Representation of LTI Systems

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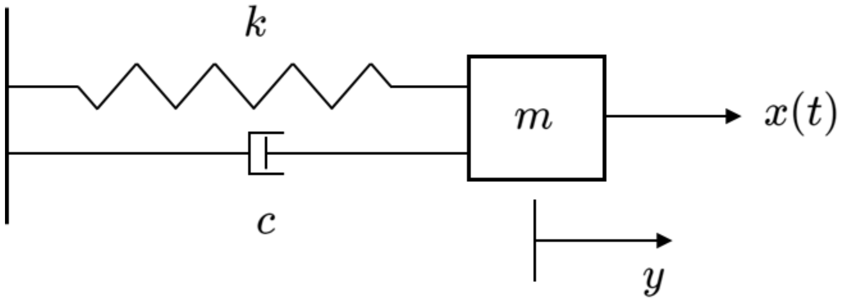
Transfer Function

$$G(s) = \frac{Y(s)}{X(s)}$$



- Equation of motion

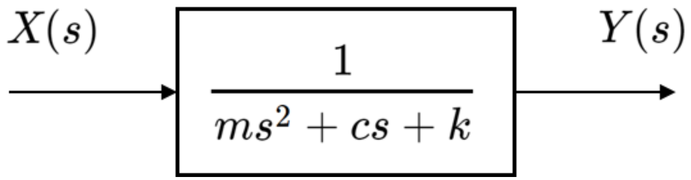
$$m\ddot{y} + c\dot{y} + ky = x(t)$$

- Laplace Transform

$$(ms^2 + cs + k)Y(s) = X(s)$$

$$\Longrightarrow \quad \frac{Y(s)}{X(s)} = \frac{1}{ms^2 + cs + k} = G(s)$$

- Block Diagram



Example

$$G(s) = \frac{s+5}{s^4 + 2s^3 + 3s^2 + 4s + 5}$$

```
num = [1,5];
den = [1,2,3,4,5];

G = tf(num,den)
```

```
G =

            s + 5
  -----------------------------
  s^4 + 2 s^3 + 3 s^2 + 4 s + 5

Continuous-time transfer function.
```

State Space Representation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

```
A = [2.25,-5,-1.25,-0.5;
     2.25,-4.25,-1.25,-0.25;
     0.25,-0.5,-1.25,-1;
     1.25,-1.75,-0.25,-0.75];

B = [4,6;
     2,4;
     2,2;
     0,2];

C = [0,0,0,1;
     0,2,0,2];

D = zeros(2,2);

G = ss(A,B,C,D)
```

```
G =

  A =
           x1      x2      x3      x4
   x1    2.25      -5   -1.25    -0.5
   x2    2.25   -4.25   -1.25   -0.25
   x3    0.25    -0.5   -1.25      -1
   x4    1.25   -1.75   -0.25   -0.75

  B =
        u1  u2
   x1    4   6
   x2    2   4
   x3    2   2
   x4    0   2

  C =
        x1  x2  x3  x4
   y1    0   0   0   1
   y2    0   2   0   2

  D =
        u1  u2
   y1    0   0
   y2    0   0

Continuous-time state-space model.
```

Three Representations of LTI Systems

Three Representations of Linear Systems

1) Time domain
2) Frequency domain
3) State space

Time Domain

$$y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau$$

$$G(s) = C(sI - A)^{-1}B + D$$

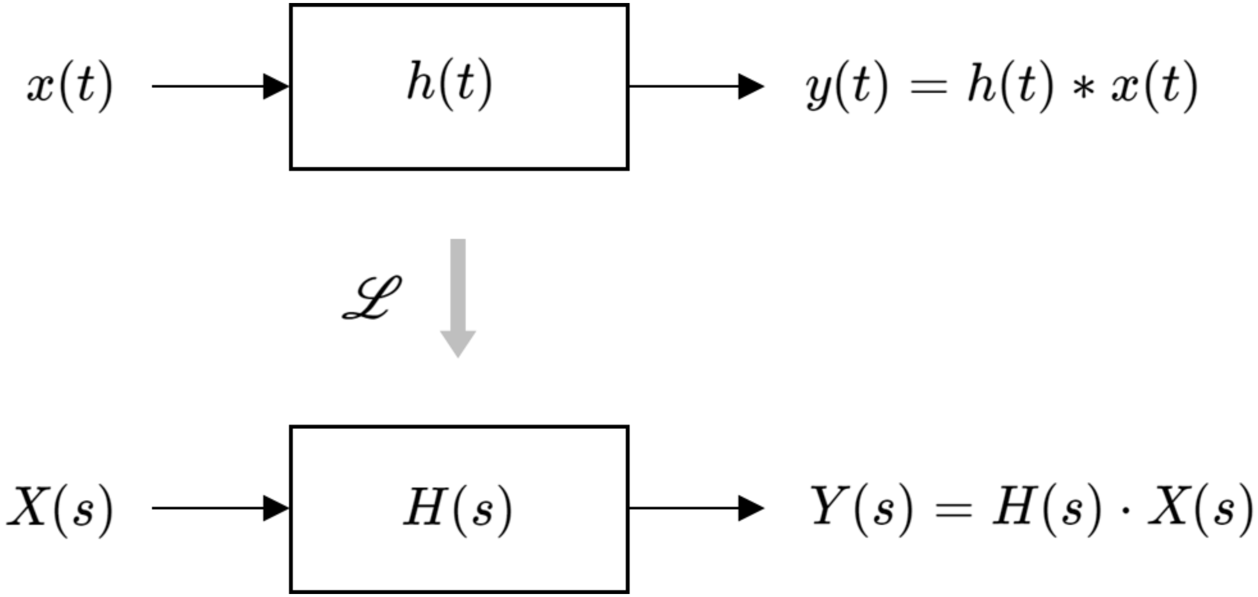
Frequency Domain

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

State Space

Time and Frequency Domains

- In linear system, convolution operation can be converted to product operation through Laplace transform



Converting from State Space to a Transfer Function

- State Space can be represented:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \xrightarrow{\text{Laplace Transform}} \begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

- Solving for $X(s)$ in the first equation Laplace transformed

$$\begin{aligned} (sI - A)X(s) &= BU(s) \\ \therefore X(s) &= (sI - A)^{-1}BU(s) \end{aligned}$$

- Substituting equation $X(s)$ into second equation Laplace transformed yields

$$\begin{aligned} Y(s) &= C(sI - A)^{-1}BU(s) + DU(s) \\ &= \left[C(sI - A)^{-1}B + D\right] U(s) \end{aligned}$$

$$\therefore\ T(s) = G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Converting from State Space to a Transfer Function

$$\begin{aligned} Y(s) &= C(sI - A)^{-1}BU(s) + DU(s) \\ &= \left[C(sI - A)^{-1}B + D\right] U(s) \end{aligned}$$

$$\therefore \; T(s) = G(s) = \frac{Y(s)}{U(s)} = \underline{C(sI - A)^{-1}B} + D$$

- We call the matrix $[C(sI - A)^{-1}B + D]$ the transfer function matrix

- Note
  - The output in time

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Laplace Transform of Matrix Exponential

- Series expansion of $(I - C)^{-1}$

$$(I - C)^{-1} = I + C + C^2 + C^3 + \cdots \quad \text{(if series converges)}$$

- Series expansion of $(sI - A)^{-1}$

$$(sI - A)^{-1} = \left(\frac{1}{s}\right)\left(I - \frac{A}{s}\right)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots$$

- Inverse Laplace transform of $(sI - A)^{-1}$

$$\mathcal{L}^{-1}\left(\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots\right) = I + At + \frac{(At)^2}{2!} + \cdots = e^{At}$$

$$\mathcal{L}\left(e^{At}\right) = (sI - A)^{-1}$$

Laplace Transform of Matrix Exponential

$$\therefore\ T(s) = G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$\mathcal{L}\left(e^{At}\right) = (sI - A)^{-1}$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Transformation of State-Space

- State space representations are not unique because we have a lot of freedom in choosing the state vector.
  - Selection of the state is quite arbitrary, and not that important
- In fact, given one model, we can transform it to another model that is equivalent in terms of its input-output properties
- To see this, define model of $G_1(s)$ as

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Now introduce the new state vector z related to the first state x through the transformation $x = Tz$
- T is an invertible (similarity) transform matrix

$$\begin{aligned}
\dot{z} &= T^{-1}\dot{x} = T^{-1}(Ax + Bu) \\
&= T^{-1}(ATz + Bu) \\
&= (T^{-1}AT)z + T^{-1}Bu = \bar{A}z + \bar{B}u \\
y &= Cx + Du = CTz + Du = \bar{C}z + \bar{D}u
\end{aligned}$$

The new model of $G_1(s)$

$$\dot{z} = \bar{A}z + \bar{B}u$$
$$y = \bar{C}z + \bar{D}u$$

Same Transfer Function ?

- Consider the two transfer functions

$$G_1(s) = C(sI - A)^{-1}B + D$$
$$G_2(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$$

- Does $G_1(s) = G_2(s)$?

$$\begin{aligned} G_1(s) &= C(sI - A)^{-1}B + D \\ &= C\left(TT^{-1}\right)(sI - A)^{-1}\left(TT^{-1}\right)B + D \\ &= (CT)\left[T^{-1}(sI - A)^{-1}T\right]\left(T^{-1}B\right) + \bar{D} \\ &= \left(\bar{C}\right)\left[T^{-1}(sI - A)T\right]^{-1}\left(\bar{B}\right) + \bar{D} \\ &= \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = G_2(s) \end{aligned}$$

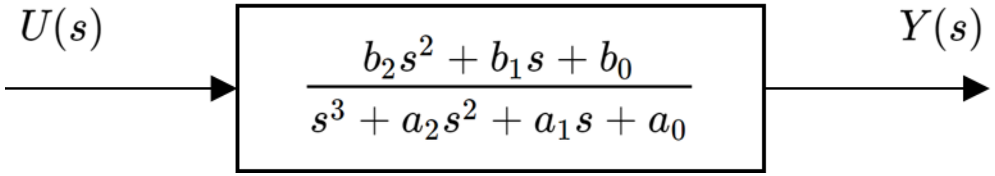
- So the transfer function is not changed by putting the state-space model through a similarity transformation

Decoupled LTI System

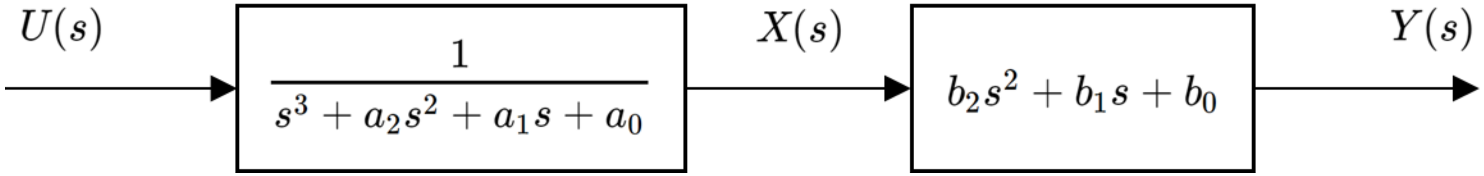
- If $T = S$, transformation to diagonal matrix

Converting a Transfer Function to State Space

- How to convert the transfer function to state space?



- We can redraw block diagram like the below



$$U(s) = \left(s^3 + a_2 s^2 + a_1 s + a_0\right) X(s)$$

$$Y(s) = \left(b_2 s^2 + b_1 s + b_0\right) X(s)$$

Converting a Transfer Function to State Space

- Reverse Laplace transform

$$u(t) = \dddot{x} + a_2\ddot{x} + a_1\dot{x} + a_0 x$$

$$y(t) = b_2\ddot{x} + b_1\dot{x} + b_0 x$$

- Choose state variable:
  - A convenient way to choose state variables is to choose the output, $y(t)$, and its $(n-1)$ derivatives as the state variables

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

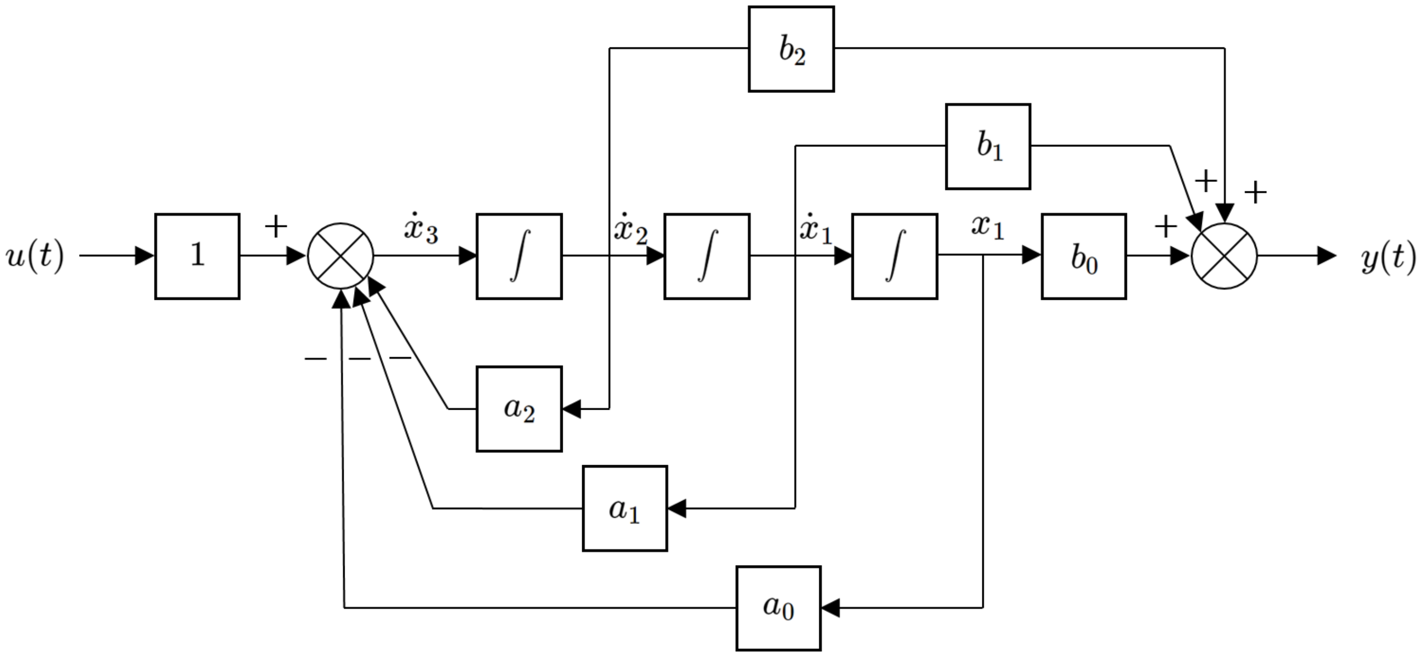
$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Converting a Transfer Function to State Space

- Draw this into a block diagram

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



MATLAB Implementation

Step Response

- Start with a step response example

$$\dot{y} + 5y = u(t), \qquad y(0) = 0$$

- The solution is given:

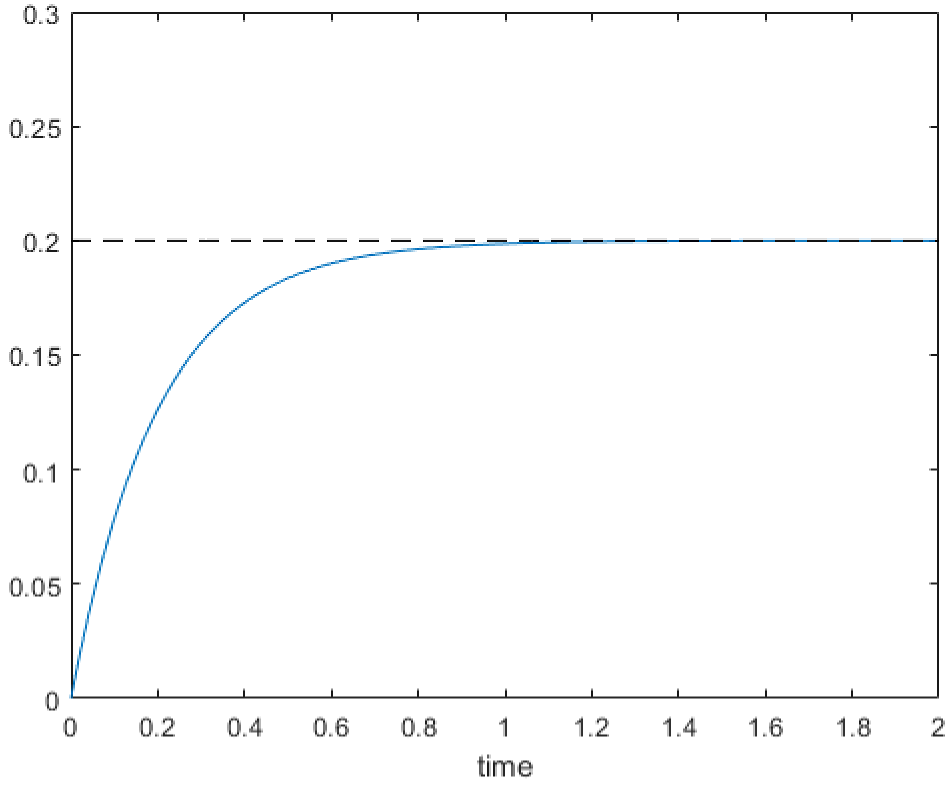
$$y(t) = \frac{1}{5}\left(1 - e^{-5t}\right)$$

```
num = 1;
den = [1 5];

G = tf(num,den);

[y,tout] = step(G,2);

plot(tout,y,tout,0.2*ones(size(tout)),'k--')
ylim([0,0.3])
xlabel('time')
```



Step Response

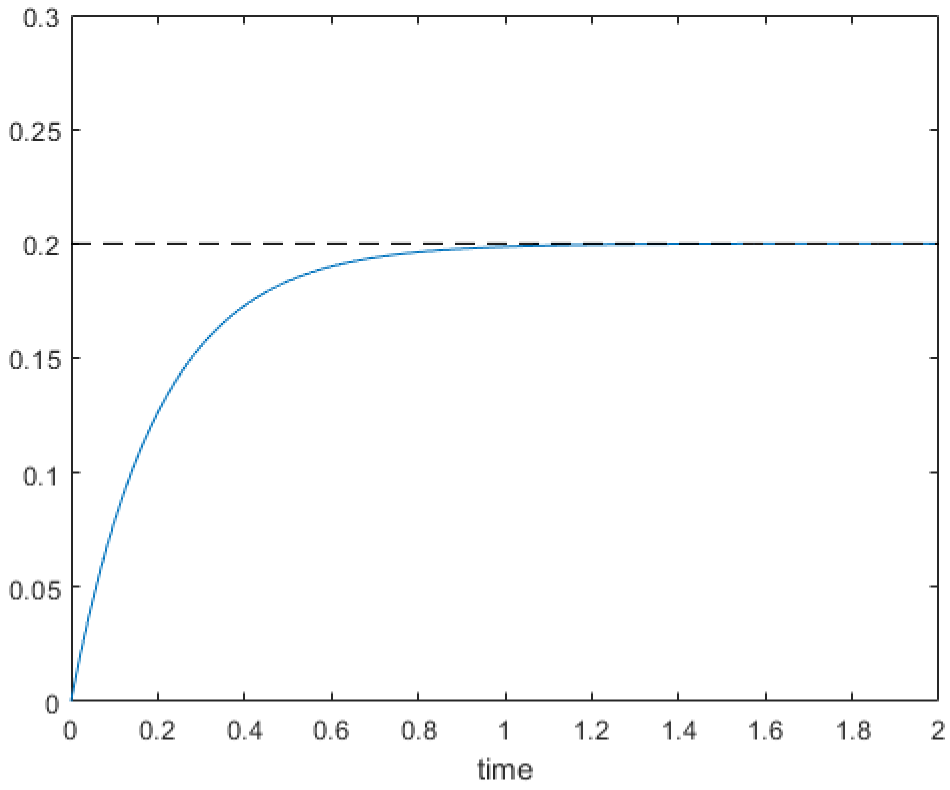
$$\dot{x} = -5x + u$$
$$y = x$$

```
A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

t = linspace(0,2,100);
u = ones(size(t));
x0 = 0;

[y,tout] = lsim(G,u,t,x0);

plot(tout,y,tout,0.2*ones(size(tout)),'k--')
ylim([0,0.3])
xlabel('time')
```



Step Response

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \omega_n^2 u(t)$$

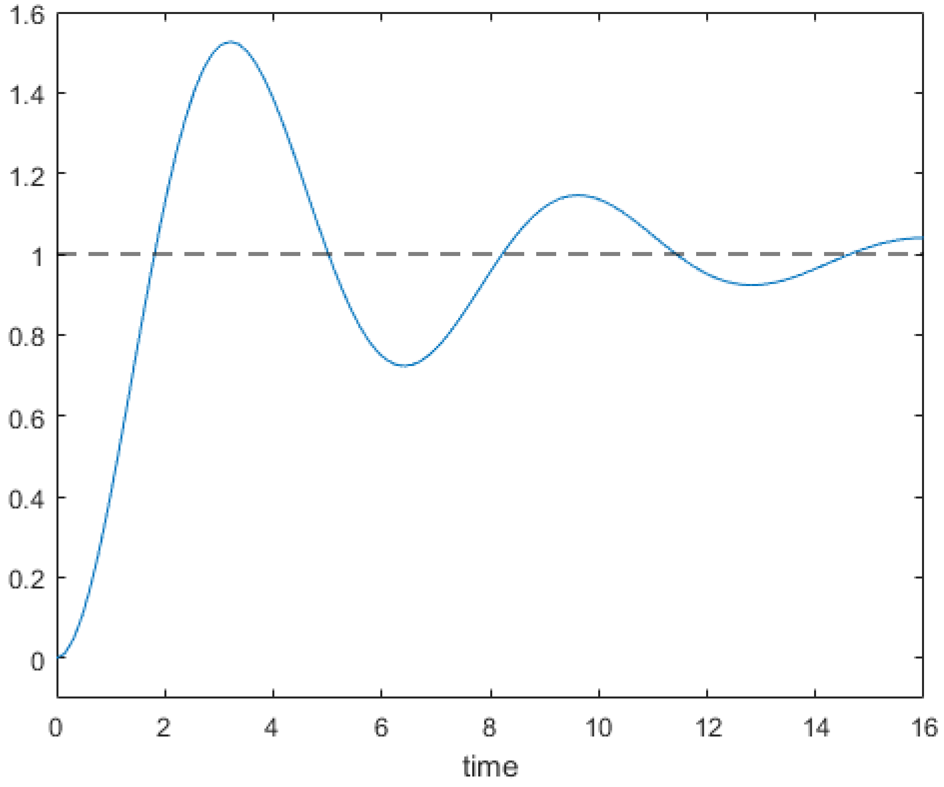
$$\Longrightarrow \quad G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

```
z = 0.2;
wn = 1;

G = tf(wn^2,[1,2*z*wn,wn^2]);

[y,tout] = step(G,16);

plot(tout,y,tout,ones(size(tout)),'k--')
ylim([-0.1 1.6])
xlabel('time')
```



Step Response

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

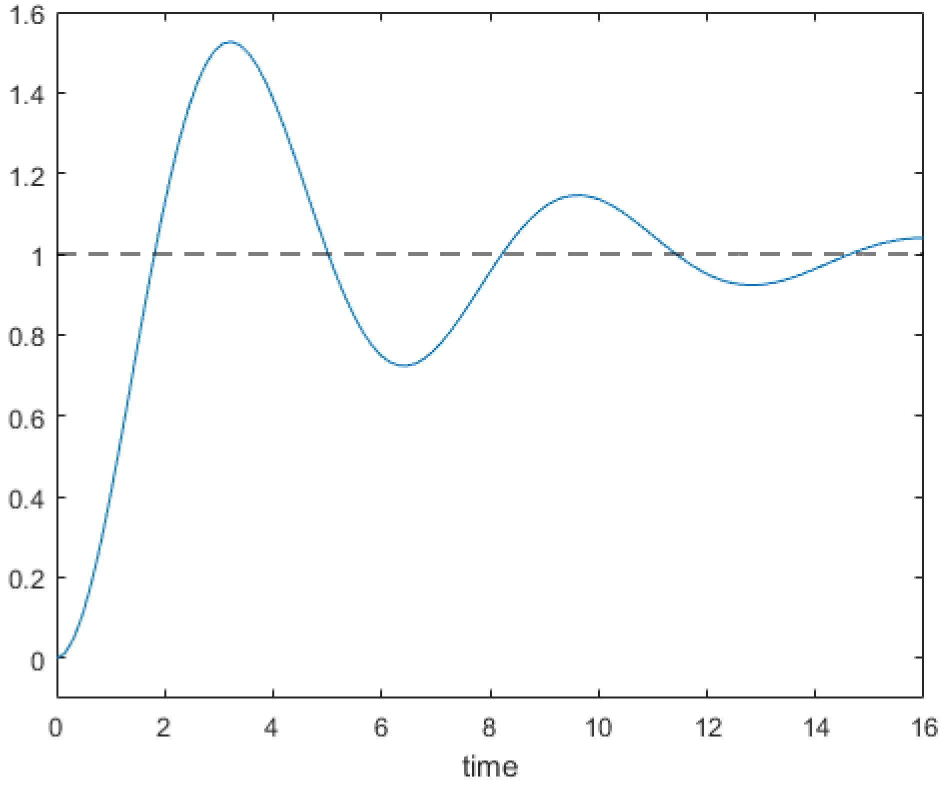
```
zeta = 0.2;
wn = 1;

A = [0, 1;-wn^2, -2*zeta*wn];
B = [0; wn^2];
C = [1, 0];
D = 0;
G = ss(A,B,C,D);

t = linspace(0,16,100);
u = ones(size(t));
x0 = [0; 0];

[y,tout] = lsim(G,u,t,x0);

plot(tout,y,tout,ones(size(tout)),'k--')
ylim([-0.1 1.6])
xlabel('time')
```



Impulse Response

- Now think about the impulse response

$$\dot{y} + 5y = \delta(t), \qquad y(0) = 0$$

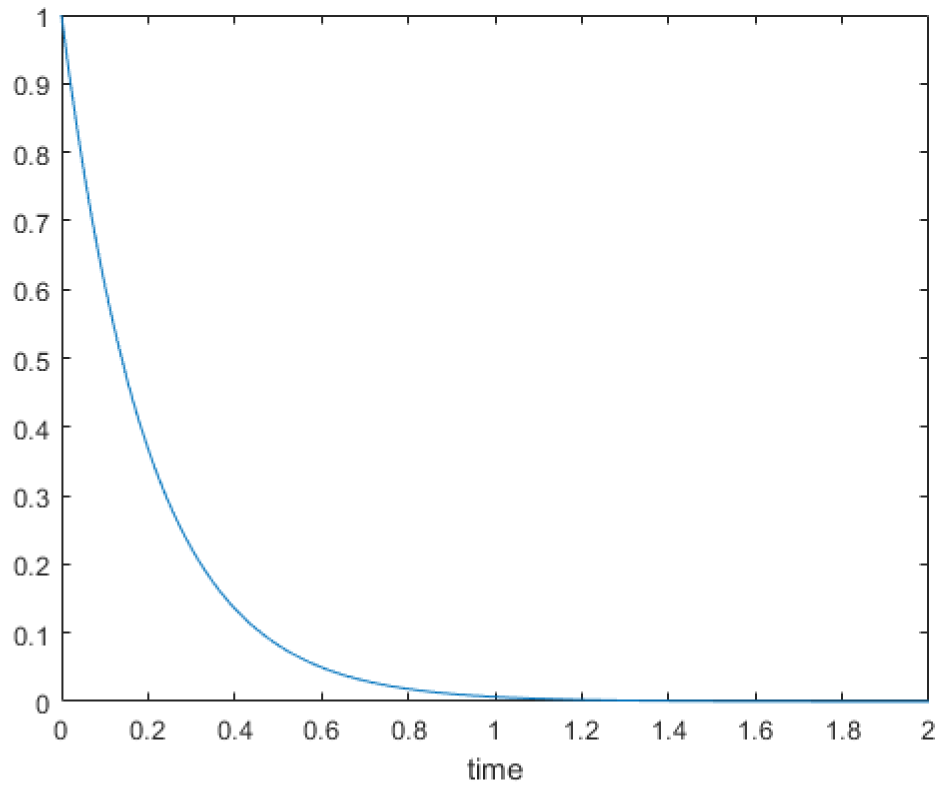
- The solution is given:

$$y(t) = h(t) = e^{-5t}, \quad t \geq 0$$

```
num = 1;
den = [1 5];
G = tf(num,den);

[h,tout] = impulse(G,2);

plot(tout,h), ylim([0,1])
xlabel('time')
```



Impulse Response

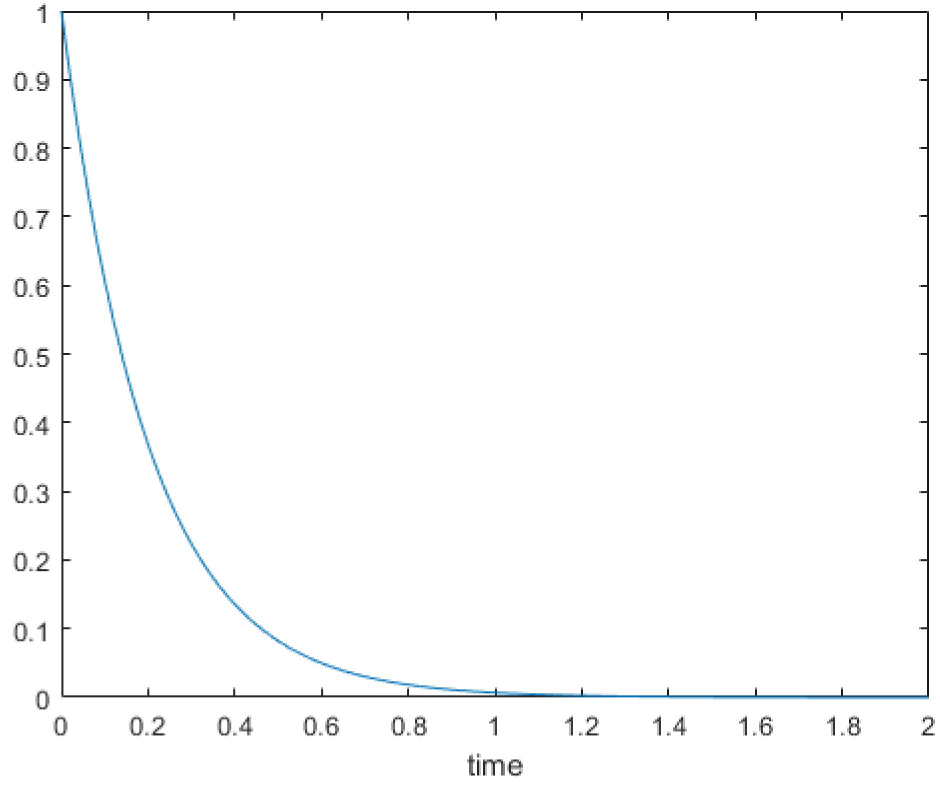
$$\dot{x} = -5x + u$$
$$y = x$$

```
A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

t = linspace(0,2,100);
u = zeros(size(t));
x0 = 1;

[h,tout] = lsim(G,u,t,x0);

plot(tout,h), ylim([0,1])
xlabel('time')
```



Impulse Response

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2\delta(t)$$

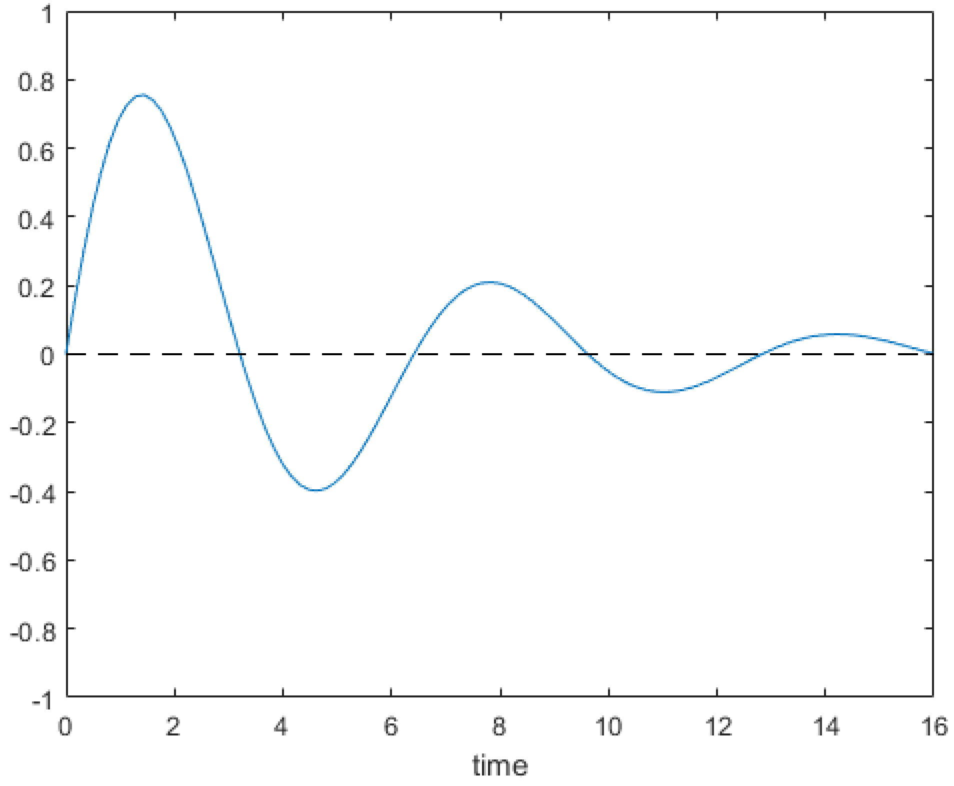
$$\implies G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

```
z = 0.2;
wn = 1;

G = tf(wn^2,[1,2*z*wn,wn^2]);

[y,tout] = impulse(G,16);

plot(tout,y,tout,zeros(size(tout)),'k--')
ylim([-1 1])
xlabel('time')
```



Response to a General Input

- Response to a general input

$$\dot{x} + 5x = u(t), \qquad x(0) = 0$$

$$\dot{x} = -5x + u$$
$$y = x$$

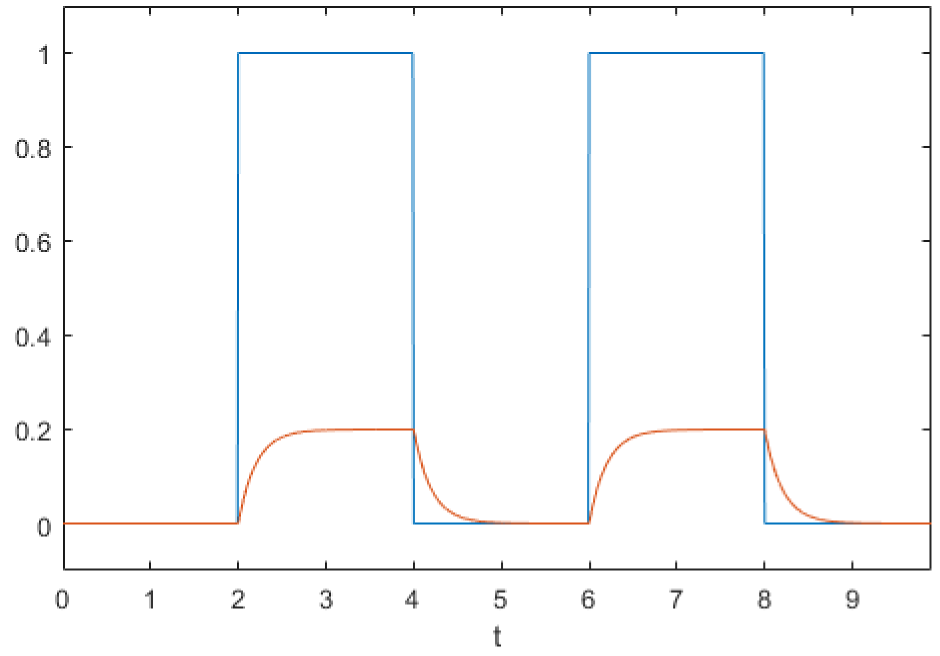
- The solution is given:

$$x(t) = h(t) * u(t), \quad t \geq 0$$

```
A = -5;
B = 1;
C = 1;
D = 0;
G = ss(A,B,C,D);

x0 = 0;
[f,t] = gensig('square',4,10,0.01);
[y,tout] = lsim(G,f,t,x0);

plot(t,f), hold on
plot(tout,y), hold off, axis([0,9.9,-0.1,1.1])
xlabel('t')
```



Model Conversion in MATLAB

State Space ↔ Transfer Function

```
A = [0 1 0 0;
    0 0 -1 0;
    0 0 0 1;
    0 0 5 0];
B = [0 1 0 -2]';
C = [1 0 0 0];
D = 0;

Gss = ss(A,B,C,D)
Gtf = tf(Gss)
```

```
[num,den] = ss2tf(A,B,C,D)
```

```
[A,B,C,D] = tf2ss(num,den)
```

Summary

- LTI Systems
  - In time
  - In Laplace (or Frequency)
  - In state space

- MATLAB Implementation