Probability for Machine Learning

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Random Variable (= r.v.)

- (Rough) Definition: Variable with a probability

- Probability that $x = a$

$$\triangleq \; P_X(x=a) \;=\; P(x=a) \;\Longrightarrow\; \begin{cases} 1) \; P(x=a) \geq 0 \\ 2) \; \sum_{\text{all}} P(x) = 1 \end{cases}$$

- $\begin{cases} \text{continuous r.v.} & \text{if } x \text{ is continuous} \\ \text{discrete r.v.} & \text{if } x \text{ is discrete} \end{cases}$

Example

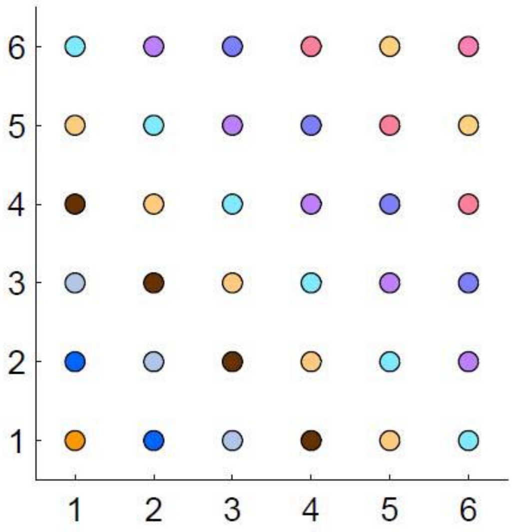
- x: die outcome

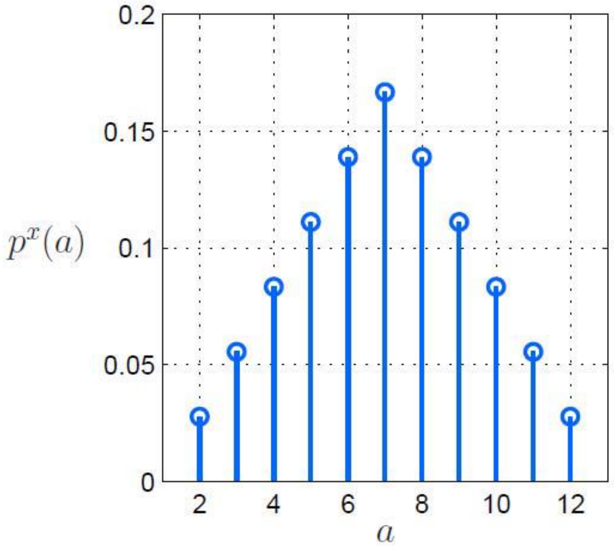
$$P(X=1) = P(X=2) = \cdots = P(X=6) = \frac{1}{6}$$

- Question

$$y = x_1 + x_2 : \quad \text{sum of two dice}$$

$$P_Y(y=5) = ?$$





Random Variable (= r.v.)

- Expectation = mean

$$E[x] = \begin{cases} \sum_x xP(x) & \text{discrete} \\ \int_x xP(x)dx & \text{continuous} \end{cases}$$

- Example

$$\text{Sample mean} \quad E[x] = \sum_x x \cdot \frac{1}{m} \quad (\because \ \text{uniform distribution assumed})$$

$$\text{Variance} \quad \text{var}[x] = E\left[(x - E[x])^2\right] \quad : \text{mean square deviation from mean}$$

Random Vectors (Multivariate R.V.)

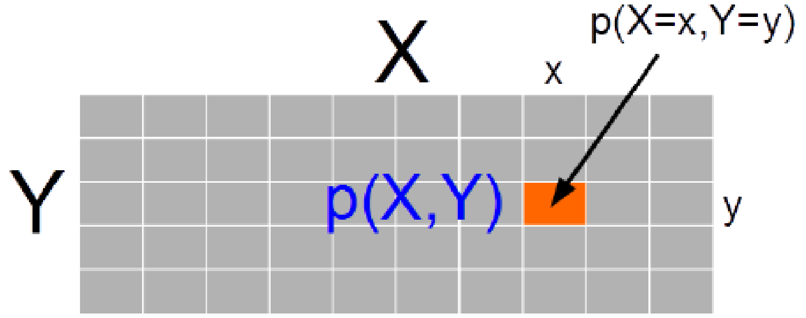
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad n \text{ random variables}$$

Joint Density Probability

- Joint density probability models probability of co-occurrence of many r.v.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad n \text{ random variables}$$

$$P_{X_1,\cdots,X_n}(X_1 = x_1, \cdots, X_n = x_n)$$



Marginal Density Probability

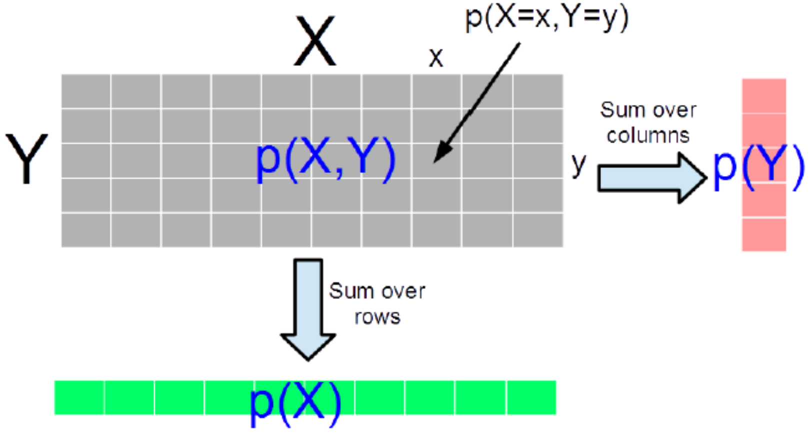
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad n \text{ random variables}$$

$$\begin{gathered} P_{X_1}(X_1 = x_1) \\ P_{X_2}(X_2 = x_2) \\ \vdots \\ P_{X_n}(X_n = x_n) \end{gathered}$$

- For two r.v.

$$P(X) = \sum_y P(X, Y = y)$$

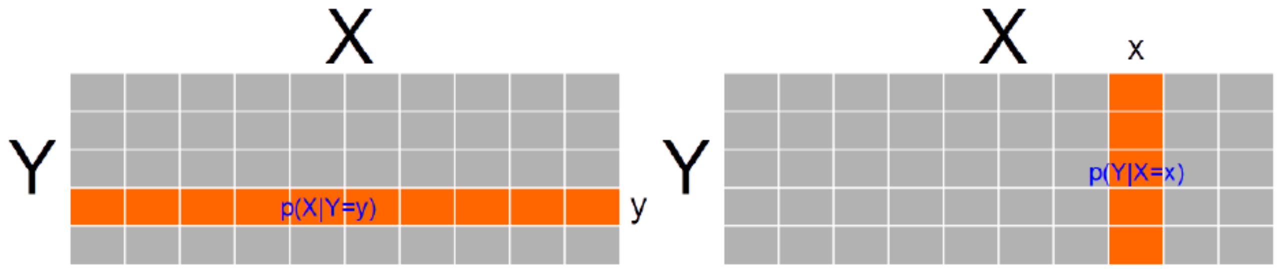
$$P(Y) = \sum_x P(X = x, Y)$$



Conditional Probability

- Probability of one event when we know the outcome of the other
- Conditional probability of x_1 given x_2

$$P_{X_1|X_2}(X_1 = x_1 \mid X_2 = x_2) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_2 = x_2)}$$



Independent Random Variables

- When one tells nothing about the other

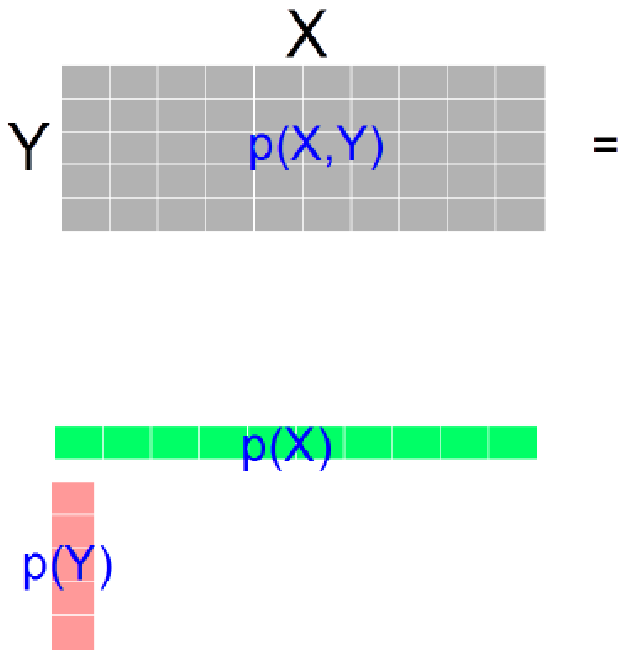
$$P(X_1 = x_1 \mid X_2 = x_2) = P(X_1 = x_1)$$

$$\updownarrow$$

$$P(X_2 = x_2 \mid X_1 = x_1) = P(X_2 = x_2)$$

$$\updownarrow$$

$$P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1)P(X_2 = x_2)$$



Example: Conditional Probability

- Four dice $\omega_1, \omega_2, \omega_3, \omega_4$

$$x = \omega_1 + \omega_2 \quad : \text{sum of the first two dice}$$

$$y = \omega_1 + \omega_2 + \omega_3 + \omega_4 \quad : \text{sum of all four dice}$$

$$\text{probability of } \begin{bmatrix} x \\ y \end{bmatrix} = ?$$

Example: Conditional Probability

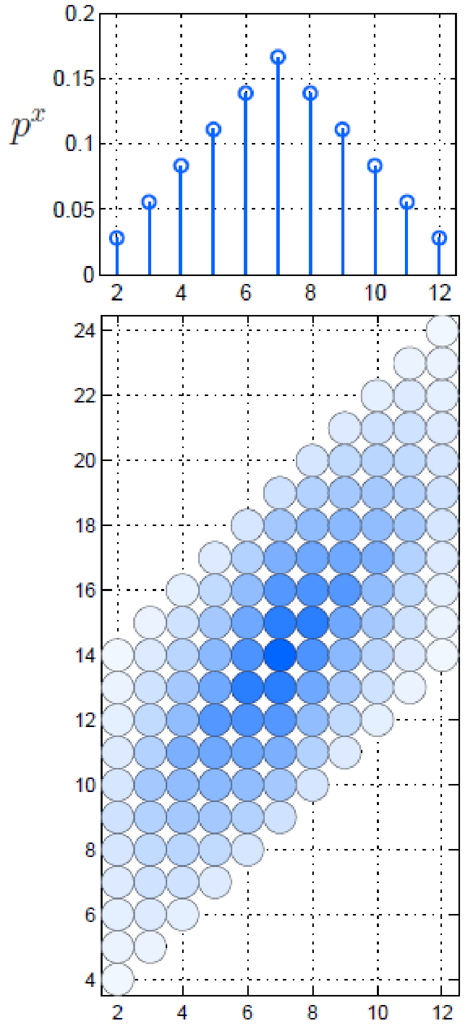
- Marginal probability

$$P_X(x) = \sum_y P_{XY}(x, y)$$

$x = \omega_1 + \omega_2$: sum of the first two dice

$y = \omega_1 + \omega_2 + \omega_3 + \omega_4$: sum of all four dice

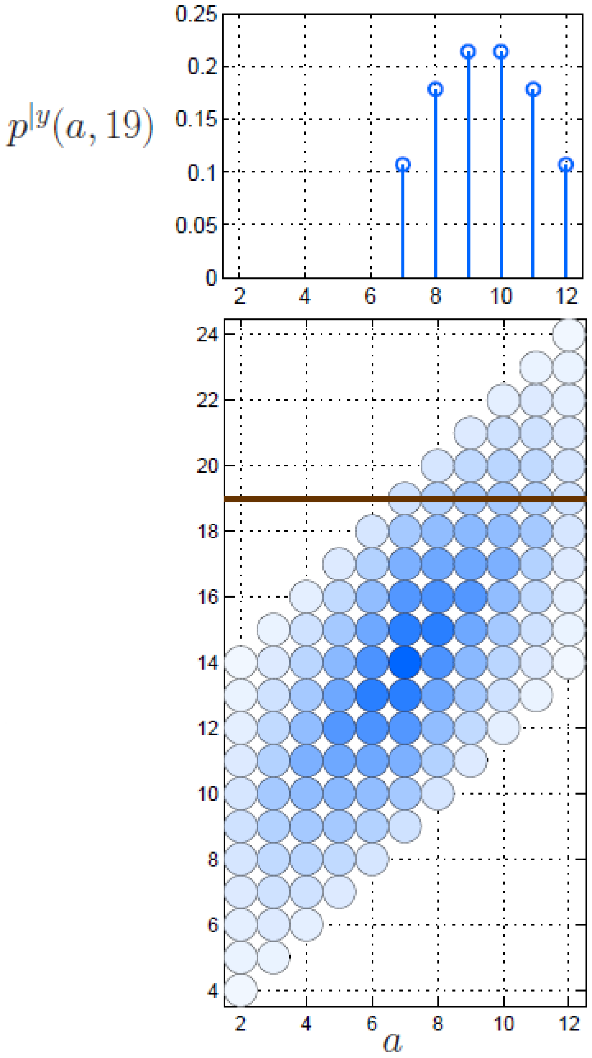
probability of $\begin{bmatrix} x \\ y \end{bmatrix} = ?$



Example: Conditional Probability

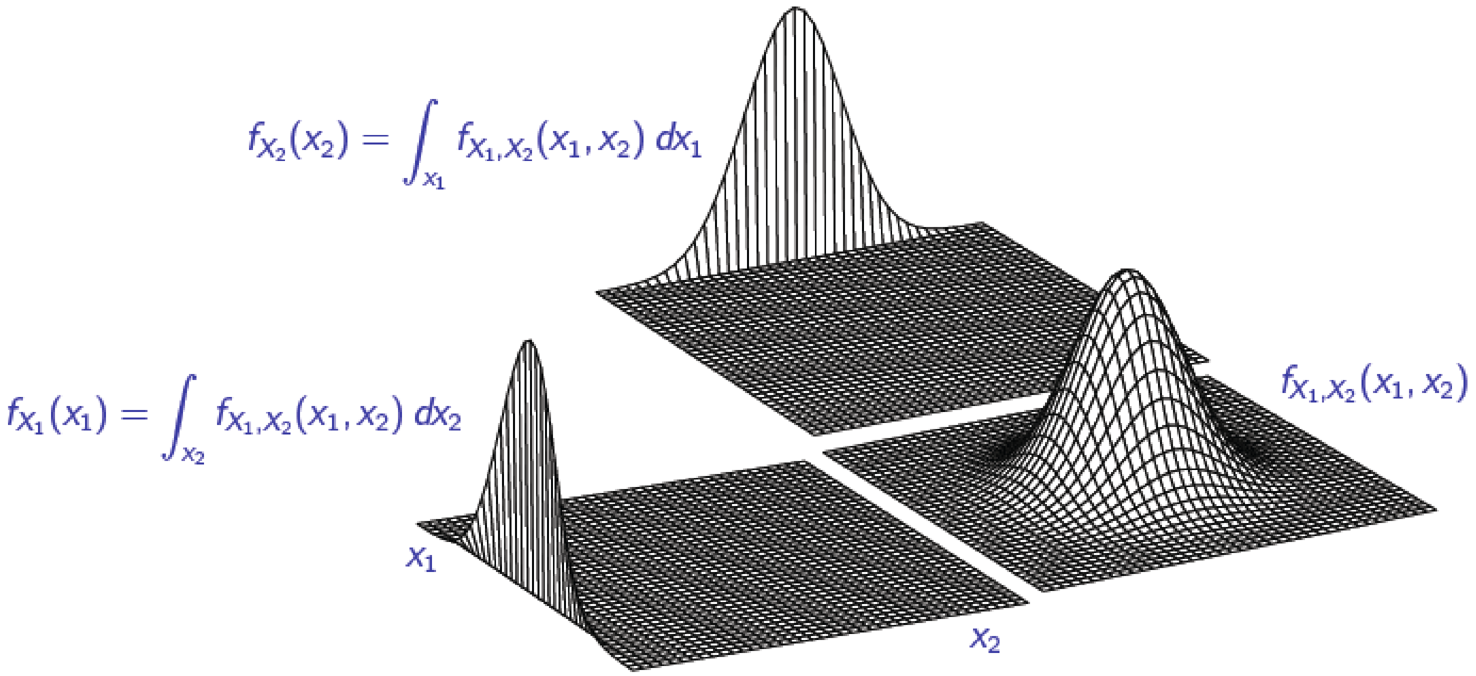
- Conditional probability
  - Suppose we measured $y = 19$

$$P_{X|Y}(x \mid y = 19) = ?$$



Pictorial Explanation

- Marginal probability for continuous r.v.



Pictorial Explanation

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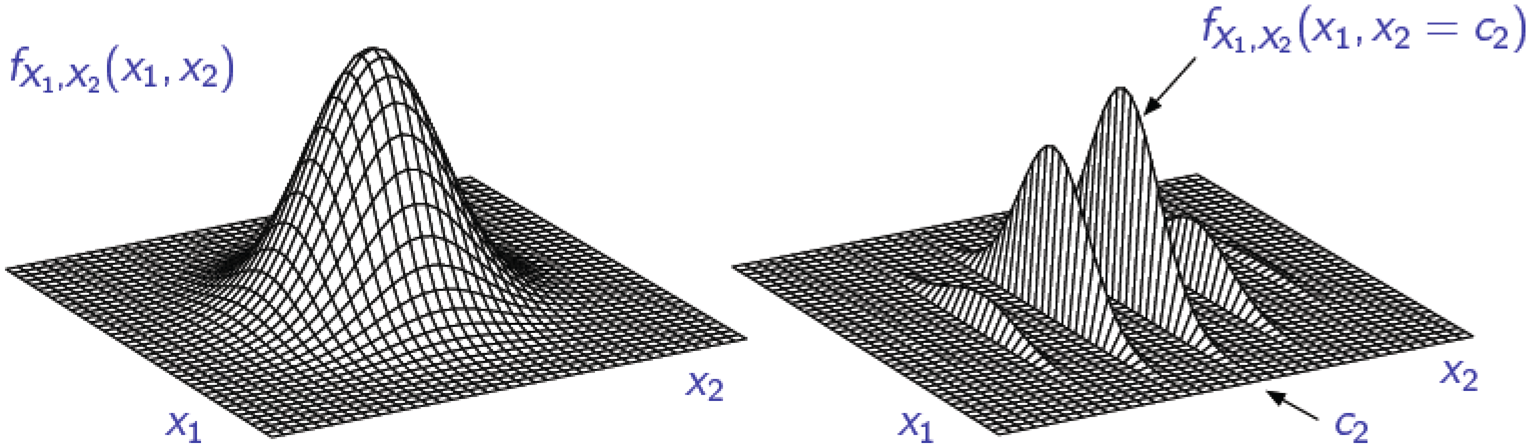


Example: Conditional Probability

- Suppose we have three bins, labeled A, B, and C.
- Two of the bins have only white balls, and one bin has only black balls

1) We take one ball, what is the probability that it is white? (white = 1)

$$P(X_1 = 1) = \frac{2}{3}$$

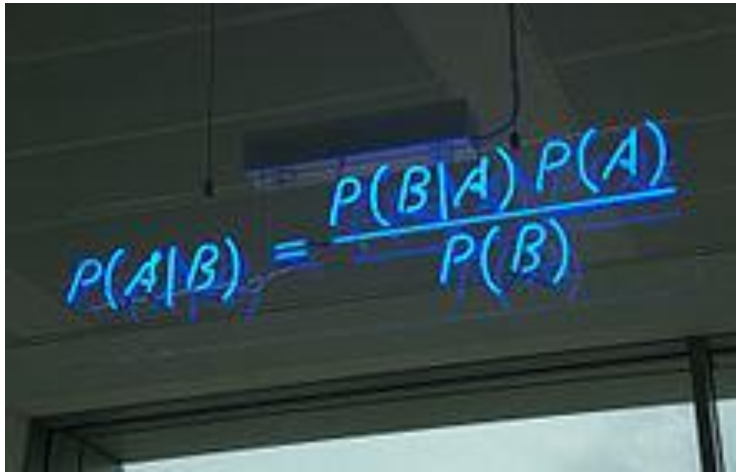
2) When a white ball has been drawn at the first, what is the probability of drawing a white ball from a different bin at the second?

$$P(X_2 = 1 \mid X_1 = 1) = \frac{1}{2}$$

3) When two balls have been drawn from two different bins, what is the probability of drawing two white balls?

$$P(X_1 = 1, X_2 = 1) = P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Bayes Rule



- Enables us to swap A and B in conditional probability

$$P(X_2, X_1) = P(X_2 \mid X_1)P(X_1) = P(X_1 \mid X_2)P(X_2)$$

$$\therefore \quad P(X_2 \mid X_1) = \frac{P(X_1 \mid X_2)P(X_2)}{P(X_1)}$$

Example: Bayes Rule

- Suppose that in a group of people, 40% are male and 60% are female.
- 50% of the males are smokers, 30% of the females are smokers.
- Find the probability that a smoker is male

$$\begin{aligned} x &= \text{ M or F} \\ y &= \text{ S or N} \end{aligned} \qquad \begin{aligned} &P(x = \mathrm{M}) = 0.4 \\ &P(x = \mathrm{F}) = 0.6 \\ &P(y = \mathrm{S} \mid x = \mathrm{M}) = 0.5 \\ &P(y = \mathrm{S} \mid x = \mathrm{F}) = 0.3 \end{aligned}$$

$$P(x = \mathrm{M} \mid y = \mathrm{S}) = \,?$$

- Bayes rule + conditional probability

Example: Bayes Rule

- Bayes rule + conditional probability

$$x = \text{M or F}$$
$$y = \text{S or N}$$

$$P(x = \text{M}) = 0.4$$
$$P(x = \text{F}) = 0.6$$
$$P(y = \text{S} \mid x = \text{M}) = 0.5$$
$$P(y = \text{S} \mid x = \text{F}) = 0.3$$

$$P(x = \text{M} \mid y = \text{S}) = ?$$

$$P(x = \text{M} \mid y = \text{S}) = \frac{P(y = \text{S} \mid x = \text{M})P(x = \text{M})}{P(y = \text{S})} = \frac{0.20}{0.38} \approx 0.53$$

$$P(y = \text{S}) = P(y = \text{S} \mid x = \text{M})P(x = \text{M}) + P(y = \text{S} \mid x = \text{F})P(x = \text{F})$$

$$= 0.5 \times 0.4 + 0.3 \times 0.6 = 0.38$$

Linear Transformation of Random Variables

Linear Transformation For Single R. V.

$$X \mapsto Y = aX$$

$$\begin{aligned} E[aX] &= aE[X] \\ \mathrm{var}(aX) &= a^2\mathrm{var}(X) \end{aligned}$$

$$\begin{aligned} \mathrm{var}(X) &= E[(X - E[X])^2] = E[(X - \bar{X})^2] = E[X^2 - 2X\bar{X} + \bar{X}^2] \\ &= E[X^2] - 2E[X\bar{X}] + \bar{X}^2 = E[X^2] - 2E[X]\bar{X} + \bar{X}^2 \\ &= E[X^2] - E[X]^2 \end{aligned}$$

Sum of Two Random Variables X and Y

$$Z = X + Y \text{ (still univariate)}$$

$$\begin{aligned}
E[X+Y] &= E[X] + E[Y] \\
\mathrm{var}(X+Y) &= E[(X+Y-E[X+Y])^2] = E[((X-\bar{X}) + (Y-\bar{Y}))^2] \\
&= E[(X-\bar{X})^2] + E[(Y-\bar{Y})^2] + 2E[(X-\bar{X})(Y-\bar{Y})] \\
&= \mathrm{var}(X) + \mathrm{var}(Y) + 2\mathrm{cov}(X,Y)
\end{aligned}$$

$$\begin{aligned}
\mathrm{cov}(X,Y) &= E[(X-\bar{X})(Y-\bar{Y})] = E[XY - X\bar{Y} - \bar{X}Y + \bar{X}\bar{Y}] \\
&= E[XY] - E[X]\bar{Y} - \bar{X}E[Y] + \bar{X}\bar{Y} = E[XY] - E[X]E[Y]
\end{aligned}$$

- Note: quality control in manufacturing process

$$\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y) + 2\mathrm{cov}(X,Y)$$

Sum of Two Random Variables X and Y

- Remark
  - Variance for univariable
  - Covariance for bivariable

- Covariance for two r.v.

$$\mathrm{cov}(x, y) = E[(x-\mu_x)(y-\mu_y)]$$

- Covariance matrix for random vectors

$$\begin{aligned}\mathrm{cov}(X) = E[(X-\mu)(X-\mu)^T] &= \begin{bmatrix} \mathrm{cov}(X_1, X_1) & \mathrm{cov}(X_1, X_2) \\ \mathrm{cov}(X_2, X_1) & \mathrm{cov}(X_2, X_2) \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{var}(X_1) & \mathrm{cov}(X_1, X_2) \\ \mathrm{cov}(X_2, X_1) & \mathrm{var}(X_2) \end{bmatrix}\end{aligned}$$

- Moments: provide rough clues on probability distribution

$$\int x^k P_x(x)dx \quad \text{or} \quad \sum x^k P_x(x)dx$$

Affine Transformation of Random Vectors

$$y = Ax + b$$

$$\begin{aligned} E[y] &= AE[x] + b \\ \text{cov}(y) &= A\,\text{cov}(x)\,A^T \end{aligned}$$

- IID random variables
  - identically distributed
  - independent

- Suppose $x_1, x_2, \cdots, x_m$ are IID with mean μ and variance σ^2

$$\text{Let } x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \quad \text{then } E[x] = \begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix}, \quad \text{cov}(x) = \begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{bmatrix}$$

Affine Transformation of Random Vectors

- Sum of IID random variables (→ single r.v.)

$$S_m = \frac{1}{m}\sum_{i=1}^{m} x_i \implies S_m = Ax \;\; \text{where } A = \frac{1}{m}\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

$$E[S_m] = AE[x] = \frac{1}{m}\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} \mu \\ \vdots \\ \mu \end{bmatrix} = \frac{1}{m} m\mu = \mu$$

$$\operatorname{var}(S_m) = A\operatorname{cov}(x)\,A^T = A\begin{bmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \ddots & \\ & & & \sigma^2 \end{bmatrix} A^T = \frac{\sigma^2}{m}$$

- Reduce the variance by a factor of $m \implies$ Law of large numbers or central limit theorem

$$\bar{x} \longrightarrow N\left(\mu, \left(\frac{\sigma}{\sqrt{m}}\right)^2\right)$$