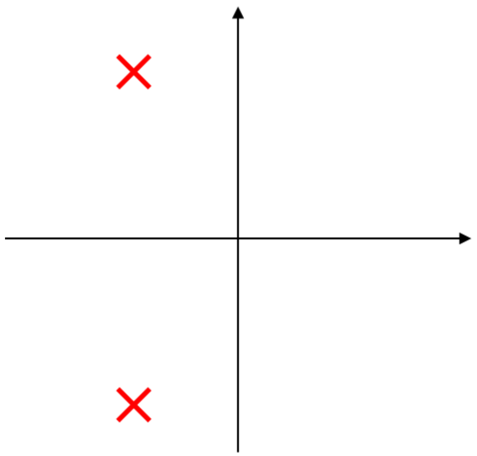
Root Locus (Stability in Time)

$$1 + KG(s) = 0 \implies KG(s) = -1 = 1\angle(2k+1)\pi, \quad k = 0, \pm 1, \pm 2, \cdots$$

- A value of s^* is a closed loop pole if

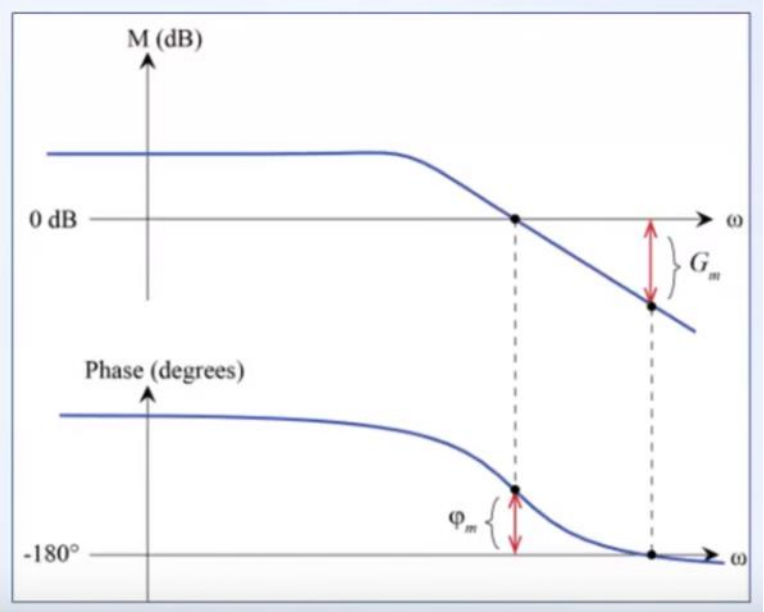
$$\begin{cases} |KG(s^*)| = 1 \quad \implies \quad K = \frac{1}{|G(s^*)|} \\ \angle KG(s^*) = (2k+1)\pi \end{cases}$$

- Closed-loop poles in the LHP indicate stability
  – The closeness of the poles to the RHP indicate how near to instability the system is

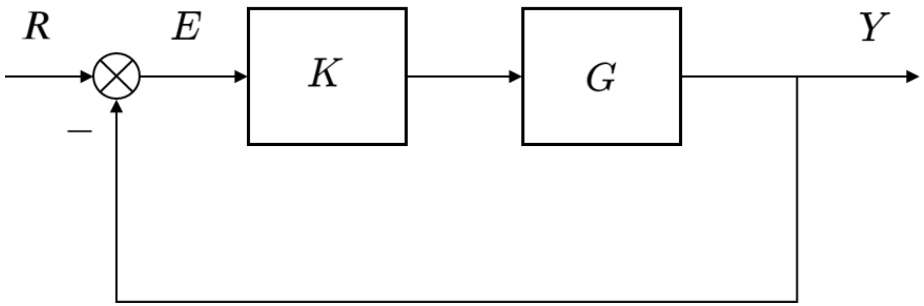


Relative Stability (Stability in Frequency)

- Suppose the Bode plot of the open-loop transfer function is given.

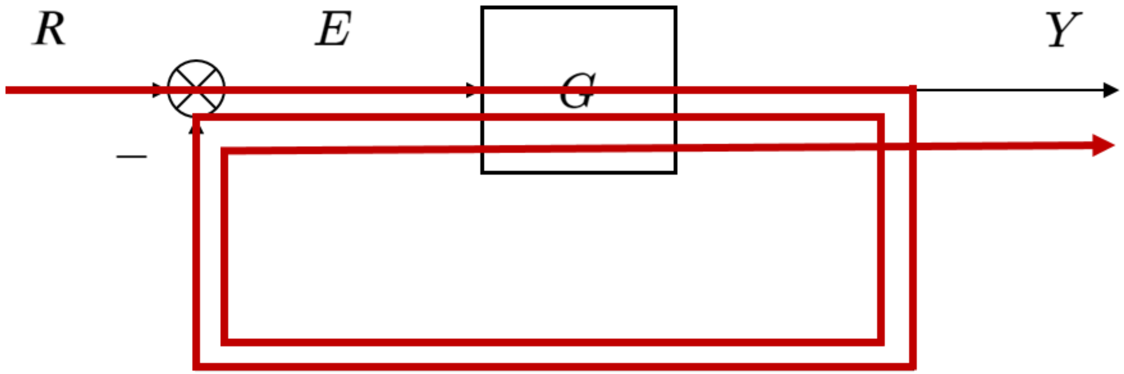


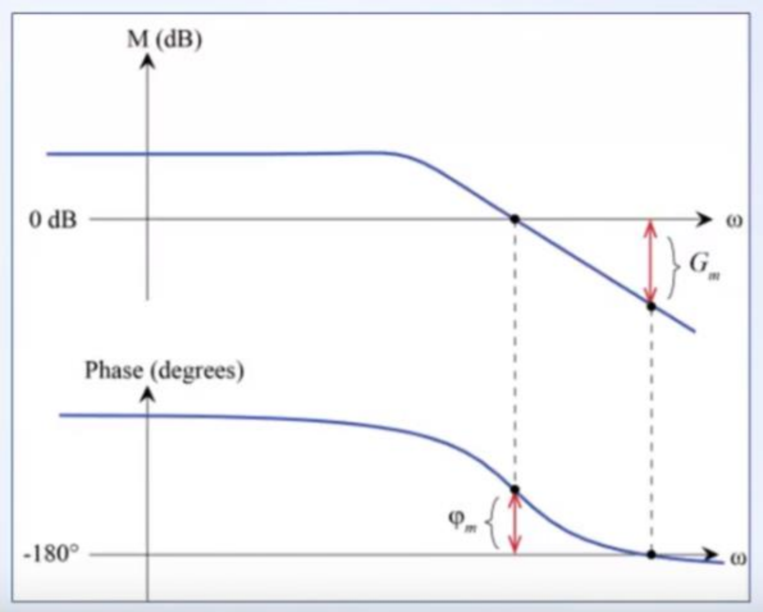
- Question:
  – tell the stability of a closed-loop system from the open-loop frequency response



Relative Stability (Stability in Frequency)

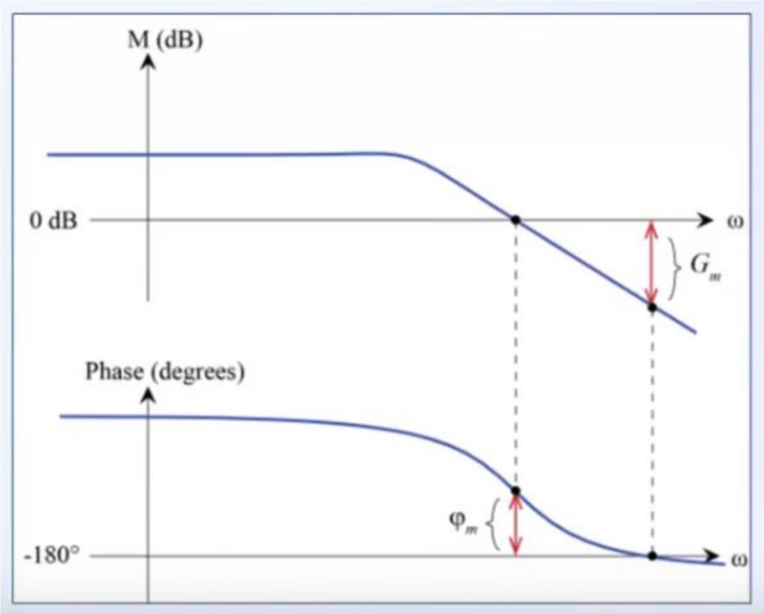
- At 180° of phase lag of the loop, the reference and feedback signal are added.
  - If the magnitude of the loop is greater than 1 the error grows exponentially (unstable)





Relative Stability

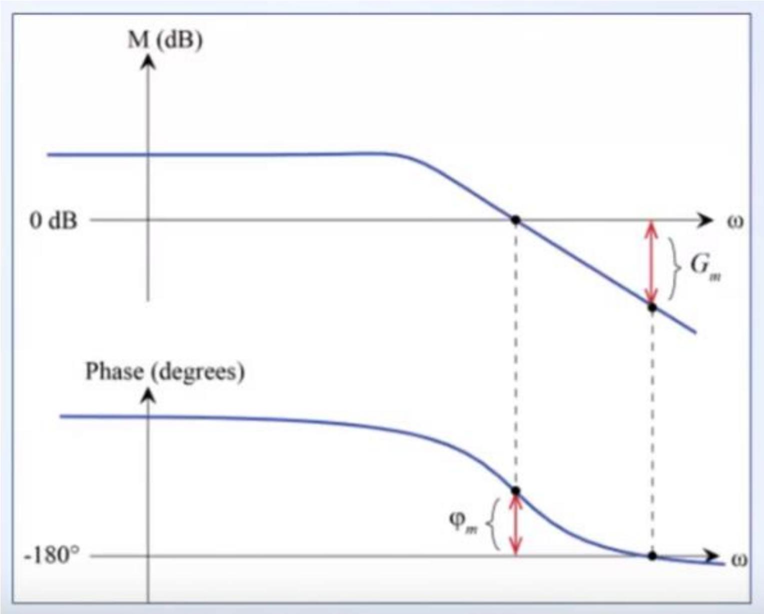
- Relative stability is indicated by how close the open-loop frequency response is to the point of 180^o of phase lag and a magnitude of 1
- More specifically,
  - Gain margin is the distance from a magnitude of 1 (0 dB) at the frequency where $\phi = 180^o$ (phase crossover frequency)
  - Phase margin is the distance from a phase of -180^o at the frequency where M = 0 dB (gain crossover frequency)



Relative Stability

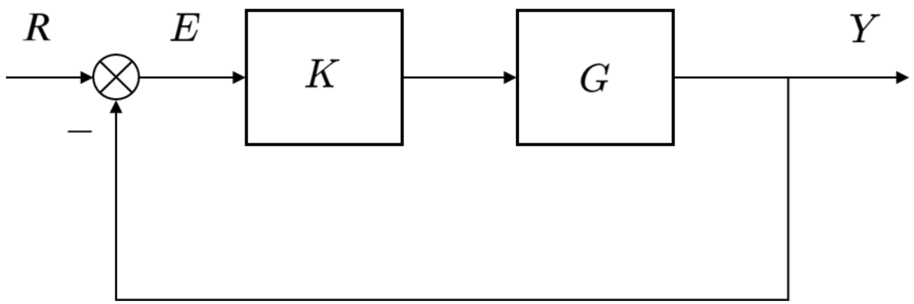
- In order to be stable, both gain and phase margin must be positive

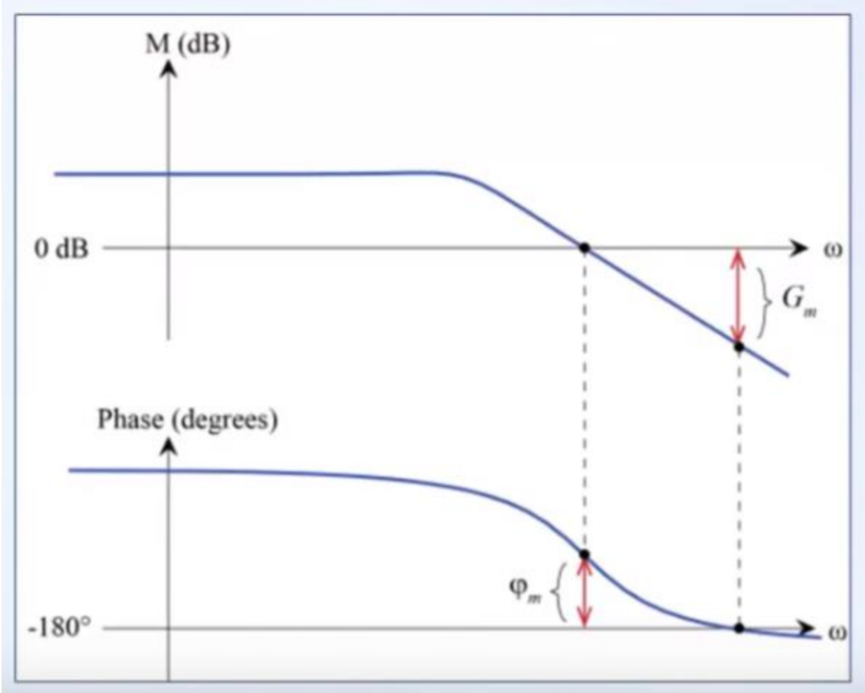
- Gain and phase margins tell how stable the system would be in closed-loop
  - These quantities can be read from the open-loop data



Relative Stability (Stability in Frequency)

- What if K proportional controller is implemented?





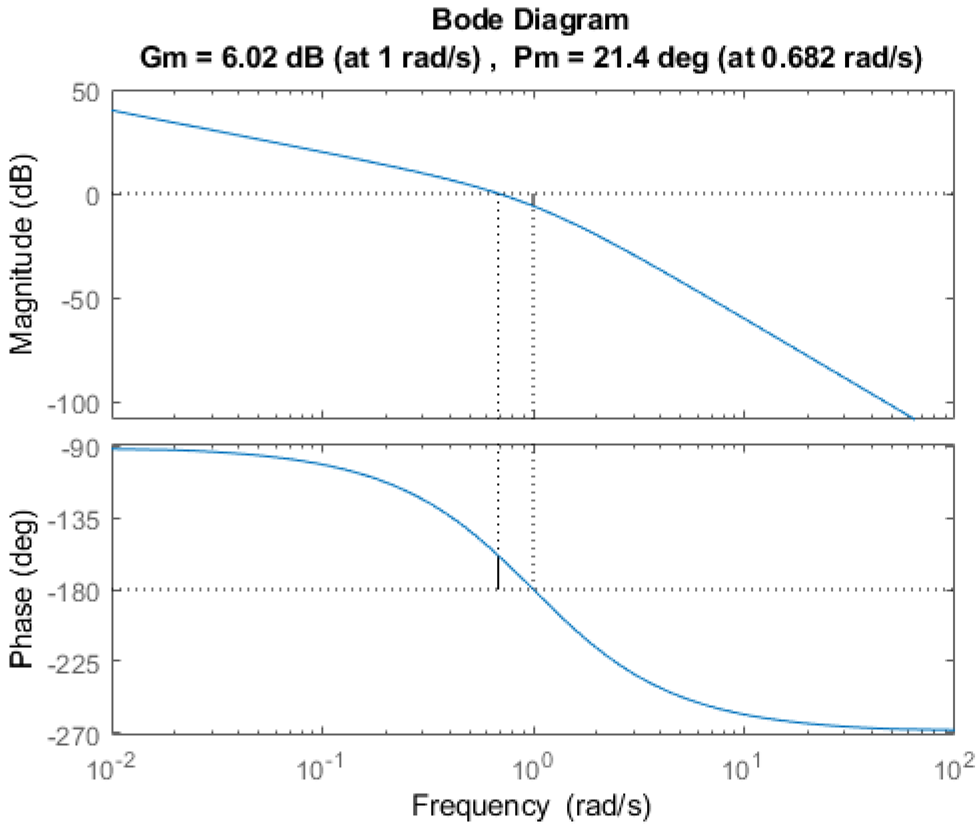
- More intuitively,
  - Gain margin indicates how much you can increase the loop gain K before the system goes unstable
  - Phase margin indicates the amount of phase lag (time delay) you can add before the system goes unstable

Relative Stability in MATLAB

$$G(s) = \frac{1}{s^3 + 2s^2 + s} = \frac{1}{s(s+1)^2}$$

```
margin(G)
[Gm,Pm,Wcg,Wcp] = margin(G)
```

```
Gm =

     2


Pm =

   21.3877
```



Relative Stability in MATLAB

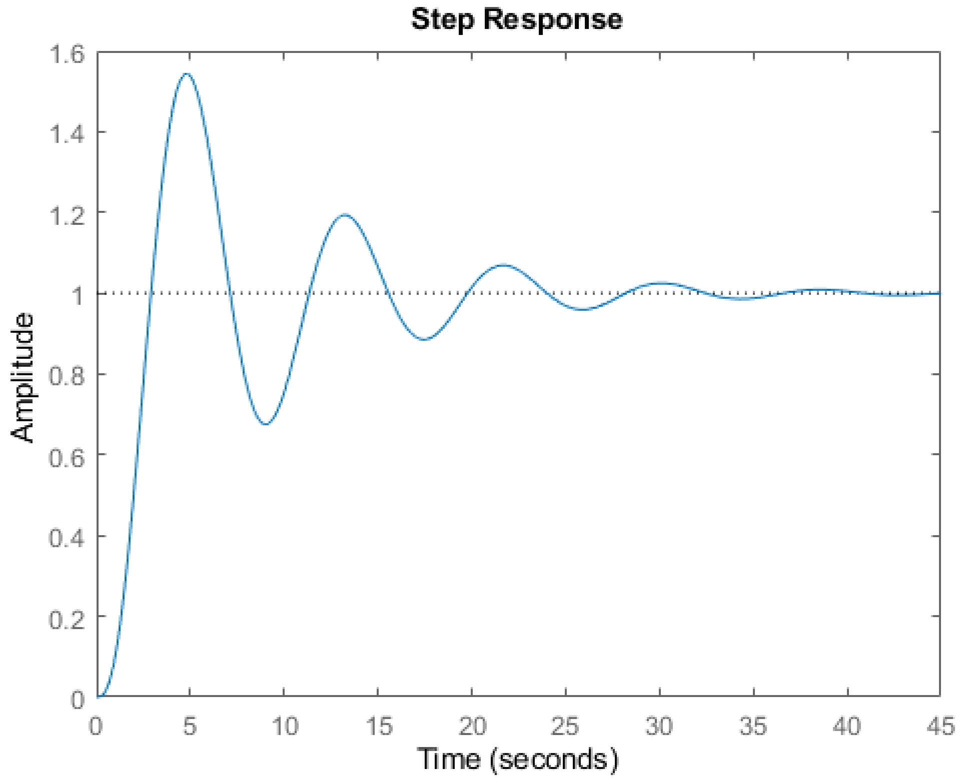
$$G(s) = \frac{1}{s^3 + 2s^2 + s} = \frac{1}{s(s+1)^2}$$

```
margin(G)
[Gm,Pm,Wcg,Wcp] = margin(G)
```

```
Gm =

     2


Pm =

   21.3877
```

- Is stable the closed-loop system with a unity negative feedback?

```
step(feedback(G,1,-1))
```

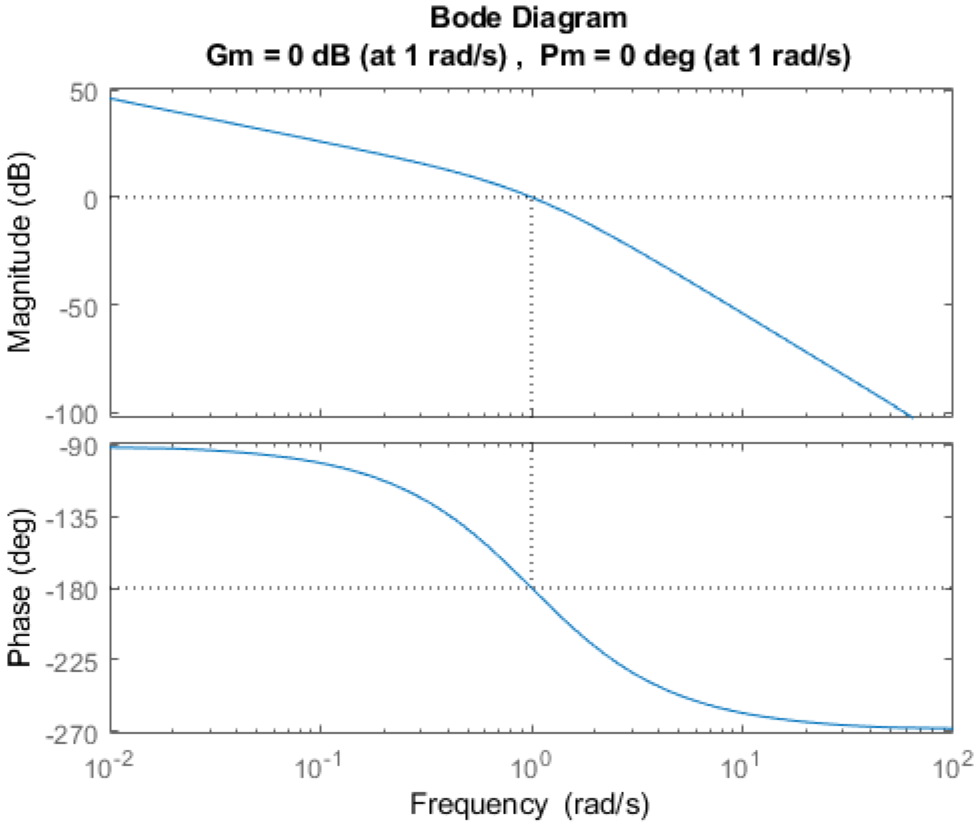


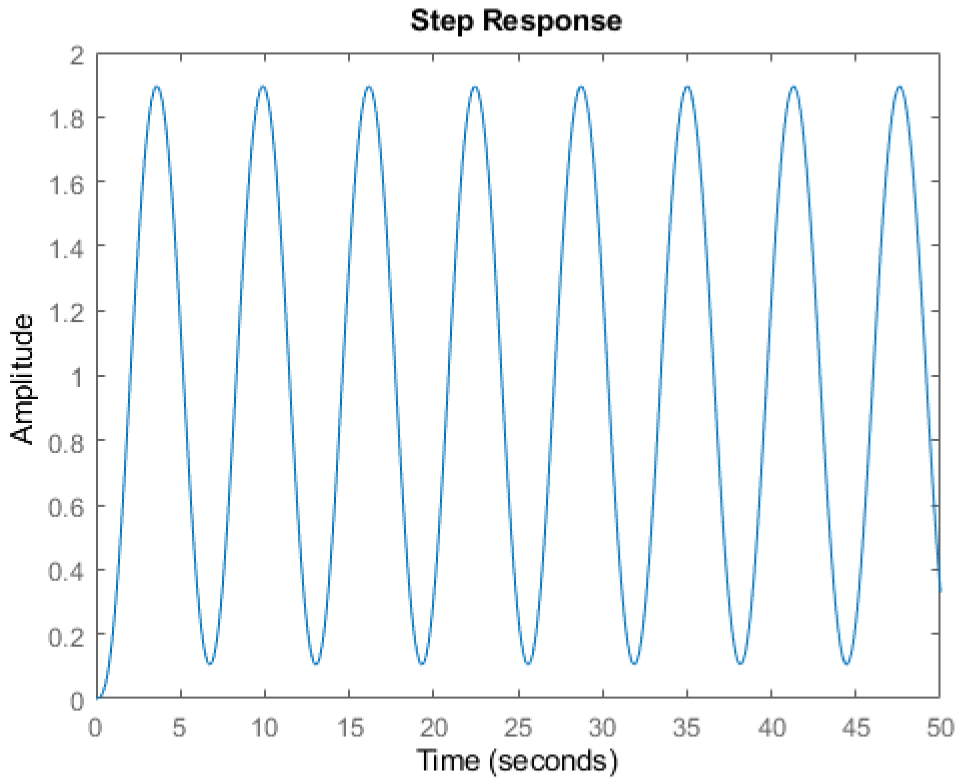
Gain Margin

```
GmG = Gm*G;
margin(GmG)
```

$$CG(s) = 2 \times \frac{1}{s^3 + 2s^2 + s}$$

```
step(feedback(GmG,1,-1))
xlim([0,50])
```



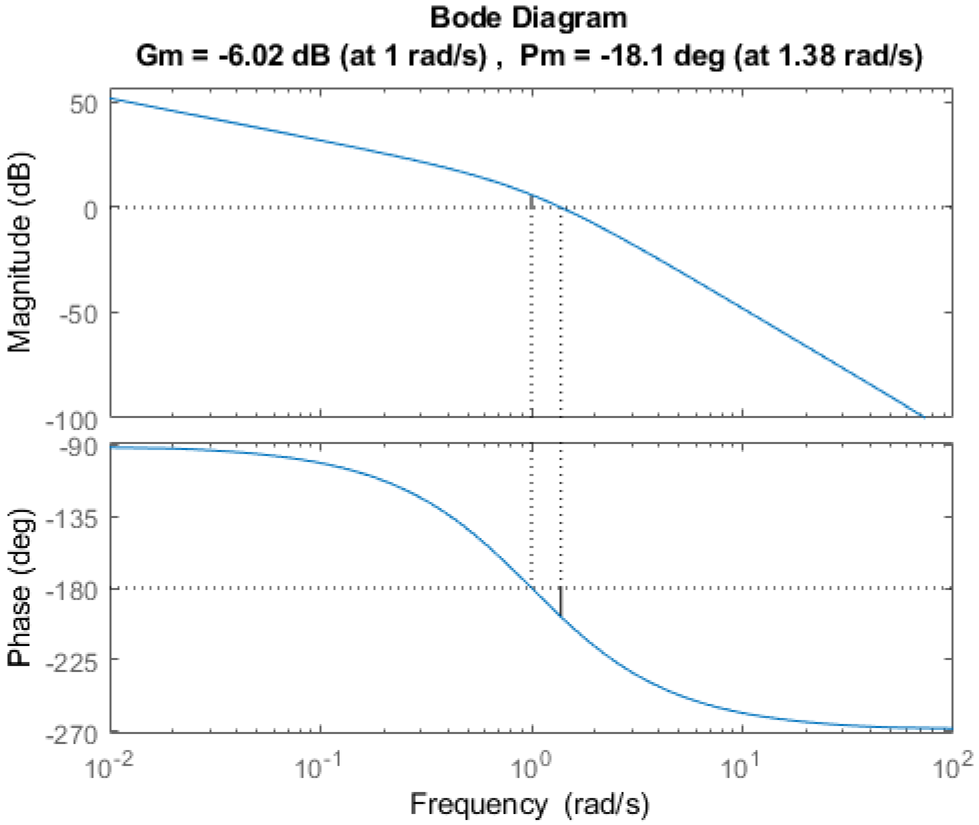


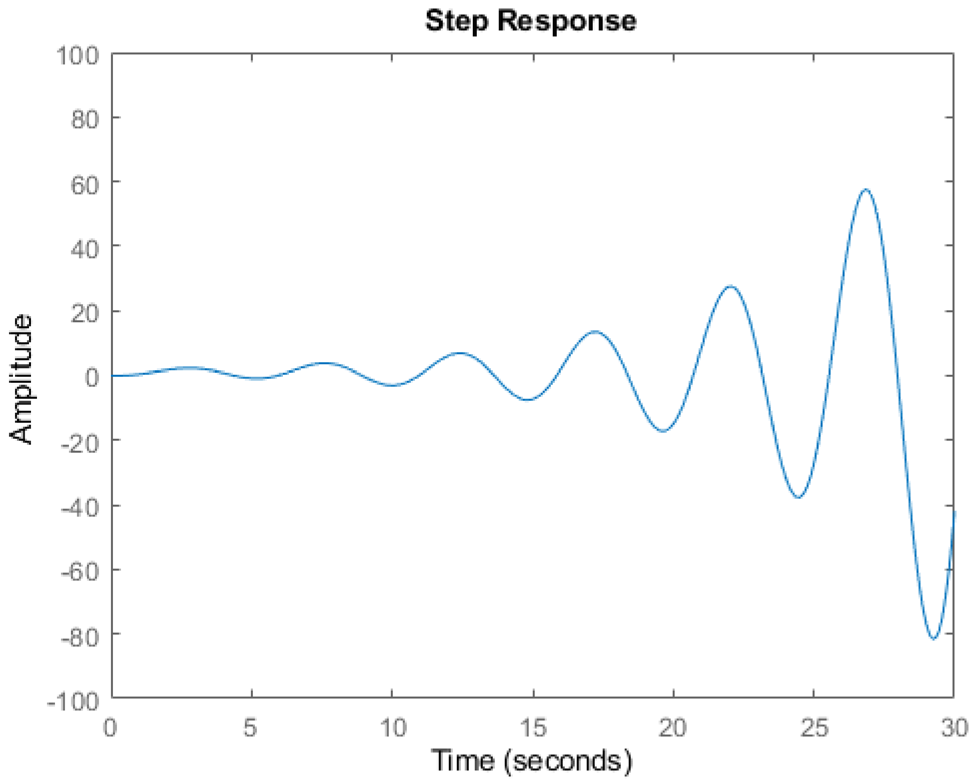
Gain Margin

```
GmG = Gm*G;
margin(GmG)
```

$$CG(s) = 4 \times \frac{1}{s^3 + 2s^2 + s}$$

```
step(feedback(GmG,1,-1))
xlim([0,50])
```



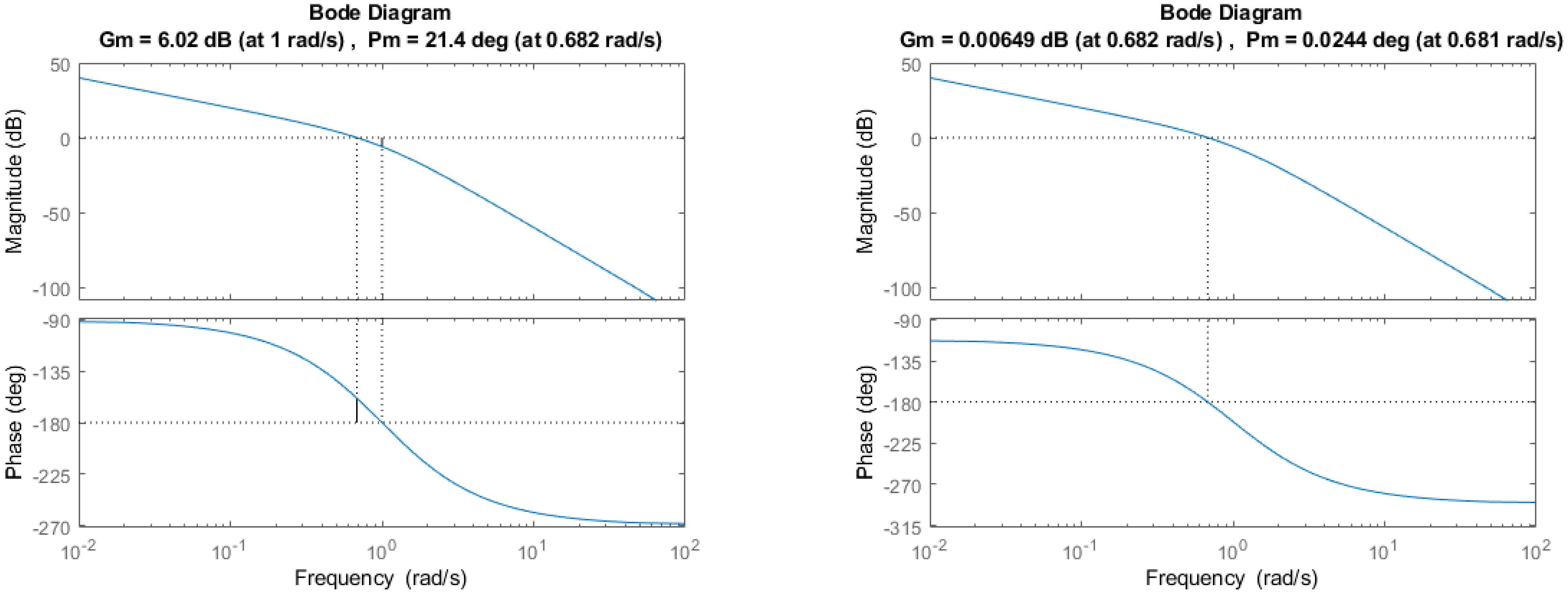


Phase Margin

- Add more delay

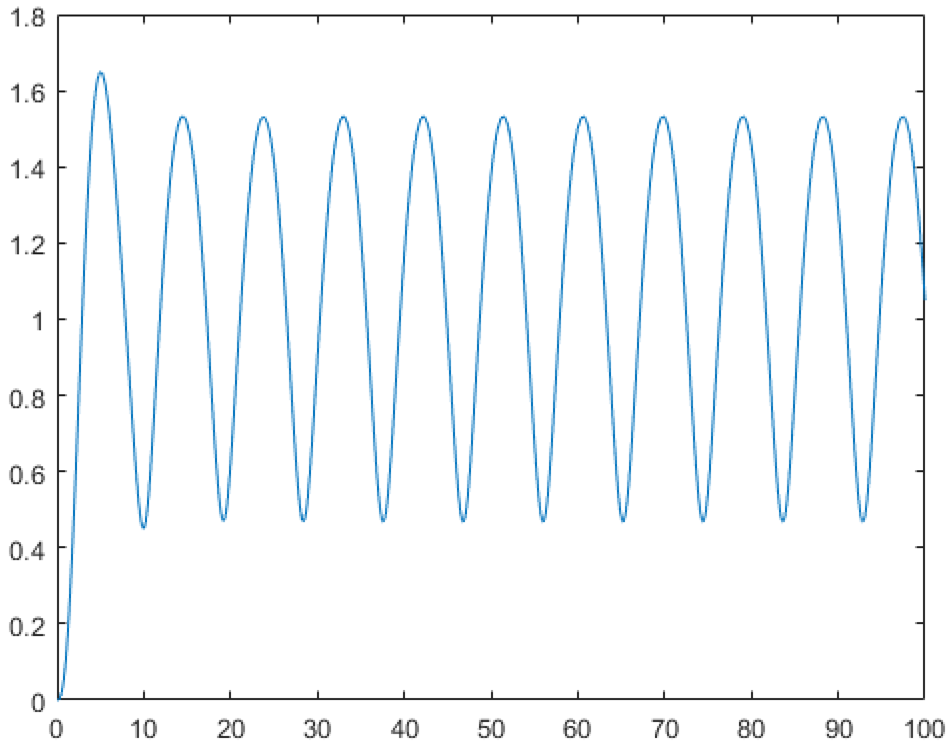
$$CG(s) = e^{-j\Phi} \times \frac{1}{s^3 + 2s^2 + s}$$

```
PmG = exp(-j*Pm/180*pi)*G
margin(PmG)
```



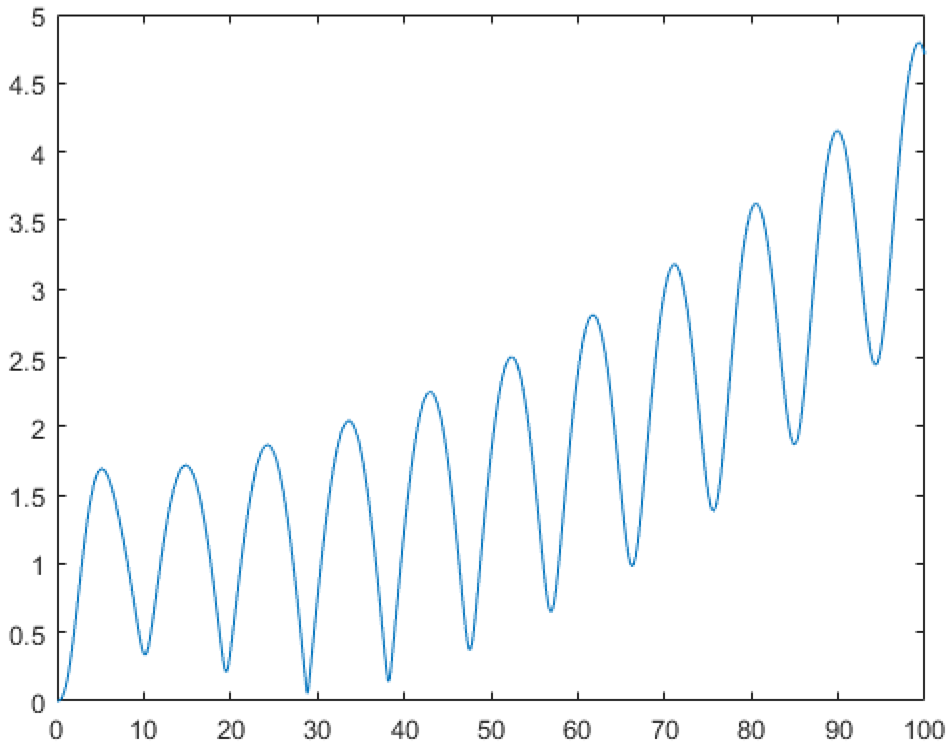
Phase Margin

```
FG = feedback(PmG,1,-1);

t = linspace(0,100,1000);
u = ones(size(t));

[y, tout] = lsim(FG,u,t,0);
plot(tout, abs(y))
```



Phase Margin

```
PmG2 = exp(-j*25/180*pi)*G
FG = feedback(PmG2,1,-1);

t = linspace(0,100,1000);
u = ones(size(t));

[y, tout] = lsim(FG,u,t,0);
plot(tout, abs(y))
```

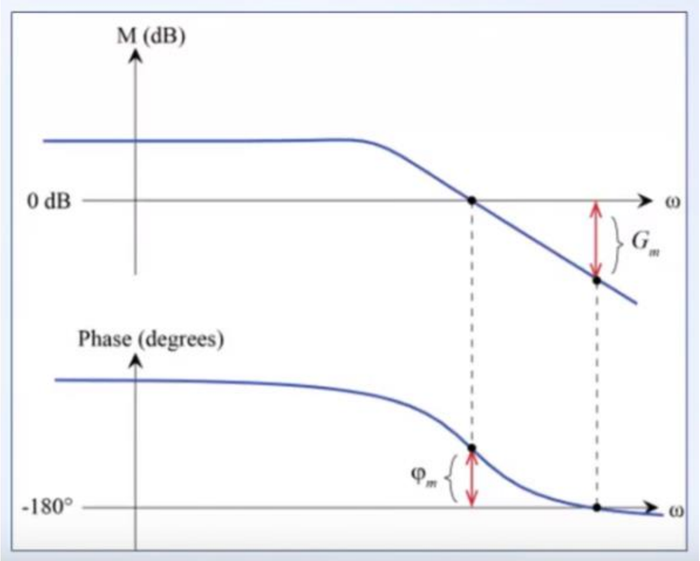


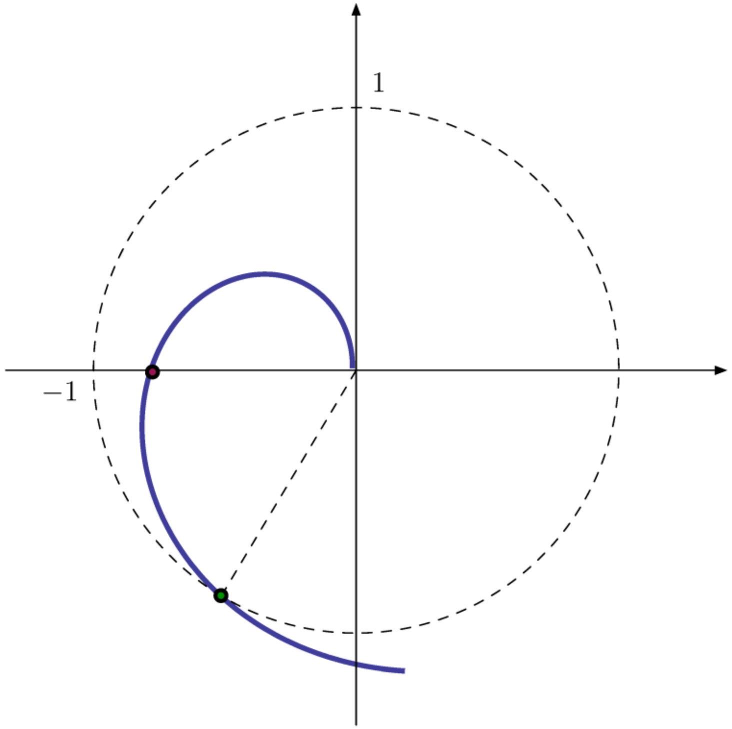
Stability in Nyquist Plot

Stability in Nyquist Plot

- The gain margin, $K_m = \frac{1}{|G(j\omega)|}$ when $\angle G(j\omega) = 180^o$
  - K_m is the maximum stable gain in closed loop
  - It is easy to find the maximum stable gain from the Nyquist plot

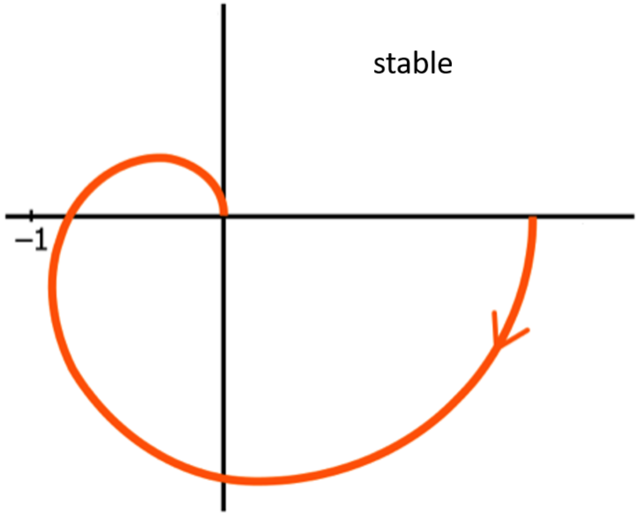
- The phase margin, Φ_m is the uniform phase change required to destabilize the system under unitary feedback

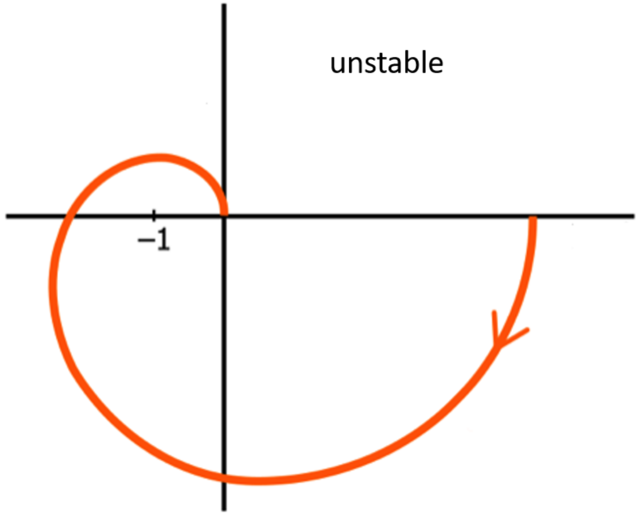




Stability in Nyquist Plot

- What if K proportional controller is implemented?
  - Gain margin indicates how much you can increase the loop gain K before the system goes unstable
  - Phase margin indicates the amount of phase lag (time delay) you can add before the system goes unstable



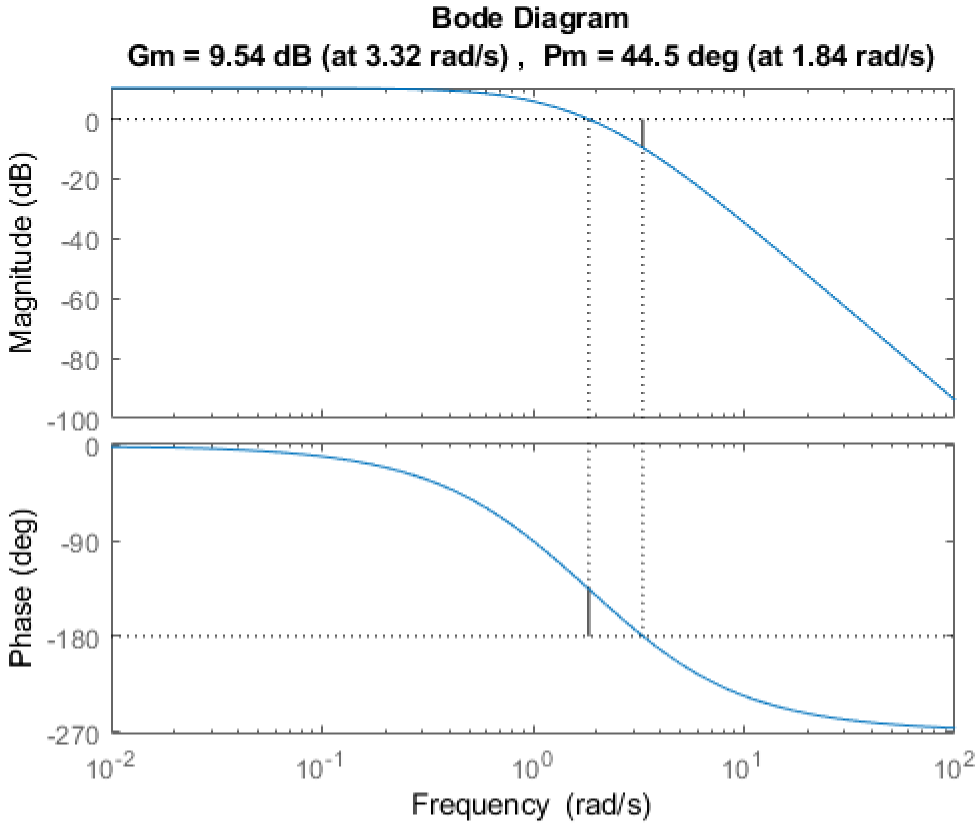


Nyquist Stability in MATLAB

$$G(s) = \frac{20}{(s+1)(s+2)(s+3)}$$

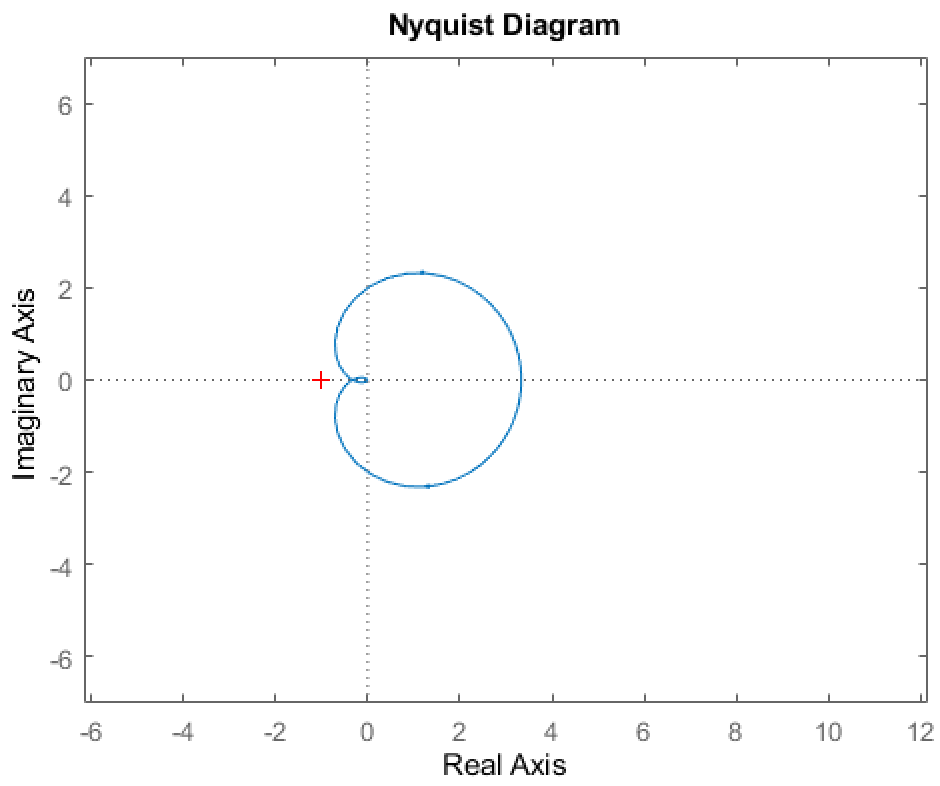
```
margin(G)
[Gm,Pm,Wcg,Wcp] = margin(G)
```

```
Gm =

    3.0000


Pm =

   44.4630
```

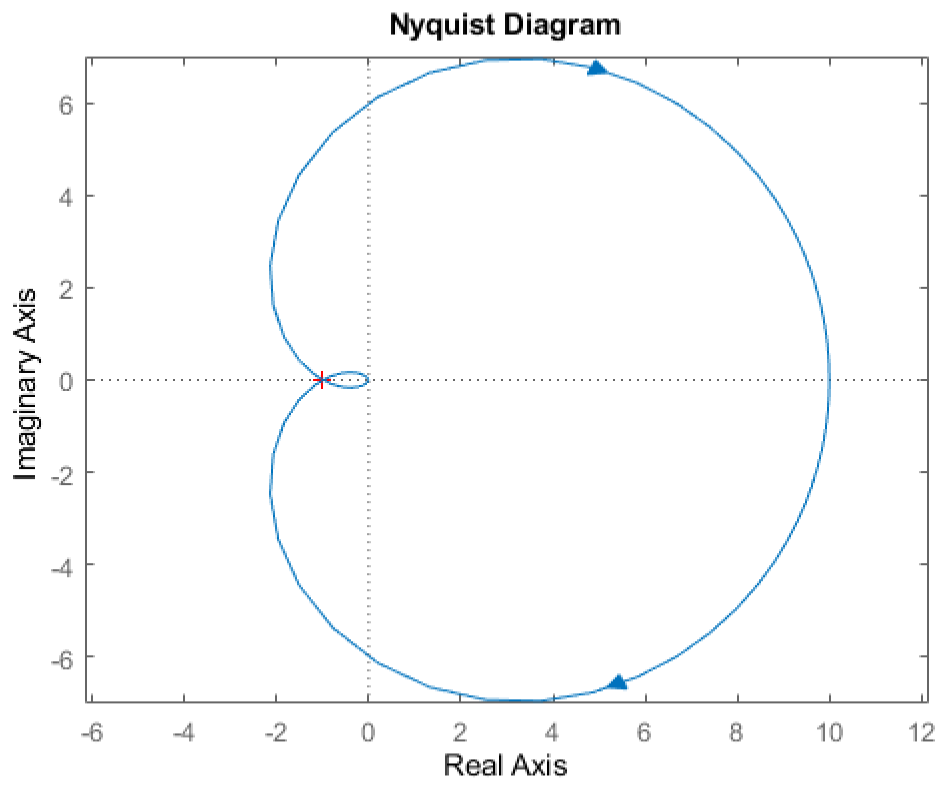


Nyquist Stability in MATLAB

```
nyquist(G)
```

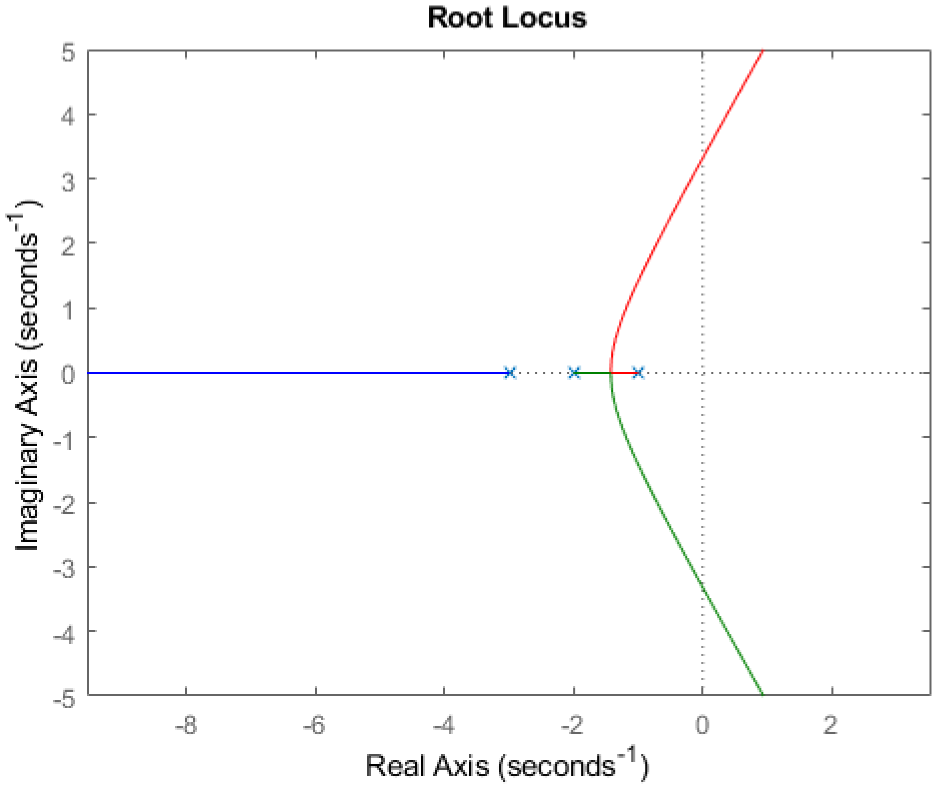



```
nyquist(Gm*G)
```



Nyquist Stability in MATLAB

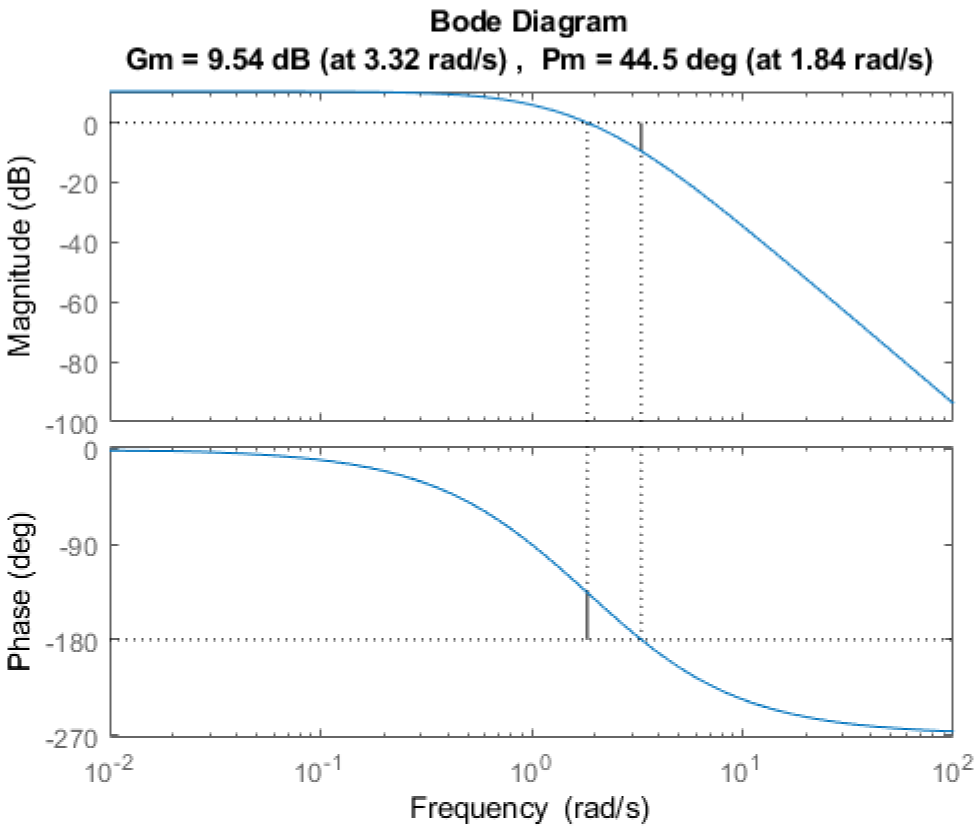
```
% expect instability for large Km

rlocus(G)
```

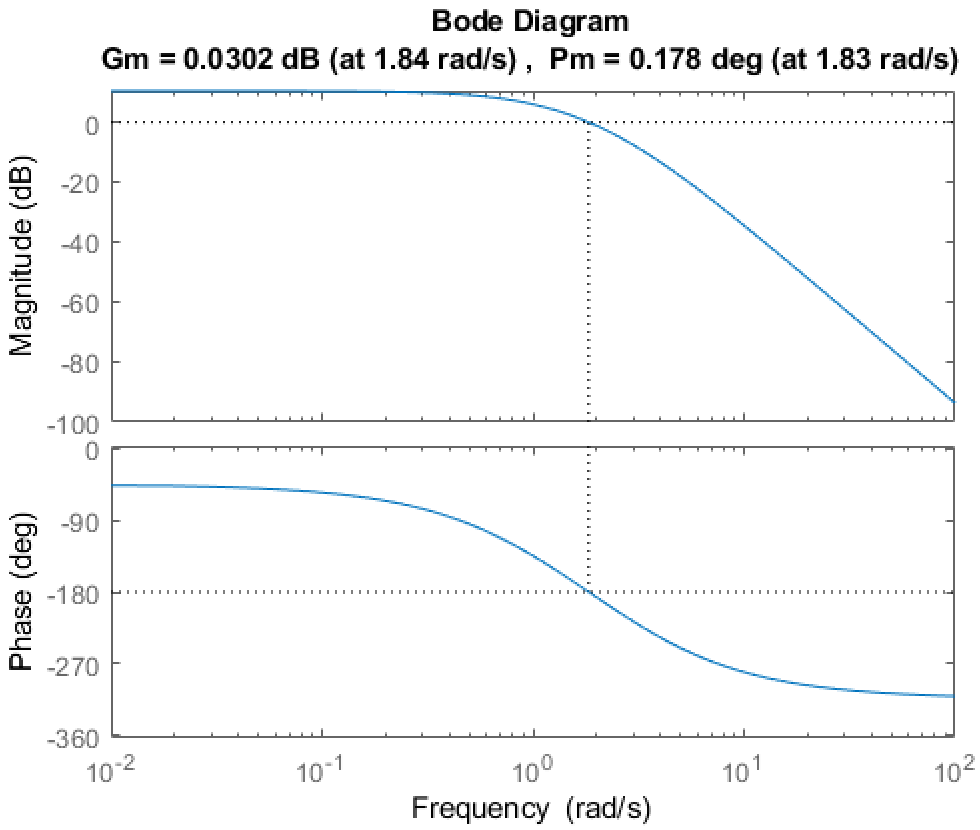


Nyquist Stability in MATLAB

```
margin(G)
[Gm,Pm,Wcg,Wcp] = margin(G)
```

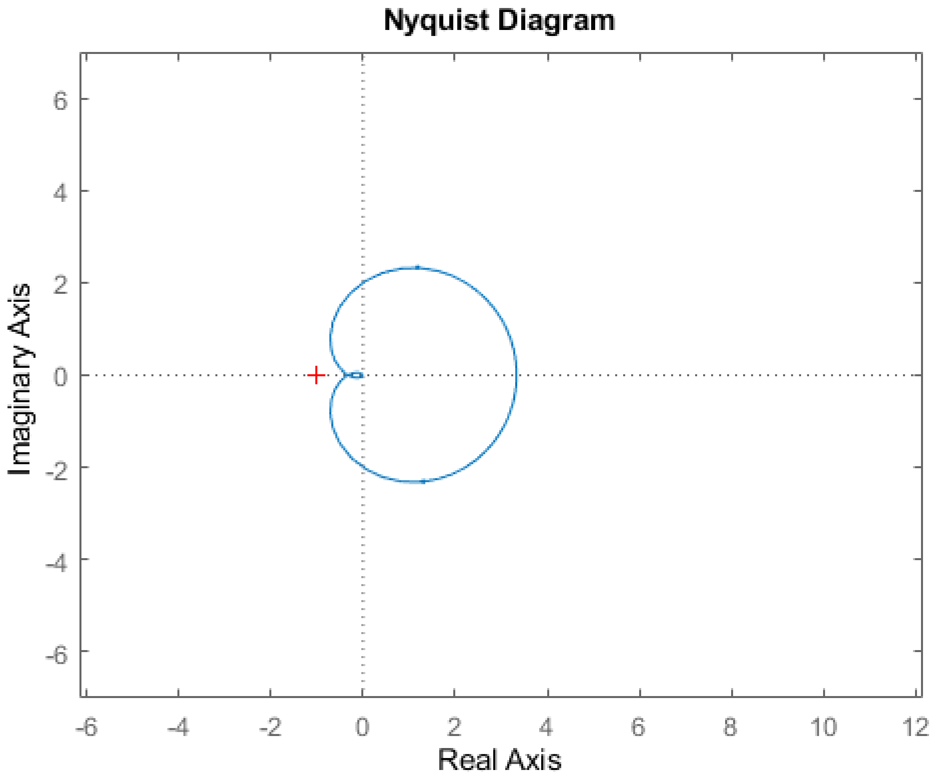
```
PmG = exp(-j*Pm/180*pi)*G

margin(PmG)
```





Nyquist Stability in MATLAB

```
nyquist(G)
```




```
nyquist(PmG)
```