Matrix Exponential

Prof. Seungchul Lee
Industrial AI Lab.

From Single Variable to Multivariate Variables

- Starting from single variable (scalar case)

$$\dot{u} = au \qquad \Longrightarrow \qquad u(t) = e^{at}u(0)$$

- Extending to multivariate case (matrix)

$$\dot{\vec{u}} = A\vec{u} \qquad \Longrightarrow \qquad \vec{u}(t) = \underbrace{e^{At}}_{\text{matrix exponential}} \vec{u}(0)$$

POSTECH

Diagonal Matrix Exponential

- Matrix exponential with diagonal matrix (intuitive)

$$e^{\Lambda t} = \exp\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} t \right)$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}$$

Similarity Transformation

- Revisit

$$S = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \quad \text{where } \vec{x}_i \text{ is eigenvectors}$$

$$\begin{aligned} AS &= S\Lambda \\ \Longrightarrow \quad \Lambda &= S^{-1}AS \\ \Longrightarrow \quad A &= S\Lambda S^{-1} \end{aligned}$$

Matrix Exponential

- Think about

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$$
$$e^0 = 1$$

$$\frac{d}{dx}e^x = 0 + 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!} + \cdots$$
$$= e^x$$

$$\therefore \;\; e^x \text{ is a solution of } \dot{y}(x) = y(x)$$

- In a similar fashion

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

Matrix Exponential

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$$

- Derivative

$$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = 0 + \sum_{k=1}^{\infty} \frac{kA^k t^{k-1}}{k!} = A \sum_{k=1}^{\infty} \frac{(At)^{k-1}}{(k-1)!} = A \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = Ae^{At}$$

Similar Transformation

- Similarity transformation

$$
\begin{aligned}
e^{A} = e^{S\Lambda S^{-1}} &= I + S\Lambda S^{-1} + \frac{S\Lambda^2 S^{-1}}{2!} + \frac{S\Lambda^3 S^{-1}}{3!} + \cdots \\
&= S\left[I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + \cdots\right] S^{-1} \\
&= S e^{\Lambda} S^{-1}
\end{aligned}
$$

$$
e^{At} = e^{S\Lambda S^{-1}t} = S e^{\Lambda t} S^{-1}
$$

MATLAB Implementation

- Simulation with $\omega_n = 2$ and $\zeta = 0.1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

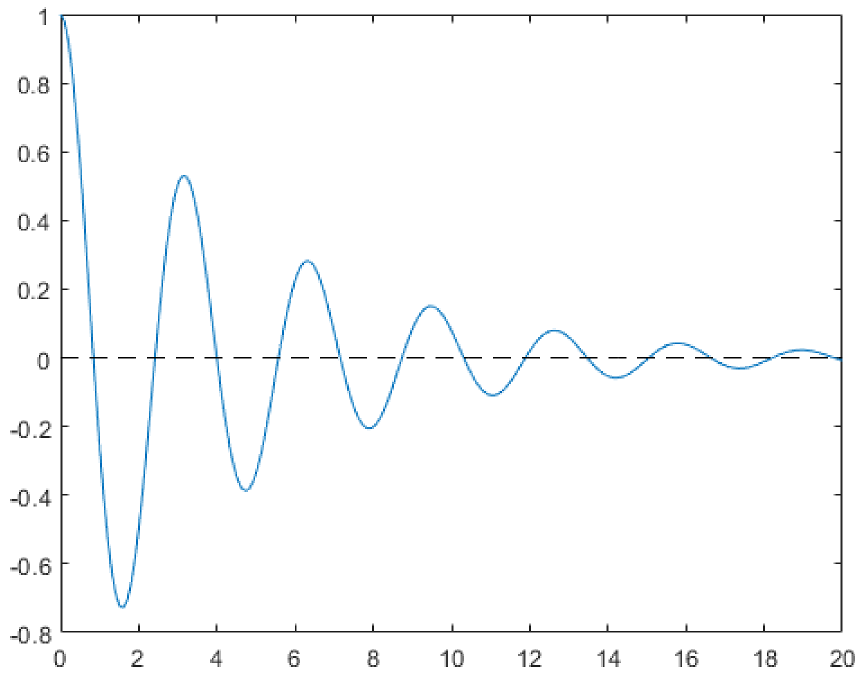
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

```
wn = 2; zeta = 0.1;
A = [0 1;-wn^2 -2*zeta*wn];

x0 = [1; 0];
t = linspace(0,20,200);
x = [];

for i = 1:length(t)
    x = [x, expm(A*t(i))*x0];
end

plot(t,x(1,:),t,zeros(size(t)),'--k')
```



Summary

- Matrix exponential