# Classification: Perceptron

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# Classification

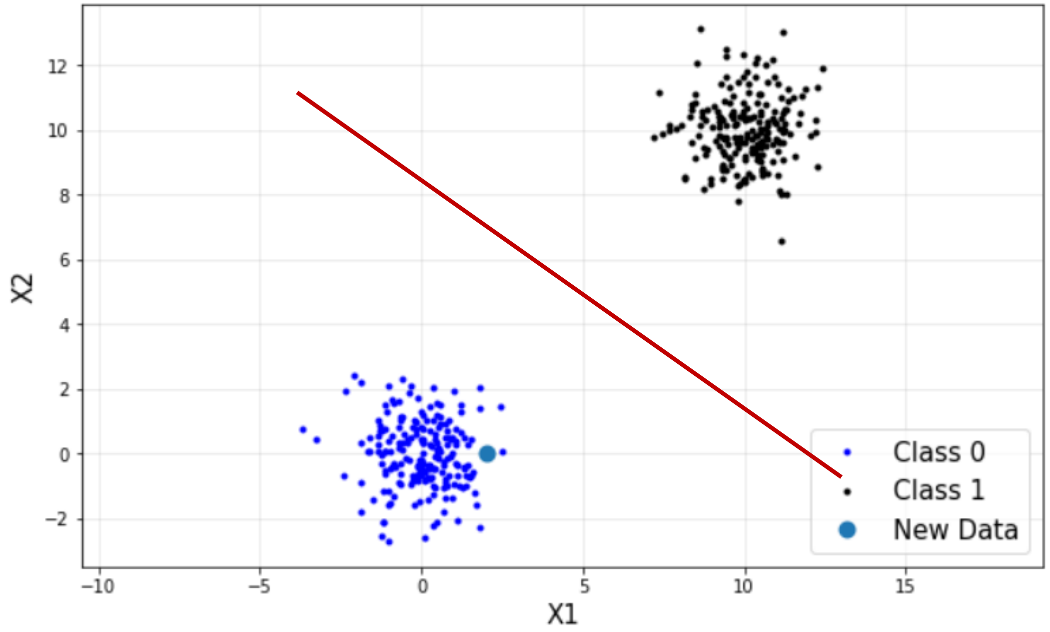
- Where $y$ is a discrete value
  - Develop the classification algorithm to determine which class a new input should fall into
- Start with a binary class problem
  - Later look at multiclass classification problem, although this is just an extension of binary classification
- We could use linear regression
  - Then, threshold the classifier output (i.e. anything over some value is yes, else no)
  - linear regression with thresholding seems to work

# Classification

- We will learn
  - Perceptron
  - Support vector machine (SVM)
  - Logistic regression

- To find

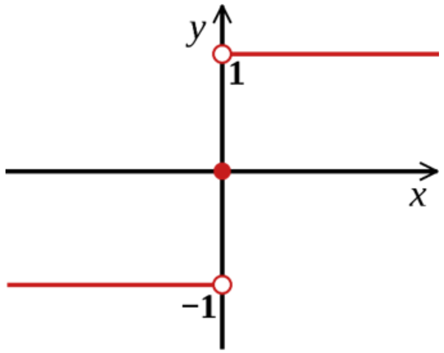
  a classification boundary



# Perceptron

- For input $x = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$ 'attributes of a customer'
- Weights $\omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_d \end{bmatrix}$

$$\text{Approve credit if } \sum_{i=1}^{d} \omega_i x_i > \text{threshold,}$$
$$\text{Deny credit if } \sum_{i=1}^{d} \omega_i x_i < \text{threshold.}$$



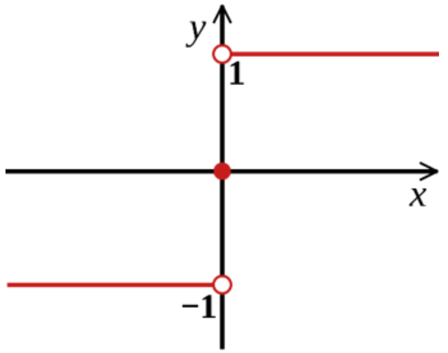

$$h(x) = \text{sign}\left(\left(\sum_{i=1}^{d} \omega_i x_i\right) - \text{threshold}\right) = \text{sign}\left(\left(\sum_{i=1}^{d} \omega_i x_i\right) + \omega_0\right)$$

# Perceptron

$$h(x) = \text{sign}\left(\left(\sum_{i=1}^{d} \omega_i x_i\right) - \text{threshold}\right) = \text{sign}\left(\left(\sum_{i=1}^{d} \omega_i x_i\right) + \omega_0\right)$$

- Introduce an artificial coordinate $x_0 = 1$ :

$$h(x) = \text{sign}\left(\sum_{i=0}^{d} \omega_i x_i\right)$$



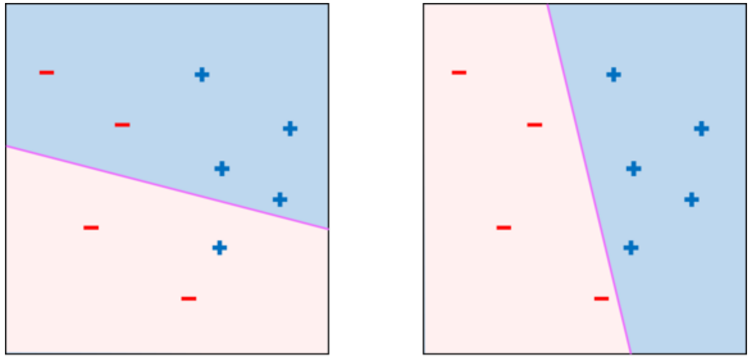

- In a vector form, the perceptron implements

$$h(x) = \text{sign}\left(\omega^T x\right)$$

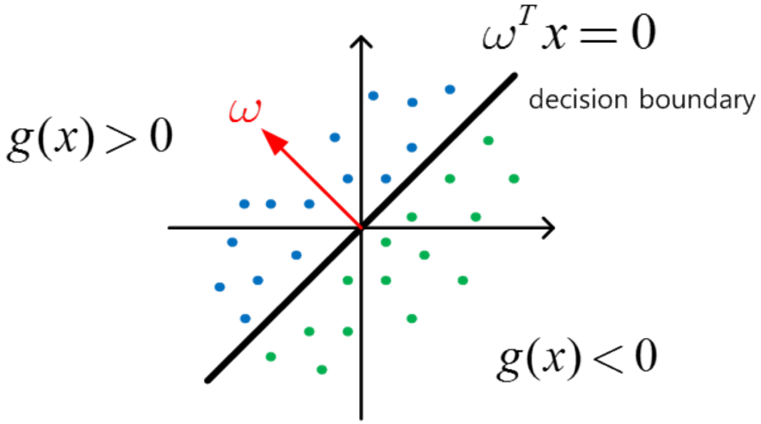
# Perceptron

- Works for linearly separable data

- Hyperplane
  - Separates a D-dimensional space into two half-spaces
  - Defined by an outward pointing normal vector
  - $\omega$ is orthogonal to any vector lying on the hyperplane
  - Assume the hyperplane passes through origin, $\omega^T x = 0$ with $x_0 = 1$

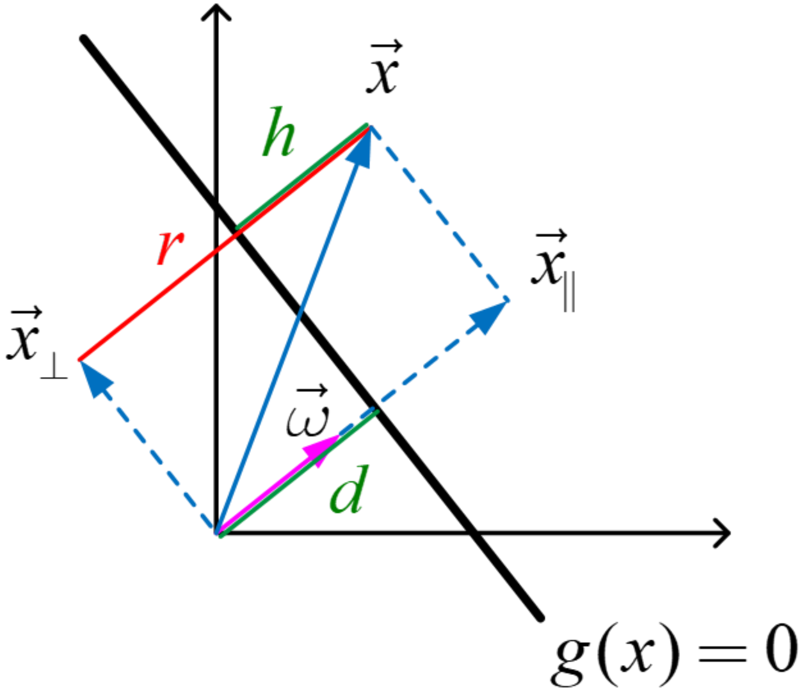


Linearly separable data



# Distance from a Line

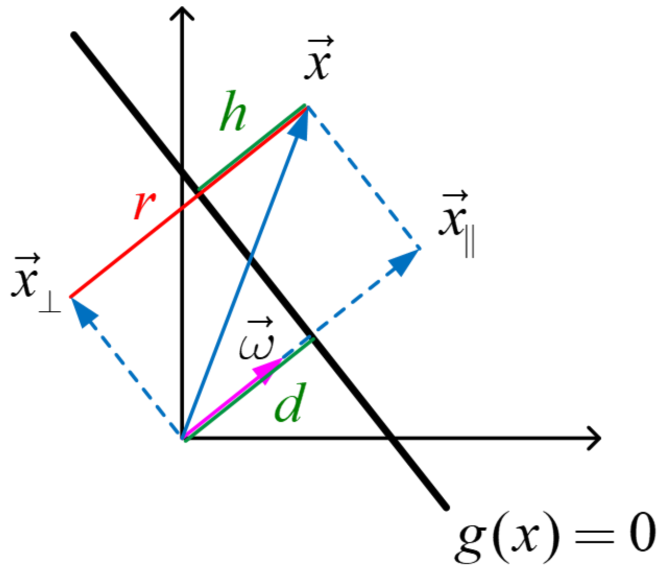
$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies g(x) = \omega^T x + \omega_0 = \omega_1 x_1 + \omega_2 x_2 + \omega_0$$



# $\omega$

- If $\vec{p}$ and $\vec{q}$ are on the decision line

$$g(\vec{p}) = g(\vec{q}) = 0 \quad \Rightarrow \quad \omega_0 + \omega^T \vec{p} = \omega_0 + \omega^T \vec{q} = 0$$
$$\Rightarrow \quad \omega^T (\vec{p} - \vec{q}) = 0$$




$$\therefore\ \omega : \text{normal to the line (orthogonal)}$$
$$\Longrightarrow \ \text{tells the direction of the line}$$

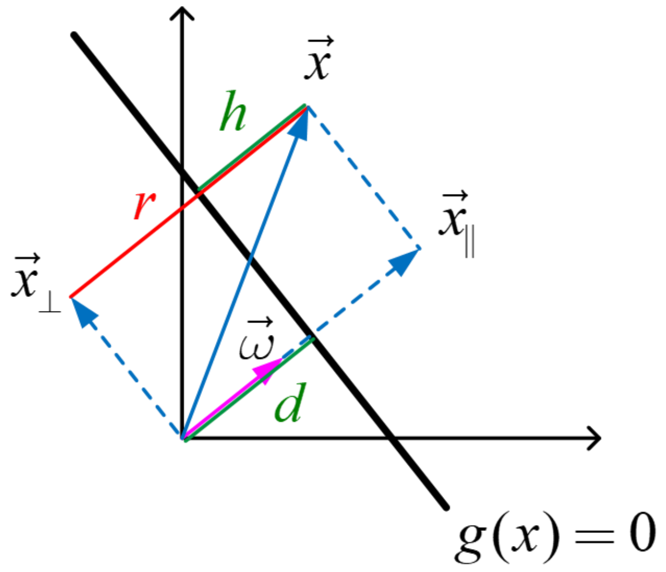
# Signed Distance $d$ from the Origin

- If $x$ is on the line and $x = d\frac{\omega}{\|\omega\|}$ (where $d$ is a normal distance from the origin to the line)

$$g(x) = \omega_0 + \omega^T x = 0$$

$$\Rightarrow \omega_0 + \omega^T d\frac{\omega}{||\omega||} = \omega_0 + d\frac{\omega^T\omega}{||\omega||} = \omega_0 + d||\omega|| = 0$$

$$\therefore d = -\frac{\omega_0}{||\omega||}$$

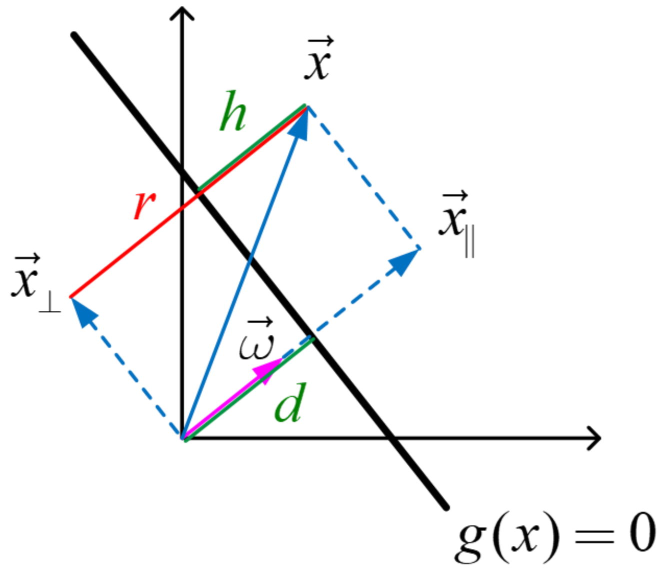


# Distance from a Line: $h$

- for any vector of $x$

$$x = x_{\perp} + r\frac{\omega}{||\omega||}$$

$$\omega^T x = \omega^T \left( x_{\perp} + r\frac{\omega}{||\omega||} \right) = r\frac{\omega^T\omega}{||\omega||} = r||\omega||$$




$$\begin{aligned} g(x) &= \omega_0 + \omega^T x \\ &= \omega_0 + r||\omega|| \qquad (r = d + h) \\ &= \omega_0 + (d+h)||\omega|| \\ &= \omega_0 + \left( -\frac{\omega_0}{||\omega||} + h \right) ||\omega|| \\ &= h||\omega|| \end{aligned}$$

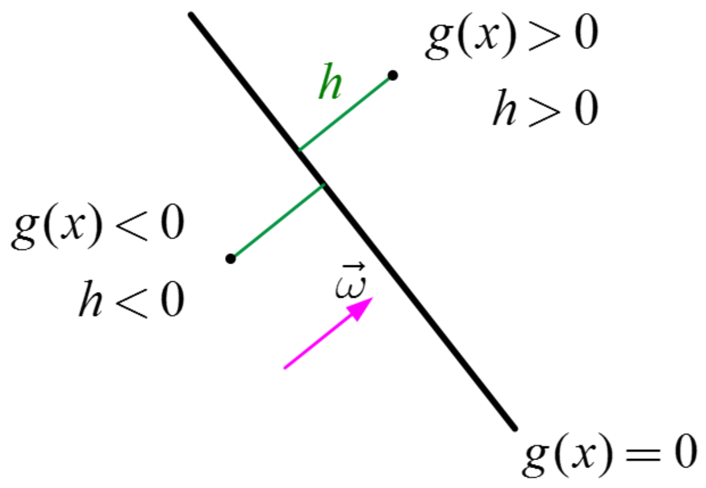
$$\therefore\ h = \frac{g(x)}{||\omega||} \implies \text{orthogonal signed distance from the line}$$

# Sign

- Sign with respect to a line

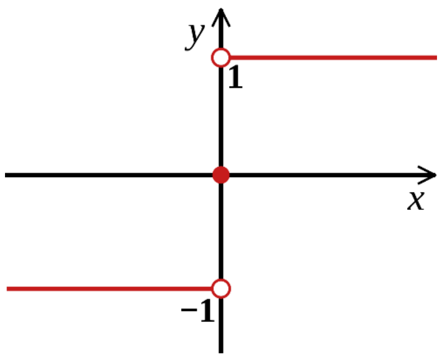
$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies g(x) = \omega_1 x_1 + \omega_2 x_2 + \omega_0 = \omega^T x + \omega_0$$

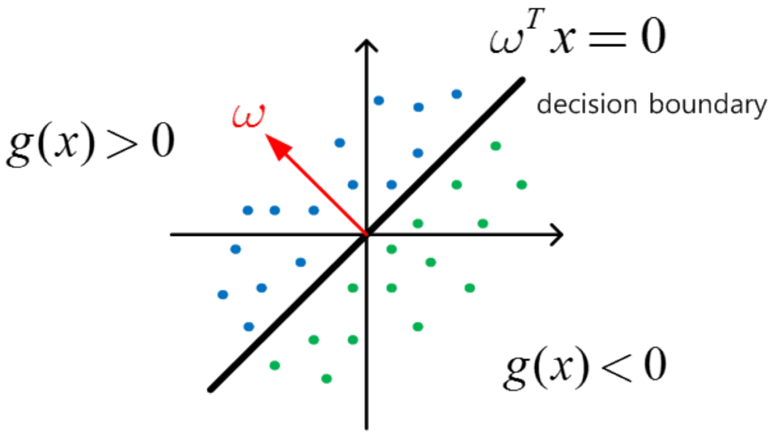
$$\omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} \implies g(x) = \omega_0 + \omega_1 x_1 + \omega_2 x_2 = \omega^T x$$



# How to Find $\omega$

- All data in class 1 ($y = 1$)
  - $g(x) > 0$
- All data in class 0 ($y = -1$)
  - $g(x) < 0$



# Perceptron Algorithm

- The perceptron implements

$$h(x) = \text{sign}\left(\omega^T x\right)$$

- Given the training set

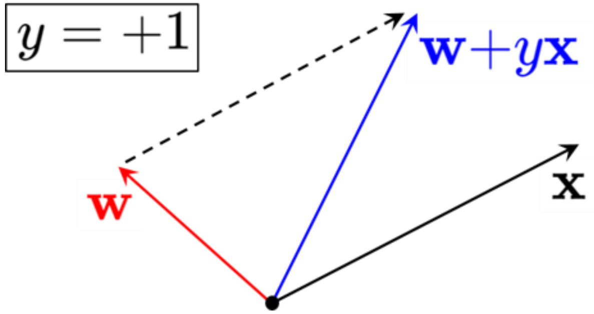
$$(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N) \quad \text{where } y_i \in \{-1, 1\}$$

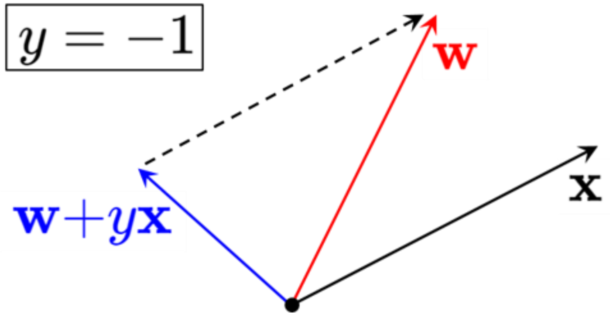
1) pick a misclassified point

$$\text{sign}\left(\omega^T x_n\right) \neq y_n$$

2) and update the weight vector

$$\omega \leftarrow \omega + y_n x_n$$



# Perceptron Algorithm

- Why perceptron updates work ?

- Let's look at a misclassified positive example $(y_n = +1)$
  - Perceptron (wrongly) thinks $\omega_{old}^T x_n < 0$

  - Updates would be

$$\omega_{new} = \omega_{old} + y_n x_n = \omega_{old} + x_n$$

$$\omega_{new}^T x_n = (\omega_{old} + x_n)^T x_n = \omega_{old}^T x_n + x_n^T x_n$$

  - Thus $\omega_{new}^T x_n$ is less negative than $\omega_{old}^T x_n$

# Iterations of Perceptron

1. Randomly assign $\omega$

2. One iteration of the PLA (perceptron learning algorithm)

$$\omega \leftarrow \omega + yx$$

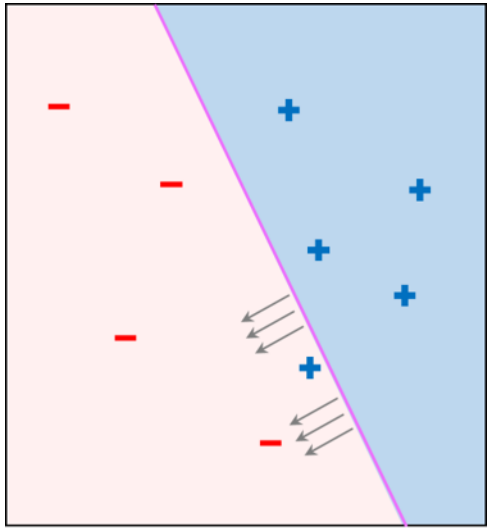
   where $(x, y)$ is a misclassified training point

3. At iteration $i = 1, 2, 3, \cdots$, pick a misclassified point from

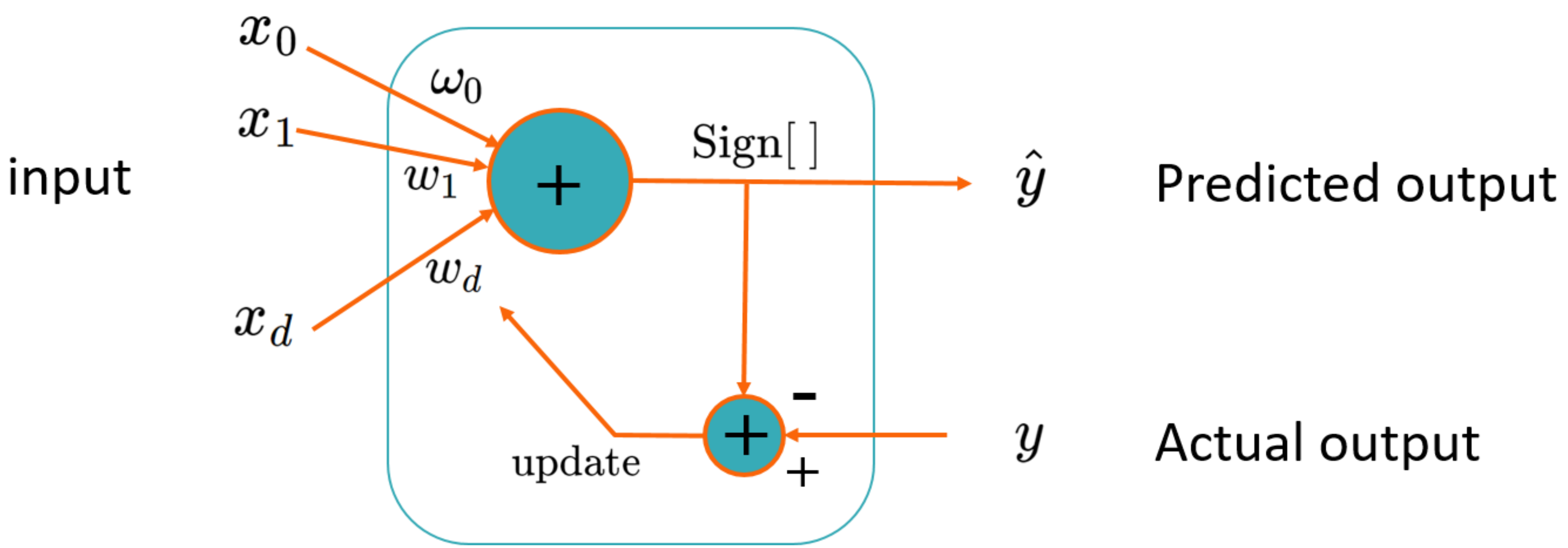
$$(x_1, y_1), (x_2, y_2), \cdots, (x_N, y_N)$$

4. And run a PLA iteration on it

5. That's it!



# Diagram of Perceptron



# Perceptron Loss Function

$$L(\omega) = \sum_{n=1}^{m} \max\left\{0, -y_n \cdot \left(\omega^T x_n\right)\right\}$$

- Loss = 0 on examples where perceptron is correct,
  *i.e.*, $y_n \cdot \left(\omega^T x_n\right) > 0$

- Loss > 0 on examples where perceptron is misclassified,
  *i.e.*, $y_n \cdot \left(\omega^T x_n\right) < 0$

- Note:
  – $\mathrm{sign}\left(\omega^T x_n\right) \neq y_n$ is equivalent to $y_n \cdot \left(\omega^T x_n\right) < 0$

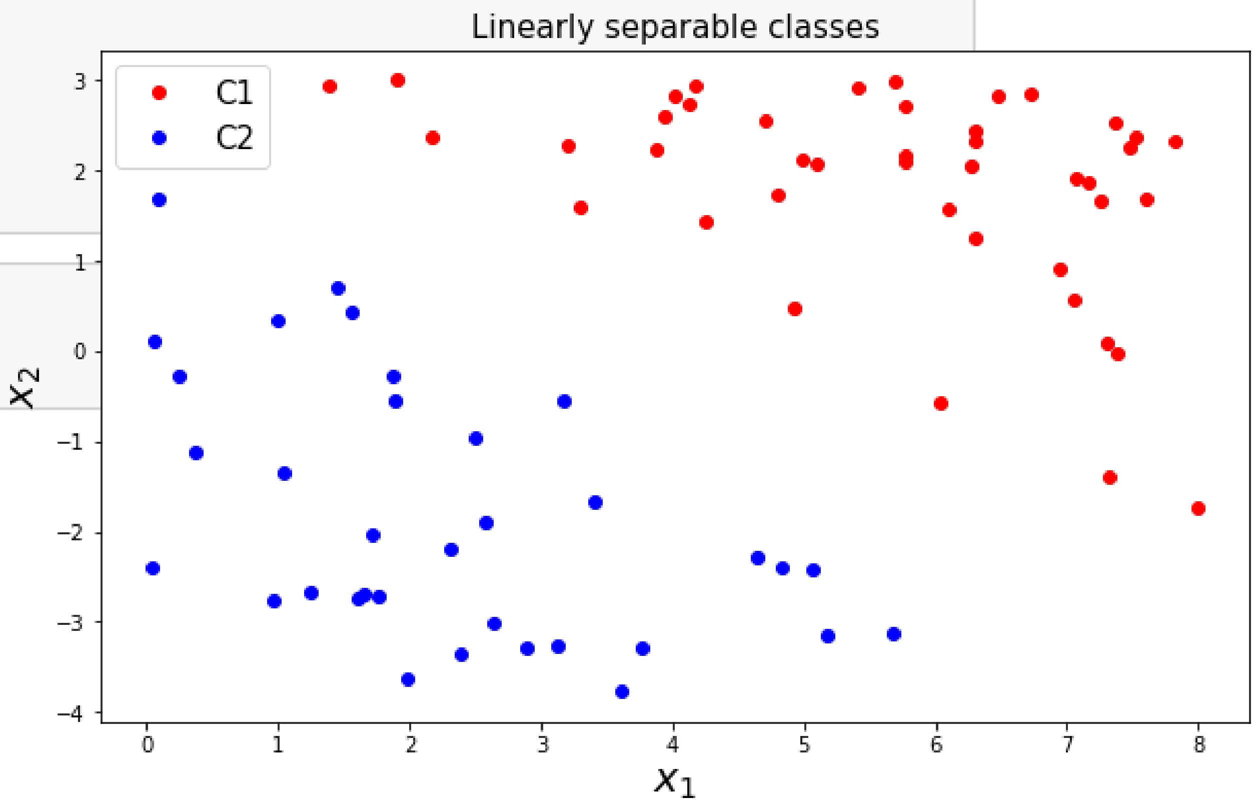
# Perceptron Algorithm in Python

```python
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

```python
#training data gerneration
m = 100
x1 = 8*np.random.rand(m, 1)
x2 = 7*np.random.rand(m, 1) - 4

g = 0.8*x1 + x2 - 3
```

```python
C1 = np.where(g >= 1)
C2 = np.where(g < -1)
print(C1)
```



# Perceptron Algorithm in Python

- Unknown parameters $\omega$

$$g(x) = \omega_0 + \omega^T x = \omega_0 + \omega_1 x_1 + \omega_2 x_2 = 0$$

$$\omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix} \qquad x = \begin{bmatrix} \left(x^{(1)}\right)^T \\ \left(x^{(2)}\right)^T \\ \left(x^{(3)}\right)^T \\ \vdots \\ \left(x^{(m)}\right)^T \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} \\ 1 & x_1^{(3)} & x_2^{(3)} \\ \vdots & \vdots & \vdots \\ 1 & x_1^{(m)} & x_2^{(m)} \end{bmatrix} \qquad y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

```
X1 = np.hstack([np.ones([C1.shape[0],1]), x1[C1], x2[C1]])
X2 = np.hstack([np.ones([C2.shape[0],1]), x1[C2], x2[C2]])
X = np.vstack([X1, X2])

y = np.vstack([np.ones([C1.shape[0],1]), -np.ones([C2.shape[0],1])])

X = np.asmatrix(X)
y = np.asmatrix(y)
```

# Perceptron Algorithm in Python

$$\omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix}$$

$\omega \leftarrow \omega + yx$ where $(x, y)$ is a misclassified training point

```
w = np.ones([3,1])
w = np.asmatrix(w)

n_iter = y.shape[0]
for k in range(n_iter):
    for i in range(n_iter):
        if y[i,0] != np.sign(X[i,:]*w)[0,0]:
            w += y[i,0]*X[i,:].T

print(w)
```

# Perceptron Algorithm in Python

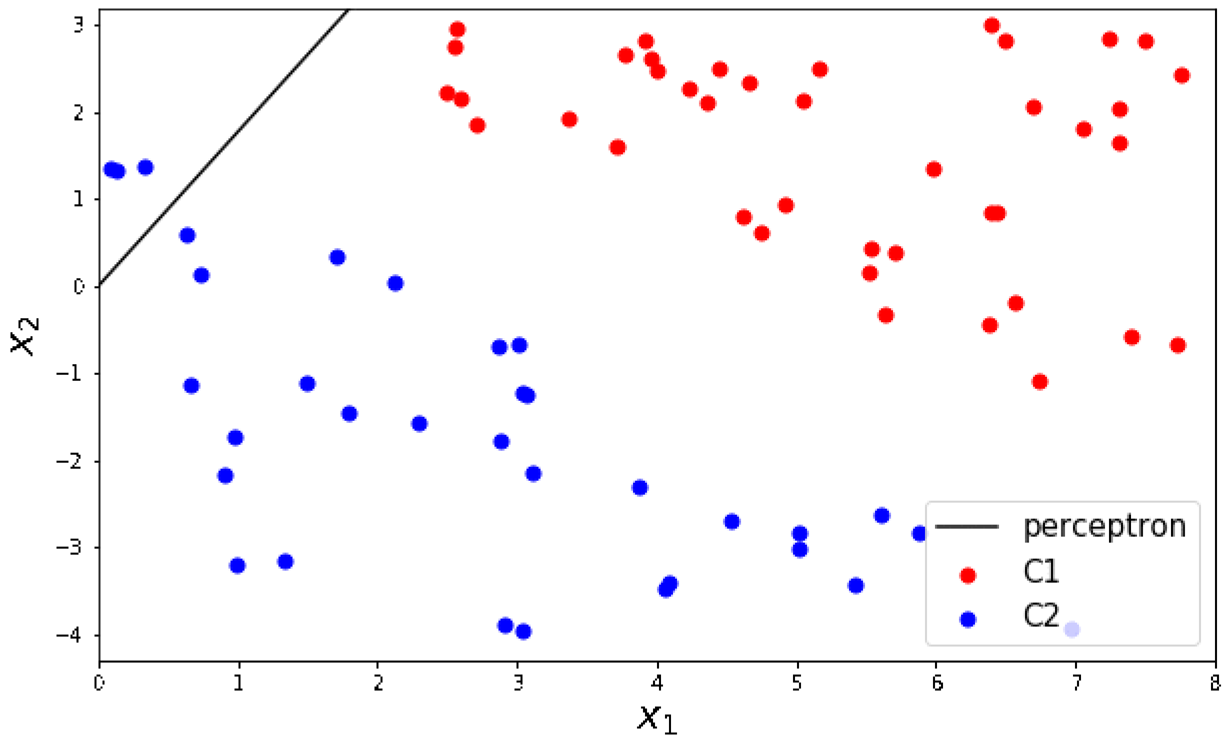
$$g(x) = \omega_0 + \omega^T x = \omega_0 + \omega_1 x_1 + \omega_2 x_2 = 0$$

$$\implies x_2 = -\frac{\omega_1}{\omega_2} x_1 - \frac{\omega_0}{\omega_2}$$

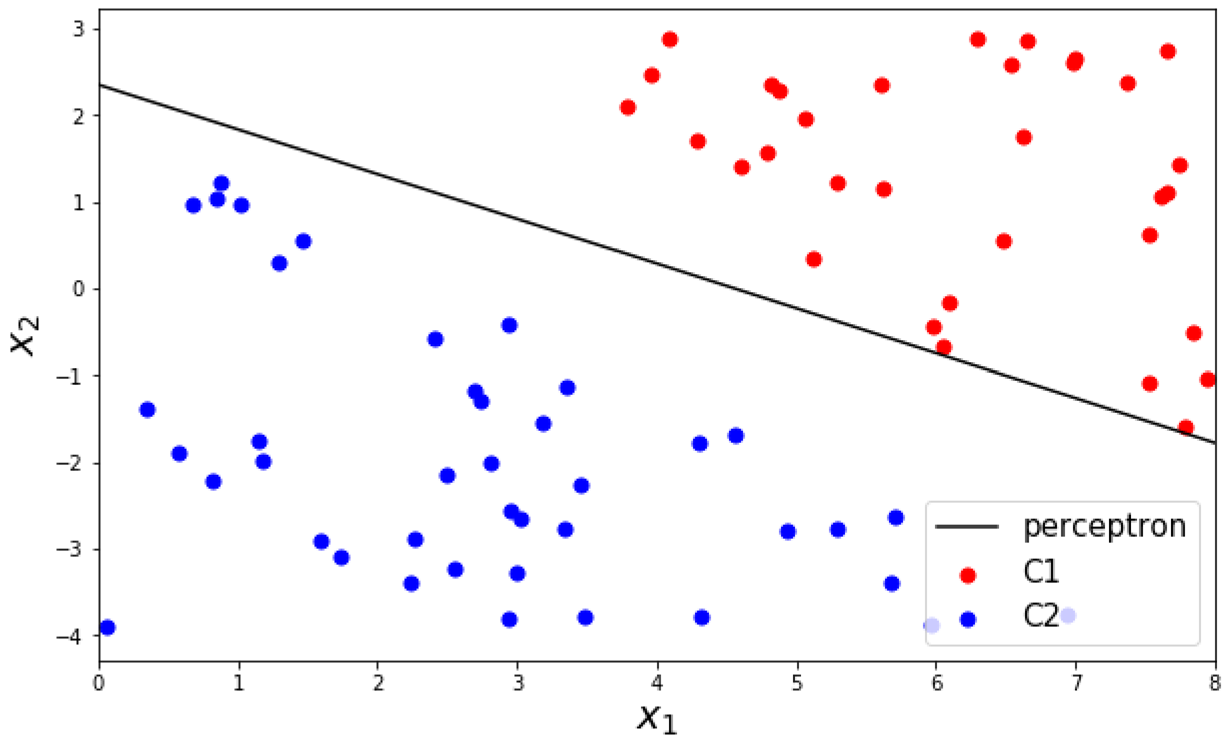
```
x1p = np.linspace(0,8,100).reshape(-1,1)
x2p = - w[1,0]/w[2,0]*x1p - w[0,0]/w[2,0]

plt.figure(figsize=(10, 6))
plt.scatter(x1[C1], x2[C1], c='r', s=50, label='C1')
plt.scatter(x1[C2], x2[C2], c='b', s=50, label='C2')
plt.plot(x1p, x2p, c='k', label='perceptron')
plt.xlim([0,8])
plt.xlabel('$x_1$', fontsize = 20)
plt.ylabel('$x_2$', fontsize = 20)
plt.legend(loc = 1, fontsize = 15)
plt.show()
```

# Perceptron Algorithm in Python



# Perceptron Algorithm in Python



# Scikit-Learn for Perceptron

```
X1 = np.hstack([x1[C1], x2[C1]])
X2 = np.hstack([x1[C2], x2[C2]])
X = np.vstack([X1, X2])

y = np.vstack([np.ones([C1.shape[0],1]), -np.ones([C2.shape[0],1])])
```

```
from sklearn import linear_model

clf = linear_model.Perceptron(tol=1e-3)
clf.fit(X, np.ravel(y))
```

```
clf.predict([[3, -2]])
```

```
array([-1.])
```

$$x = \begin{bmatrix} \left(x^{(1)}\right)^T \\ \left(x^{(2)}\right)^T \\ \left(x^{(3)}\right)^T \\ \vdots \\ \left(x^{(m)}\right)^T \end{bmatrix} = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} \\ x_1^{(2)} & x_2^{(2)} \\ x_1^{(3)} & x_2^{(3)} \\ \vdots & \vdots \\ x_1^{(m)} & x_2^{(m)} \end{bmatrix}$$

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

# The Best Hyperplane Separator?

- Perceptron finds one of the many possible hyperplanes separating the data if one exists
- Of the many possible choices, which one is the best?

- Utilize distance information
- Intuitively we want the hyperplane having the maximum margin
- Large margin leads to good generalization on the test data
  - we will see this formally when we discuss Support Vector Machine (SVM)

- Utilize distance information from all data samples
  - We will see this formally when we discuss the logistic regression

- Perceptron will be shown to be a basic unit for neural networks and deep learning later