Natural Response to Non-zero Initial Conditions

Prof. Seungchul Lee
Industrial AI Lab.

The First Order ODE

$$\frac{dx(t)}{dt} = kx(t), \qquad x(0) = x_0$$

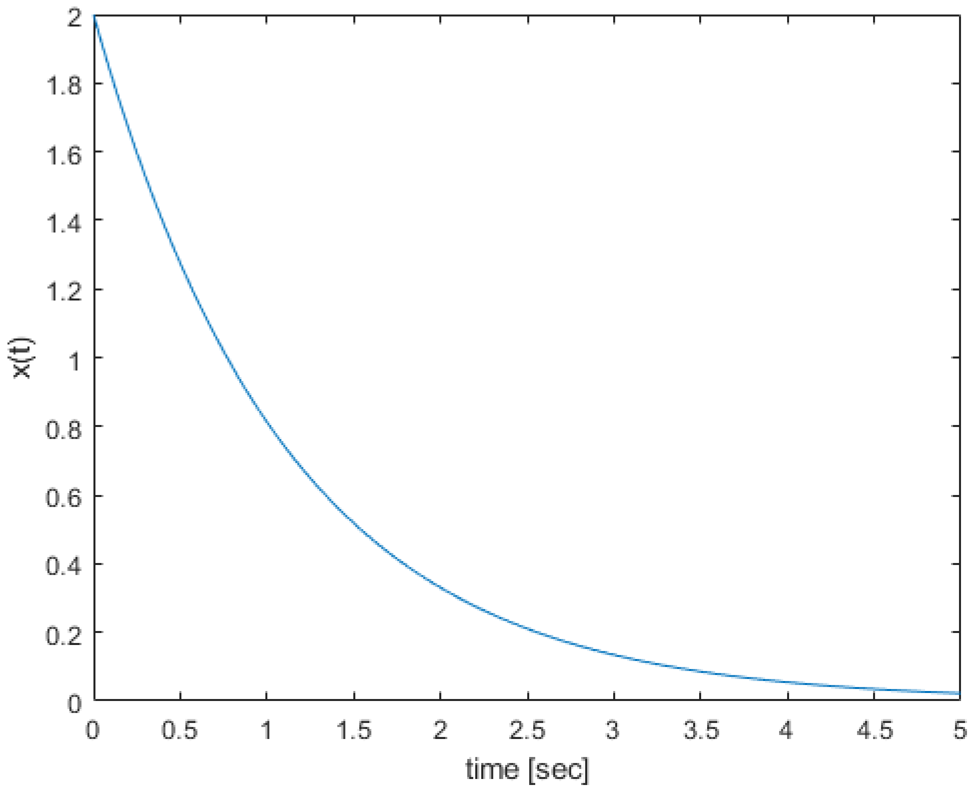
$$\rightarrow \quad x(t) = x_0 e^{kt}$$

- Solution will be exponential functions
  - Unknown coefficient determined by initial conditions

- Stability
  - unstable if $k > 0$
  - stable if $k < 0$

The First Order ODE

```
% plot an analytic solution

k = -0.9;
x0 = 2;

t = linspace(0,5,100);
x = x0*exp(k*t);

plot(t,x);
xlabel('time [sec]')
ylabel('x(t)')
ylim([0,2])

% but, all we did is just plotting (not computing)
```



The First Order ODE

$$\frac{dx(t)}{dt} = kx(t), \qquad x(0) = x_0$$

$$\rightarrow \; x(t) = x_0 e^{kt}$$

- τ: time constant
  - Large τ : slow response
  - Small τ : fast response

$$\dot{x} + \frac{1}{\tau}x = 0 \quad \Longrightarrow \quad \dot{x} = -\frac{1}{\tau}x = ax$$

$$x(t) = x(0)e^{-\frac{1}{\tau}t}$$

$$\frac{x(\tau)}{x(0)} = e^{-1} = \frac{1}{e} = 0.368\cdots$$

Two First Order ODEs (Independent)

- Suppose u_1 and u_2 are independent

$$\dot{u}_1 = \lambda_1 u_1 \implies u_1(t) = u_1(0)e^{\lambda_1 t}$$

$$\dot{u}_2 = \lambda_2 u_2 \implies u_2(t) = u_2(0)e^{\lambda_2 t}$$

- In a matrix form

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\dot{u} = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 u_1 \\ \lambda_2 u_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \Lambda u$$

ODE in Vector Form (Dependent)

- Suppose u_1 and u_2 are dependent

$$\dot{u}_1 = a_{11}u_1 + a_{12}u_2$$
$$\dot{u}_2 = a_{21}u_1 + a_{22}u_2$$

- In a matrix form

$$\dot{u} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} u = Au$$

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} =?$$

Systems of Differential Equations

Systems of Differential Equations

- Given

$$\dot{\vec{u}} = A\vec{u}, \qquad \vec{u}(0) = \vec{u}_0$$

- Superposition

$$\begin{aligned} \dot{\vec{u}}_1 &= A\vec{u}_1, \\ \dot{\vec{u}}_2 &= A\vec{u}_2 \end{aligned}$$

$$\begin{aligned} \dot{\vec{u}} &= c_1\dot{\vec{u}}_1 + c_2\dot{\vec{u}}_2 = c_1 A\vec{u}_1 + c_2 A\vec{u}_2 \\ &= A(c_1\vec{u}_1 + c_2\vec{u}_2) = A\vec{u} \end{aligned}$$

Systems of Differential Equations

- For a single ODE

$$\dot{u} = au \qquad \Longrightarrow \qquad u(t) = ce^{at}$$

- Let us try

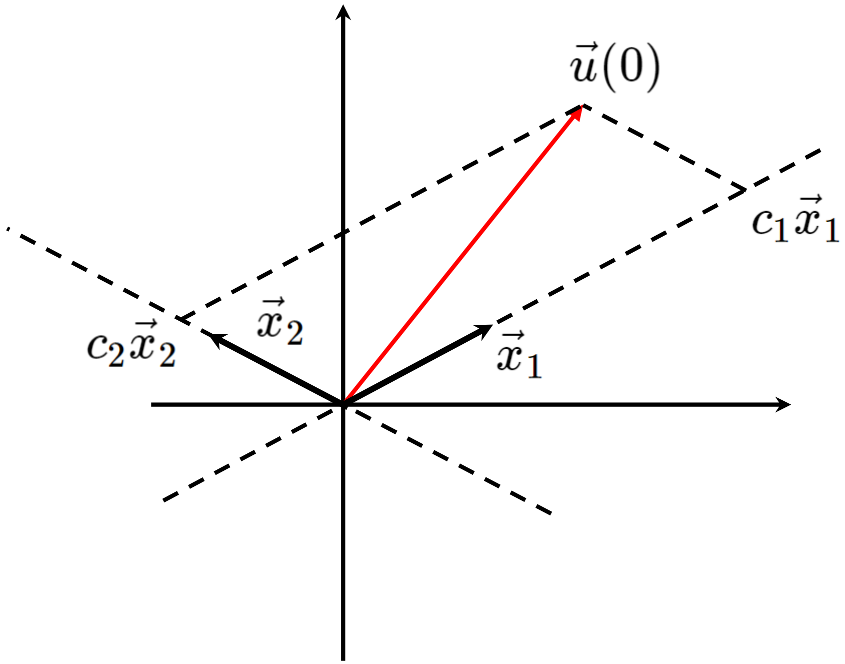
$$\vec{u}(t) = \vec{x}e^{\lambda t} \quad \text{(in a vector form)}$$

$$\dot{\vec{u}} = \vec{x}\lambda e^{\lambda t} = \underline{\lambda\vec{x}}e^{\lambda t} = A\vec{u} = \underline{A\vec{x}}e^{\lambda t} \qquad \Longleftrightarrow \qquad A\vec{x} = \lambda\vec{x} \quad \text{(eigenvalue problem)}$$

- Linear ODE = Eigenvalue problem

Eigenanalysis

$$\dot{\vec{u}} = A\vec{u}, \qquad \vec{u}(0) = \vec{u}_0$$



- Eigenanalysis

$$\begin{aligned} A\vec{x}_1 &= \lambda_1\vec{x}_1 \\ A\vec{x}_2 &= \lambda_2\vec{x}_2 \end{aligned}$$

- General solution

$$\vec{u}(t) = c_1\, e^{\lambda_1 t}\, \vec{x}_1 + c_2\, e^{\lambda_2 t}\, \vec{x}_2$$

where

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \left[\vec{x}_1 \;\; \vec{x}_2\right]^{-1} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

Eigenanalysis

$$
\begin{aligned}
Ax_1 &= \lambda_1 x_1, \qquad & x_1, x_2 &: \quad \text{eigenvectors} \\
Ax_2 &= \lambda_2 x_2, \qquad & \lambda_1, \lambda_2 &: \quad \text{eigenvalues}
\end{aligned}
$$

$$
A \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
$$

$$
S \triangleq \begin{bmatrix} x_1 & x_2 \end{bmatrix} \quad \text{eigenvector matrix}
$$

$$
\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{Diagnonal matrix}
$$

$$
\begin{aligned}
AS &= S\Lambda \\
S^{-1}AS &= \Lambda \\
A &= S\Lambda S^{-1}
\end{aligned}
$$

Eigenanalysis

- Linear Transformation

$$u = Sv$$

$$\begin{aligned} \dot{u} &= Au \\ S\dot{v} &= ASv \\ \dot{v} &= S^{-1}ASv \\ &= \Lambda v, \qquad v(0) = S^{-1}u(0) \end{aligned}$$

- Solution

$$u(t) = v_1(0)e^{\lambda_1 t}x_1 + v_2(0)e^{\lambda_2 t}x_2$$

$$\begin{aligned} u(0) &= v_1(0)x_1 + v_2(0)x_2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix} = Sv(0) \end{aligned}$$

Eigenanalysis

- v - frame is decoupled by $\vec{x}_1$ and $\vec{x}_2$

$$v(t) = \begin{bmatrix} v_1(0)e^{\lambda_1 t} \\ v_2(0)e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix}, \qquad v(0) = \begin{bmatrix} v_1(0) \\ v_2(0) \end{bmatrix}$$

$$\begin{aligned} u(t) &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} v(0) \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{-1} u(0) \\ &= S e^{\Lambda t} S^{-1} u(0) \end{aligned}$$

Real Eigenvalues

$$\vec{u}(t) = c_1\, \vec{x}_1\, e^{\lambda_1 t} + c_2\, \vec{x}_2\, e^{\lambda_2 t}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\, \vec{x}_1 \;\; \vec{x}_2 \,]^{-1} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}.$$

- Example 1

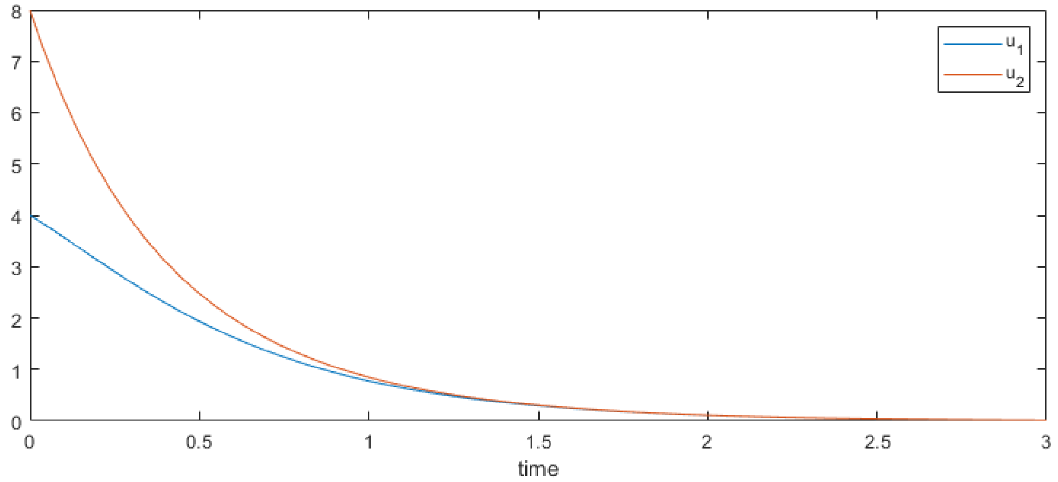
$$\dot{u_1} = -3u_1 + u_2$$
$$\dot{u_2} = u_1 - 3u_2$$

$$\dot{\vec{u}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{u} \qquad \text{where} \quad A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$$

Real Eigenvalues

$$\dot{\vec{u}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{u}, \qquad u(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

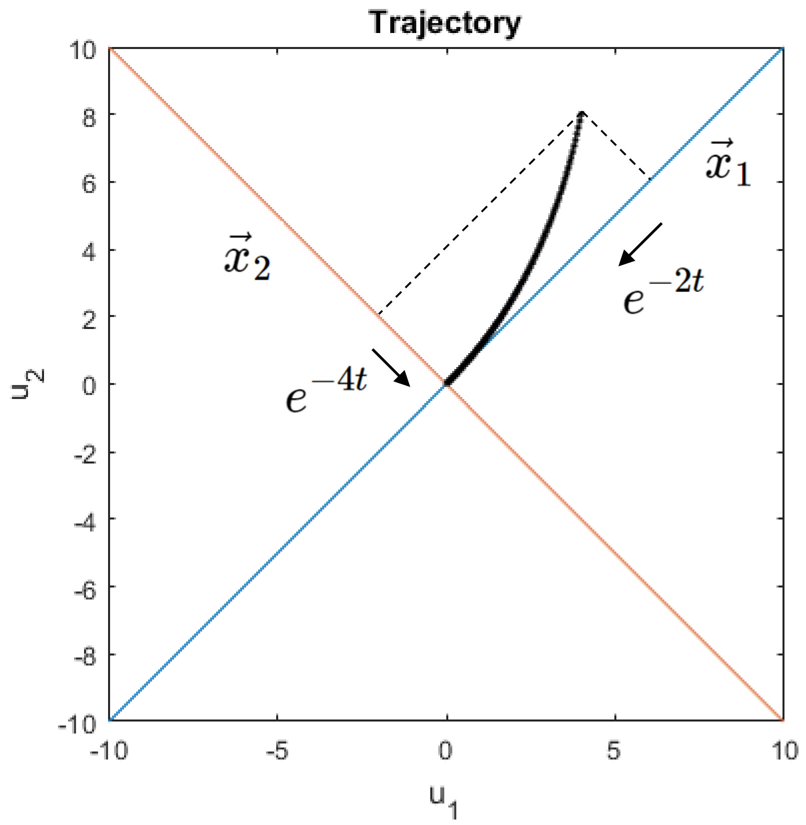
```
A = [-3  1;
      1 -3];

%% eigen-analysis
[S,D] = eig(A);

[lamb,idx] = sort(diag(D),'descend');
S = S(:,idx);

x1 = S(:,1);
x2 = S(:,2);

u0 = [4;8];
C = inv(S)*u0;

t = 0:0.01:3;
u = C(1)*x1*exp(lamb(1)*t) + C(2)*x2*exp(lamb(2)*t);

% plot u1 and u2 as a function of time
plot(t,u(1,:),t,u(2,:))
xlabel('time')
legend('u_1', 'u_2')
```



Phase Portrait

- Geometric representation of the trajectories of a dynamical system in the phase plane

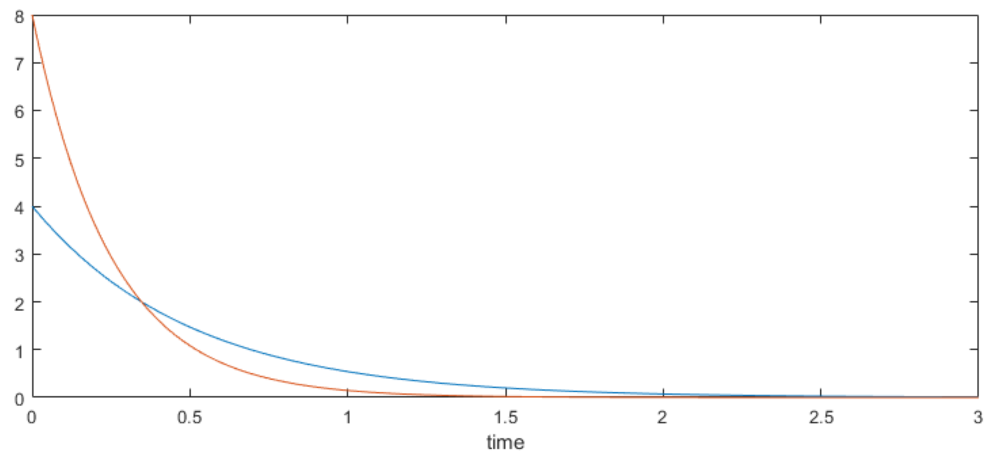
```
% plot eigenvectors (X1 and X2)
k = -20:0.1:20;
y1 = S(:,1)*k;
y2 = S(:,2)*k;

plot(y1(1,:), y1(2,:)); hold on
plot(y2(1,:), y2(2,:));
xlabel('u_1', 'fontsize', 12)
ylabel('u_2', 'fontsize', 12)
title('Trajectory', 'fontsize', 12)
axis equal
axis([-10 10 -10 10])

% plot a trajectory of u1 and u2
for i = 1:length(t)
    plot(u(1,i), u(2,i), 'k.');
end
hold off
```

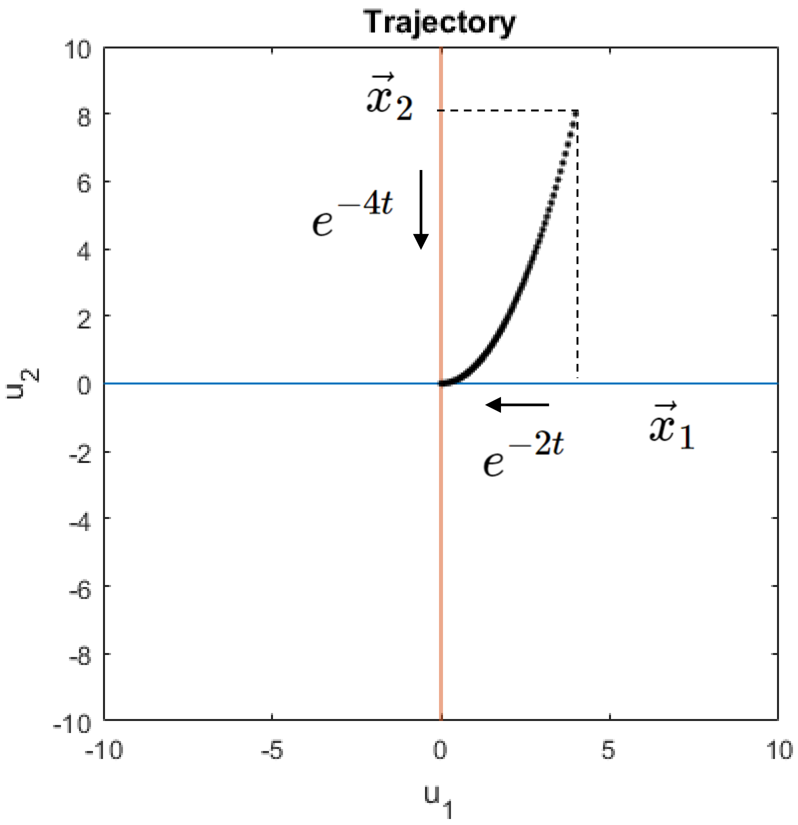


Real Eigenvalues

- Example 2

$$\dot{\vec{u}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \vec{u}, \qquad u(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

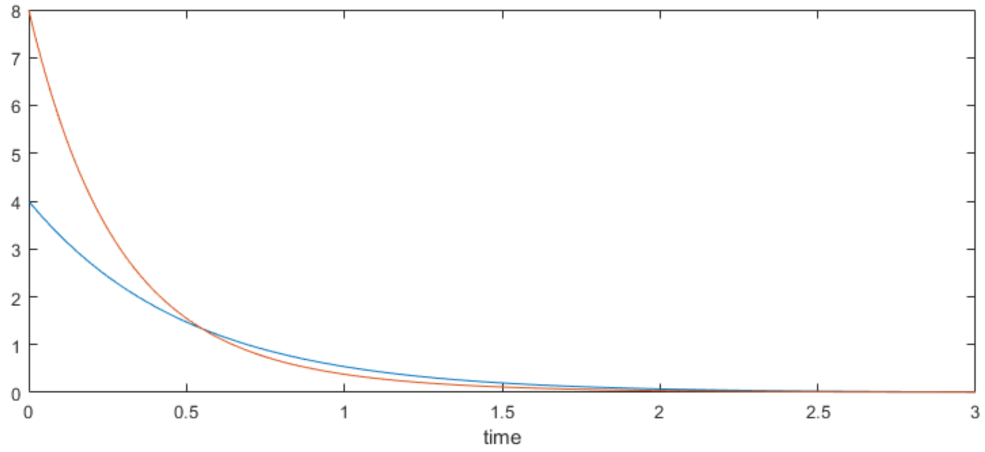


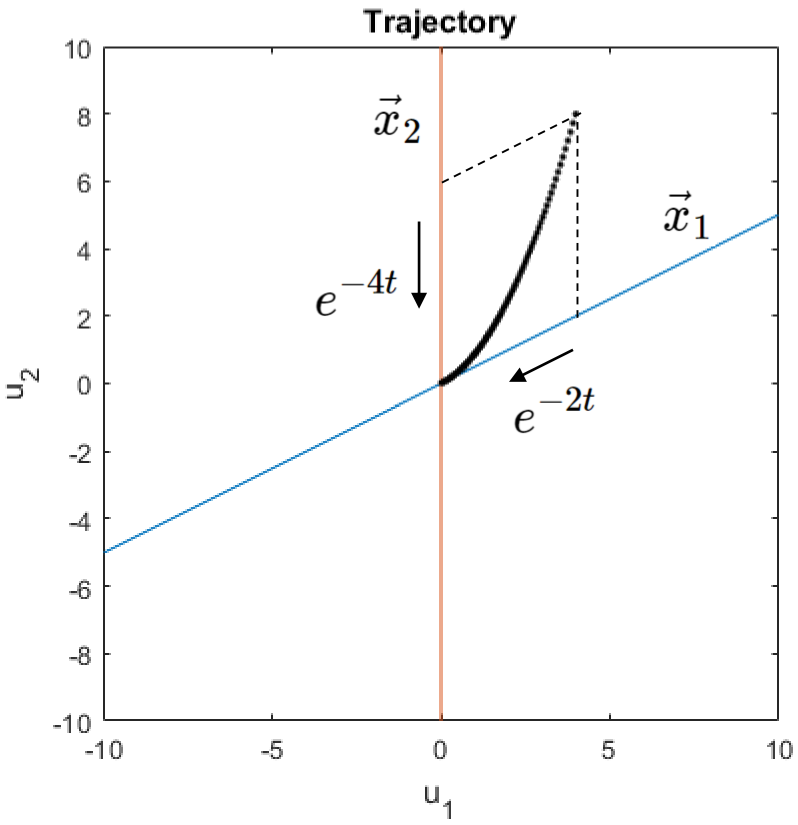


Real Eigenvalues

- Example 3

$$\dot{\vec{u}} = \begin{bmatrix} -2 & 0 \\ 1 & -4 \end{bmatrix} \vec{u}, \qquad u(0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$



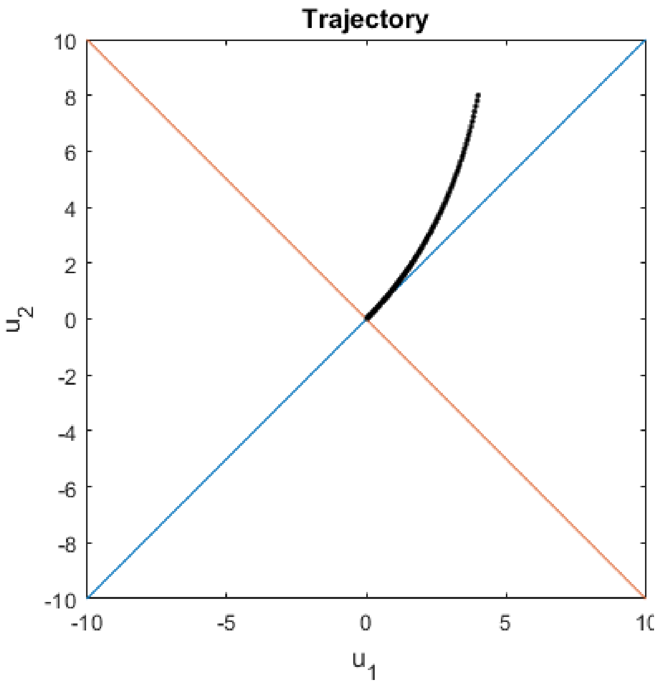


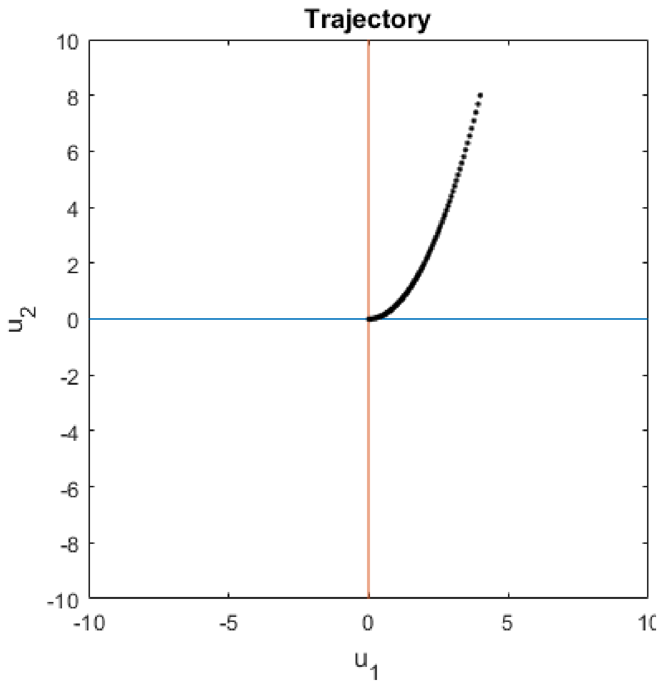
Different Eigenvectors with the Same Eigenvalues

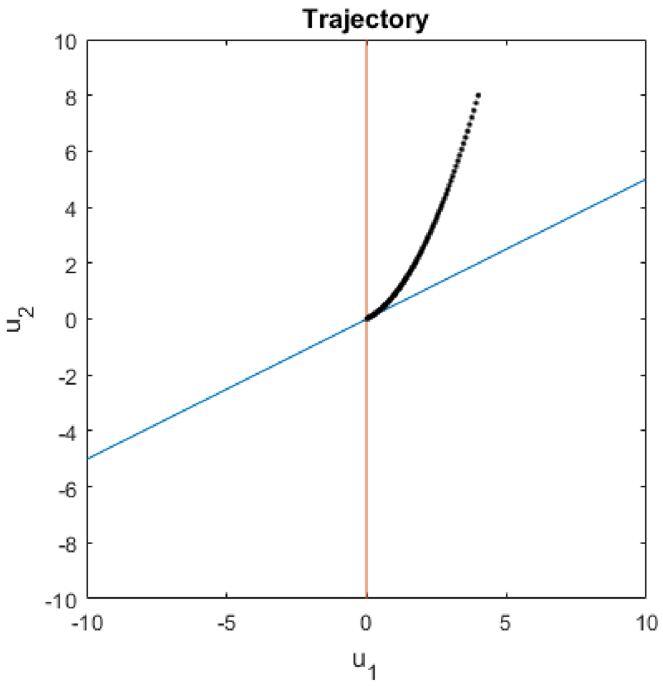
$$\dot{\vec{u}} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{u},$$

$$\dot{\vec{u}} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \vec{u},$$

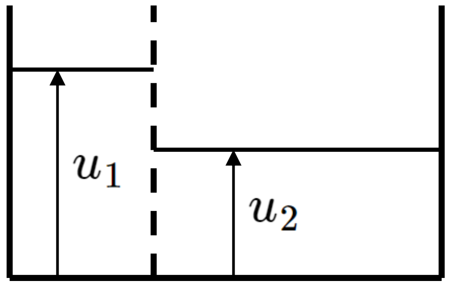
$$\dot{\vec{u}} = \begin{bmatrix} -2 & 0 \\ 1 & -4 \end{bmatrix} \vec{u},$$







De-coupling via Linear Transformation



$$\begin{aligned}\dot{u}_1 &= -2(u_1 - u_2) \\ \dot{u}_2 &= \quad (u_1 - u_2)\end{aligned} \qquad \dot{\vec{u}} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \vec{u}$$

$$\begin{vmatrix} -2-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} = (2+\lambda)(1+\lambda) - 2 = \lambda^2 + 3\lambda = \lambda(\lambda+3) = 0$$

$$\lambda_1 = 0 \quad \Longrightarrow \quad \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \vec{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

- Change variables

$\lambda_1 = 0 \quad \Longrightarrow \quad e^{0t} = 1 \quad \Longrightarrow$ does not change over time (= invariant)

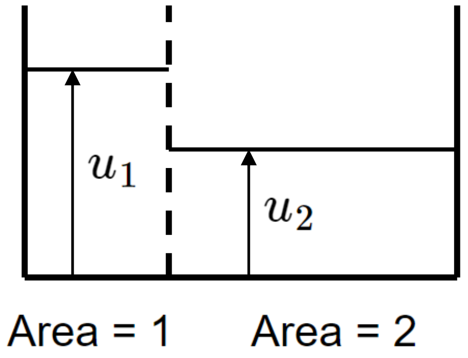
De-coupling via Linear Transformation

- Change variables
  - Total amount of water

$$v_1 = 1u_1 + 2u_2$$

  - Difference in height

$$v_2 = u_1 - u_2$$

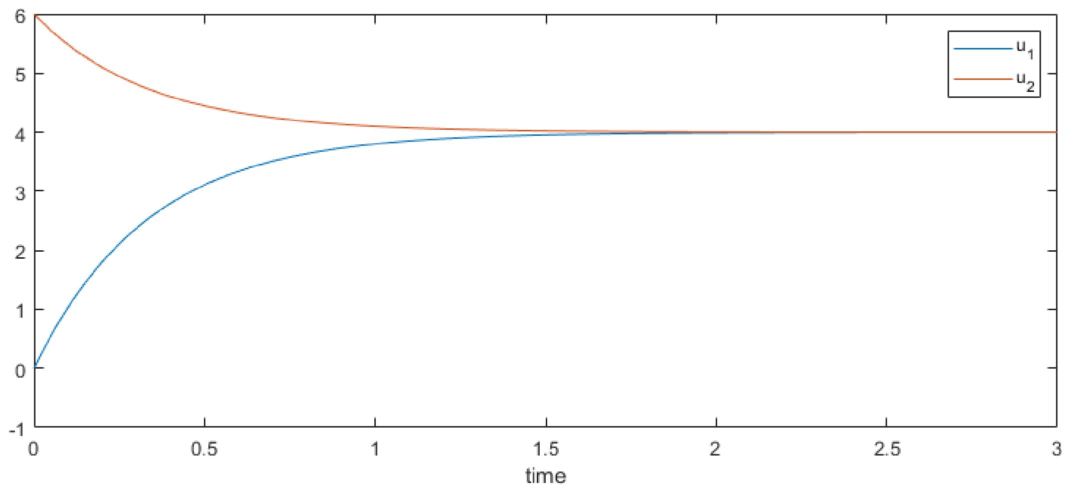


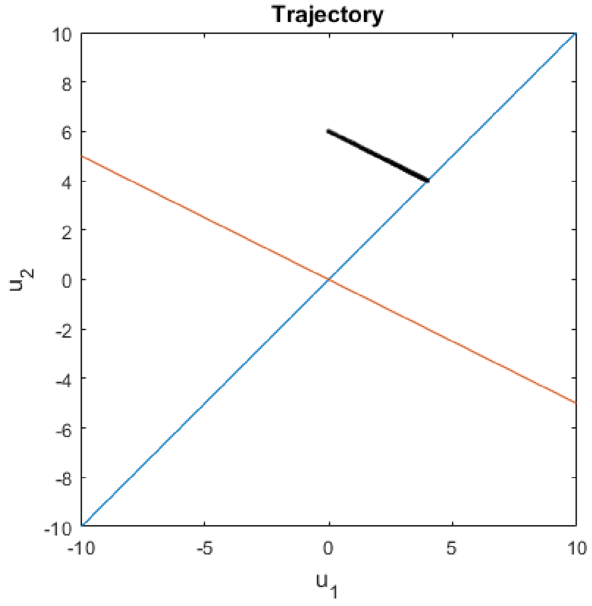
- De-coupled

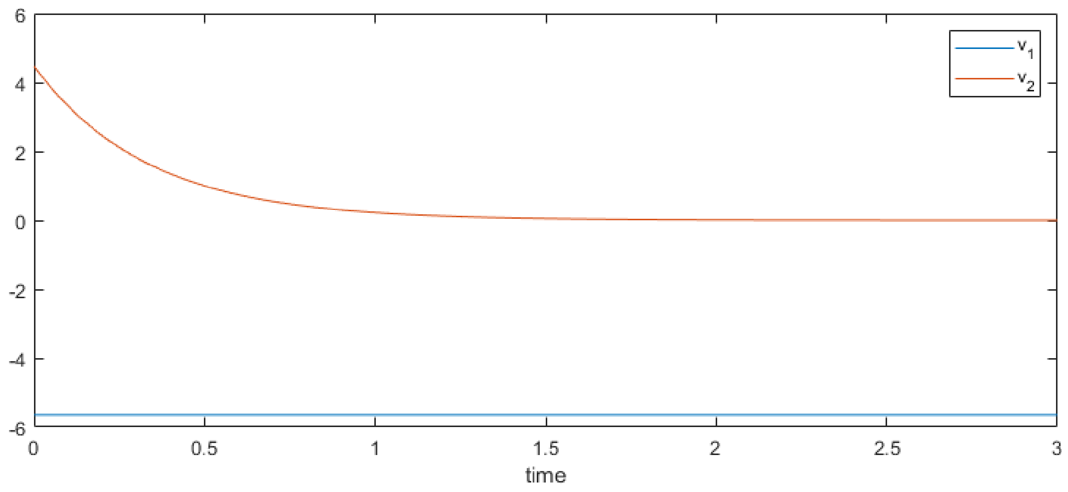
$$\begin{aligned}\dot{v}_1 &= \dot{u}_1 + 2\dot{u}_2 = -2(u_1 - u_2) + 2(u_1 - u_2) = 0\\ \dot{v}_2 &= \dot{u}_1 - \dot{u}_2 = -2(u_1 - u_2) - (u_1 - u_2) = -3(u_1 - u_2) = -3v_2\end{aligned}$$

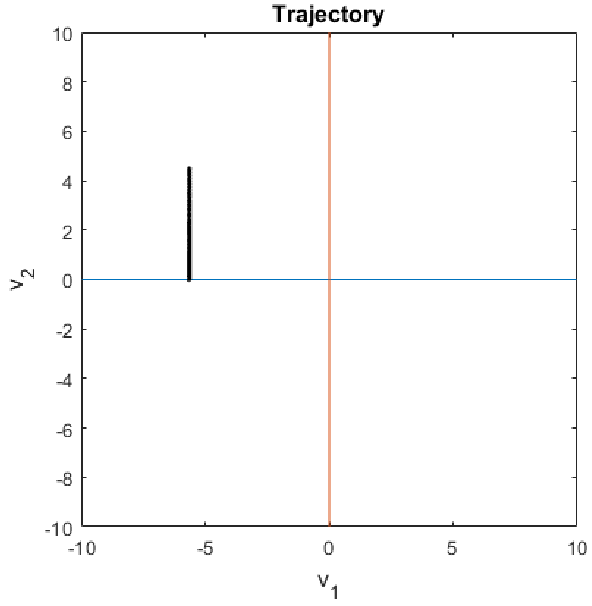
$$\begin{aligned}\dot{v}_1 &= 0 &\implies\quad& v_1(t) = v_1(0): &&\text{constant}\\ \dot{v}_2 &= -3v_2 &\implies\quad& v_2(t) = v_2(0)e^{-3t}: &&\text{decay}\end{aligned}$$

Trajectory Comparison









Systems of Differential Equations: Complex Eigenvalues

Complex Eigenvalues (Starting Oscillation)

- λ can be a complex number $\lambda = \sigma + j\omega$

$$e^{\lambda t} = e^{(\sigma + i\omega)t} = \underbrace{e^{\sigma t}}_{\text{decay}} \underbrace{e^{i\omega t}}_{\text{oscillates}}$$

$$= e^{\sigma t}(\cos \omega t + i \sin \omega t)$$

$$\text{If } \mathrm{Re}(\lambda) = \sigma < 0 \quad \Longrightarrow \quad \text{stable}$$

$$\Longrightarrow \quad \text{decay by } e^{\sigma t}$$

Complex Eigenvalues (Starting Oscillation)

- Example 1

$$\dot{\vec{u}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{u} \qquad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0, \qquad \begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned} \qquad \text{complex conjugate}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \vec{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \vec{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

vector complex conjugate

Complex Eigenvalues (Starting Oscillation)

- Example 1

$$\vec{u}(t) = c_1\vec{x}_1 e^{\lambda_1 t} + c_2\vec{x}_2 e^{\lambda_2 t}$$

$$= c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{it} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{-it}$$

$$= \begin{bmatrix} c_1(\cos t + i\sin t) + c_2(\cos t - i\sin t) \\ c_1(i\cos t - \sin t) - c_2(i\cos t + \sin t) \end{bmatrix}$$

$$\vec{u}(0) = \begin{bmatrix} c_1 + c_2 \\ ic_1 - ic_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \implies c_1 = c_2 = \frac{1}{2}$$

$$\vec{u}(t) = \begin{bmatrix} \frac{1}{2}(e^{it} + e^{-it}) \\ -\frac{1}{2i}(e^{it} - e^{-it}) \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

What is the Corresponding Physical System?

- Simple harmonic motion Revisited

$$\ddot{u} + \omega_n^2 u = 0, \qquad \text{assume } \omega_n = 1$$

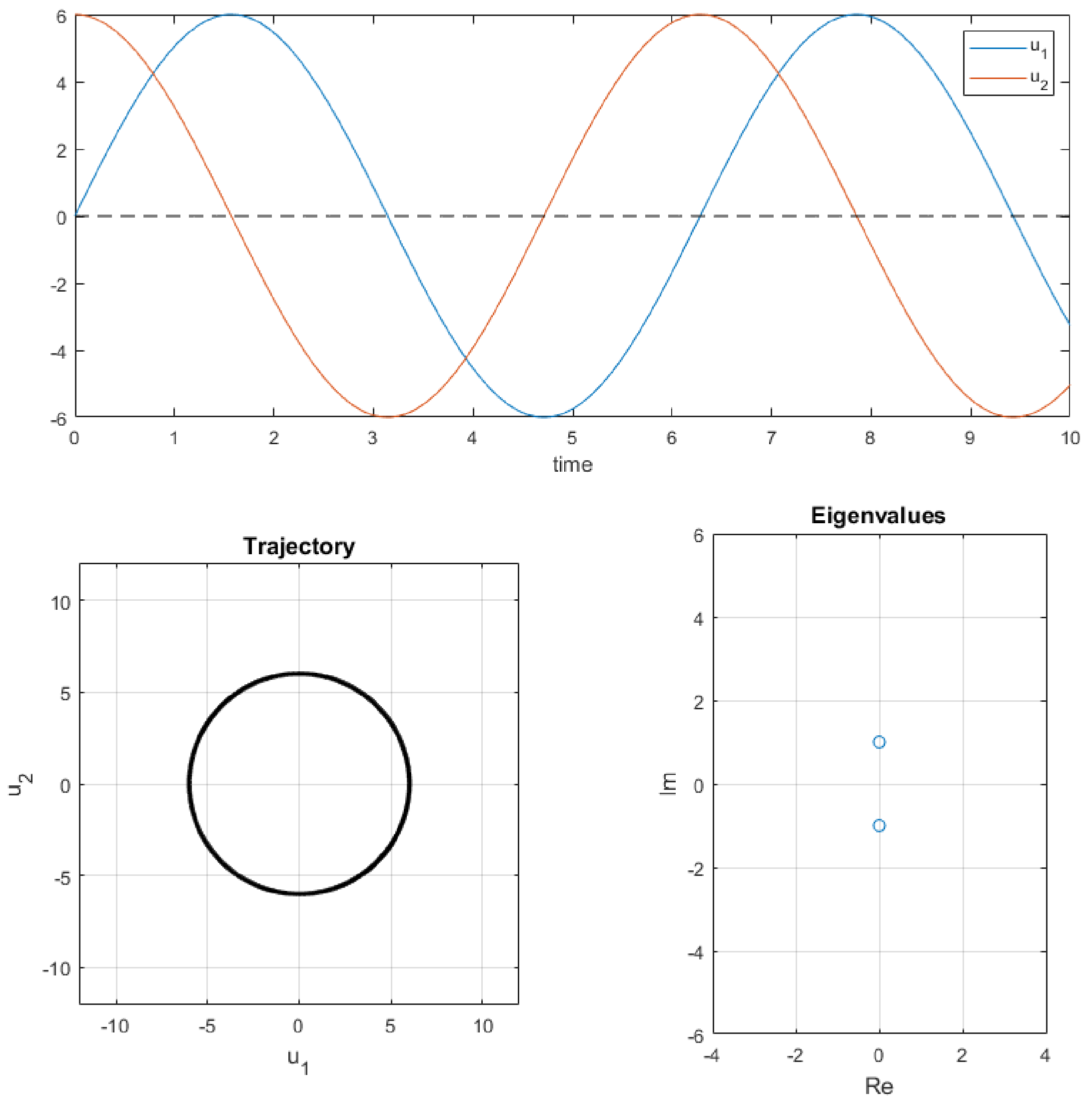
$$\ddot{u} + u = 0, \qquad \begin{aligned} u_1 &= u : \text{displacement or position} \\ u_2 &= \dot{u} : \text{velocity} \end{aligned}$$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} u_2 \\ -u_1 \end{bmatrix}$$

$$\therefore \; \dot{\vec{u}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{u}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0, \qquad \begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned} \qquad \text{two angular velocities (?)}$$

Pure Oscillation



Complex Eigenvalues

- Example 2

$$\dot{\vec{u}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \vec{u}, \qquad \vec{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -4 & \lambda \end{vmatrix} = \lambda^2 + 4 = 0, \quad \begin{matrix} \lambda_1 = 2i \\ \lambda_2 = -2i \end{matrix} \quad \Longrightarrow \quad \text{angular velocity}$$

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

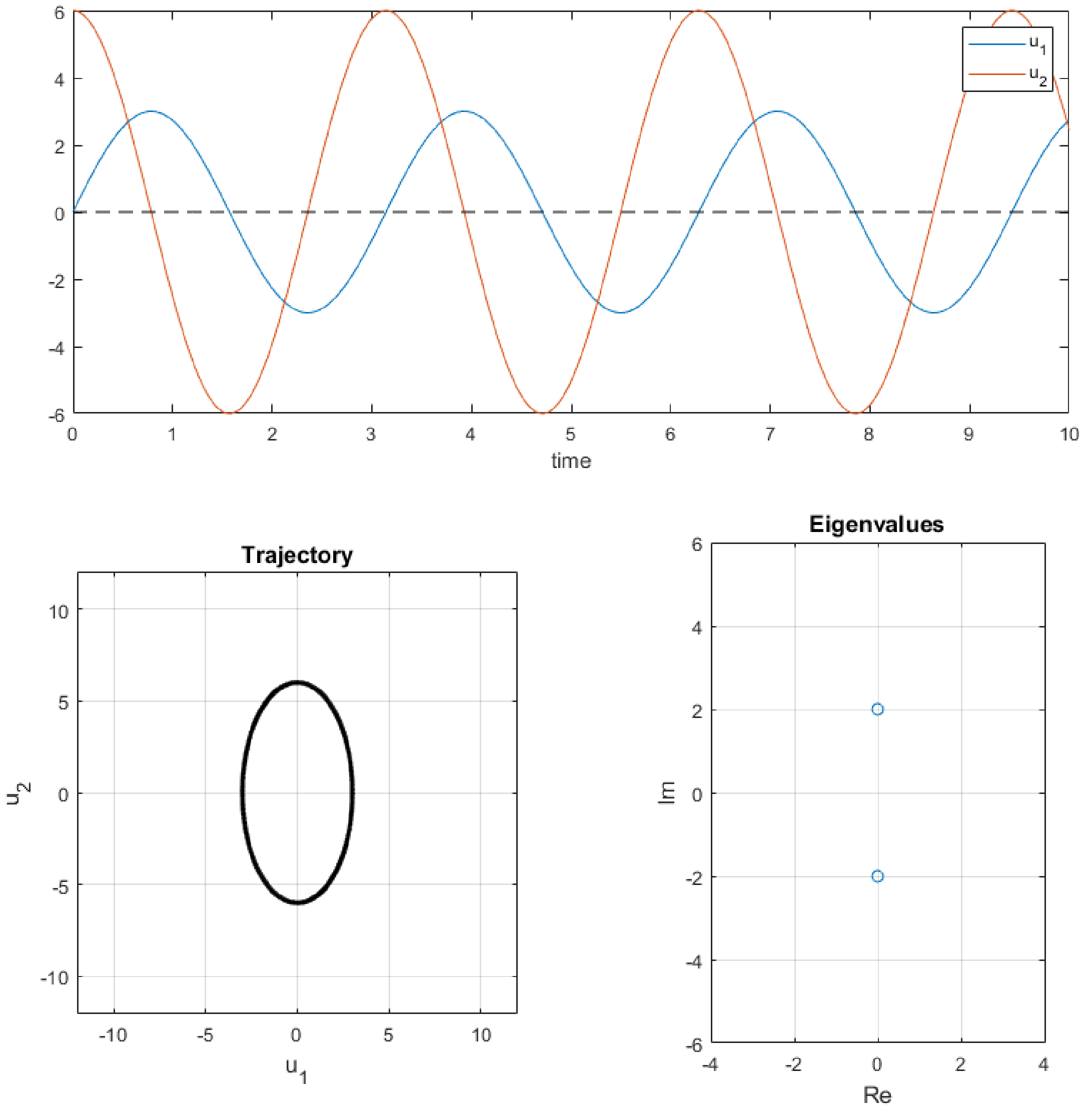
$$\begin{bmatrix} 2i & 1 \\ -4 & 2i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \vec{x}_2 = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

$$\begin{aligned} \vec{u}_t &= c_1 \vec{x}_1 e^{\lambda_1 t} + c_2 \vec{x}_2 e^{\lambda_2 t} \\ &= c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{i2t} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-i2t} \end{aligned}$$

$$\vec{u}(t) = \begin{bmatrix} \frac{1}{2}\left(e^{i2t} + e^{-i2t}\right) \\ -2\frac{1}{2i}\left(e^{i2t} - e^{-i2t}\right) \end{bmatrix} = \begin{bmatrix} \cos t \\ -2\sin t \end{bmatrix}$$

$$\vec{u}(0) = \begin{bmatrix} c_1 + c_2 \\ 2i(c_1 - c_2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad c_1 = c_2 = \frac{1}{2}$$

Pure Oscillation



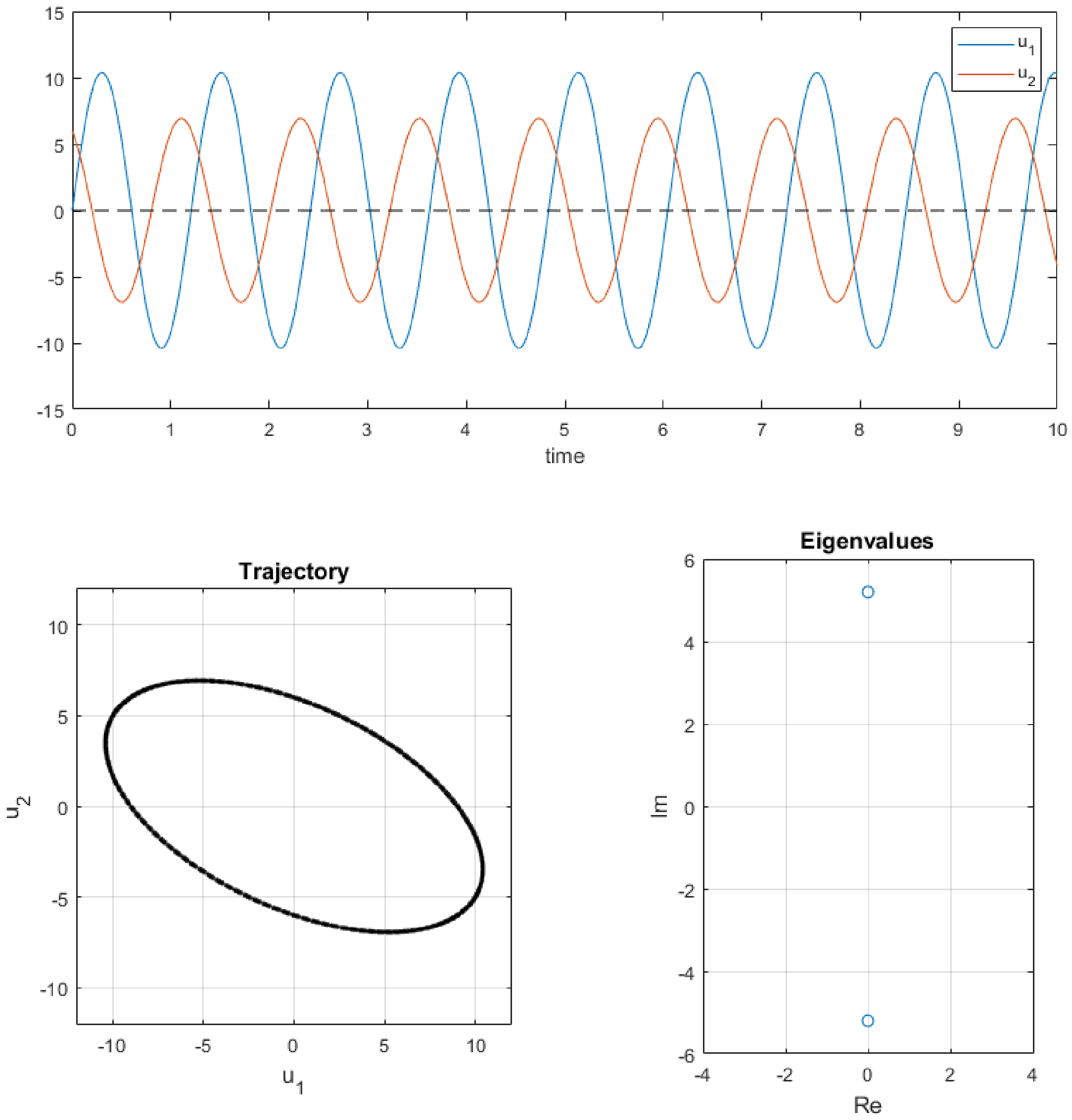
Complex Eigenvalues

- Example 3

$$\dot{u} = \begin{bmatrix} 3 & -9 \\ 4 & -3 \end{bmatrix} u$$

$$\begin{vmatrix} 3-\lambda & -9 \\ 4 & -3-\lambda \end{vmatrix} = \lambda^2 - 9 + 36 = \lambda^2 + 27 = 0, \qquad \begin{aligned} \lambda_1 &= 3\sqrt{3}i \\ \lambda_2 &= -3\sqrt{3}i \end{aligned}$$

Pure Oscillation



Complex Eigenvalues with Damping

- Example 1

$$\dot{\vec{u}} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \vec{u} \qquad \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

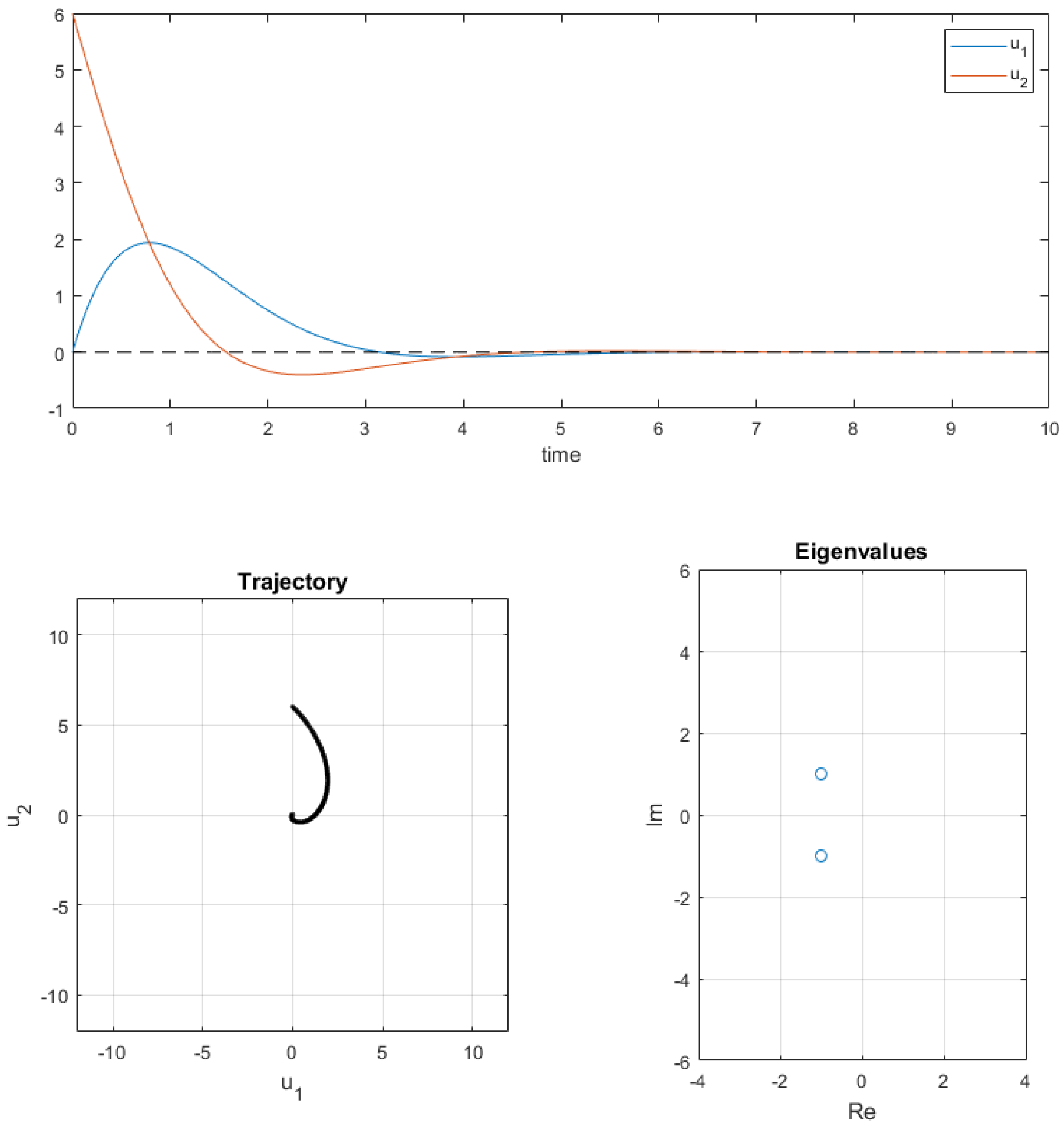
$$\begin{vmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 - 1 = \lambda^2 + 2\lambda + 2 = 0, \qquad \begin{aligned} \lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i \end{aligned}$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \vec{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \vec{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$u(t) = e^{-t}(c_1 \vec{x}_1 e^{it} + c_2 \vec{x}_2 e^{-it})$$

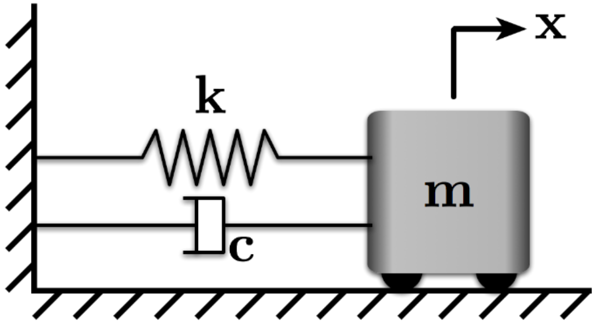
Oscillation with Damping



POSTECH

Mass-Spring-Damper System

- Mass-spring-damper system



$$a\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + cx(t) = 0, \qquad \dot{x}(0) = v_0, x(0) = x_0$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

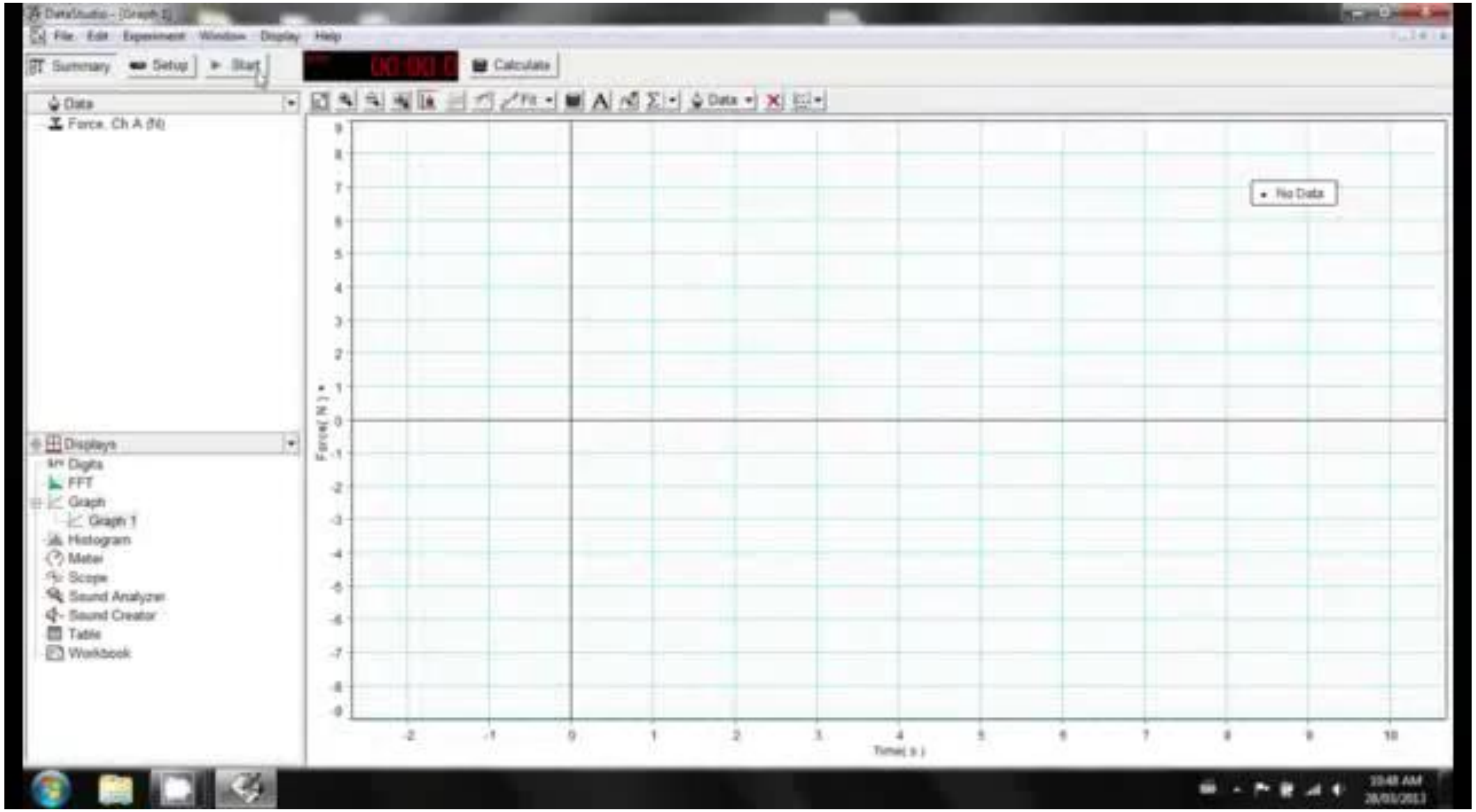
$$\boxed{x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + v_0}{\omega_d}\sin\omega_d t\right)}$$

$$\omega_n^2 = \frac{k}{m}$$

$$2\zeta\sqrt{\frac{k}{m}} = \frac{c}{m} \qquad \Longrightarrow \qquad \zeta = \frac{1}{2}\frac{c}{m}\sqrt{\frac{m}{k}} = \frac{1}{2}\frac{c}{\sqrt{mk}}$$

Mass-Spring-Damper System

$$x(t) = \underline{e^{-\zeta\omega_n t}}\left(x_0 \cos \omega_d t + \frac{\zeta\omega_n x_0 + v_0}{\omega_d} \sin \omega_d t\right)$$



Mass-Spring-Damper System

$$\omega_n^2 = \frac{k}{m}$$

State Space Representation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

- Define states

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

- State space

$$\dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \vec{x}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{\vec{x}} = A\vec{x}, \qquad A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$$

Eigenanalysis

$$\begin{vmatrix} -\lambda & 1 \\ -\omega_n^2 & -2\zeta\omega_n - \lambda \end{vmatrix} = \lambda(2\zeta\omega_n + \lambda) + \omega_n^2$$
$$= \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\therefore \lambda = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\begin{array}{lll} \text{if} \quad \zeta > 1, & \lambda < 0 \text{ and real} & \\ \text{if} \quad 0 < \zeta < 1, & \lambda \ \text{ complex number} & \Longrightarrow \ \text{start oscilating} \end{array}$$

- Physical interpretation of $0 < \zeta < 1$

$$\lambda = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

$$e^{\lambda t} = \underbrace{e^{-\zeta\omega_n t}}_{\text{decaying}} \cdot \underbrace{e^{j\omega_n\sqrt{1-\zeta^2}t}}_{\text{oscillating}}$$

Eigenvalues in S-plane

- Oscillating with damping (under damping)

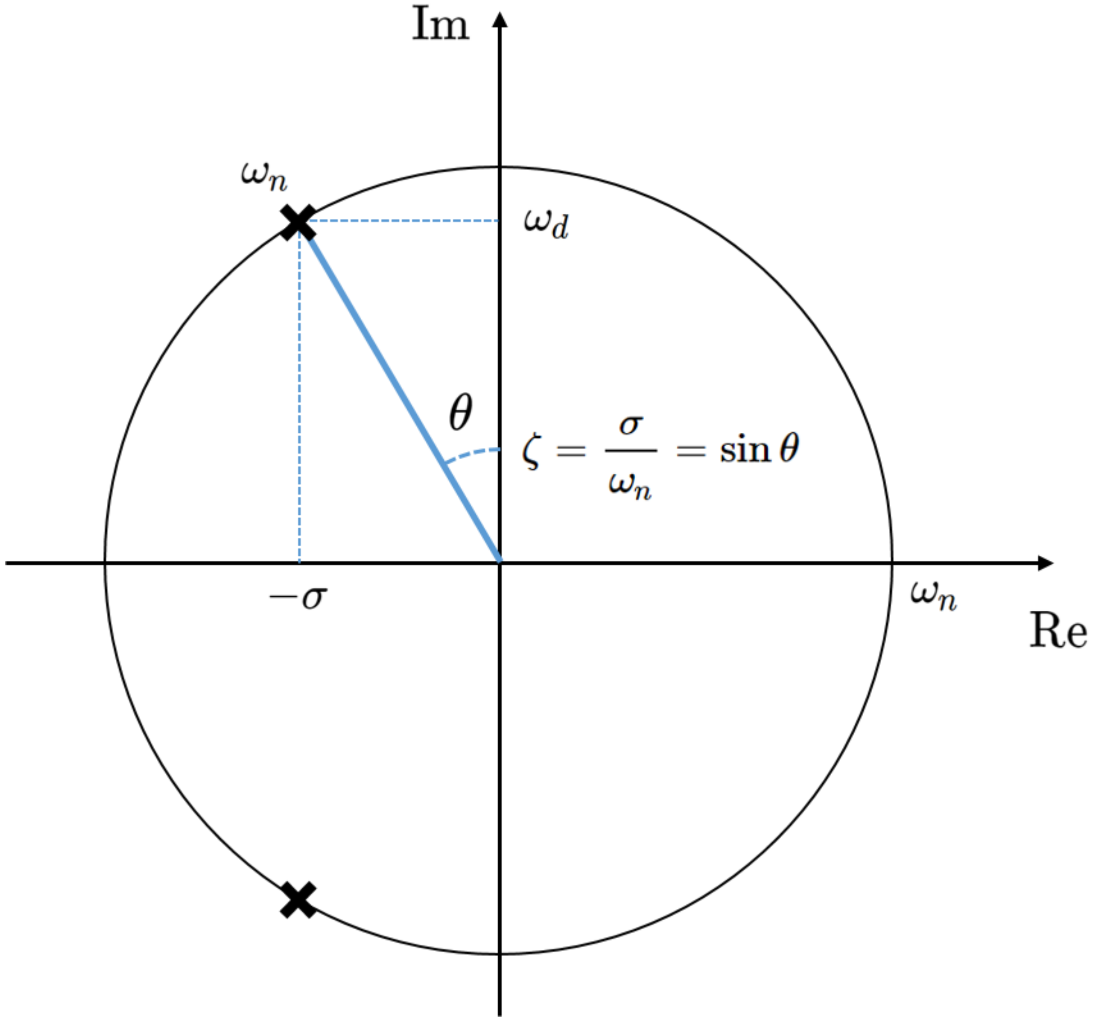
$$\lambda = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$\zeta\omega_n = \sigma$$

$$\omega_n\sqrt{1-\zeta^2} = \omega_d$$

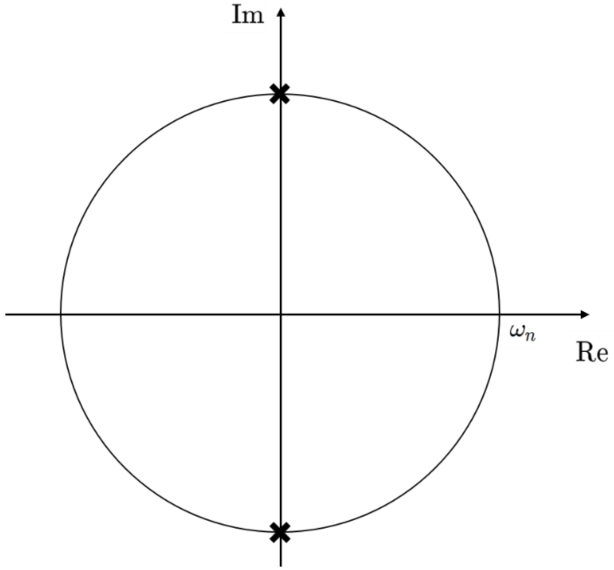
$$\zeta = \frac{\sigma}{\omega_n} = \sin\theta$$

$$\frac{\omega_d}{\omega_n} = \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\zeta^2}$$

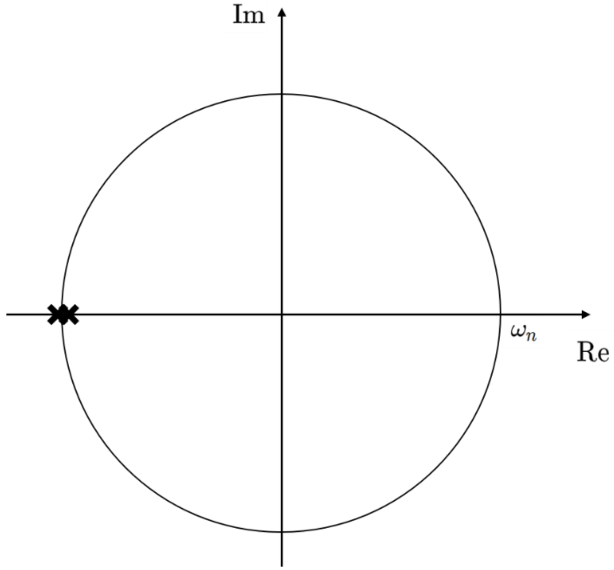


Eigenvalues in S-plane

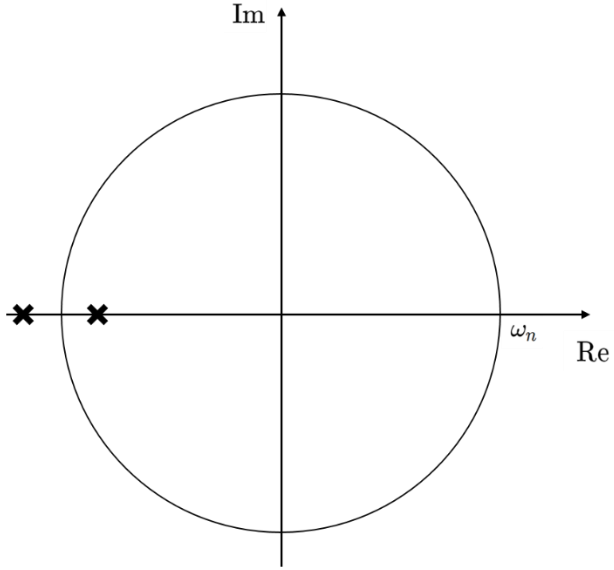
Pure oscillating



Critical damping



Over damping



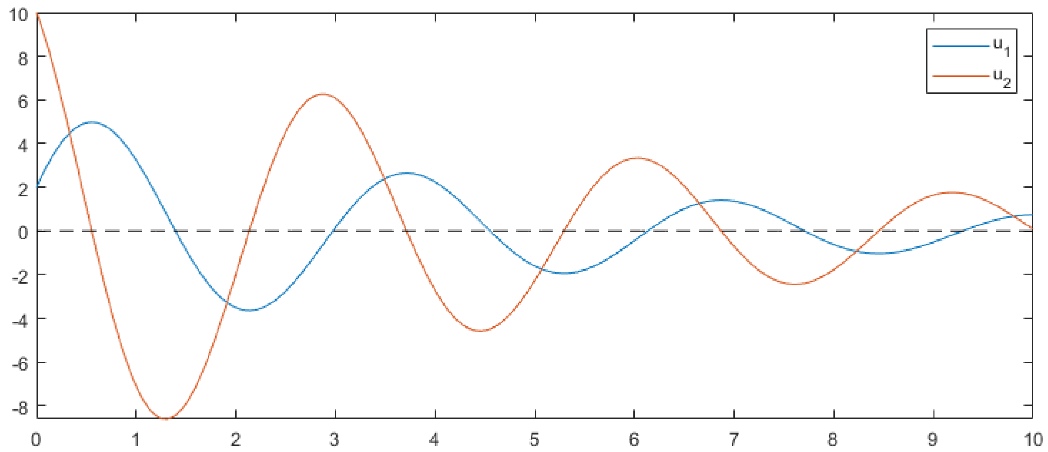
The Second Order ODE

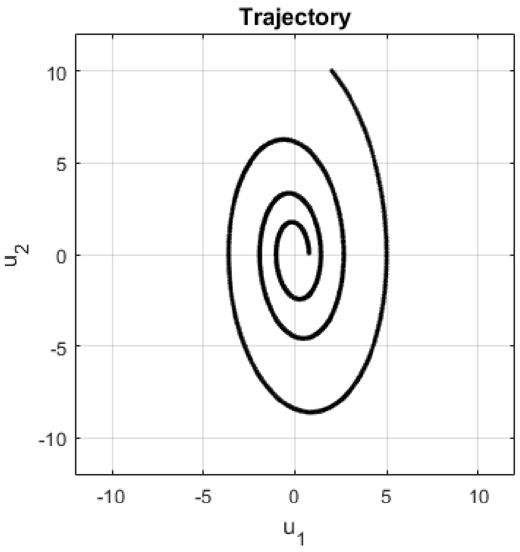
- State space representation

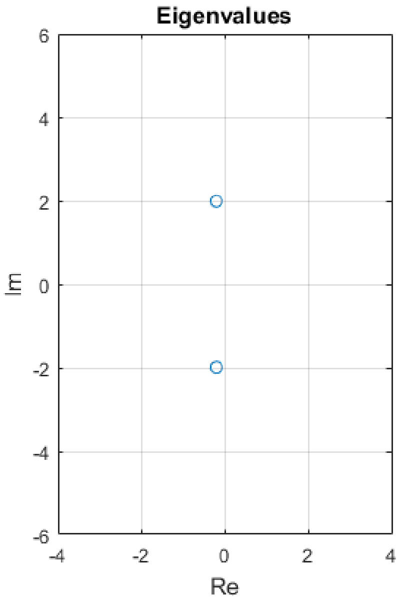
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Simulation with $\omega_n = 2$, and $\zeta = 0.1$







Stability

- Scalar systems

$$\dot{x} = ax \qquad \Longrightarrow \quad x(t) = e^{at}x(0)$$

$$\begin{cases} a > 0 : \text{unstable} \\ a < 0 : \text{asymptotically stable} \\ a = 0 : \text{critically stable} \end{cases}$$

- Matrix systems

$$Av = \lambda v$$

$$\begin{cases} \mathrm{Re}\,(\lambda) > 0 : \text{unstable} \\ \mathrm{Re}\,(\lambda) < 0 : \text{asymptotically stable} \\ \mathrm{Re}\,(\lambda) = 0 : \text{critically stable} \end{cases}$$

Summary

- Natural response with non-zero initial conditions
- Systems of differential equations
- Eigen-analysis
- Complex eigenvalues
  - Their locations in s-plane
- The second order ODE
  - Mass, spring, and damper system