Linear Algebra 1

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Some materials from linear algebra review by Prof. Zico Kolter from CMU

Linear Equations

- Set of linear equations (two equations, two unknowns)

$$
\begin{aligned}
4x_1 - 5x_2 &= -13 \\
-2x_1 + 3x_2 &= 9
\end{aligned}
$$

Linear Equations

- Solving linear equations
  - Two linear equations

$$\begin{aligned} 4x_1 - 5x_2 &= -13 \\ -2x_1 + 3x_2 &= 9 \end{aligned}$$

  - In a vector form, $Ax = b$, with

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

Linear Equations

- Solving linear equations
  - Two linear equations

$$
\begin{aligned}
4x_1 - 5x_2 &= -13 \\
-2x_1 + 3x_2 &= 9
\end{aligned}
$$

  - In a vector form, $Ax = b$, with

$$
A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}
$$

  - Solution using inverse

$$
\begin{aligned}
Ax &= b \\
A^{-1}Ax &= A^{-1}b \\
x &= A^{-1}b
\end{aligned}
$$

Linear Equations

- Solving linear equations
  - Two linear equations

$$4x_1 - 5x_2 = -13$$
$$-2x_1 + 3x_2 = 9$$

  - In a vector form, $Ax = b$, with

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

  - Solution using inverse

$$Ax = b$$
$$A^{-1}Ax = A^{-1}b$$
$$x = A^{-1}b$$

  - Don't worry here about how to compute matrix inverse
  - We will use a numpy to compute

Linear Equations in Python

$$\begin{aligned} 4x_1 - 5x_2 &= -13 \\ -2x_1 + 3x_2 &= 9 \end{aligned}$$

$$A = \begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} -13 \\ 9 \end{bmatrix}$$

```
import numpy as np
```

```
A = np.array([[4, -5],
              [-2, 3]])
b = np.array([[-13],
              [9]])

x = np.linalg.inv(A).dot(b)
print(x)
```

```
[[ 3.]
 [ 5.]]
```

```
A = np.asmatrix(A)
b = np.asmatrix(b)

x = A.I*b
print(x)
```

```
[[ 3.]
 [ 5.]]
```

System of Linear Equations

- Consider a system of linear equations

$$
\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}
$$

- Can be written in a matrix form as $y = Ax$, where

$$
y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
$$

Elements of a Matrix

- Can write a matrix in terms of its columns

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$$

- Careful, a_i here corresponds to an entire vector $a_i \in \mathbb{R}^m$
- Similarly, can write a matrix in terms of rows

$$A = \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_m^T & - \end{bmatrix}$$

- $b_i \in \mathbb{R}^n$

Vector-Vector Products

- Inner product: $x, y \in \mathbb{R}^n$

$$x^T y = \sum_{i=1}^{n} x_i \, y_i \qquad \in \mathbb{R}$$

```
x = np.array([[1],
              [1]])
y = np.array([[2],
              [3]])

print(x.T.dot(y))
```

```
[[5]]
```

```
x = np.asmatrix(x)
y = np.asmatrix(y)

print(x.T*y)
```

```
[[5]]
```

Matrix-Vector Products

- $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \Leftrightarrow Ax \in \mathbb{R}^m$

- Writing A by rows, each entry of Ax is an *inner product* between x and a row of A

$$A = \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_m^T & - \end{bmatrix}, \qquad Ax \in \mathbb{R}^m = \begin{bmatrix} b_1^T x \\ b_2^T x \\ \vdots \\ b_m^T x \end{bmatrix}$$

Matrix-Vector Products

- $A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^n \Leftrightarrow Ax \in \mathbb{R}^m$

- Writing A by columns, Ax is a *linear combination* of the columns of A, with coefficients given by x

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}, \qquad Ax \in \mathbb{R}^m = \sum_{i=1}^{n} a_i x_i$$

Symmetric Matrices

- Symmetric matrix:

$$A \in \mathbb{R}^{n \times n} \text{ with } A = A^T$$

- Arise naturally in many settings

- For $A \in \mathbb{R}^{m \times n}$,

$$A^T A \in \mathbb{R}^{n \times n} \text{ is symmetric}$$

Norms (Strength or Distance in Linear Space)

- A vector norm is any function $f: \mathbb{R}^n \Longrightarrow \mathbb{R}$ with

  1. $f(x) \geq 0$ and $f(x) = 0 \quad \Longleftrightarrow \quad x = 0$
  2. $f(ax) = |a| f(x)$ for $a \in \mathbb{R}$
  3. $f(x + y) \leq f(x) + f(y)$

- l_2 norm

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

- l_1 norm

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

- $\|x\|$ measures length of vector (from origin)

Norms in Python

```
x = np.array([[4],
              [3]])

np.linalg.norm(x, 2)
```

```
5.0
```

```
np.linalg.norm(x, 1)
```

```
7.0
```

Orthogonality

- Two vectors $x, y \in \mathbb{R}^n$ are *orthogonal* if

$$x^T y = 0$$

- They are *orthonormal* if

$$x^T y = 0 \quad \text{and} \quad \|x\|_2 = \|y\|_2 = 1$$

Angle between Vectors

- For any $x, y \in \mathbb{R}^n$,

$$|x^T y| \leq \|x\| \, \|y\|$$

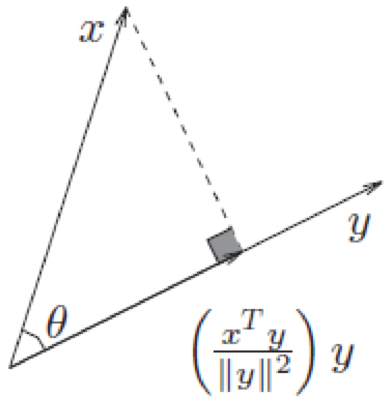
- (unsigned) angle between vectors in $\mathbb{R}^n$ defined as

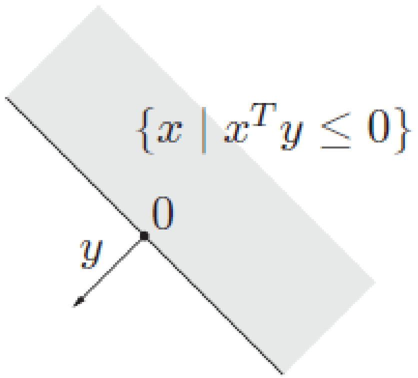
$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \, \|y\|}$$

$$\text{thus } x^T y = \|x\| \, \|y\| \cos\theta$$

Angle between Vectors

$$\theta = \angle(x, y) = \cos^{-1} \frac{x^T y}{\|x\| \|y\|}$$





- $\{x | x^T y \leq 0\}$ defines a half space with outward normal vector y, and boundary passing through 0

Linear Algebra 2

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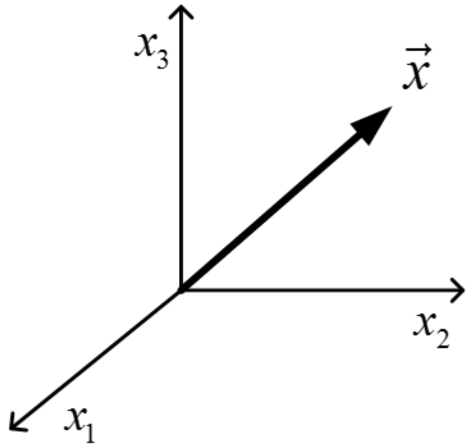
Vector

- Vector

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Matrix and (Linear) Transformation

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

$$\vec{y} = M\vec{x}$$

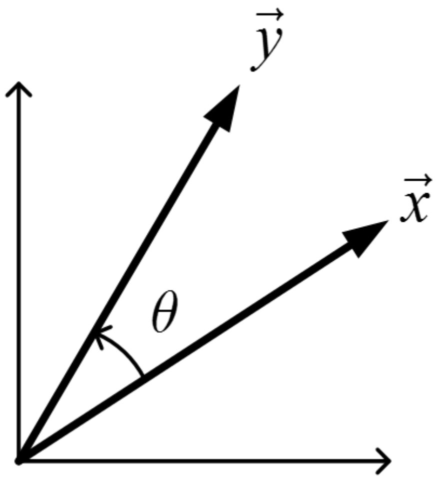
$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Given		Interpret
linear transformation	$\longrightarrow$	matrix
matrix	$\longrightarrow$	linear transformation

$\vec{x}$	linear transformation	$\vec{y}$
input	$\Longrightarrow$	output

Rotation

- Is a rotation operation linear?

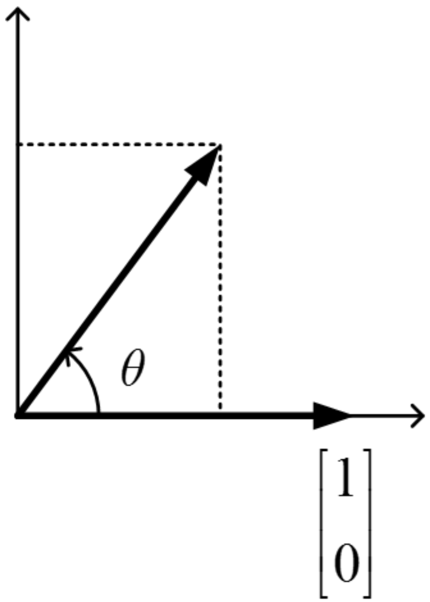


- Rotation matrix: $M = R(\theta)$
- Transformation: $\vec{y} = R(\theta)\vec{x}$

Rotation

- To find matrix $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$

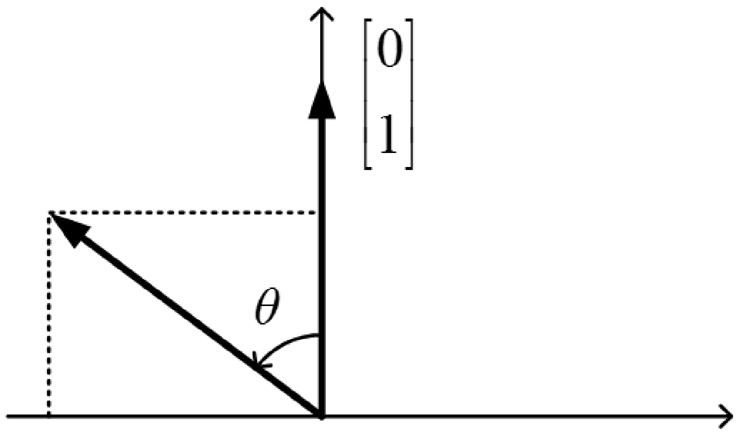


$$\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix} = R(\theta)\begin{bmatrix}1\\0\end{bmatrix}$$

Rotation

- To find matrix $M = R(\theta)$

$$\vec{y} = R(\theta)\vec{x}$$



$$\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} = R(\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Rotation

- To find matrix $M = R(\theta)$

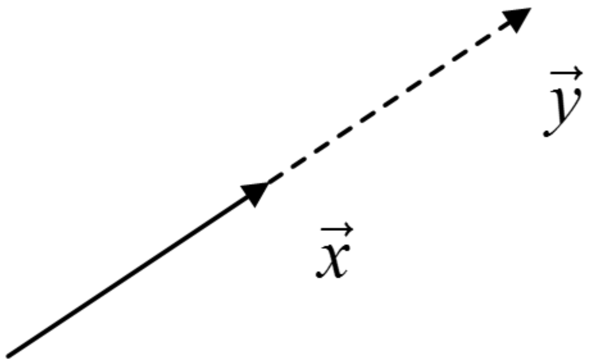
$$M\vec{x}_1 = \vec{y}_1 \qquad \qquad \Rightarrow M\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} = \begin{bmatrix} \vec{y}_1 & \vec{y}_2 \end{bmatrix}$$
$$M\vec{x}_2 = \vec{y}_2$$

$$\Longrightarrow \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = R(\theta)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R(\theta)$$

- Note on how to find a matrix from two vectors and their linearly-transformed ones

Stretch/Compress

- Stretch/Compress
  - keep the direction



$$\vec{y} = \underset{\substack{\uparrow \\ \text{scalar (not matrix)}}}{k}\vec{x}$$

$$\vec{y} = kI\vec{x} \qquad \text{where } I = \text{ Identity martix}$$

$$\vec{y} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \vec{x}$$

- Still represented by a matrix

Stretch/Compress: Example

- T: stretch by a along $\hat{x}$-direction & stretch by b along $\hat{y}$-direction
- Compute the corresponding matrix A

$$\begin{bmatrix} ax_1 \\ bx_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Longrightarrow A = ?$$
$$= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$
$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$
$$A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

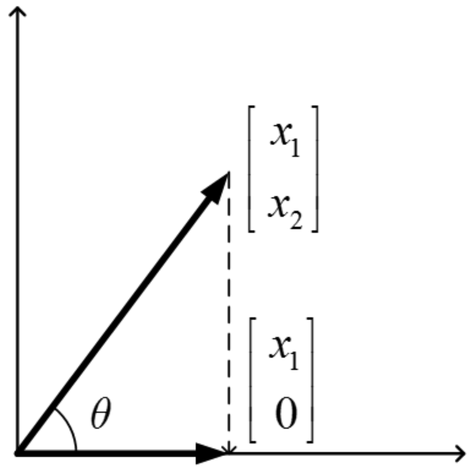
- More importantly, can you think of the corresponding transformation T by looking at $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$?

Projection

- Is a projection operation linear?

- Suppose P: Projection onto $\hat{x}$ - axis

$$\underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \overset{P}{\Longrightarrow} \underset{\vec{y}}{\begin{bmatrix} x_1 \\ 0 \end{bmatrix}}$$



$$\vec{y} = P\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Multiple Transformations

- T_1: transformation 1 of matrix M_1
- T_2: transformation 2 of matrix M_2
- T : Do transformation 1, followed by transformation 2

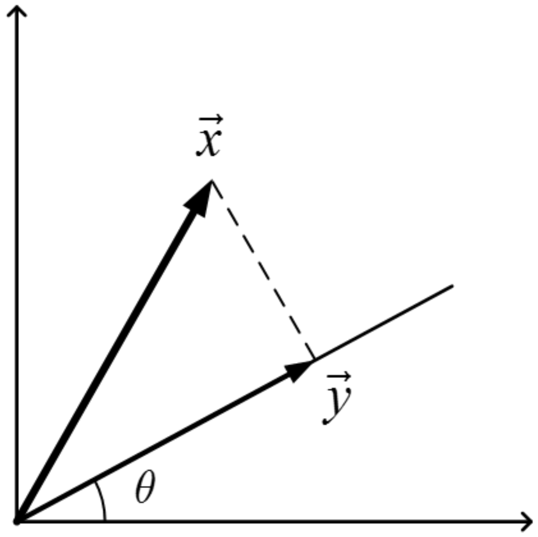
$$\vec{x} \xrightarrow{T_1} \vec{y} \xrightarrow{T_2} \vec{z}$$

$$\begin{aligned}
\vec{y} &= M_1\vec{x} \\
\vec{z} &= M_2\vec{y} = M_2M_1\vec{x} \\
&= M\vec{x}
\end{aligned}$$

$$\therefore\ M = M_2M_1$$

$$\longleftarrow$$

Example: Projection onto Vector = $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$



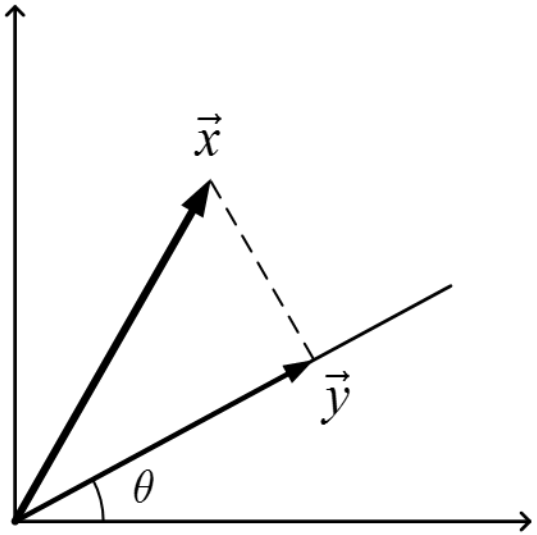
$$P\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ \cos\theta\sin\theta \end{bmatrix}$$

$$P\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$$

$$P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Example: Projection onto Vector = $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

- Another way to find this projection matrix



$$\vec{x} \xrightarrow{R(-\theta)} \vec{x}' \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \vec{x}'' \xrightarrow{R(\theta)} \vec{y}$$

$$\begin{aligned}
\vec{y} &= R(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R(-\theta)\vec{x} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}
\end{aligned}$$

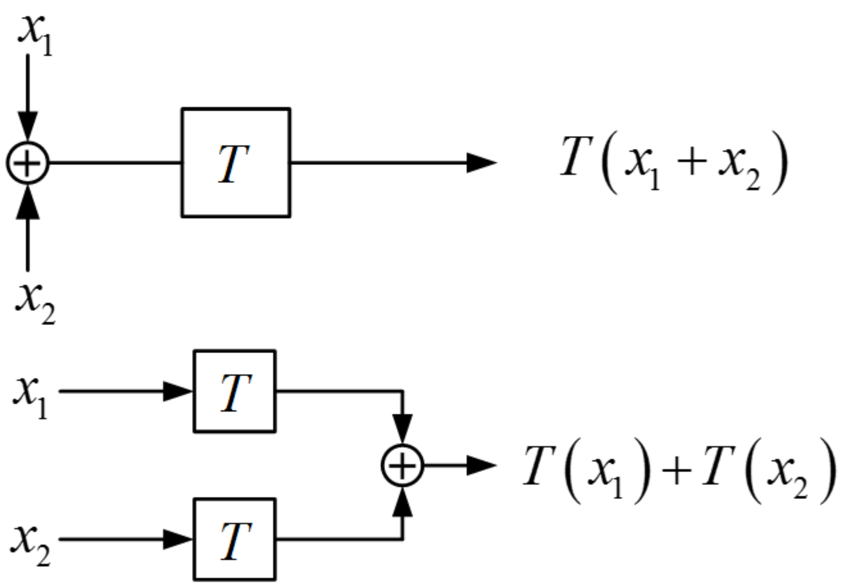
Linear Transformation

- See if the given transformation is linear
  - A linear system makes our life much easier

- Superposition
- Homogeneity

Linear Transformation

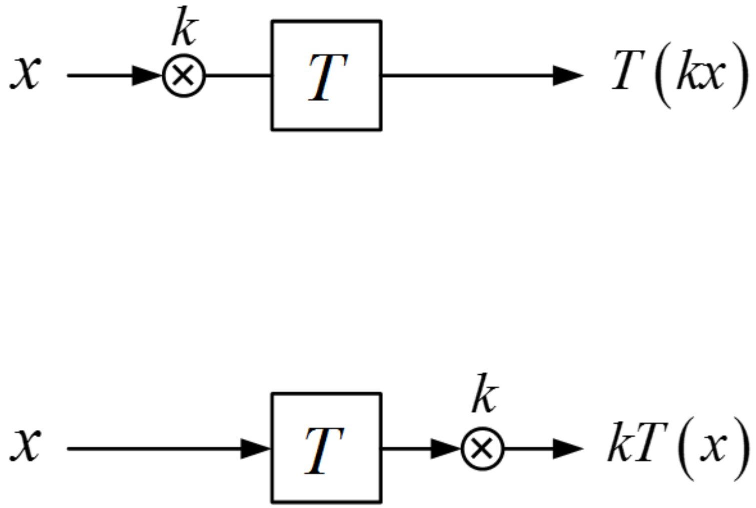
- Superposition



$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

Linear Transformation

- Homogeneity



$$T(kx) = kT(x)$$

Linear Transformation

- Linear vs. Non-linear

linear	non-linear
$f(x) = 0$	$f(x) = x + c$
$f(x) = kx$	$f(x) = x^2$
$f(x(t)) = \frac{dx(t)}{dt}$	$f(x) = \sin x$
$f(x(t)) = \int_a^b x(t)dt$	

Linear Transformation

- If $\vec{v}_1$ and $\vec{v}_2$ are basis, and we know $T(\vec{v}_1) = \vec{\omega}_1$ and $T(\vec{v}_2) = \vec{\omega}_2$

- Then, for any $\vec{x}$

$$
\begin{aligned}
\vec{x} \quad &= a_1\vec{v}_1 + a_2\vec{v}_2 \qquad\qquad (a_1 \text{ and } a_2 \text{ unique}) \\
T(\vec{x}) \quad &= T(a_1\vec{v}_1 + a_2\vec{v}_2) \\
&= a_1T(\vec{v}_1) + a_2T(\vec{v}_2) \\
&= a_1\vec{\omega}_1 + a_2\vec{\omega}_2
\end{aligned}
$$

- This is why a linear system makes our life much easier
- Only thing that we need is to observe how basis are linearly-transformed

Eigenvalue and Eigenvector

$$A\vec{v} = \lambda\vec{v}$$

$A\vec{v}$ parallel to $\vec{v}$

$$\lambda \quad = \quad \begin{cases} \text{positive} \\ 0 \\ \text{negative} \end{cases}$$

$\lambda\vec{v}$: stretched vector (same direction with $\vec{v}$)

$A\vec{v}$: linearly transformed vector (generally rotate + stretch)

Linear Transformation

- If $\vec{v}_1$ and $\vec{v}_2$ are basis and eigenvectors, and we know $T(\vec{v}_1) = \vec{\omega}_1 = \lambda_1\vec{v}_1$and $T(\vec{v}_2) = \vec{\omega}_2 = \lambda_2\vec{v}_2$

- Then, for any $\vec{x}$

$$\begin{aligned}
\vec{x} &= a_1\vec{v}_1 + a_2\vec{v}_2 \qquad\qquad (a_1 \text{ and } a_2 \text{ unique}) \\
\\
T(\vec{x}) &= T(a_1\vec{v}_1 + a_2\vec{v}_2) \\
&= a_1T(\vec{v}_1) + a_2T(\vec{v}_2) \\
&= a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2 \\
&= \lambda_1a_1\vec{v}_1 + \lambda_2a_2\vec{v}_2
\end{aligned}$$

- This is why a linear system makes our life much easier
- Only thing that we need is to observe how each basis is independently scaled

How to Compute Eigenvalue and Eigenvector

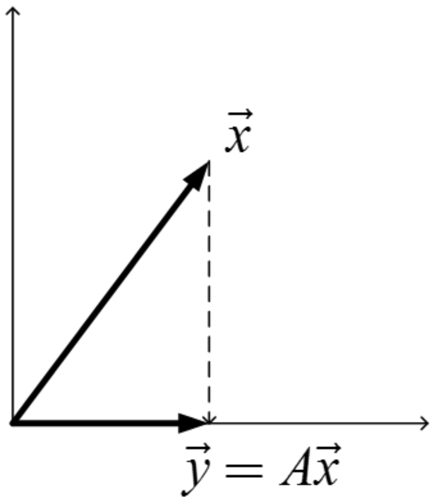
$$A\vec{v} = \lambda\vec{v} = \lambda I\vec{v}$$
$$A\vec{v} - \lambda I\vec{v} = (A - \lambda I)\vec{v} = 0$$

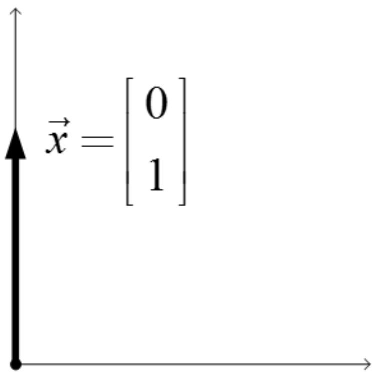
$$\Longrightarrow \quad \begin{aligned} &A - \lambda I = 0 \text{ or} \\ &\vec{v} = 0 \text{ or} \\ &(A - \lambda I)^{-1} \text{ does not exist} \end{aligned}$$

$$\Longrightarrow \quad \det(A - \lambda I) = 0$$

Example: Eigen Analysis of $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

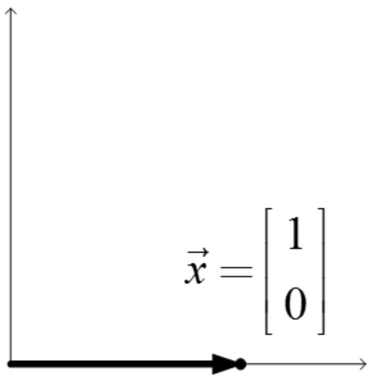
- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$: projection onto $\hat{x}$ - axis
- Find eigenvalues and eigenvectors of A.





$$\vec{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = A\vec{x} = 0 \cdot \vec{x}$$

$$\lambda_1 = 0 \text{ and } \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



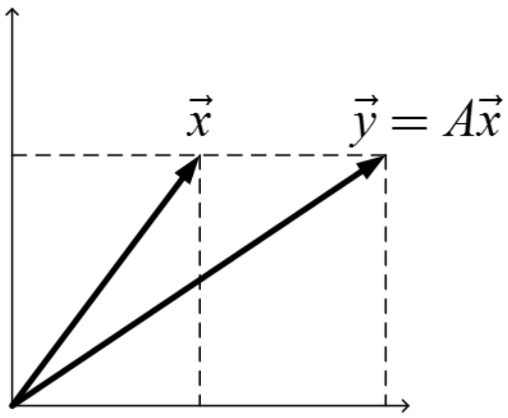
$$\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A\vec{x} = 1 \cdot \vec{x}$$

$$\lambda_2 = 1 \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

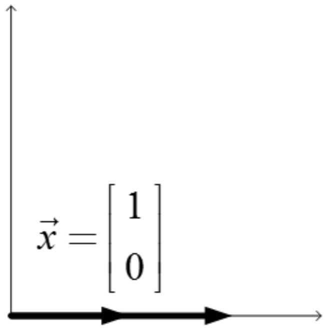
Example: Eigen Analysis of $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

- $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$: stretch by 2 along $\vec{x}$- axis

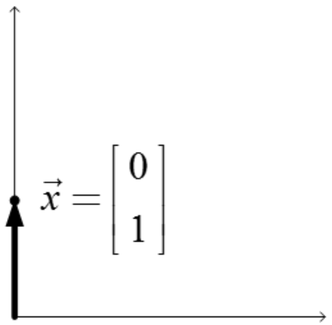
  stretch by 1 along $\vec{y}$- axis



- Find eigenvalues and eigenvectors.



$\lambda_1 = 2$ and $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$\lambda_2 = 1$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

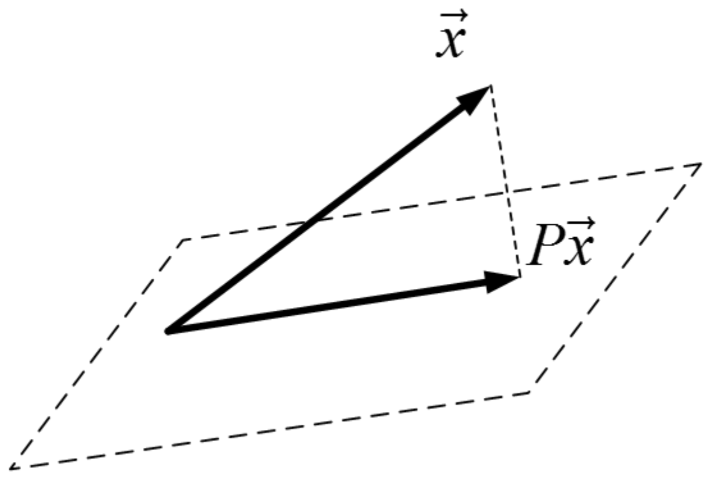
Eigen Analysis in Python

```python
A = np.array([[2, 0],
              [0, 1]])
D, V = np.linalg.eig(A)

idx = np.argsort(-D)
D = D[idx]
V = V[idx]

print('D :', D)
print('V :', V)
```

```
D : [ 2.  1.]
V : [[ 1.  0.]
 [ 0.  1.]]
```

Example: Eigen Analysis of Projection

- Projection onto the plane
- Find eigenvalues and eigenvectors

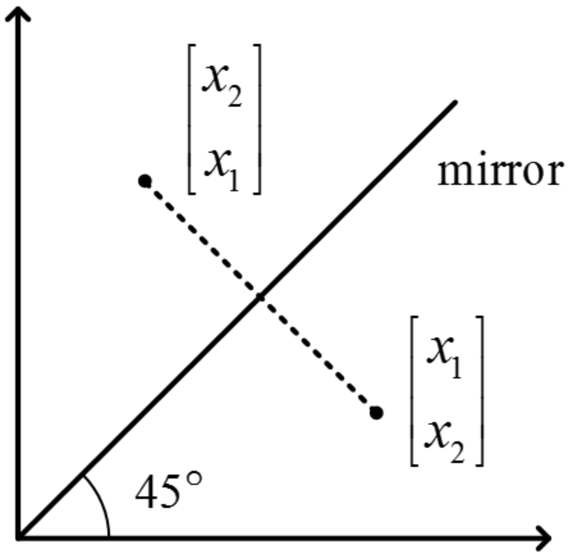


- For any $\vec{x}$ in the plane, $P\vec{x} = \vec{x} \rightarrow \lambda = 1$
- For any $\vec{x}$ perpendicular to the plane, $P\vec{x} = \vec{0} \rightarrow \lambda = 0$

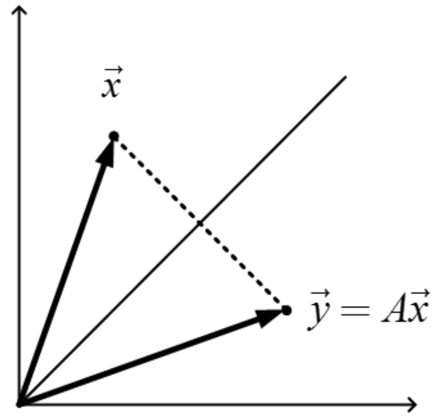
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- What kind of a linear transformation?

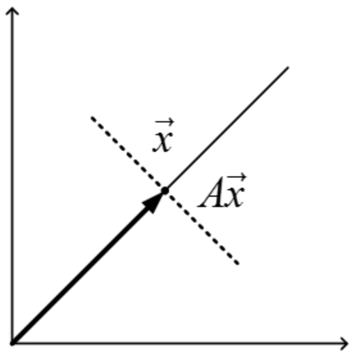
$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



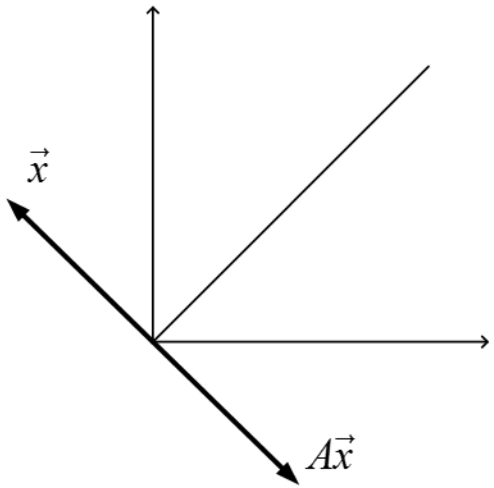
Example: Eigen Analysis of Mirror



- Eigenvalues and eigenvectors?
  - can $\vec{x}$ be an eigenvector?



$$A\vec{x} = \vec{x}, \quad \lambda = 1$$

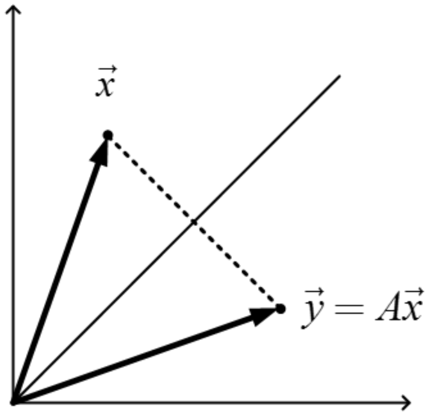


$$A\vec{x} = -\vec{x}, \quad \lambda = -1$$

Example: Eigen Analysis of Mirror

- Side note: Matrix A can be seen as a multiple transformations

$$A = R(45)MR(-45)$$



$$R(45) = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$M \quad : \text{ mirror along } \hat{x}\text{- axis, } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

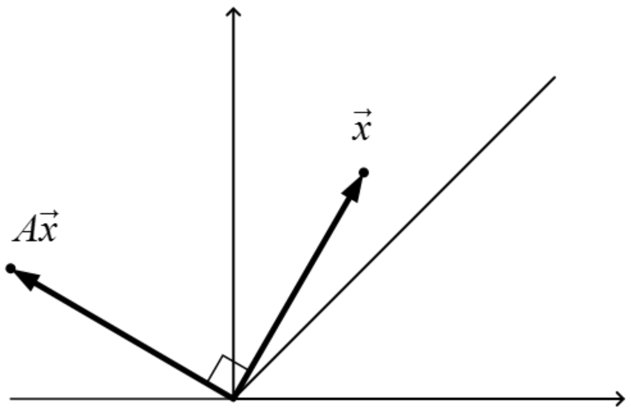
$$A = \left(\frac{1}{\sqrt{2}}\right)^2 \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- What kind of a linear transformation?

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = R\left(\frac{\pi}{2}\right) = R(90^\circ) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$$



$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- What kind of a linear transformation?

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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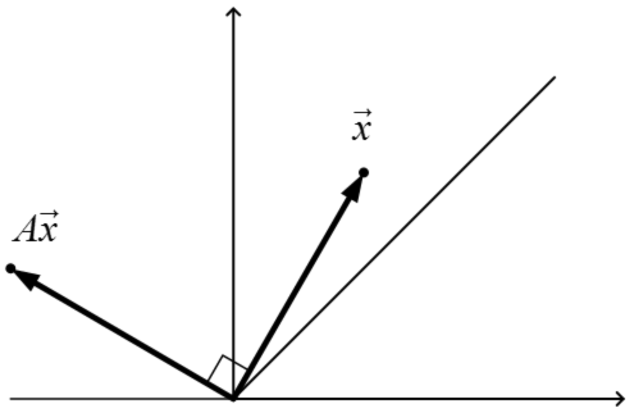
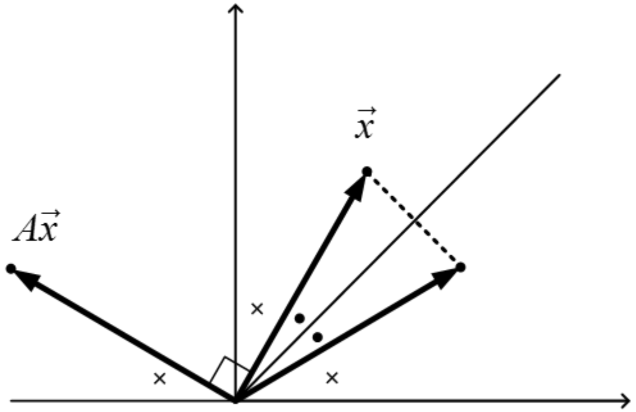


- Multiple transformations

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$



Example: Eigen Analysis of Rotation

- What kind of a linear transformation?

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A = R\left(\frac{\pi}{2}\right) = R(90^\circ) = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$$

- Eigenvalues: complex numbers

$$\begin{aligned} \Rightarrow \quad \det(A - \lambda I) &= 0 \\ \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} &= \lambda^2 + 1 = 0 \\ &\therefore \ \lambda = \pm i \end{aligned}$$

- What is the physical meaning?

Linear Transformation and Eigenvectors

- If $\vec{v}_1$ and $\vec{v}_2$ are basis and eigenvectors, and we know $T(\vec{v}_1) = \vec{\omega}_1 = \lambda_1 \vec{v}_1$ and $T(\vec{v}_2) = \vec{\omega}_2 = \lambda_2 \vec{v}_2$

- Then, for any $\vec{x}$

$$
\begin{aligned}
\vec{x} &= a_1\vec{v}_1 + a_2\vec{v}_2 && (a_1 \text{ and } a_2 \text{ unique}) \\
T(\vec{x}) &= T(a_1\vec{v}_1 + a_2\vec{v}_2) \\
&= a_1 T(\vec{v}_1) + a_2 T(\vec{v}_2) \\
&= a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2 \\
&= \lambda_1 a_1\vec{v}_1 + \lambda_2 a_2\vec{v}_2
\end{aligned}
$$

- This is why a linear system makes our life much easier
- Only thing that we need is to observe how each basis is independently scaled
- (optional) Fourier transform
  - Sinusoids are orthonormal basis and eigenvectors for functions (or signals)

Linear Algebra 3

Industrial AI Lab.

Prof. Seungchul Lee

System of Linear Equations

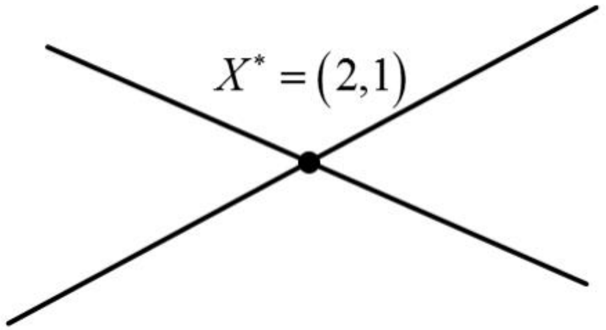
- Well-determined linear systems
- Under-determined linear systems
- Over-determined linear systems

Well-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1^* &= 2 \\ x_2^* &= 1 \end{aligned}$$

- Geometric point of view



Well-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1^* &= 2 \\ x_2^* &= 1 \end{aligned}$$

- Matrix form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{aligned} \quad \xRightarrow{\text{Matrix form}} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$AX = B$$

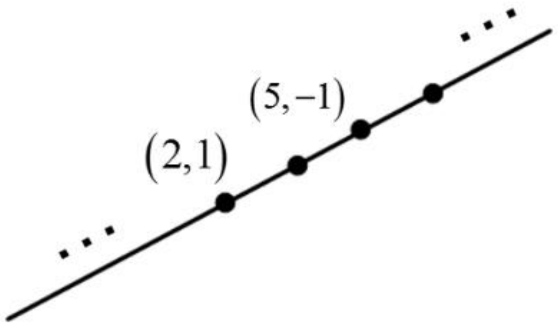
$$\therefore \quad X^* = A^{-1}B \quad \text{if } A^{-1} \text{ exists}$$

Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \quad \Longrightarrow \quad \text{Many solutions}$$

- Geometric point of view



Under-Determined Linear Systems

- System of linear equations

$$2x_1 + 3x_2 = 7 \quad \Longrightarrow \quad \text{Many solutions}$$

- Matrix form

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \overset{\text{Matrix form}}{\Longrightarrow} \quad \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

$$AX = B$$

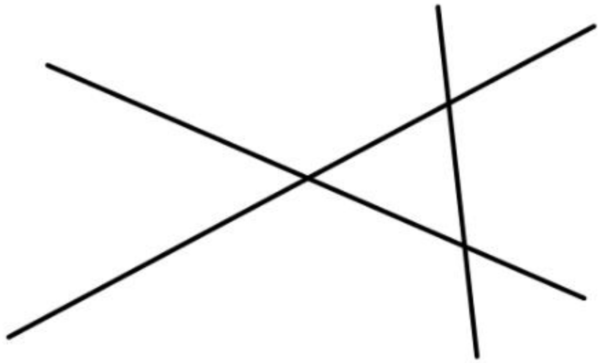
$$\therefore \ \text{Many Solutions when } A \text{ is fat}$$

Over-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \\ x_1 + x_2 &= 4 \end{aligned} \quad \Longrightarrow \quad \text{No solutions}$$

- Geometric point of view



Over-Determined Linear Systems

- System of linear equations

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 + 4x_2 &= 6 \\ x_1 + x_2 &= 4 \end{aligned} \implies \text{No solutions}$$

- Matrix form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \\ a_{31}x_1 + a_{32}x_2 &= b_3 \end{aligned} \quad \overset{\text{Matrix form}}{\implies} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$AX = B$$

$$\therefore \text{ No Solutions when } A \text{ is skinny}$$

Summary of Linear Systems

$$AX = B$$

- Square: Well-determined

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- Fat: Under-determined

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1$$

- Skinny: Over-determined

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Least-Norm Solution

- For under-determined linear system

$$\begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b_1 \quad \text{or} \quad AX = B$$

- Find the solution of $AX = B$ that minimize $\|X\|$ or $\|X\|^2$

- *i.e.*, optimization problem

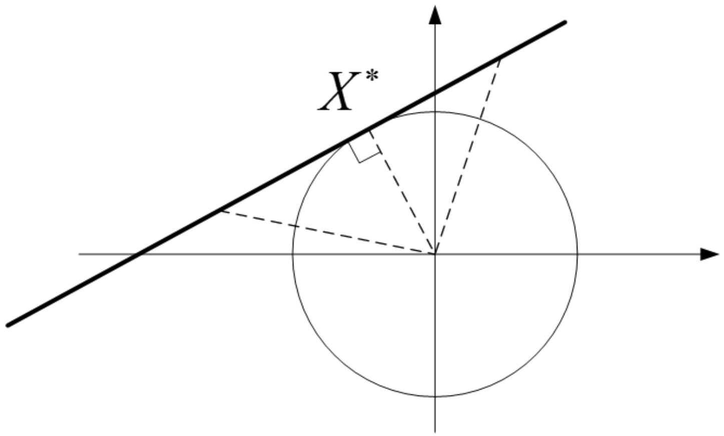
$$\begin{aligned} \min \quad & \|X\|^2 \\ \text{s. t.} \quad & AX = B \end{aligned}$$

Least-Norm Solution

- Optimization problem

$$\begin{aligned} &\min \quad \|X\|^2 \\ &\text{s. t. } AX = B \end{aligned}$$

- Geometric interpretation



- Select one solution among many solutions
- Often control problem

$$X^* = A^T\left(AA^T\right)^{-1}B \quad \text{Least norm solution}$$

Least-Square Solution

- For over-determined linear system

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{or} \quad AX \neq B$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \neq \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- Find X that minimizes $\|E\|$ or $\|E\|^2$ (error)

- *i.e.* optimization problem

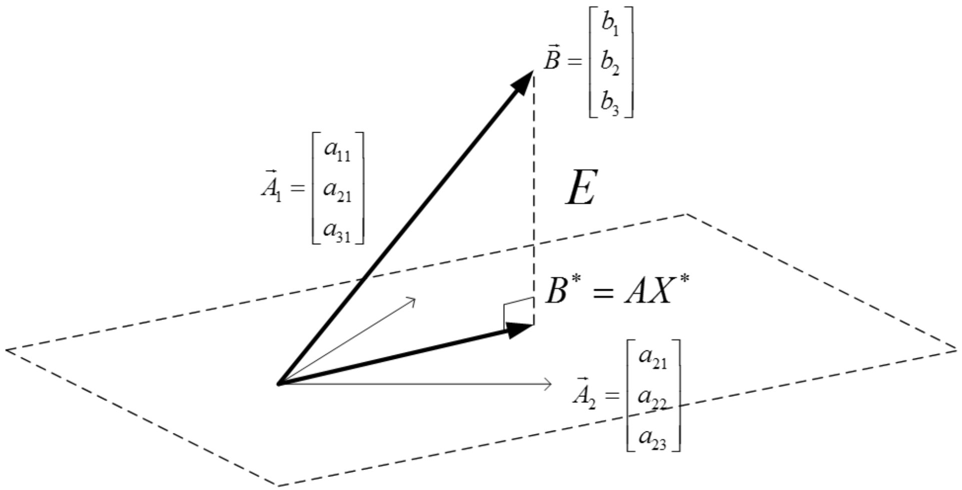
$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$

Least-Square Solution

- *i.e.* optimization problem

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$
$$X^* = \left(A^T A\right)^{-1} A^T B$$
$$B^* = AX^* = A\left(A^T A\right)^{-1} A^T B$$

- Geometric interpretation



- Often estimation problem

Vector Projection onto Y

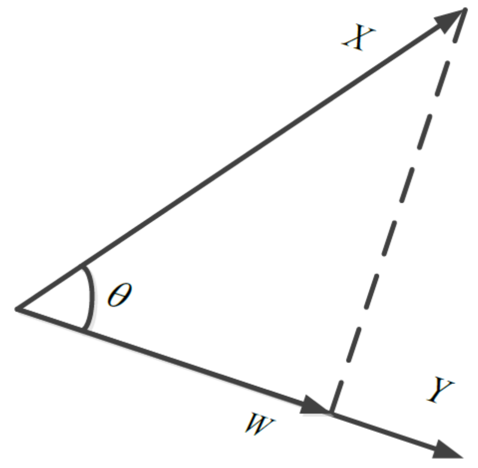
- The vector projection of a vector X on (or onto) a nonzero vector Y is the orthogonal projection of X onto a straight line parallel to Y

$$W = \omega \hat{Y} = \omega \frac{Y}{\|Y\|}, \text{ where } \omega = \|W\|$$

$$\omega = \|X\| \cos\theta = \|X\| \frac{X \cdot Y}{\|X\|\|Y\|} = \frac{X \cdot Y}{\|Y\|}$$

$$W = \omega \hat{Y} = \frac{X \cdot Y}{\|Y\|} \frac{Y}{\|Y\|} = \frac{X \cdot Y}{\|Y\|\|Y\|} Y = \frac{X^T Y}{Y^T Y} Y = \frac{\langle X, Y \rangle}{\langle Y, Y \rangle} Y$$

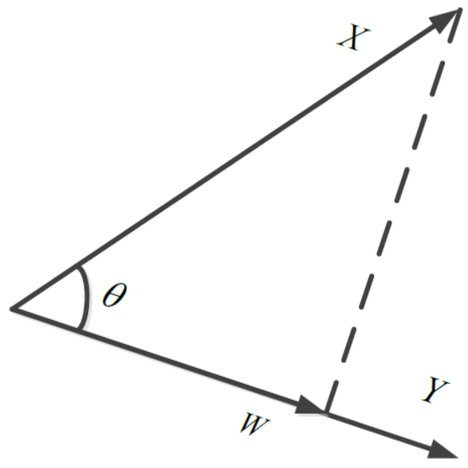
$$= Y \frac{X^T Y}{Y^T Y} = Y \frac{Y^T X}{Y^T Y} = \frac{Y Y^T}{Y^T Y} X = PX$$



Vector Projection onto Y

- Another way of computing ω and W

$$
\begin{aligned}
& Y \perp (X - W) \\
\Longrightarrow\ & Y^T(X - W) = Y^T\left(X - \omega\frac{Y}{\|Y\|}\right) = 0 \\
\Longrightarrow\ & \omega = \frac{Y^T X}{Y^T Y}\|Y\| \\
& W = \omega\frac{Y}{\|Y\|} = \frac{Y^T X}{Y^T Y}Y = \frac{\langle X, Y\rangle}{\langle Y, Y\rangle}Y
\end{aligned}
$$



Orthogonal Projection onto a Subspace

- Projection of B onto a subspace U of span of A_1 and A_2
- Orthogonality

$$\min_X \|E\|^2 = \min_X \|AX - B\|^2$$
$$X^* = \left(A^T A\right)^{-1} A^T B$$
$$B^* = AX^* = A\left(A^T A\right)^{-1} A^T B$$

$$A \perp (AX^* - B)$$
$$A^T (AX^* - B) = 0$$
$$A^T A X^* = A^T B$$
$$X^* = \left(A^T A\right)^{-1} A^T B$$
$$B^* = AX^* = A\left(A^T A\right)^{-1} A^T B$$