Root Locus

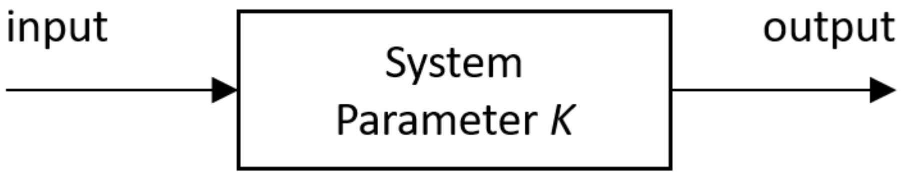
Prof. Seungchul Lee
Industrial AI Lab.

POSTECH

Most Slides from the Root Locus Method by Brian Douglas

Motivation for Root Locus

- For example



$$\text{System} = \frac{s^2 + s + 1}{s^3 + 4s^2 + Ks + 1}$$

- Unknown parameter affects poles

- Poles of system are values of s when

$$s^3 + 4s^2 + \color{red}{K}s + 1 = 0$$

Motivation for Root Locus

- What value of K should I choose to meet my system performance requirement?

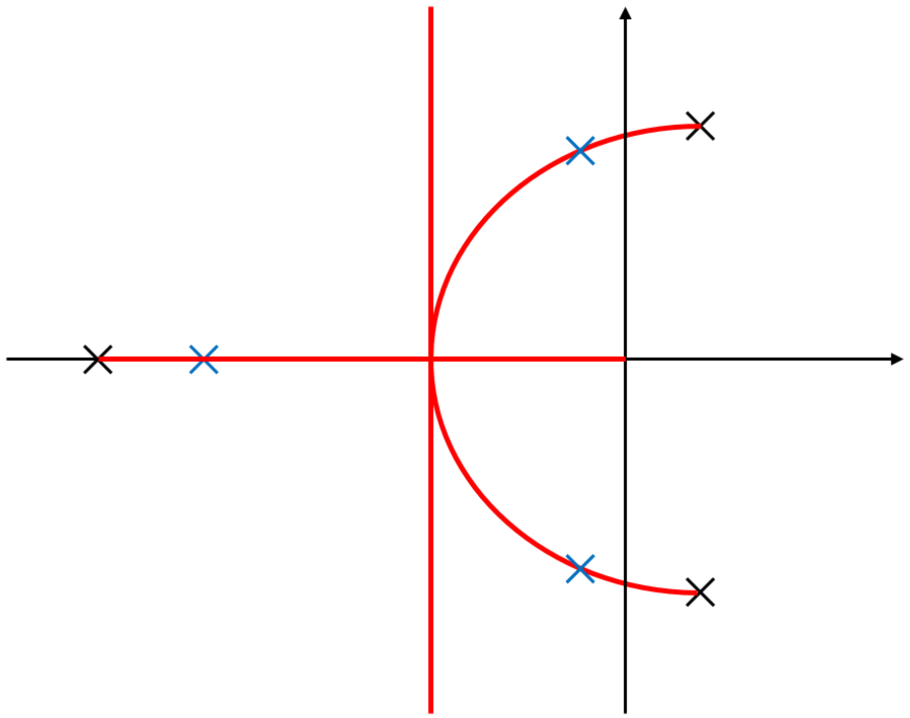
$$s^3 + 4s^2 + Ks + 1 = 0$$

$$s^3 + 4s^2 + 0s + 1 = 0$$

$$s^3 + 4s^2 + 1s + 1 = 0$$

$$s^3 + 4s^2 + 2s + 1 = 0$$

$$\vdots$$

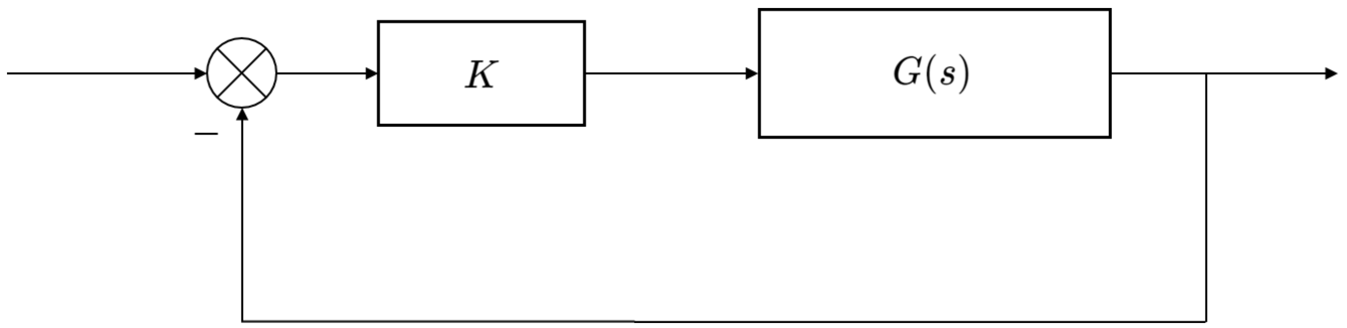


Root Locus

Definition: Root Locus

- Given the plant transfer function $G(s)$, the typical closed-loop transfer function is

$$\frac{KG(s)}{1+KG(s)}$$



- The root locus of an (open-loop) transfer function $G(s)$ is a plot of the locations (locus) of all possible closed-loop poles with some parameter, often a proportional gain K, varied between 0 and ∞.

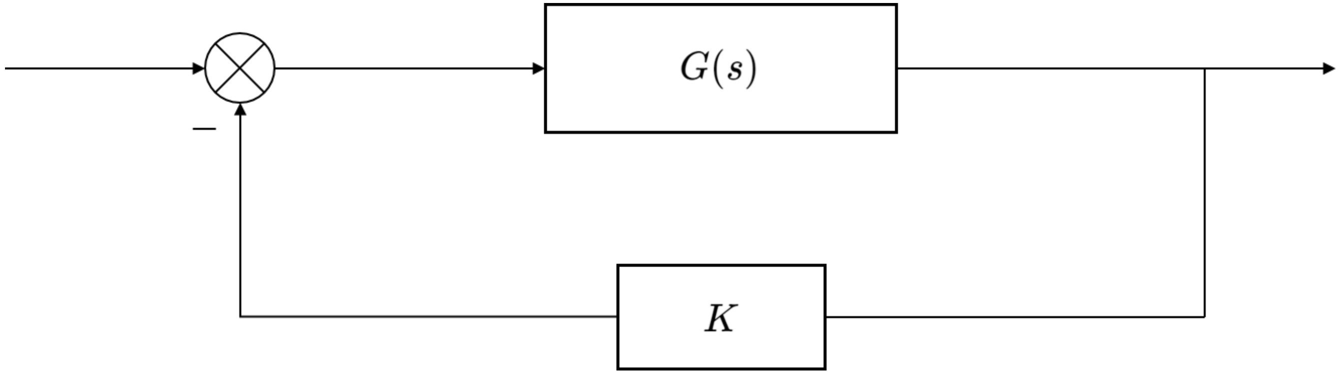
Root Locus in MATLAB

- The basic form for drawing the root locus

$$1 + KG(s) = 0$$

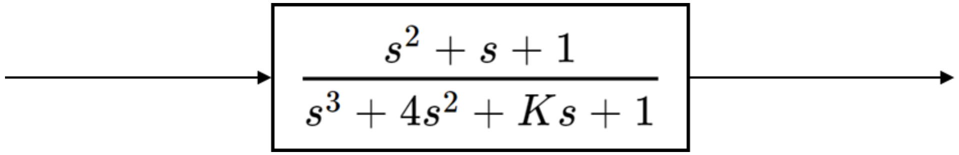
- In MATLAB, rlocus(G(s))
  - The same denominator system is

$$\frac{G(s)}{1 + KG(s)}$$



Standard Form for Root Locus

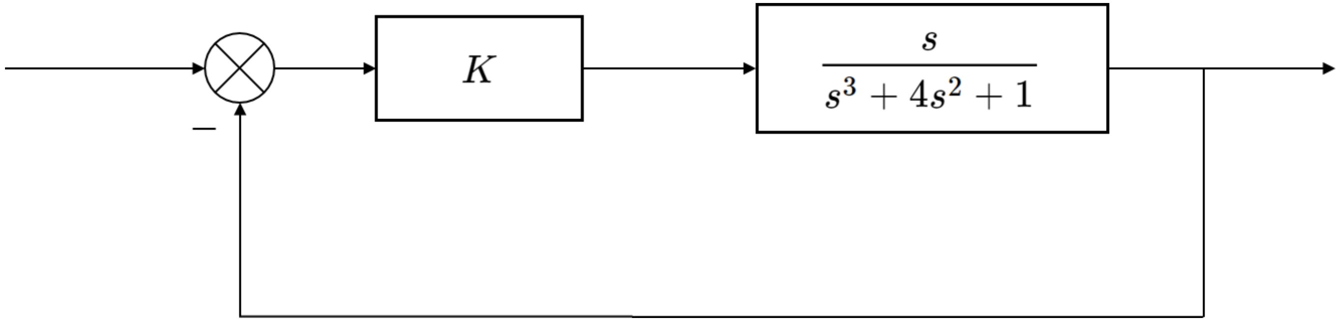
- But you noticed that in the previous example I used



$$s^3 + 4s^2 + Ks + 1 = 0, \qquad \text{not in the correct form}$$

$$1 + K\frac{s}{s^3+4s^2+1} = 1 + KG(s) = 0$$

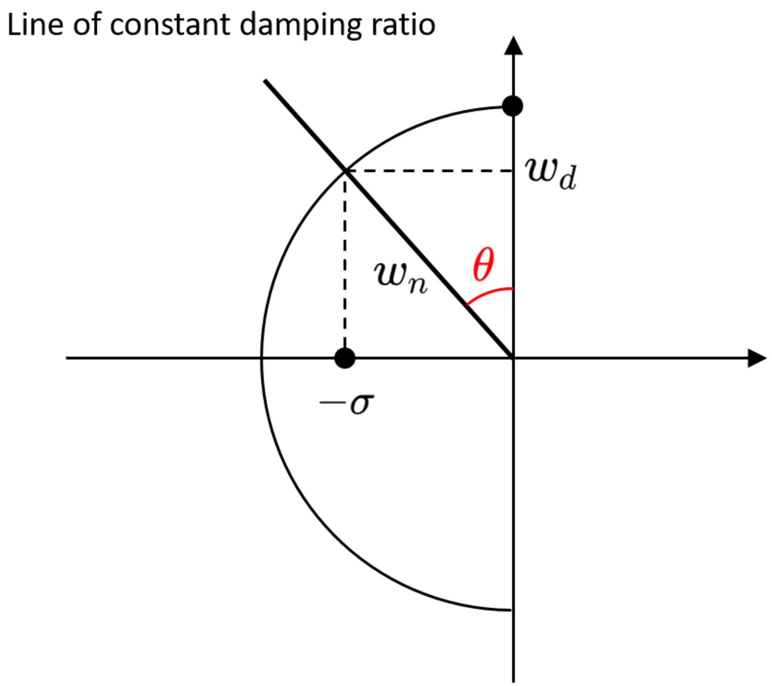
- Equivalent $G(s)$ and the closed loop system



Graphical Representation of Closed Loop Poles

- Root-Locus: a graphical representation of closed-loop poles as K varied

- Based on root-locus graph, we can choose the parameter K for stability and the desired transient response.

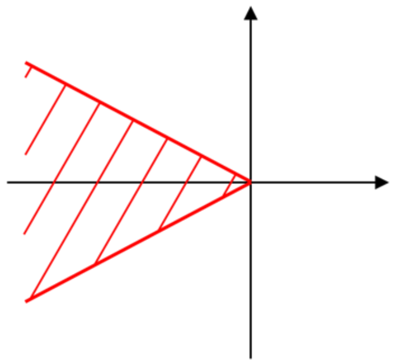


$$w_d = \sqrt{1-\zeta^2} w_n$$
$$\sigma = \zeta w_n$$
$$\zeta = \sin\theta$$
$$\sqrt{1-\zeta^2} = \cos\theta$$

Pole Locations for Closed Loop

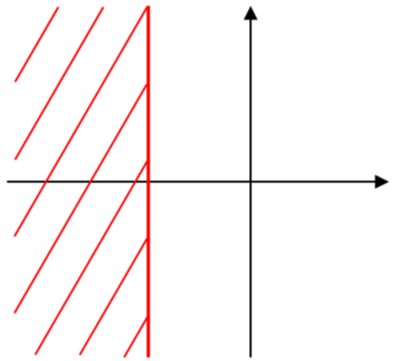
- So why should we care about this?

Damping Ratio



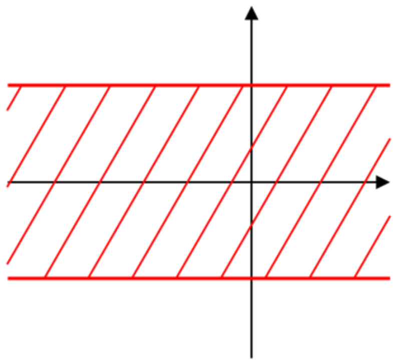
Roots must be in cone

Exponential decay



Roots must be left of line

Natural frequency



Roots must be inside lines

- Now that we understand how pole locations affect the system

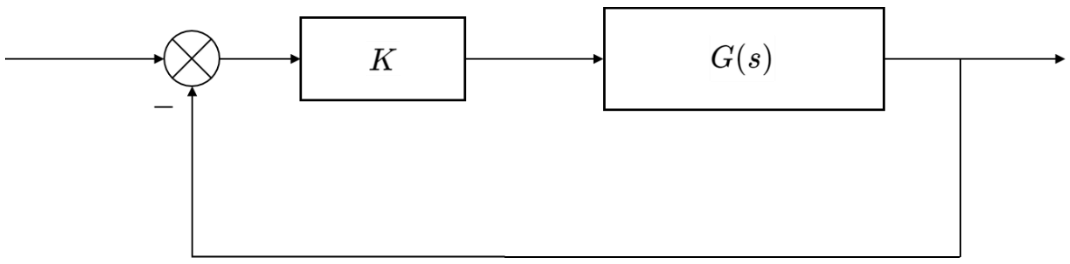
How to Draw Root Locus

- Question: how do we draw root locus ?
  - for more complex system and
  - without calculating poles

- We will be able to make a rapid sketch of the root locus for higher-order systems without having to factor the denominator of the closed-loop transfer function.
- You might not use an exact sketch very often in practice, but you will use an approximated one!
- What does closed loop root locus look like from open loop?

- The closed loop system is

$$H(s) = \frac{KG}{1 + KG}$$



How to Draw Root Locus

- A pole exists when the characteristic polynomial in the denominator becomes zero

$$H(s) = \frac{KG}{1+KG} \qquad 1 + KG(s) = 0 \implies KG(s) = -1 = 1\angle(2k+1)\pi, \quad k = 0, \pm 1, \pm 2, \cdots$$

- A value of s^* is a closed loop pole if

$$\begin{cases} |KG(s^*)| = 1 \qquad \implies \qquad K = \frac{1}{|G(s^*)|} \\ \angle KG(s^*) = (2k+1)\pi \end{cases}$$

8 Rules for Root Locus

Rule 1

- There will be 8 rules to drawing a root locus

$$1 + KG(s) = 1 + K\frac{Q(s)}{P(s)} = 0$$

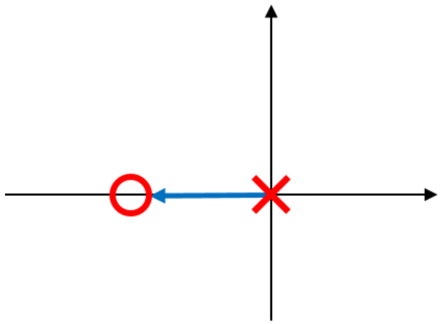
- Rule 1: There are n lines (loci) where n is the degree of Q or P, whichever greater.

Rule 2 (1/2)

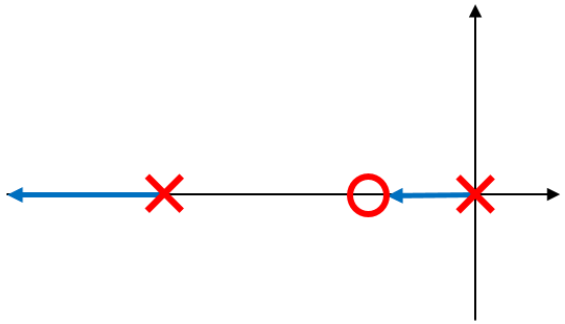
- Rule 2: As K increases from 0 to ∞, the closed loop roots move from the pole of $G(s)$ to the zeros of $G(s)$

$$P(s) + KQ(s) = 0$$

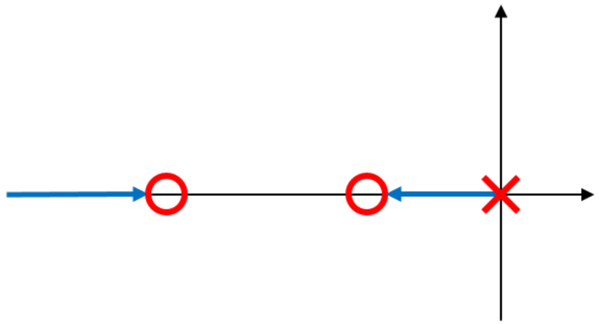
  - Poles of $G(s)$ are when $P(s) = 0, K = 0$
  - Zeros of $G(s)$ are when $Q(s) = 0$, as $K \to \infty, P(s) + \infty Q(s) = 0$
  - So closed loop poles travel from poles of $G(s)$ to zeros of $G(s)$



$\# P(s) = \# Q(s)$



$\# P(s) > \# Q(s)$



$\# P(s) < \# Q(s)$

Rule 2 (2/2)

- Poles and zeros at infinity
  - $G(s)$ has a zero at infinity if $G(s \to \infty) \to 0$
  - $G(s)$ has a pole at infinity if $G(s \to \infty) \to \infty$

- Example

$$KG(s) = \frac{K}{s(s+1)(s+2)}$$

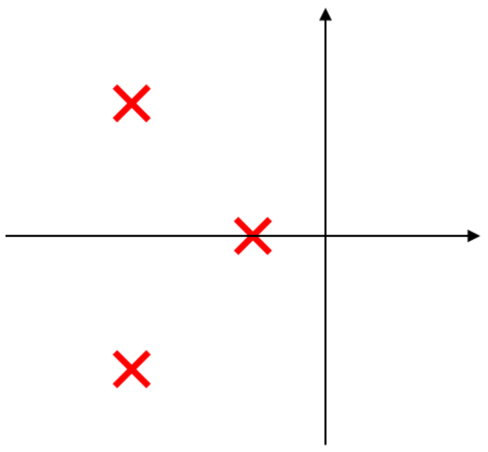
  - Clearly, this open loop transfer function has three poles 0, -1, -2. It has not finite zeros.
  - For large s, we can see that

$$KG(s) \approx \frac{K}{s^3}$$

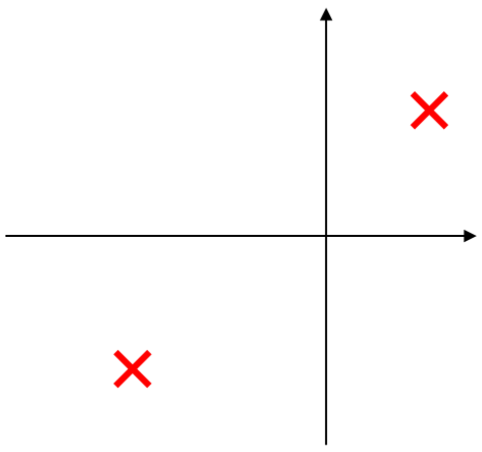
  - So this open loop transfer function has three zeros at infinity

Rule 3

- Rule 3: When roots are complex, they occur in conjugate pairs (= symmetric about real axis)



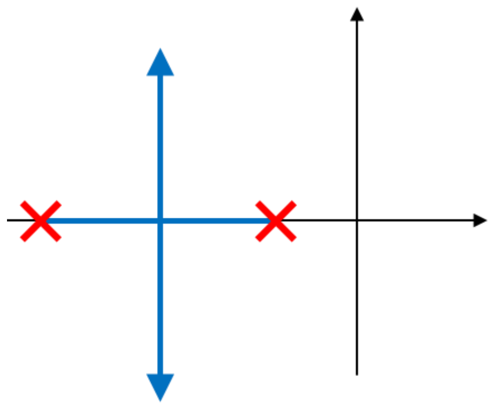
Correct



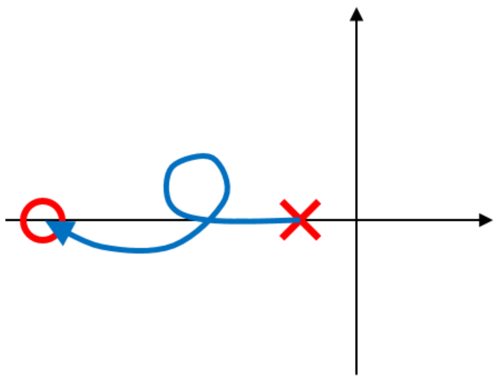
Wrong

Rule 4

- Rule 4: At no time will the same root cross over its path



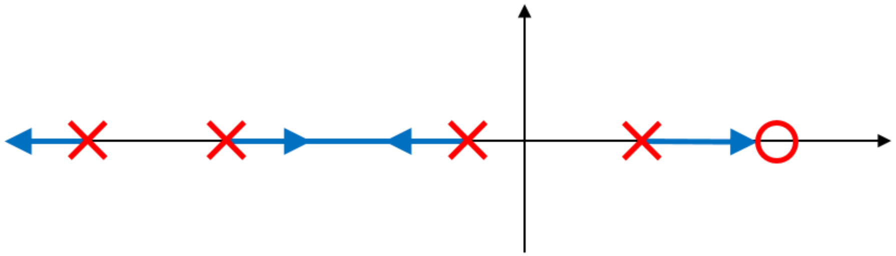
possible



impossible

Rule 5 (1/2)

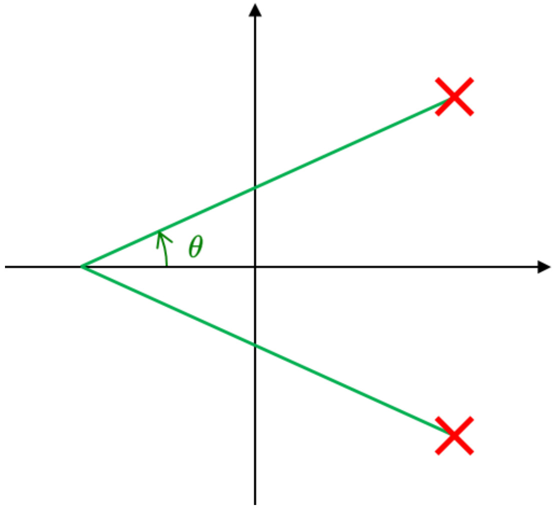
- Rule 5: The portion of the real axis to the left of an odd number of open loop poles and zeros are part of the loci
  - which parts of real line will be a part of root locus?



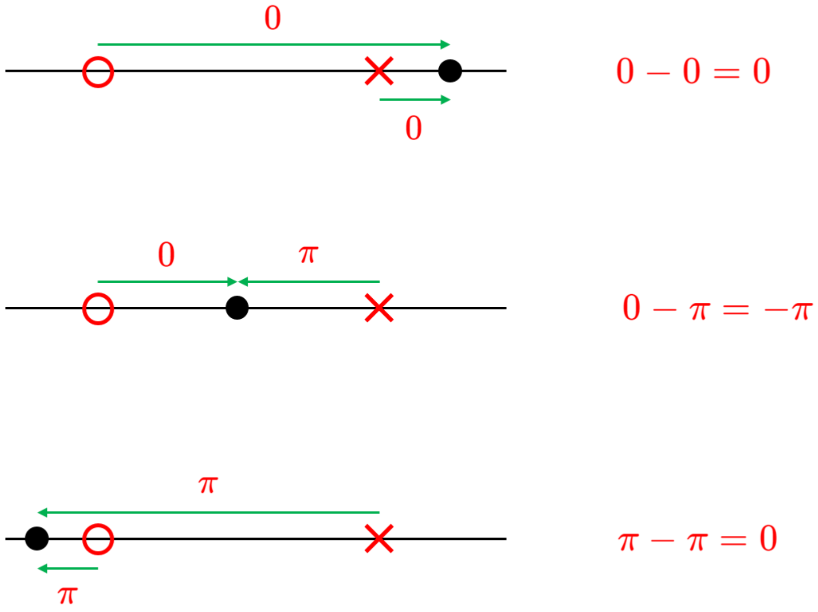
Rule 5 (2/2)

$$G(s) = \frac{Q(s)}{P(s)} = \frac{\prod(s - z_i)}{\prod(s - p_j)}$$

- For complex conjugate zero and pole pair
  $\Rightarrow \angle G(\cdot) = 0$

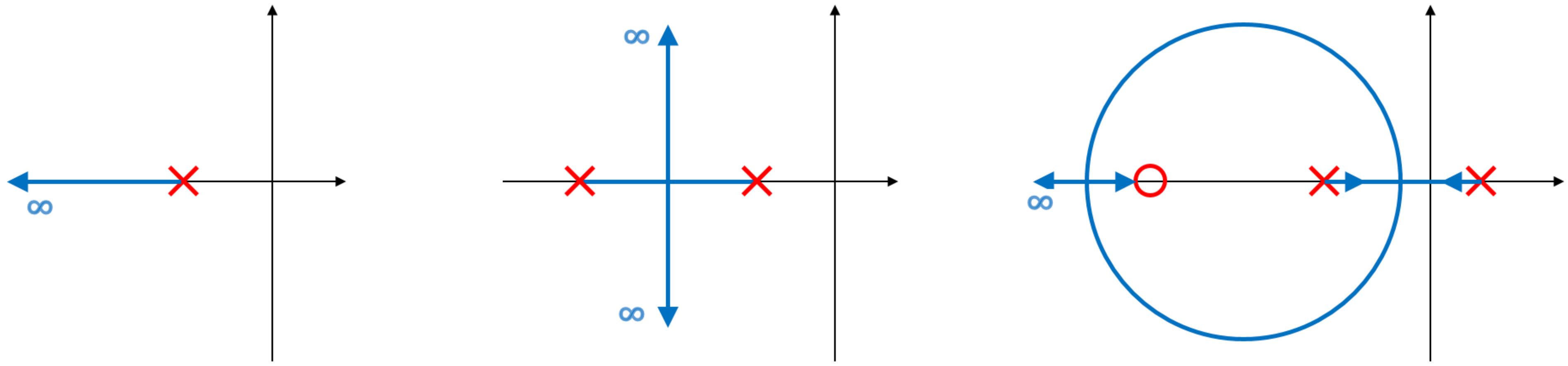


- For real zeros or poles



Rule 6 and Rule 7

- Rule 6: Lines leave (break out) and enter (break in) the real axis at $90°$
- Rule 7: If there are not enough poles and zeros to make a pair, then the extra lines go to or come from infinity.



Rule 8 (1/3)

- Rule 8 : Lines go to infinity along asymptotes

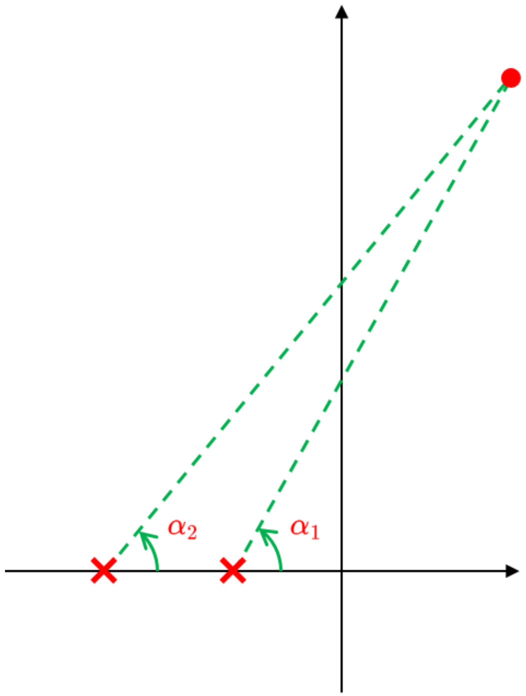
$n - m$ = # poles - # zeros = number of lines that go to infinity.

  - The angles of the asymptotes

$$\phi_A = \frac{2k+1}{n-m}180^\circ \quad \text{where } k = 0, 1, \cdots, n-m-1$$

  - The centroid of the asymptotes on the real axis

$$\sigma_A = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{n-m}$$



Rule 8 (2/3)

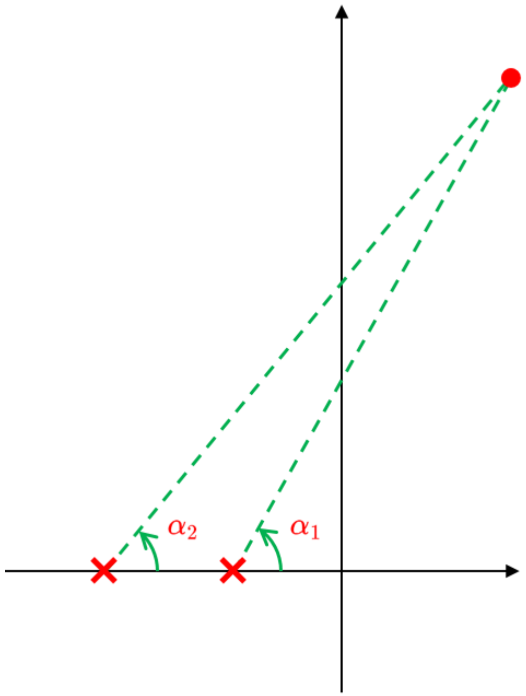
- Lines go to infinity along asymptotes

$$G(s) = \frac{1}{(s+1)(s+2)} \approx \frac{1}{s} \cdot \frac{1}{s}$$

$$\begin{aligned} \angle G(s) &= \pi + 2k\pi \\ &\approx 0 - 2\alpha = \begin{cases} \pi \\ -\pi \end{cases} \end{aligned}$$

$$\therefore \alpha = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

$$\phi_A = \frac{(2k+1)\pi}{n-m}$$



Rule 8 (3/3)

- The centroid of the asymptotes on the real axis

$$\sigma_A = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{n-m}$$

$$G(s) = \beta \frac{(s-z_1)(s-z_2)\cdots}{(s-p_1)(s-p_2)\cdots} = \beta \frac{s^m - \left(\sum z_i\right) s^{m-1} + \cdots}{s^n - \left(\sum p_i\right) s^{n-1} + \cdots}, \quad \text{assume } \; n > m$$

$$G(s) = \beta \frac{s^m - \left(\sum z_i\right) s^{m-1} + \cdots}{s^n - \left(\sum p_i\right) s^{n-1} + \cdots} \approx \beta \frac{1}{(s-\sigma_A)^{n-m}} = \beta \frac{1}{s^{n-m} - (n-m)\sigma_A s^{n-m-1} + \cdots}$$

$$\left(s^m - \left(\sum z_i\right) s^{m-1} + \cdots\right)\left(s^{n-m} - (n-m)\sigma_A s^{n-m-1} + \cdots\right) \approx s^n - \left(\sum p_i\right) s^{n-1} + \cdots$$

$$s^n - \left(\sum z_i + (n-m)\sigma_A\right) s^{n-1} + \cdots \approx s^n - \left(\sum p_i\right) s^{n-1} + \cdots$$

$$\sum z_i + (n-m)\sigma_A = \sum p_i$$

$$\therefore \; \sigma_A = \frac{\sum p_i - \sum z_i}{n-m}$$

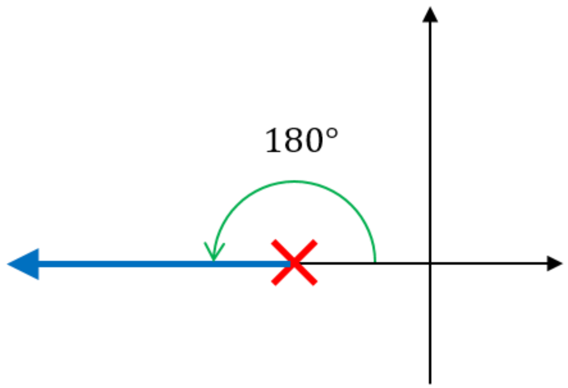
Rule 8

- If $n - m = 1$

$$\phi_A = \frac{2k+1}{n-m}180^\circ \quad \text{where } k = 0, 1, \cdots, n-m-1$$

$$\sigma_A = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{n-m}$$

$$\phi_A = \frac{2 \cdot 0 + 1}{1}180 = 180^\circ$$



Rule 8

- If $n-m=2$

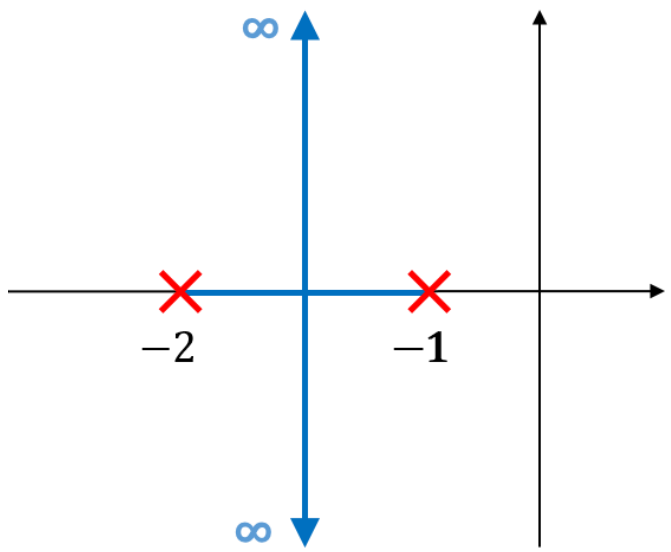
$$\phi_A = \frac{2k+1}{n-m} 180^\circ \quad \text{where } k = 0, 1, \cdots, n-m-1$$

$$\sigma_A = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{n-m}$$

$$\phi_{A1} = \frac{2 \cdot 0 + 1}{2} 180 = 90^\circ$$

$$\phi_{A2} = \frac{2 \cdot 1 + 1}{2} 180 = 270^\circ$$

$$\sigma_A = \frac{(-2-1)-(0)}{2} = -1.5$$



Rule 8

- If $n - m = 3$

$$\phi_A = \frac{2k+1}{n-m} 180^\circ \quad \text{where } k = 0, 1, \cdots, n-m-1$$

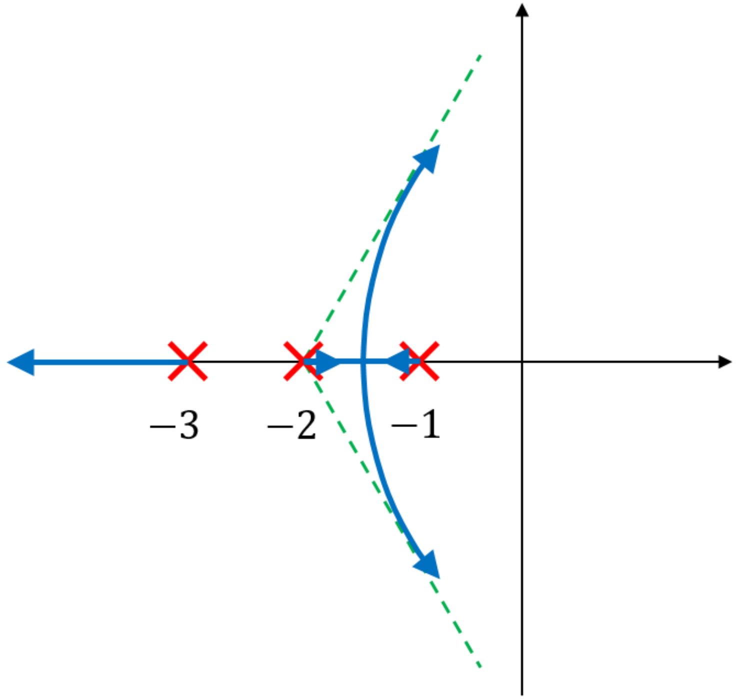
$$\sigma_A = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{n-m}$$

$$\phi_{A1} = \frac{2 \cdot 0 + 1}{3} 180 = 60^\circ$$

$$\phi_{A2} = \frac{2 \cdot 1 + 1}{3} 180 = 180^\circ$$

$$\phi_{A3} = \frac{2 \cdot 2 + 1}{3} 180 = 300^\circ$$

$$\sigma_A = \frac{(-1-2-3)-(0)}{3} = -2$$



Rule 8

- If $n - m = 4$

$$\phi_A = \frac{2k+1}{n-m} 180^\circ \quad \text{where } k = 0, 1, \cdots, n-m-1$$

$$\sigma_A = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{n-m}$$

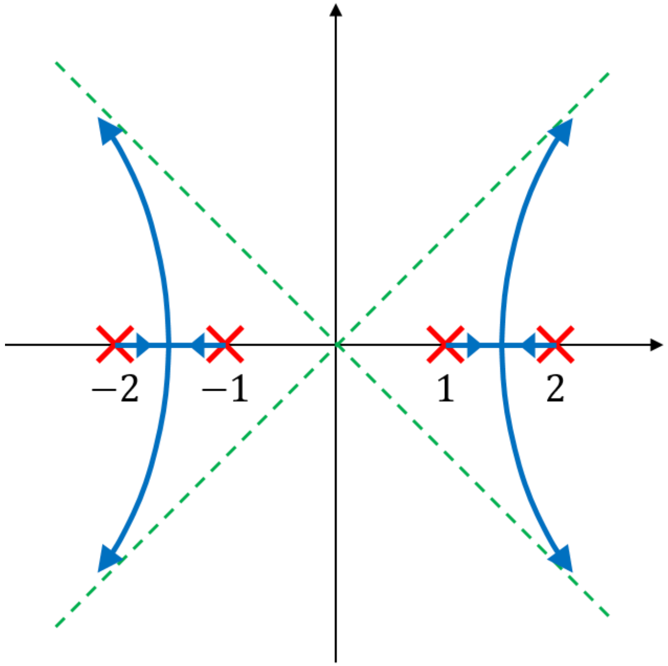
$$\phi_{A1} = \frac{2 \cdot 0 + 1}{4} 180 = 45^\circ$$

$$\phi_{A2} = \frac{2 \cdot 1 + 1}{4} 180 = 135^\circ$$

$$\phi_{A3} = \frac{2 \cdot 2 + 1}{4} 180 = 225^\circ$$

$$\phi_{A4} = \frac{2 \cdot 3 + 1}{4} 180 = 315^\circ$$

$$\sigma_A = \frac{(1 + 2 - 1 - 2) - (0)}{4} = 0$$



Break-away, Break-in Points

- Break-away is the point where loci leave the real axis.
- Break-in is the point where loci enter the real axis.

- The method is to maximize and minimizes the gain K using differential calculus.

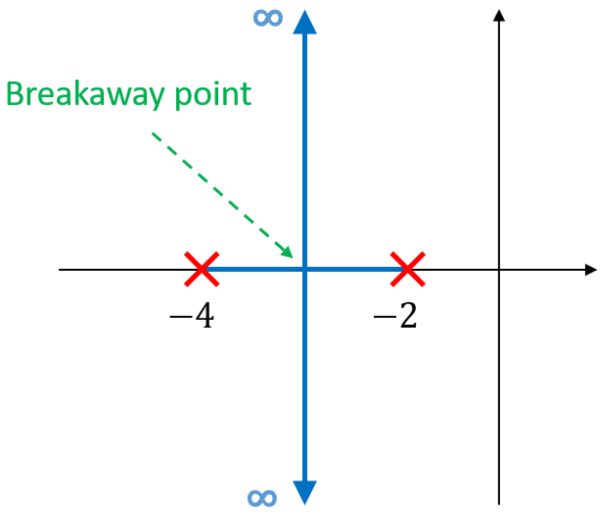
- For all points on the root locus,

$$K = -\frac{1}{G(s)}$$

Break-away, Break-in Points

- Determine the breakaway points

$$G(s) = \frac{1}{(s+2)(s+4)}$$



$$1 + K\frac{1}{(s+2)(s+4)} = 0 \Rightarrow s^2 + 6s + 8 + K = 0$$

$$\Rightarrow s = -3 \pm \sqrt{9 - (8+K)} = -3 \pm \sqrt{1-K}$$

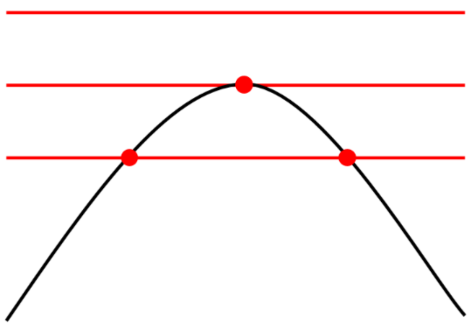
  – When $K < 1$: two real solutions, overdamped
  – When $K > 1$: two complex numbers, underdamped

Break-away, Break-in Points

- With respect to K, (as value of K changes)
- When $\frac{dK}{ds} = 0$, K is Break-away and Break-in.
- The number of solutions changes $0 \rightarrow 1 \rightarrow 2$ or $2 \rightarrow 1 \rightarrow 0$

$$\frac{d}{ds}\frac{1}{G(s)} = 0$$

$$1 + KG(s) = 0 \implies K = -\frac{1}{G(s)}$$



$$\frac{dK}{ds} = 0 \text{ at a breakaway point}$$

$$K = -(s+2)(s+4) = -(s^2 + 6s + 8)$$

$$\frac{dK}{ds} = -(2s+6) = 0$$

$$\therefore s = -3$$

Find Angles of Departure/Arrival for Complex Poles/Zeros

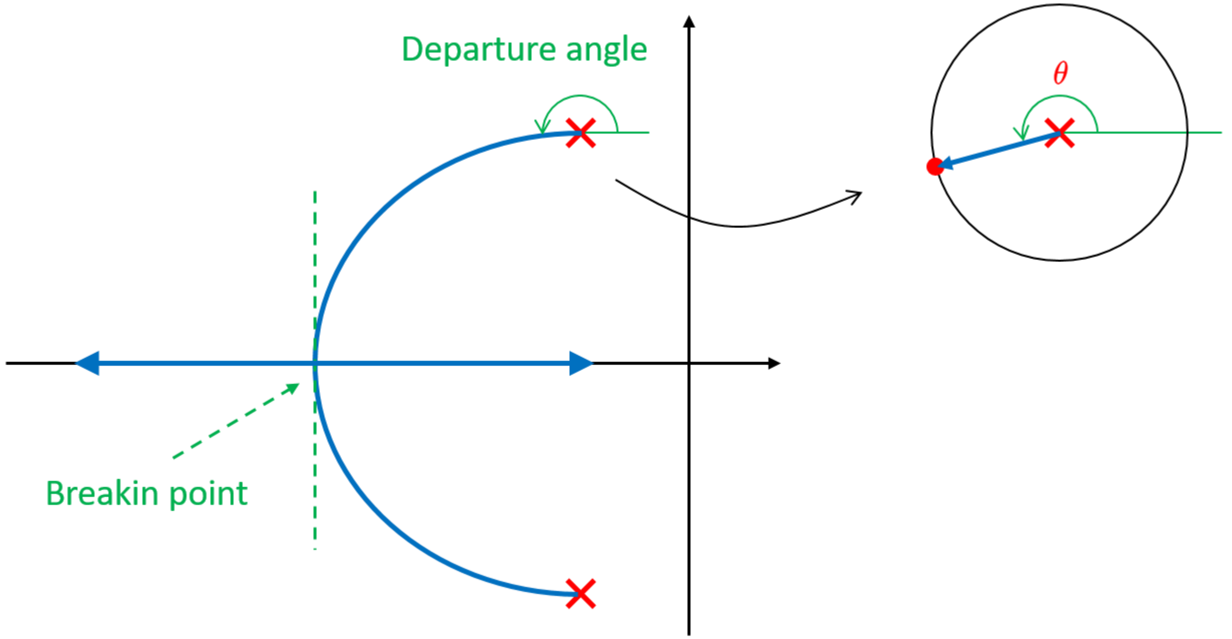
- Loot at a very small region around the departure point

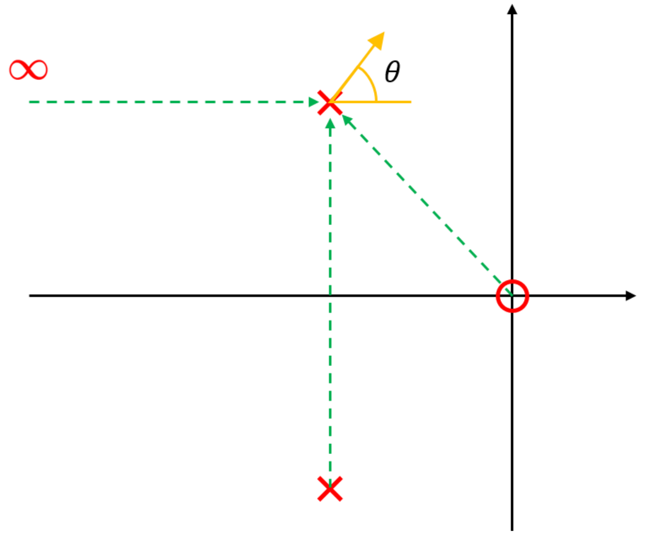
$$G(s) = \frac{s-0}{(s-(-1+i))(s-(-1-i))}$$

$$\angle G(p) = \pi + 2k\pi$$

$$\underbrace{\left(\frac{3}{4}\pi + 0\right)}_{\text{zeros}} - \underbrace{\left(\frac{\pi}{2} + \theta\right)}_{\text{poles}} = \pi$$

$$\therefore \theta = -\frac{3}{4}\pi$$

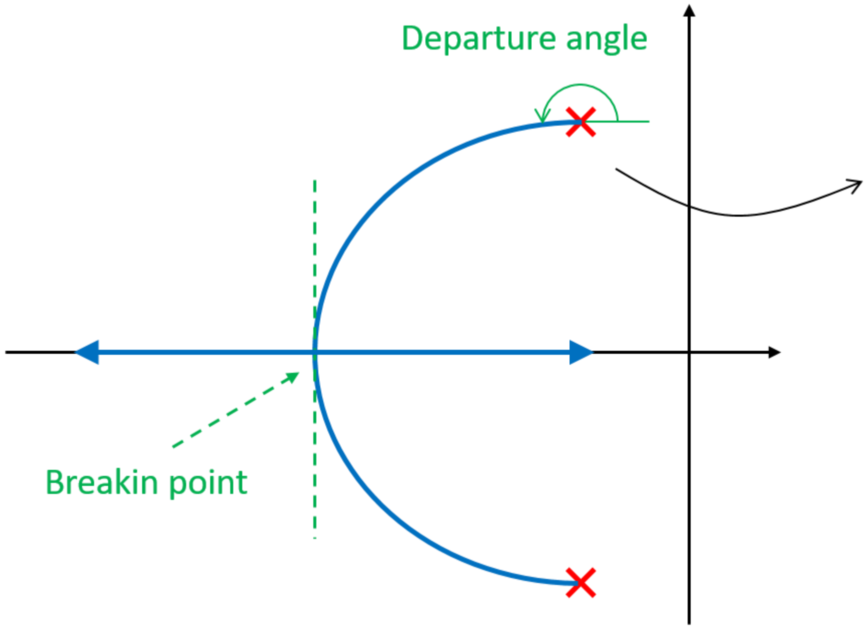


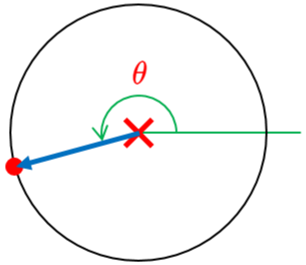


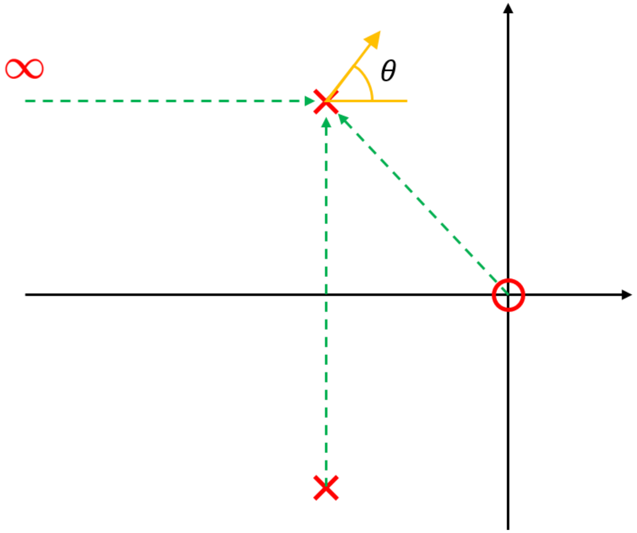
Rule 6: Lines leave (break out) and enter (break in) the real axis at 90°

- Revisit

$$G(s) = \frac{s-0}{(s-(-1+i))(s-(-1-i))}$$







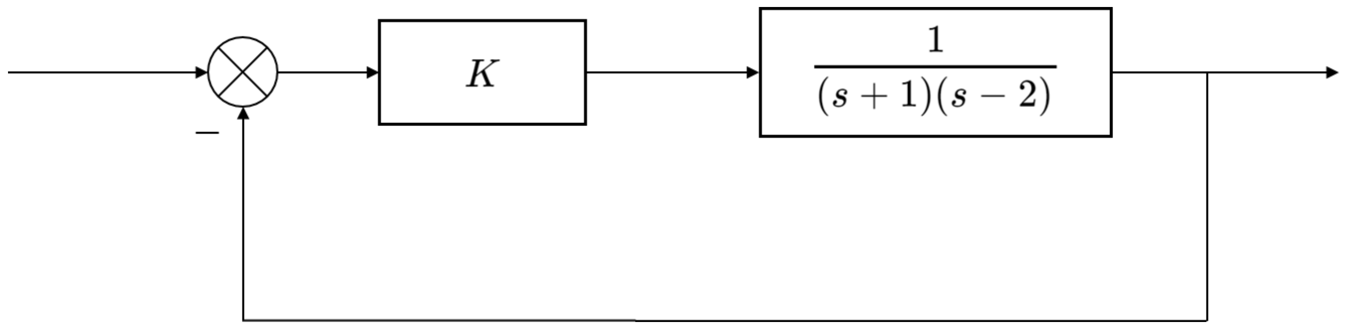
Root Locus for Stability

Root Locus for Stability Evaluation

- Consider the following unstable plant.

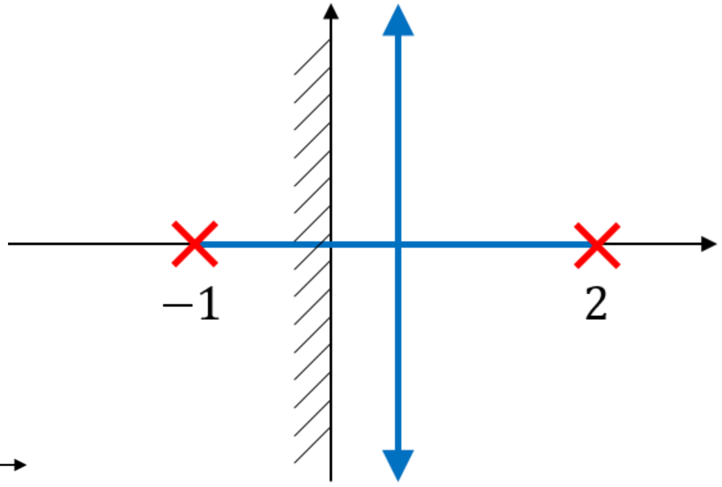
$$G(s) = \frac{1}{(s+1)(s-2)}$$

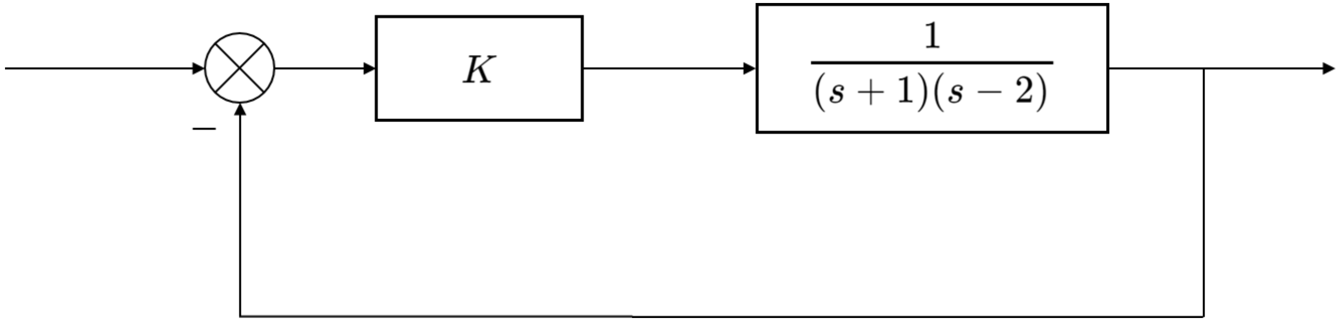
- Try a proportional controller K to stabilize the system



Root Locus for Stability Evaluation

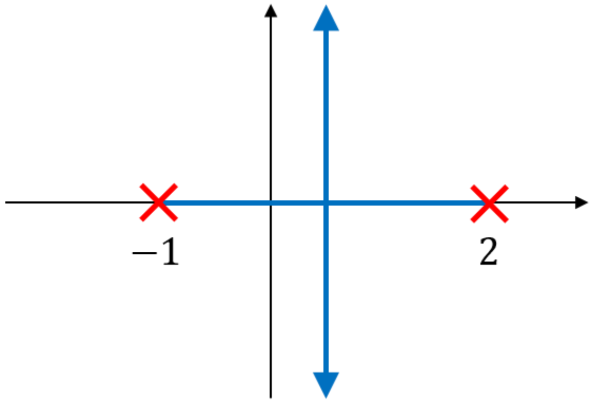
- It turns out that we cannot solve this problem with K (proportional controller only)
- At least one root is always in RHP $\Rightarrow$ unstable

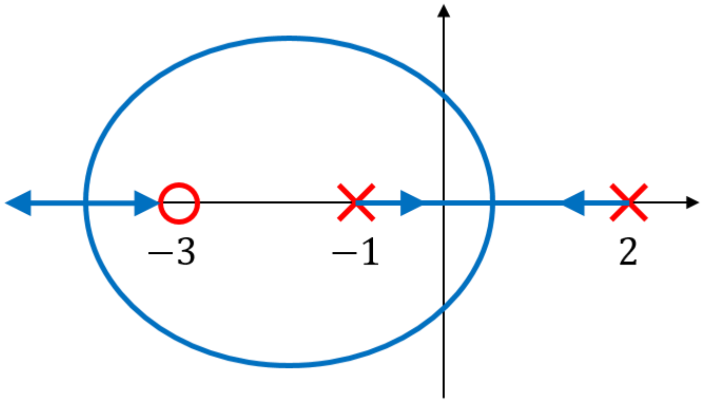


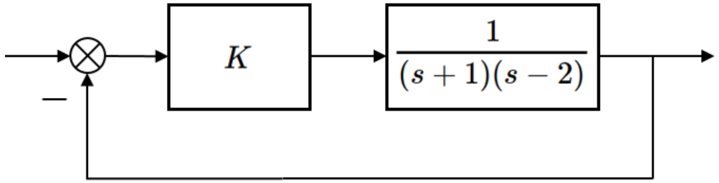


Root Locus for Stability Evaluation

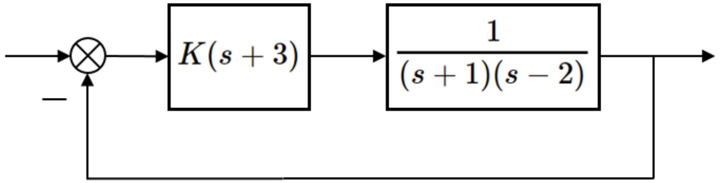
- How can we make this stable?







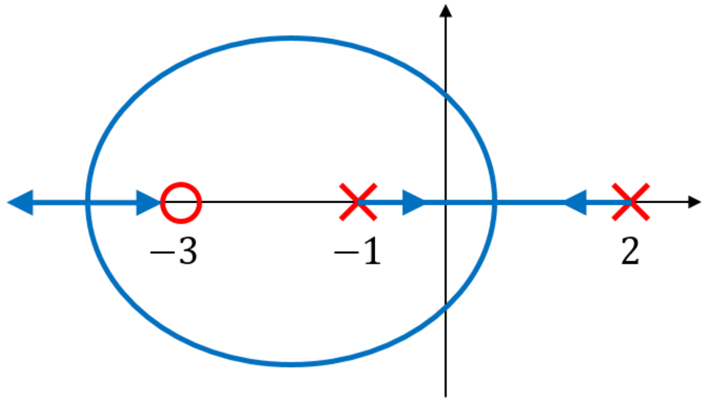
Unstable

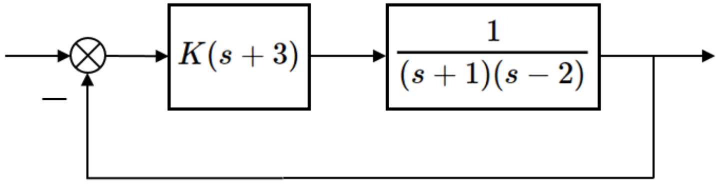


PD controller !

$j\omega$ Axis Crossings

- When poles of closed loop are crossing $j\omega$ axis, the system stability changes
- Use Routh-Hurwitz to find $j\omega$ axis crossings
  - When we have $j\omega$ axis crossings, the Routh-table has all zeros at a row.





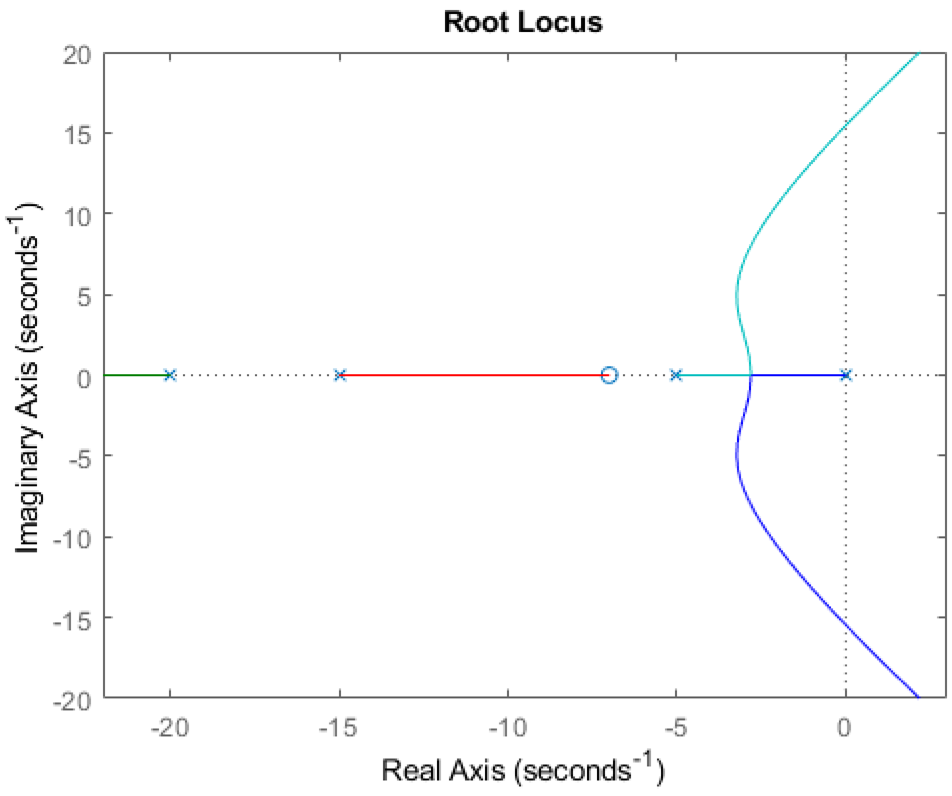
PD controller !

Root Locus in MATLAB

Root Locus in MATLAB

```
rlocus(G)
```

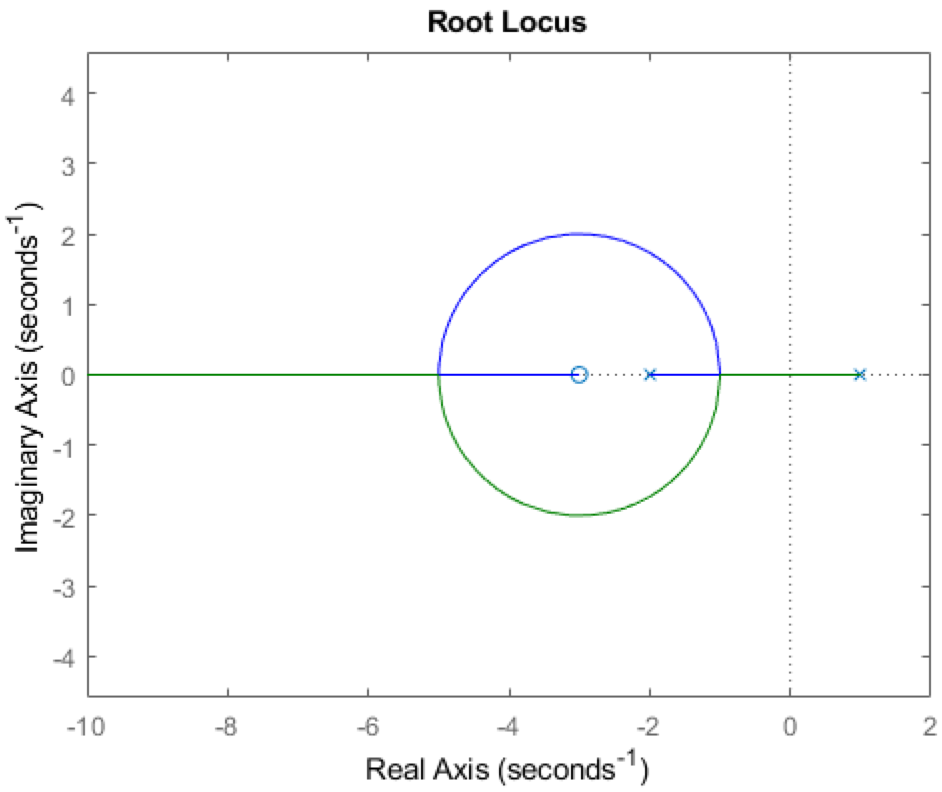
$$G(s) = \frac{s+7}{s(s+5)(s+15)(s+20)}$$



Root Locus in MATLAB

- Example 1: Lines leave the real axis at 90 degrees

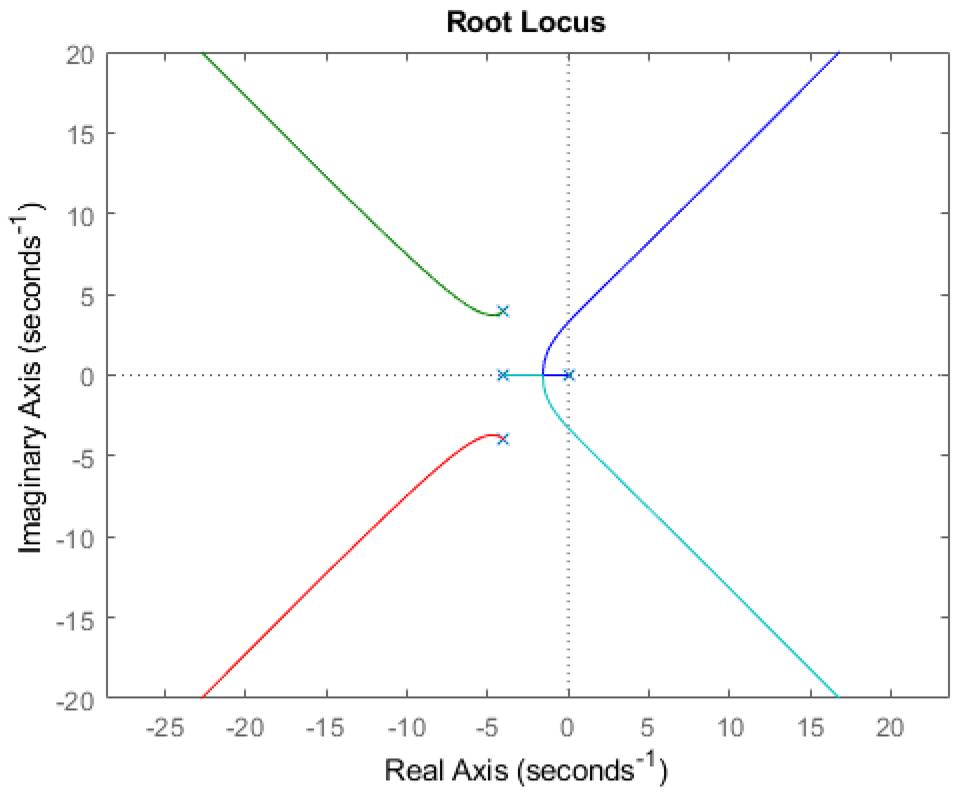
$$G(s) = \frac{s+1}{s^2+s-2}$$



Root Locus in MATLAB

- Example 2: Asymptotes

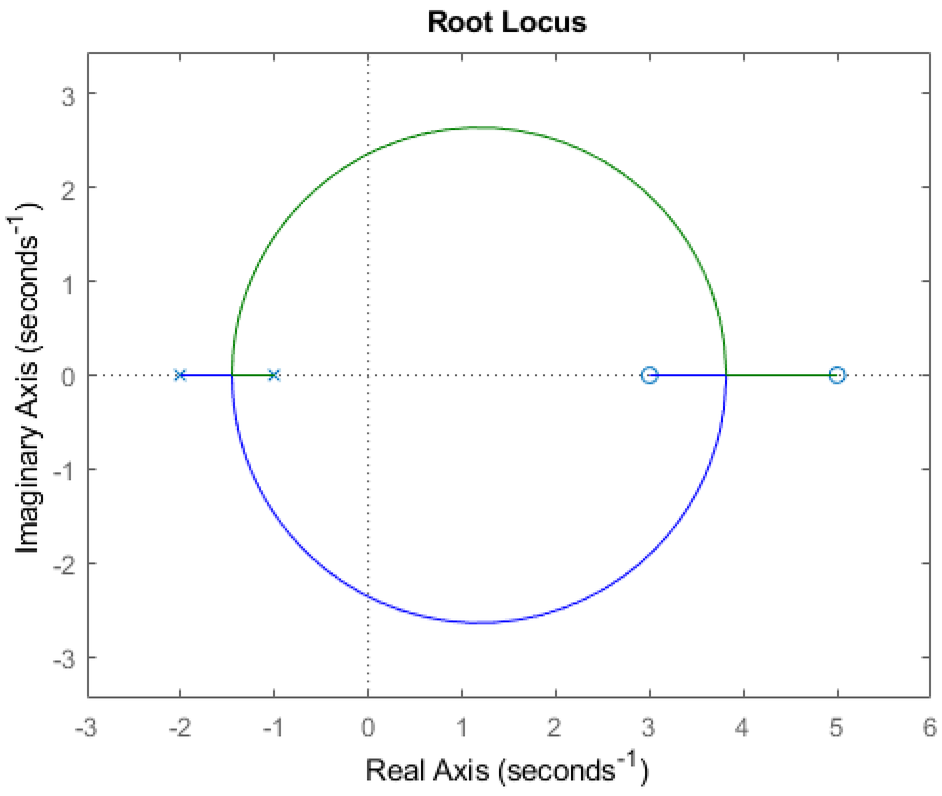
$$G(s) = \frac{1}{s^4 + 12s^3 + 64s^2 + 128s}$$



Root Locus in MATLAB

- Example 3: determining the breakaway points

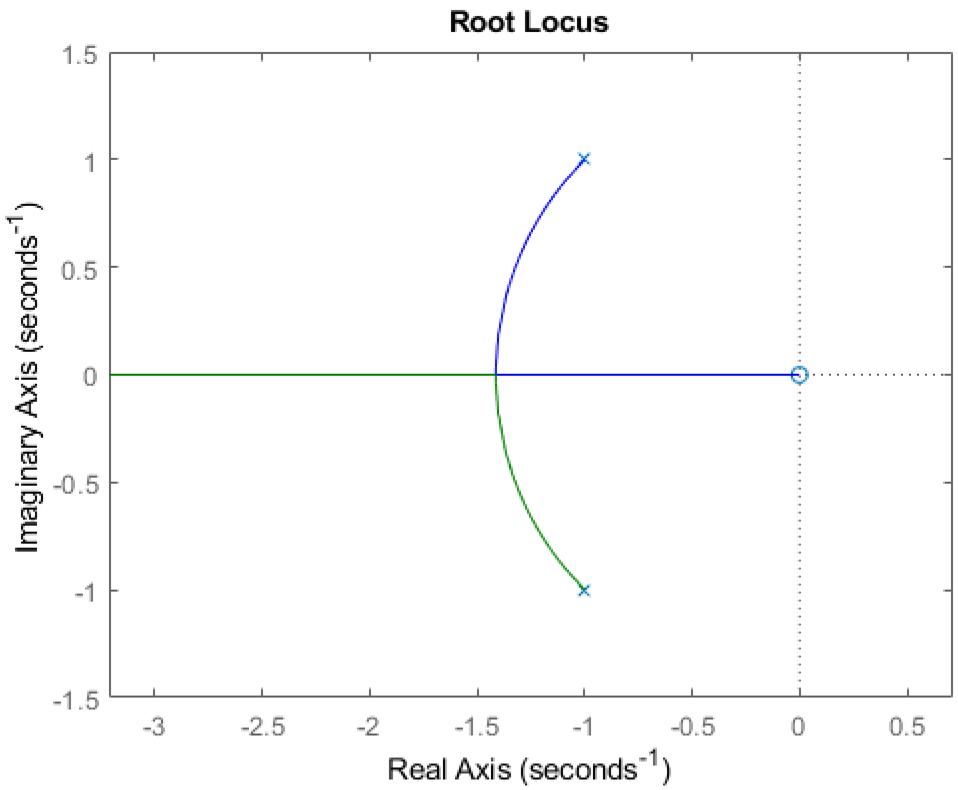
$$G(s) = \frac{(s-3)(s-5)}{(s+1)(s+2)}$$



Root Locus in MATLAB

- Example 4: Departure angle

$$G(s) = \frac{s}{(s-(1+i))(s-(1-i))}$$



Root Locus in MATLAB

- Example 4: Departure angle

$$G(s) = \frac{s+1}{(s+2)(s^2+4s+8)}$$