Discrete Signals

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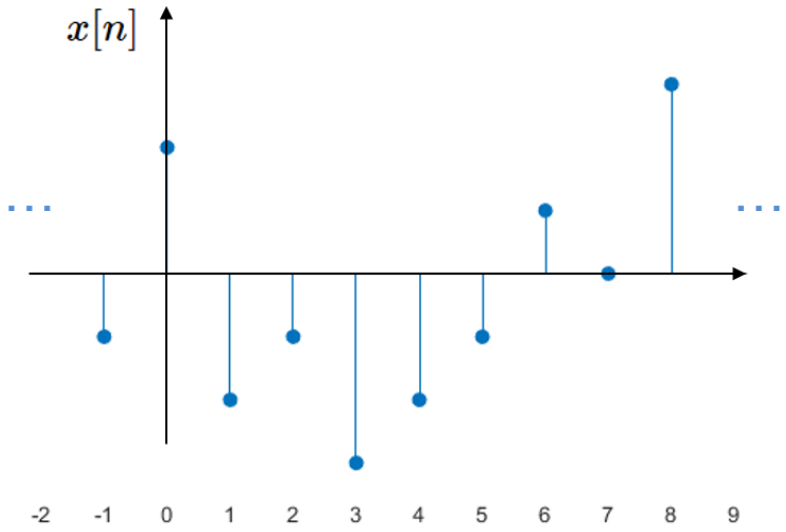
Most slides from
(edX) Discrete Time Signals and Systems by Prof. Richard G. Baraniuk

Discrete Time Signals

- A signal $x[n]$ is a function that maps an independent variable to a dependent variable.

- We will focus on discrete-time signals $x[n]$
  – Independent variable is an integer: $n \in \mathbb{Z}$
  – Dependent variable is a real or complex number: $x[n] \in \mathbb{R}$ or $\mathbb{C}$

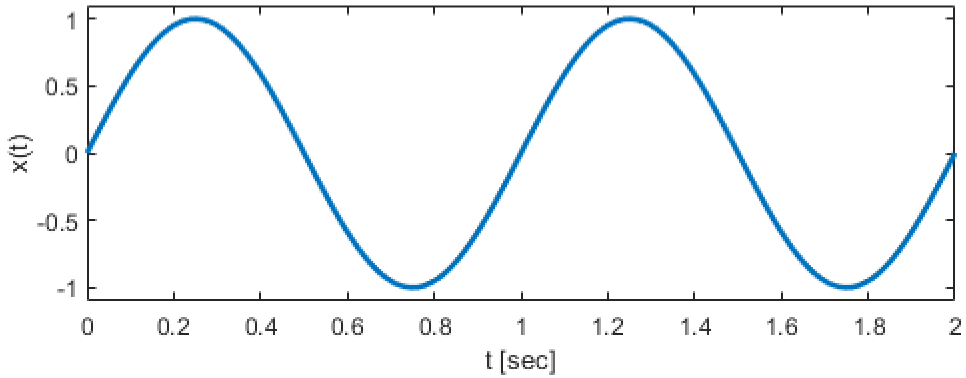


Plot Real Signals

- Continuous signal

$$x(t) = \sin(2\pi t)$$

```
t = 0:0.01:2;
x = sin(2*pi*t);

plot(t, x, 'linewidth', 2);
ylim([-1.1 1.1]);
xlabel('t [sec]');
ylabel('x(t)');
```

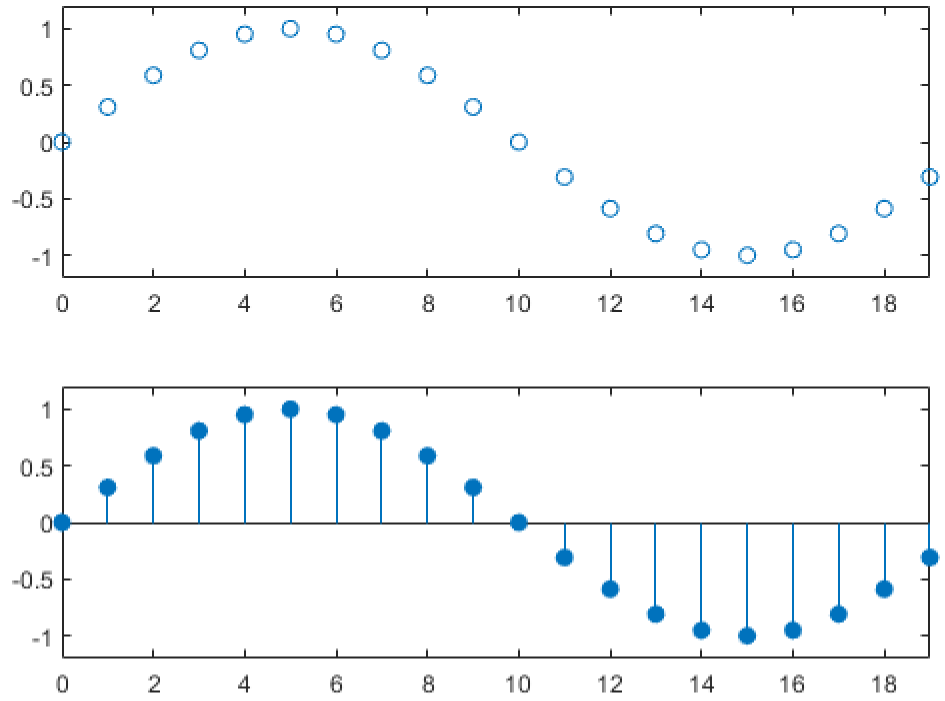


Plot Real Signals

- Discrete signals

$$x(n) = \sin\left(\frac{2\pi}{N}n\right)$$

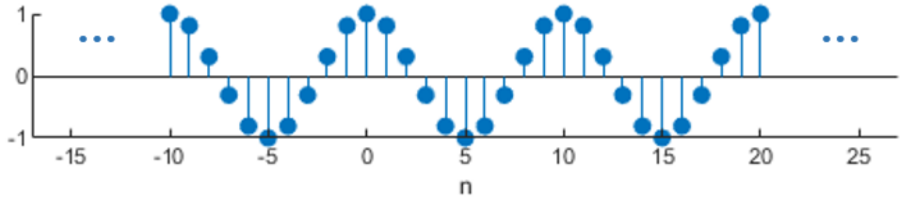
```
N = 20;
n = 0:N-1;
x = sin(2*pi/N*n);

subplot(2,1,1)
plot(n,x,'o'), axis tight, ylim([-1.2, 1.2])
subplot(2,1,2)
stem(n,x,'filled'), axis tight, ylim([-1.2, 1.2])
```



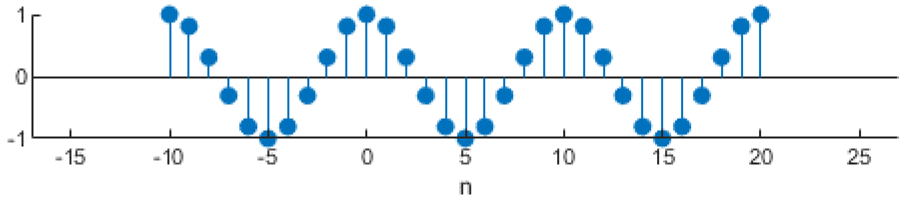
Discrete Signal Properties

Finite/Infinite Length Signals

- An infinite-length discrete-time signal $x[n]$ is defined for all $n \in \mathbb{Z}$, i.e., $-\infty < n < \infty$



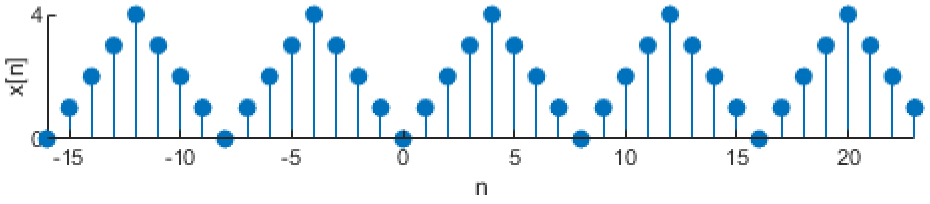
- An finite-length discrete-time signal $x[n]$ is defined only for a finite range of $N_1 \leq n \leq N_2$



Periodic Signals

- A discrete-time signal is periodic if it repeats with period $N \in \mathbb{Z}$

$$x[n + mN] = x[n], \qquad \forall m \in \mathbb{Z}$$



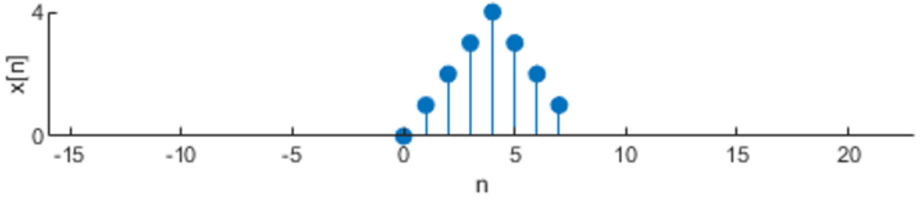
- The period N must be an integer
- A periodic signal is infinite in length

Periodic Signals

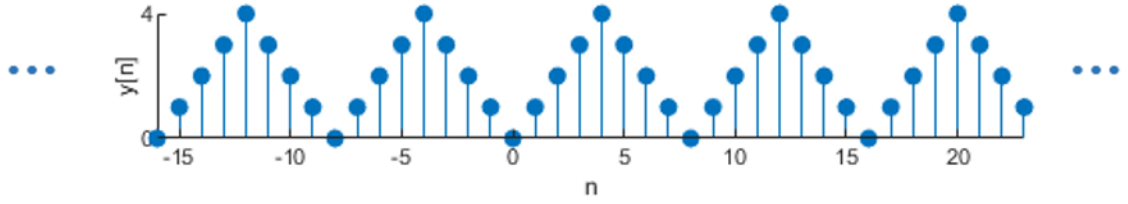
- Convert a finite-length signal $x[n]$ defined for $N_1 \leq n \leq N_2$ into an infinite-length signal by either
  - (infinite) zero padding, or
  - periodization with period N

$$y[n] = \begin{cases} 0 & n < N_1 \\ x[n] & N_1 \leq n \leq N_2 \\ 0 & N_2 < n \end{cases}$$

$$\begin{aligned} y[n] &= \sum_{m=-\infty}^{\infty} x[n - mN] \\ &= \cdots + x[n+N] + x[n] + x[n-N] + \cdots \end{aligned}$$



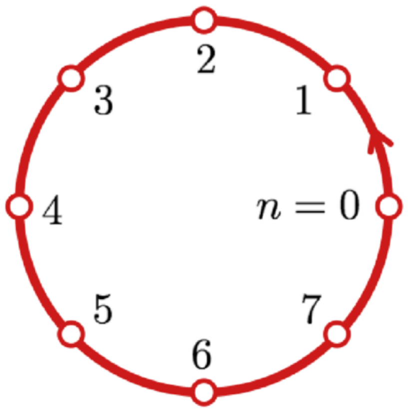
$y[n]$ with period $N = 8$



Modular Arithmetic

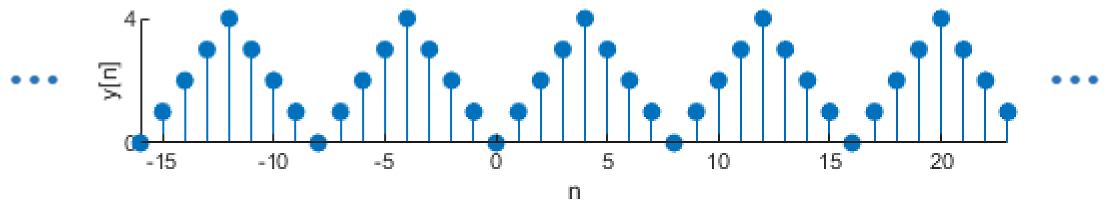
- Modular arithmetic with modulus N takes place on a clock with N
  - Modular arithmetic is inherently periodic

$$\cdots = (-12)_8 = (-4)_8 = (4)_8 = (12)_8 = (20)_8 = \cdots$$



- Periodization via Modular Arithmetic
  - Consider a length-N signal $x[n]$ defined for $0 \le n \le N-1$
  - A convenient way to express periodization with period N is $y[n] = x[(n)_N]$
  - Important interpretation
    - Infinite-length signals live on the (infinite) number line
    - Periodic signals live on a circle

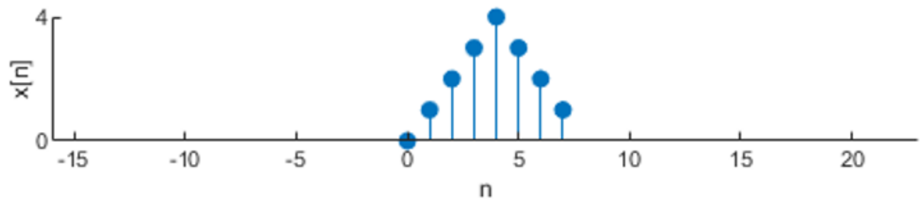
```
N = 8;
n = 0:N-1;
x = [0 1 2 3 4 3 2 1];
```

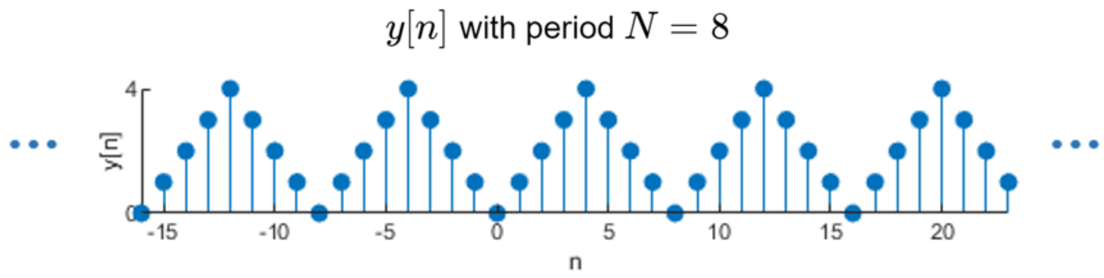



```
%% periodic using mod
y = [];
n = -16:23;

for i = 1:length(n)
    y(i) = x(mod(n(i),N)+1);
end
```

Finite-Length and Periodic Signals

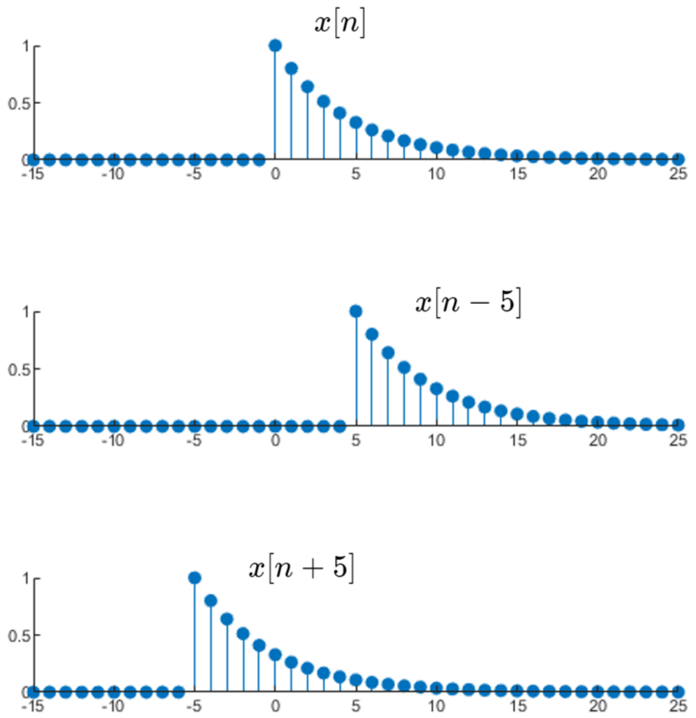
- Finite-length and periodic signals are equivalent
  - All of the information in a periodic signal is contained in one period (of finite length)
  - Any finite-length signal can be periodized
  - Conclusion: We will think of finite-length signals and periodic signals interchangeably





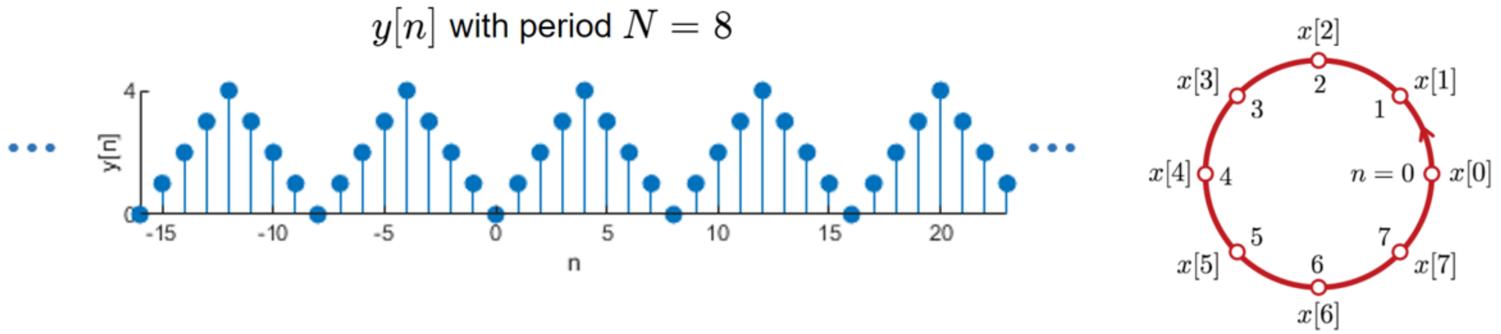
Finite-Length and Periodic Signals

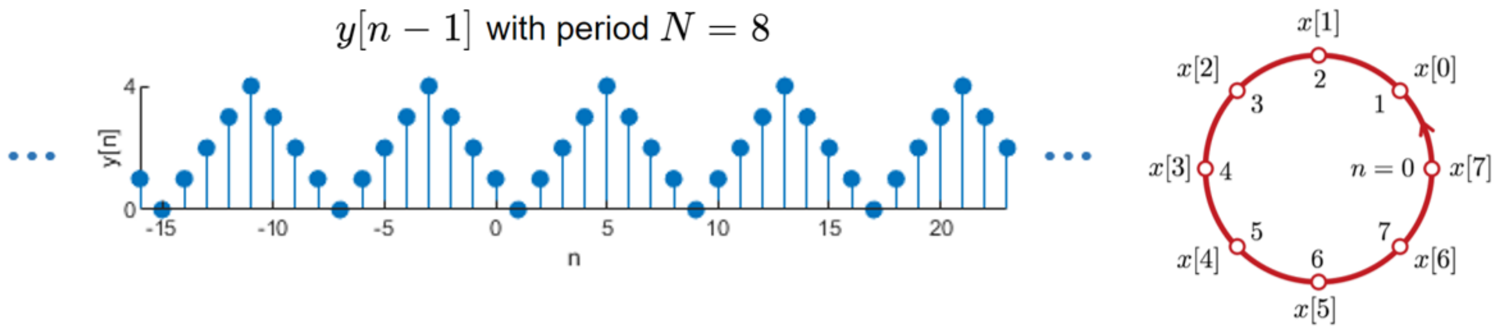
- Shifting infinite-length signals
  - Given an infinite-length signal $x[n]$, we can shift back and forth in time via $x[n-m]$
  - When $m > 0$, $x[n-m]$ shifts to the right (forward in time, delay)
  - When $m < 0$, $x[n-m]$ shifts to the left (back in time, advance)



Finite-Length and Periodic Signals

- Shifting periodic signals
  - Periodic signals can also be shifted; consider $y[n] = x[(n)_N]$
  - Shift one sample into the future: $y[n-1] = x[(n-1)_N]$





Shifting Finite-Length Signals

- Consider finite-length signals x and y defined for $0 \leq n \leq N-1$ and suppose $y[n] = x[n-1]$

$$
\begin{aligned}
y[0] &= ? \\
y[1] &= x[0] \\
y[2] &= x[1] \\
y[3] &= x[2] \\
&\vdots \\
y[N-1] &= x[N-2] \\
? &= x[N-1]
\end{aligned}
$$

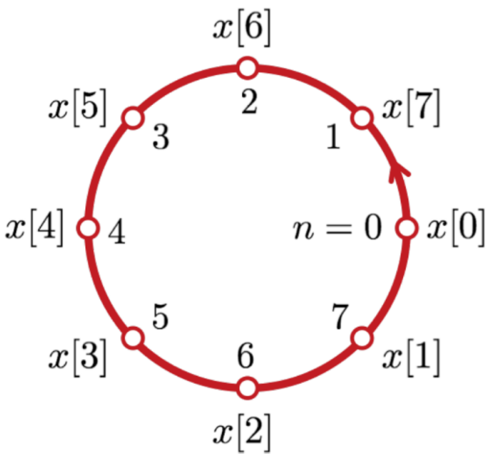
- What to put in $y[0]$? What to do with $x[N-1]$? We do not want to invent/lose information
- Elegant solution: Assume x and y are both periodic with period N; then $y[n] = x[(n-1)_N]$
- This is called a periodic or circular shift

Circular Time Reversal

$$y[n] = x[(-n)_N]$$

- Example with $N = 8$

$$\begin{aligned} y[0] &= x[0] \\ y[1] &= x[7] \\ y[2] &= x[6] \\ y[3] &= x[5] \\ &\vdots \\ y[7] &= x[1] \end{aligned}$$

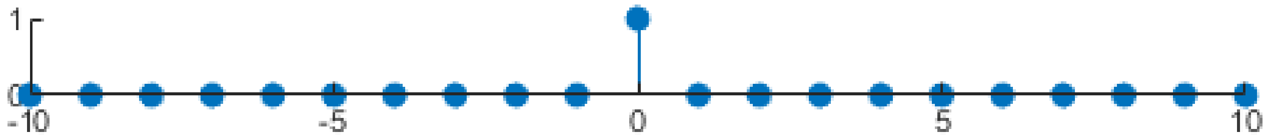


Key Discrete Signals

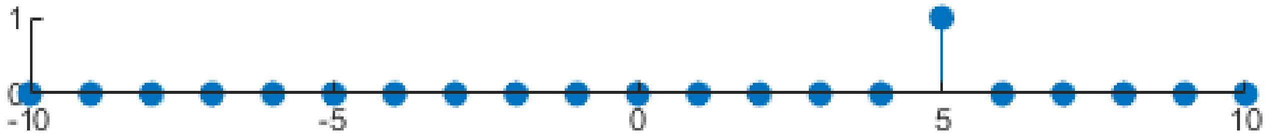
Delta Function

- Delta function = unit impulse = unit sample

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$



- The shifted delta function $\delta[n-m]$ peaks up at $n = m$, (here $m = 5$)

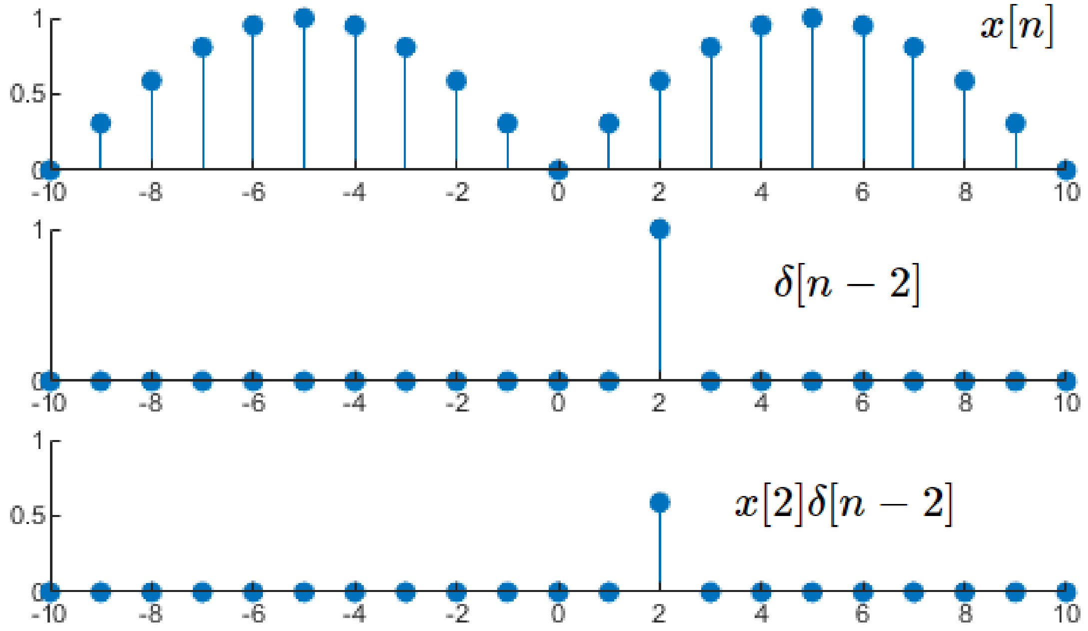


Delta Function Sample

- Multiplying a signal by a shifted delta function picks out one sample of the signal and sets all other samples to zero

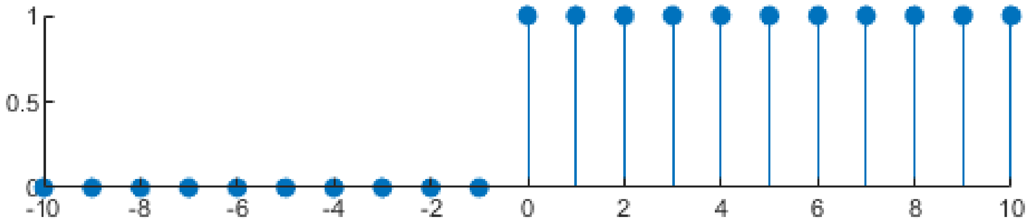
$$y[n] = x[n]\delta[n-m] = x[m]\delta[n-m]$$

- Important: m is a fixed constant, and so $x[m]$ is a constant (and not a signal)

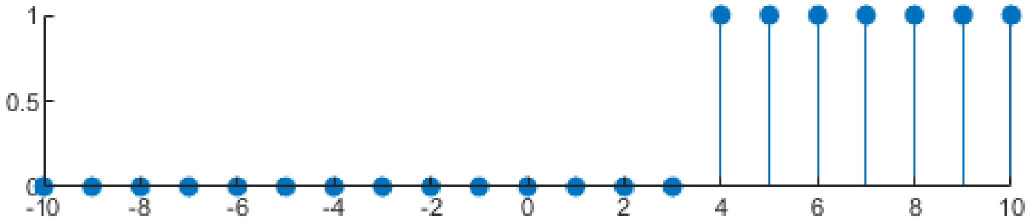


Unit Step Function

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



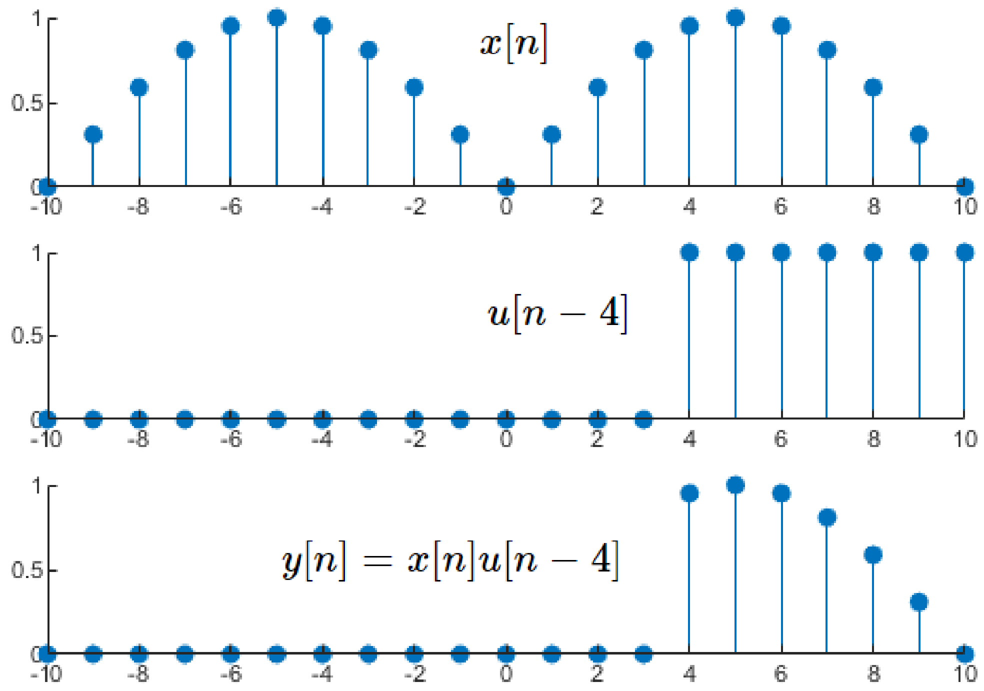
- The shifted unit step $u[n-m]$ jumps from 0 to 1 at $n = m$, (here $m = 4$)



Unit Step Selects Part of a Signal

- Multiplying a signal by a shifted unit step function zeros out its entries for $n < m$

$$y[n] = x[n]u[n-m]$$



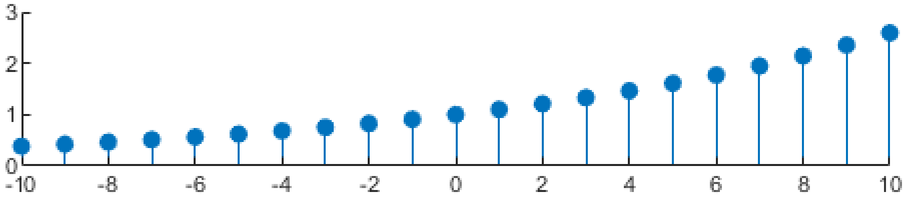
POSTECH

Real Exponential

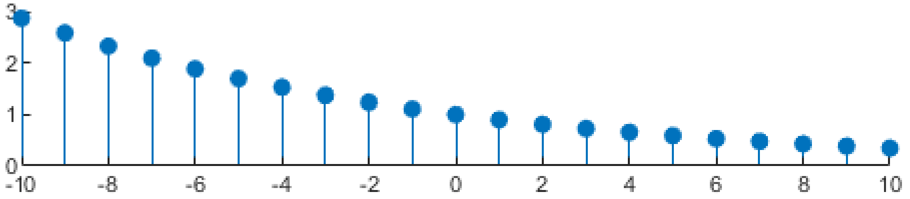
- The real exponential

$$x[n] = a^n, \quad n \in \mathbb{R}$$

- For $a > 1$, $x[n]$ grows to the right



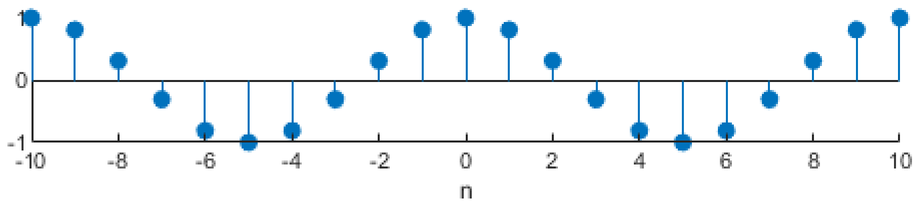
- For $0 < a < 1$, $x[n]$ shrinks to the right



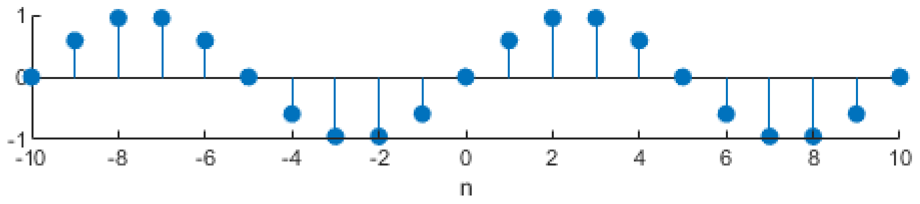
Sinusoid Signals

- There are two natural real-valued sinusoids: $\cos(\omega n+\phi)$ and $\sin(\omega n+\phi)$
  - Frequency: ω (units: radians/sample)
  - Phase: ϕ (units: radians)

  - $\cos(\omega n)$

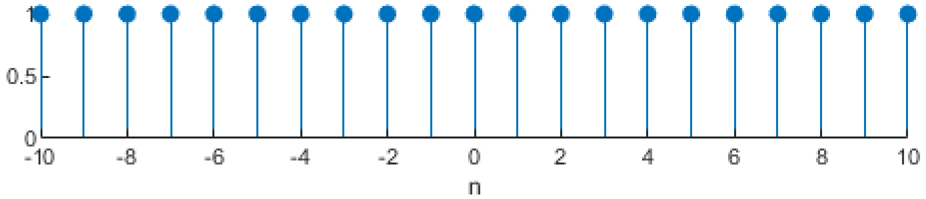


  - $\sin(\omega n)$

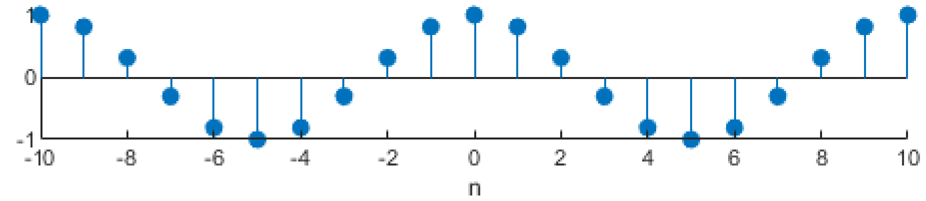


Sinusoid Signals

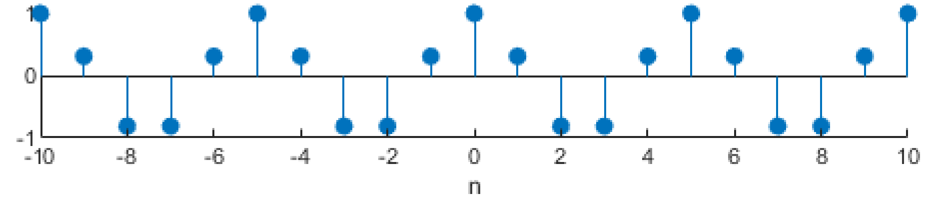
- $\cos(0n)$



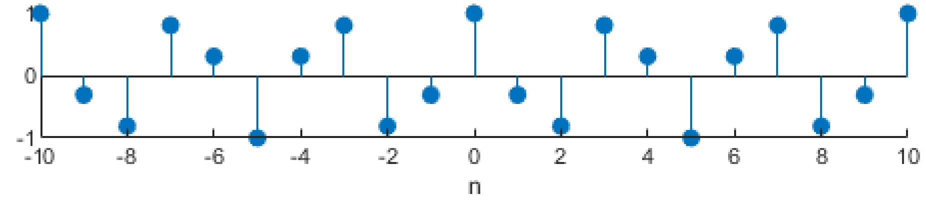
- $\cos\left(\frac{2\pi}{10}n\right)$



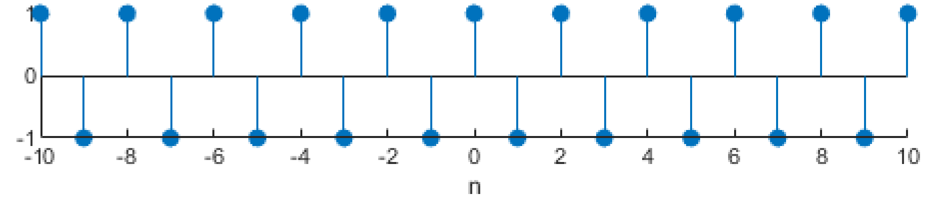
- $\cos\left(\frac{4\pi}{10}n\right)$



- $\cos\left(\frac{6\pi}{10}n\right)$

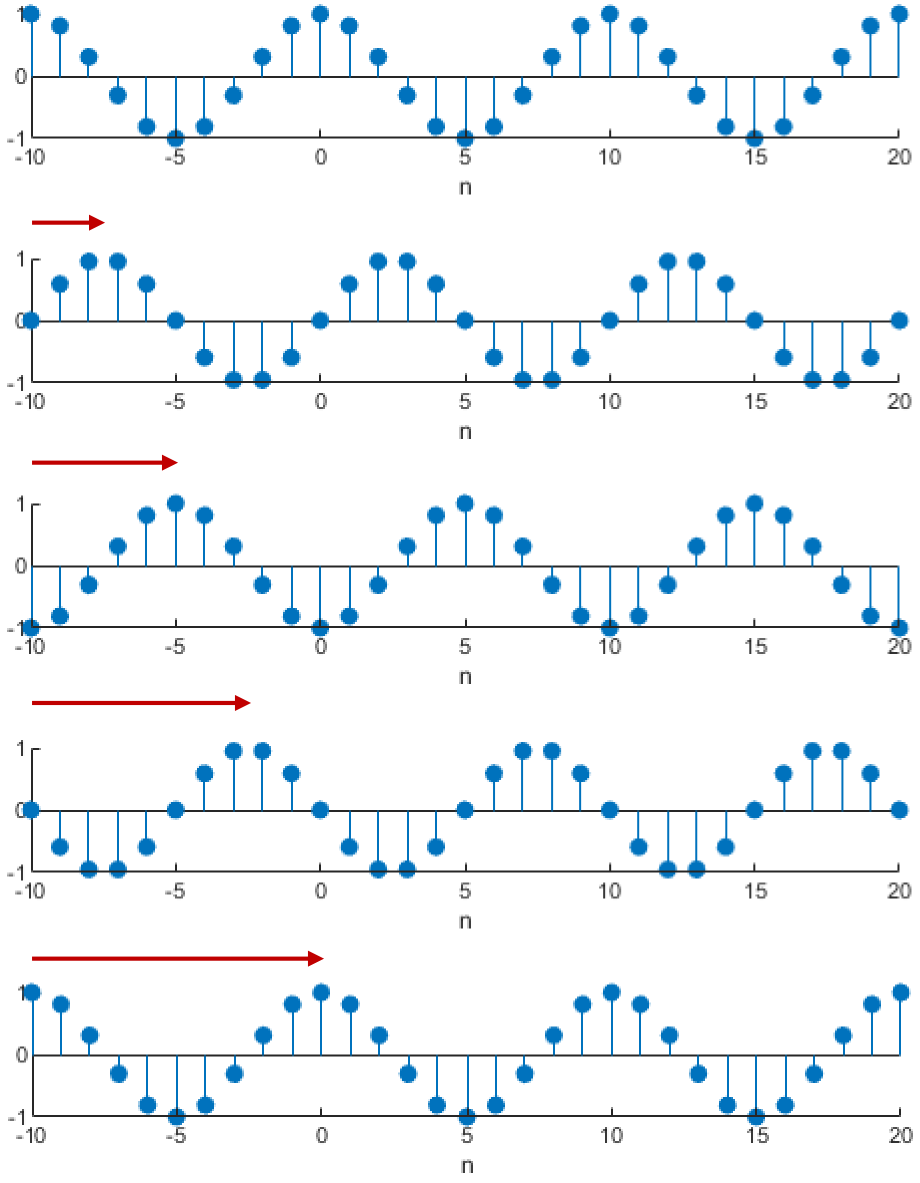


- $\cos\left(\frac{10\pi}{10}n\right) = \cos(\pi n)$



Phase of Sinusoid

- $\cos\left(\frac{2\pi}{10}n\right)$
- $\cos\left(\frac{2\pi}{10}n-\frac{\pi}{2}\right)$
- $\cos\left(\frac{2\pi}{10}n-\frac{2\pi}{2}\right)$
- $\cos\left(\frac{2\pi}{10}n-\frac{3\pi}{2}\right)$
- $\cos\left(\frac{2\pi}{10}n-\frac{4\pi}{2}\right)=\cos\left(\frac{2\pi}{10}n\right)$



Complex Sinusoid

Complex Number

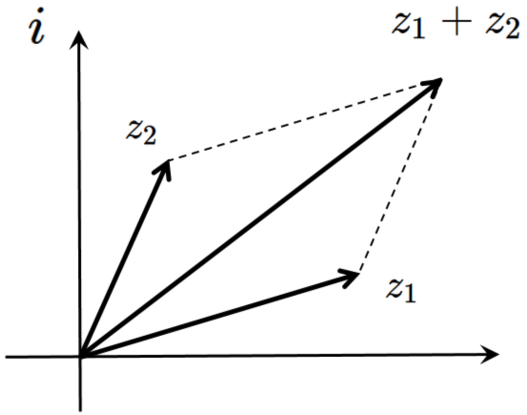
$$z_1 = a_1 + b_1 i, \qquad \vec{z}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

$$z_2 = a_2 + b_2 i, \qquad \vec{z}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

- Adding

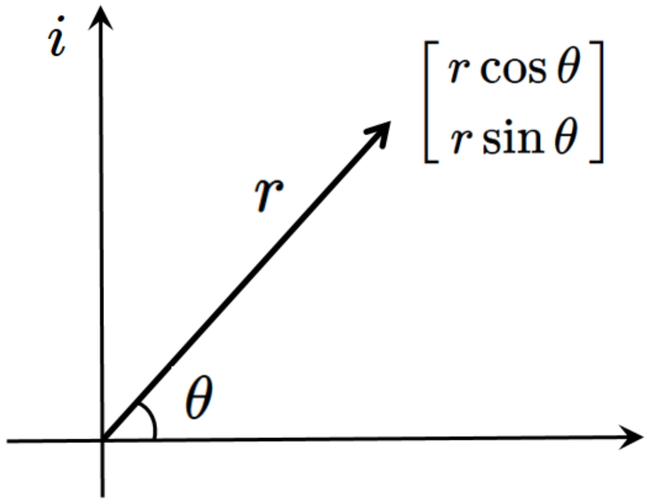
$$z = z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$$

$$\vec{z} = \vec{z}_1 + \vec{z}_2 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$$



Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$



$$\vec{z} = r\cos\theta + i\,r\sin\theta$$

$$\vec{z} = re^{i\theta} = r\,(\cos\theta + i\sin\theta)$$

r : magnitude (length)

θ : phase (angle)

Complex Number

$$z_1 = a_1 + b_1 i, \qquad \vec{z}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

$$z_2 = a_2 + b_2 i, \qquad \vec{z}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

- Multiplying

$$\begin{aligned} z_1 &= r_1 e^{i\theta_1} \\ z_2 &= r_2 e^{i\theta_2} \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} z_1 \cdot z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \end{aligned}$$

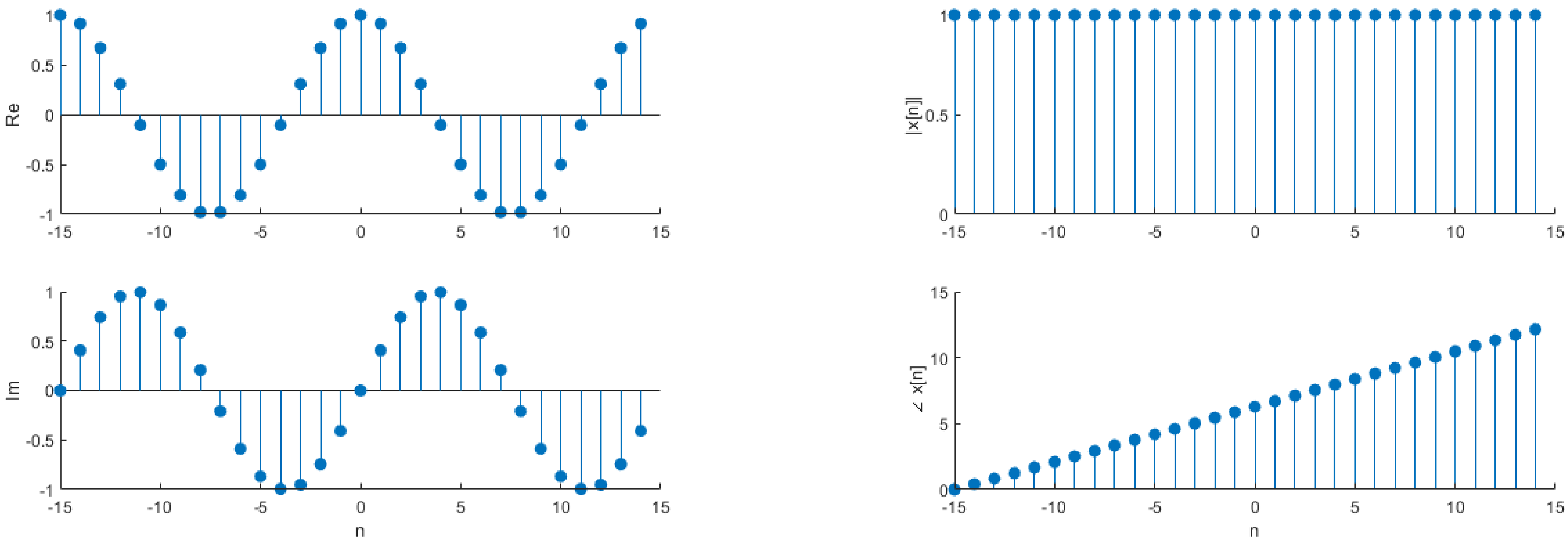
Plot Complex Signals

- When $x[n] \in \mathbb{C}$, we can use two signal plots

$$x[n] = \mathrm{Re}\{x[n]\} + j\mathrm{Im}\{x[n]\} \qquad \text{Rectangular form}$$

$$x[n] = |x[n]|e^{j\angle x[n]} \qquad \text{Polar form}$$

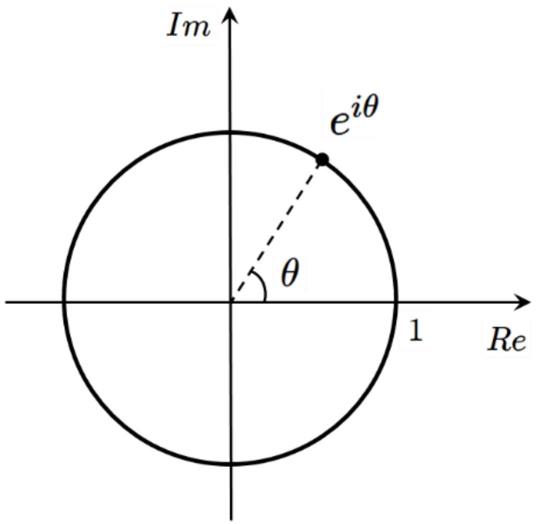
- For $e^{j\omega n}$



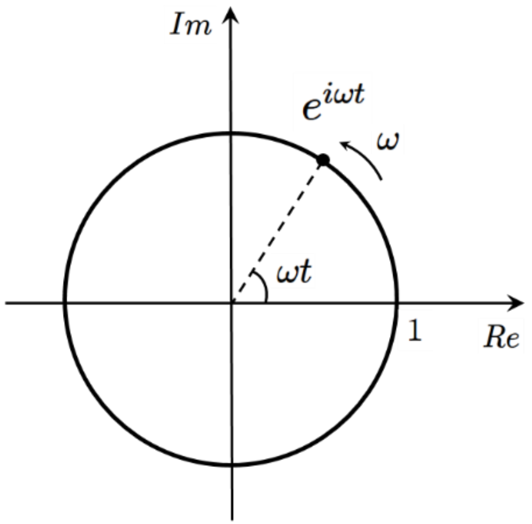
phase

Geometrical Meaning of $e^{i\theta}$

- $e^{i\theta}$: point on the unit circle with angle of θ



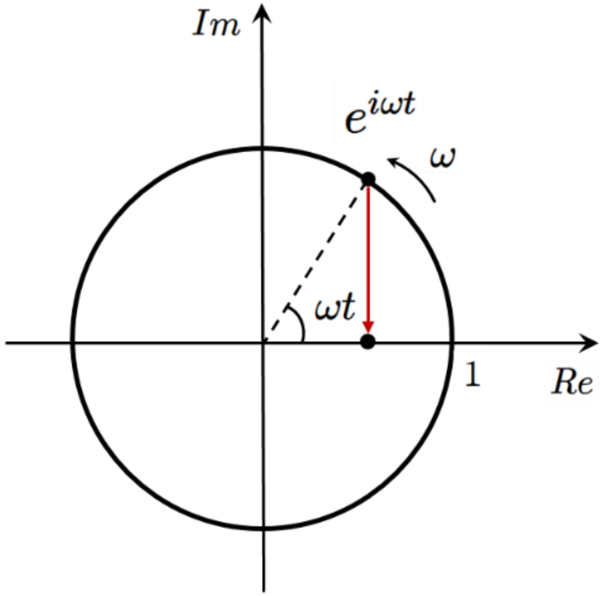
- $\theta = \omega t$
- $e^{i\omega t}$: rotating on an unit circle with angular velocity of ω



- Question: what is the physical meaning of $e^{-i\omega t}$?

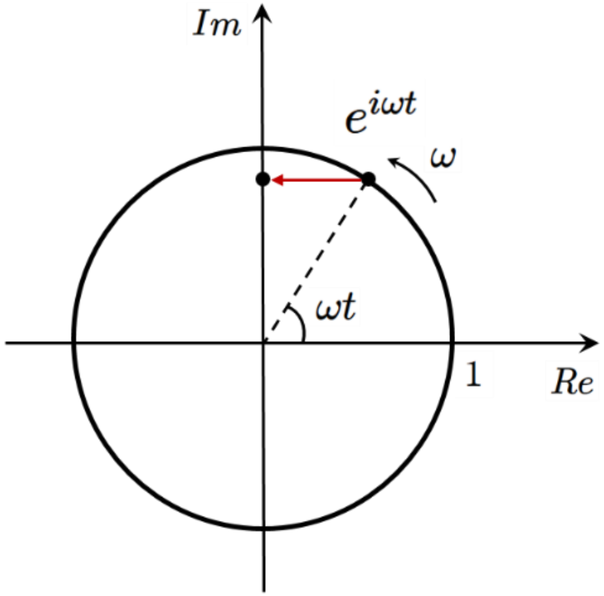
Sinusoidal Functions from Circular Motions

projection of $e^{j\omega t}$ onto Re-axis



$$\mathrm{Re}\left(e^{i\omega t}\right) = \cos\omega t$$

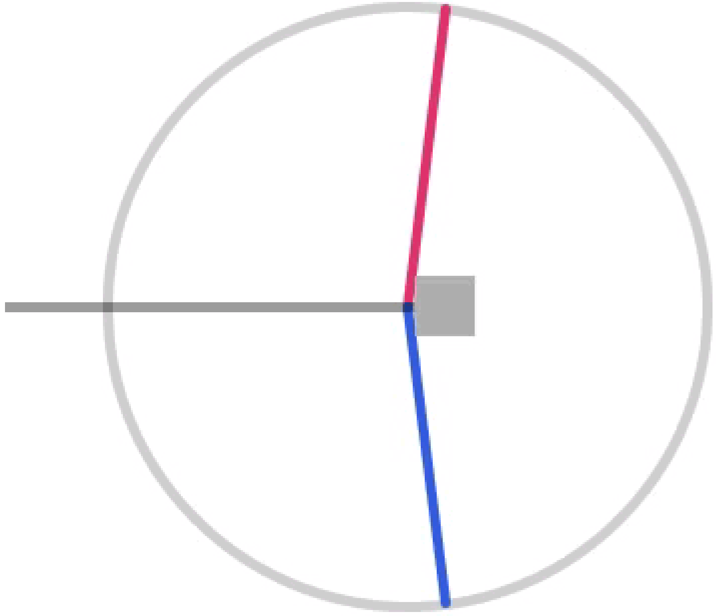
projection of $e^{j\omega t}$ onto Im-axis

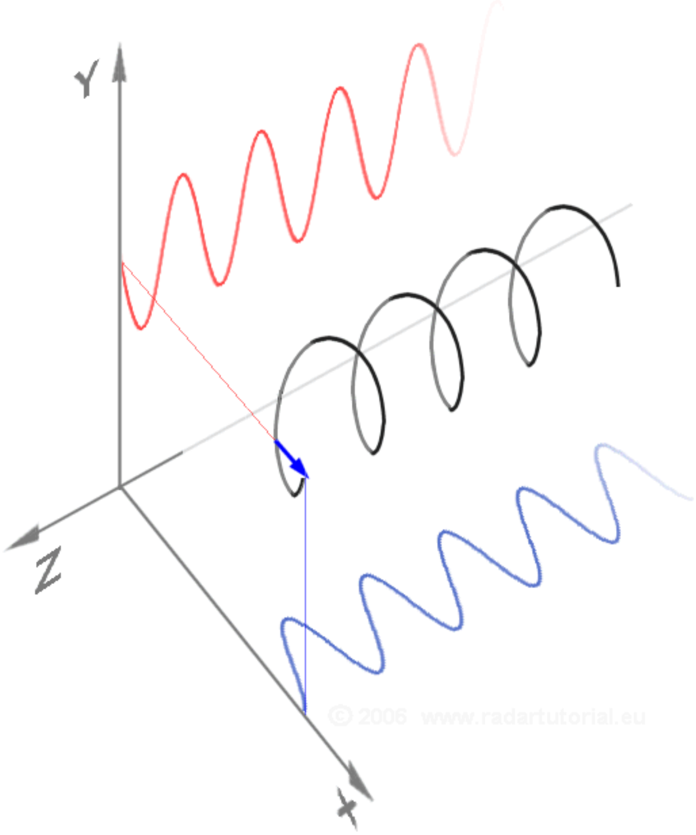


$$\mathrm{Im}\left(e^{i\omega t}\right) = \sin\omega t$$

Sinusoidal Functions from Circular Motions

$$\cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$$





Discrete Sinusoids

- Discrete Sinusoids

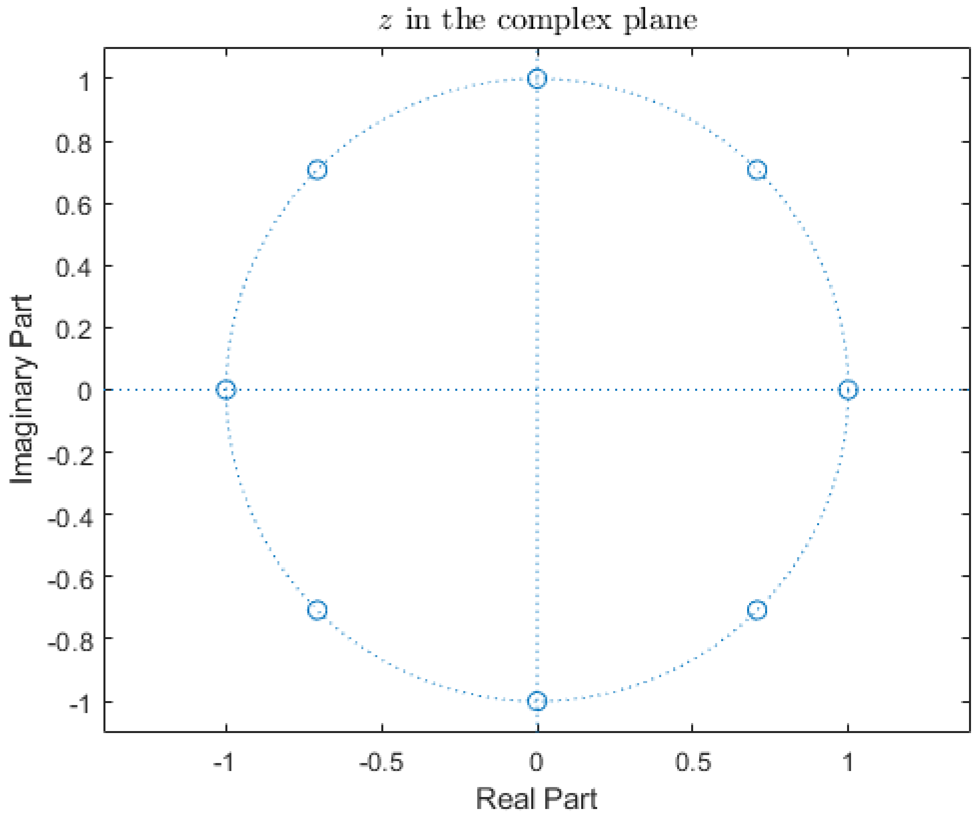
$$x[n] = A\cos(\omega_0 n + \phi) \quad \text{or}$$

$$x[n] = Ae^{j(\omega_0 n + \phi)} \qquad \text{where} \;\; \omega_0 = \frac{2\pi}{N}k$$

Visualize the Discrete Sinusoidals

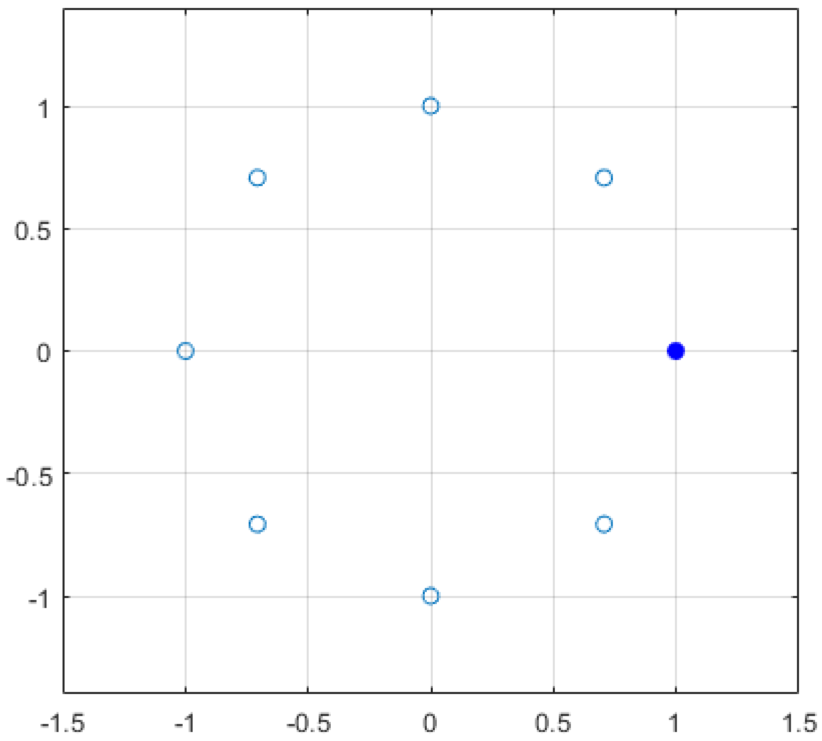
$$x[n] = e^{j\omega n}, \qquad \omega = \frac{2\pi}{N} k$$

```
N = 8;
k = 1;
z1 = exp(1j*2*pi/N*k);

n = 0:N-1;
z = (z1.^n).';
```



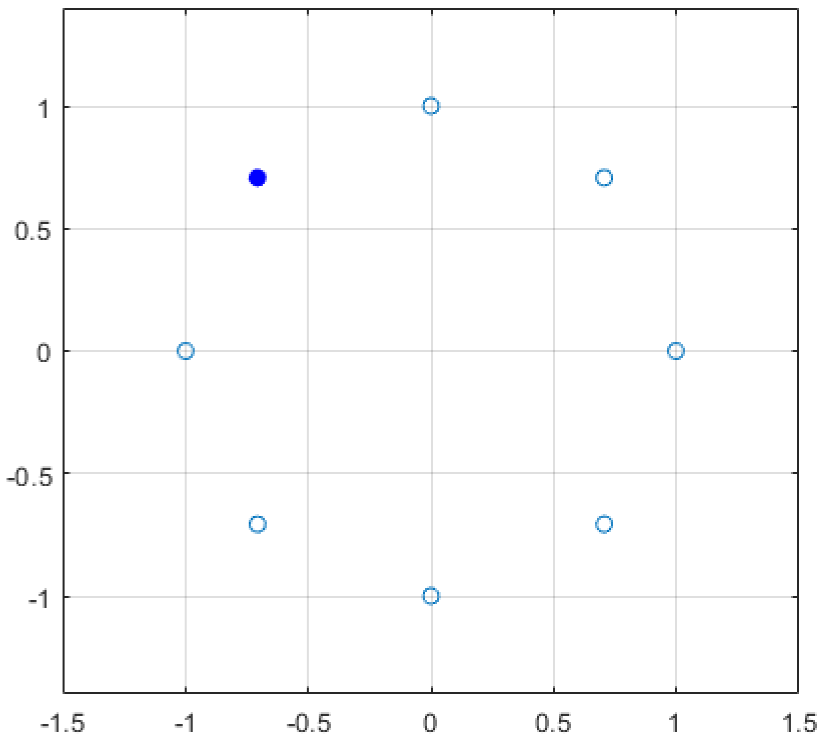
Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3, \; e^{j\omega 0}$



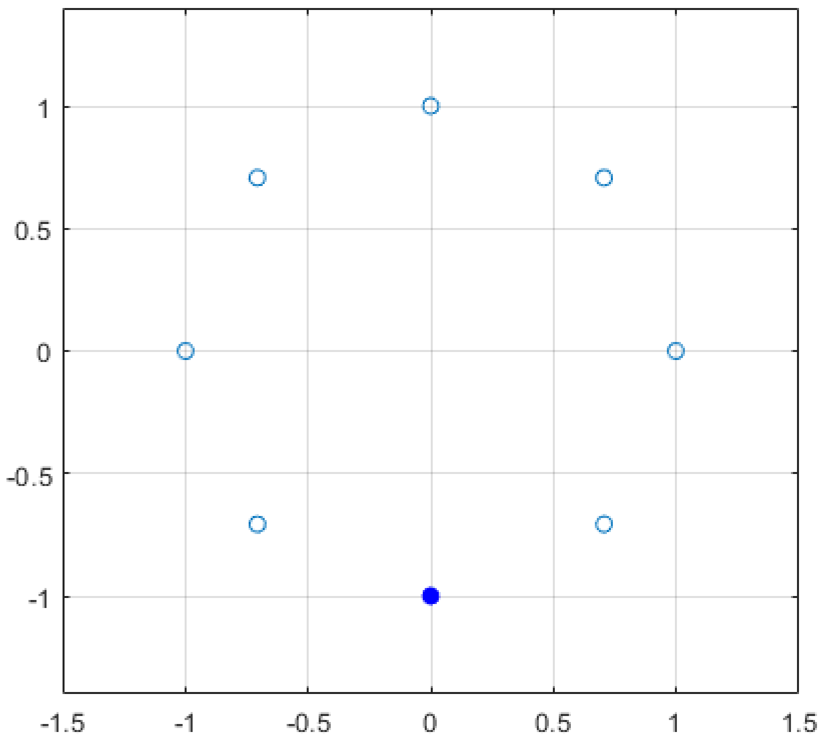
Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3,\ e^{j\omega 1}$



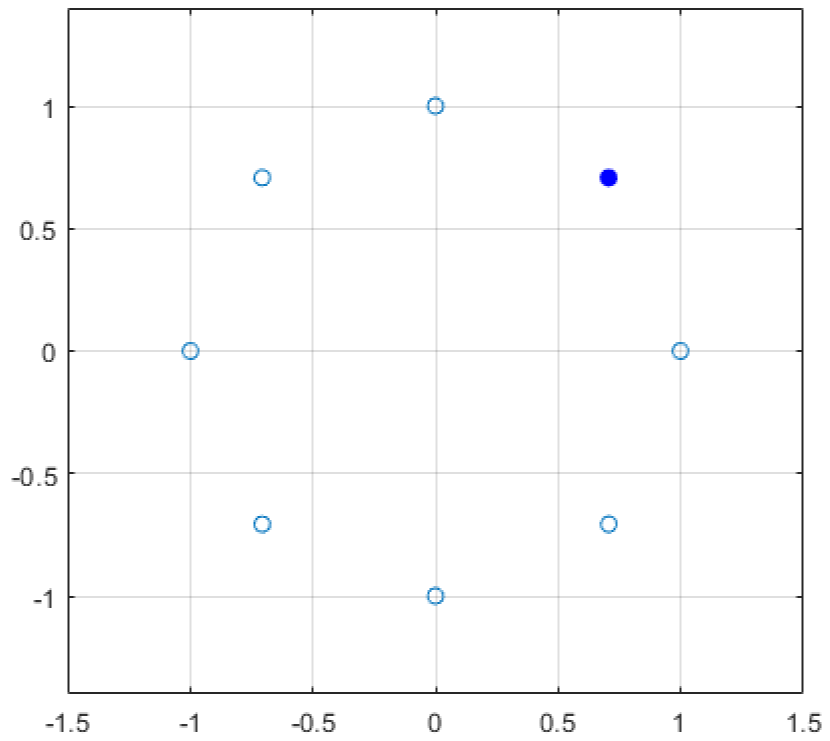
Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3,\ e^{j\omega 2}$



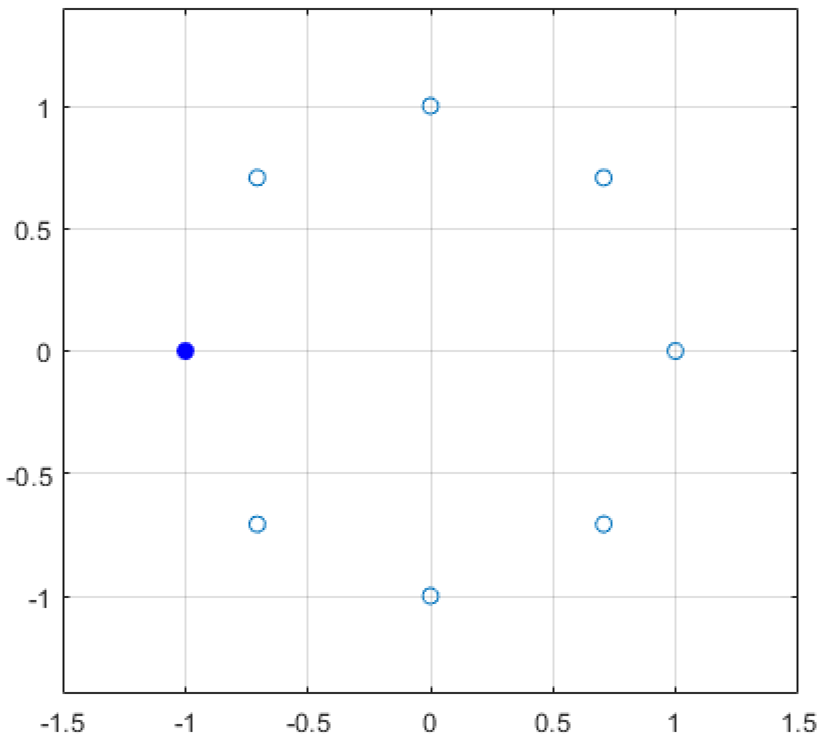
Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3,\ e^{j\omega 3}$



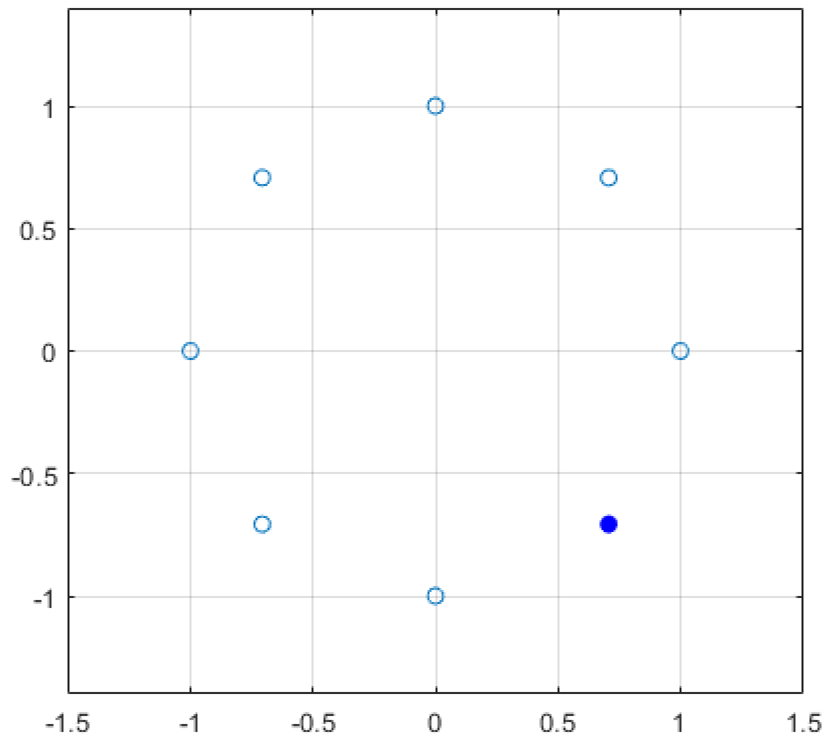
Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3,\ e^{j\omega 4}$



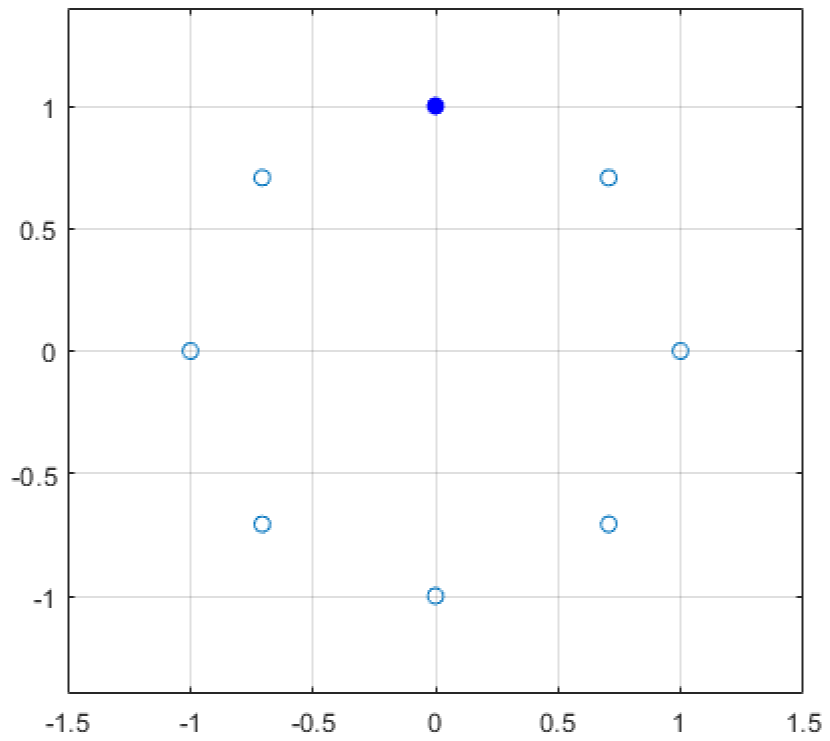
Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3,\ e^{j\omega 5}$



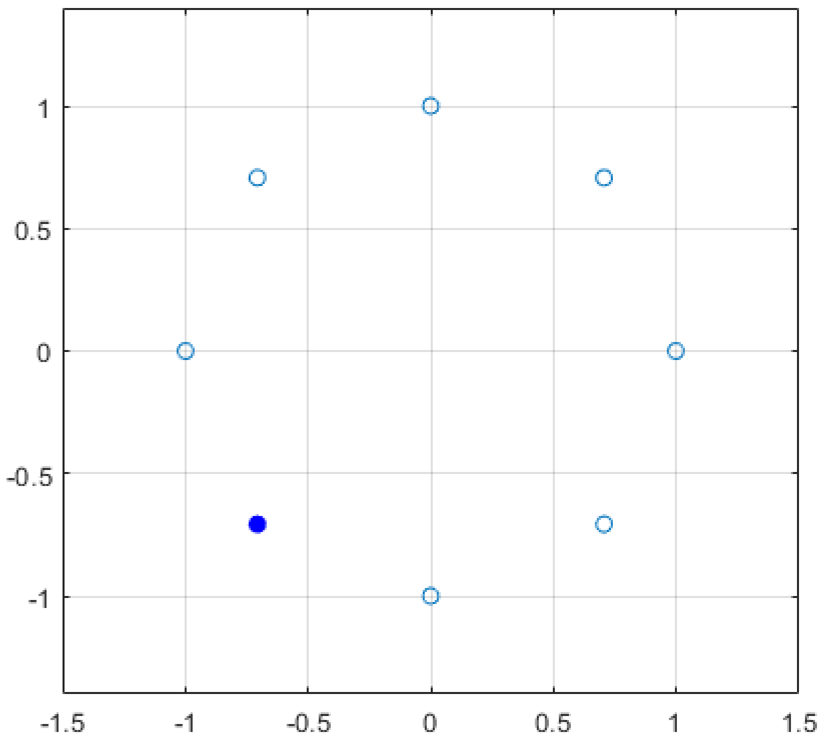
Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3, \ e^{j\omega 6}$



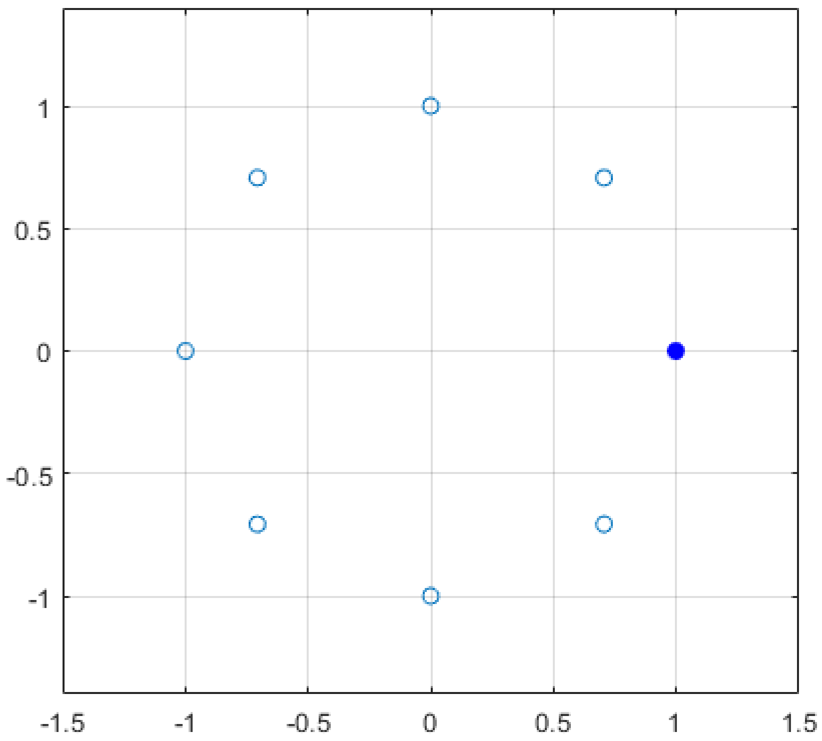
Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3,\ e^{j\omega 7}$



Visualize the Discrete Sinusoidals

- $\omega = \frac{2\pi}{8}3,\ e^{j\omega 8}$



Aliasing

Aliasing of Discrete Sinusoids

- Consider two sinusoids with two different frequencies

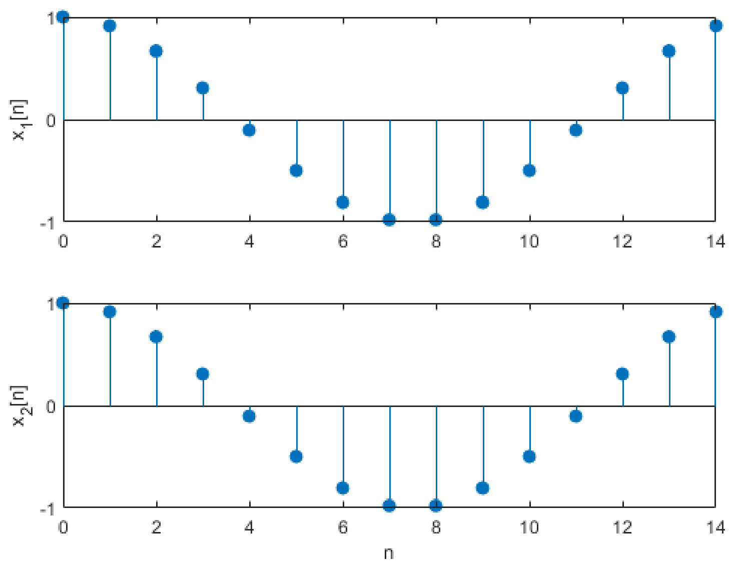
$$\begin{aligned} \omega \quad &\Longrightarrow \quad x_1[n] = e^{j(\omega n+\phi)} \\ \omega + 2\pi \quad &\Longrightarrow \quad x_2[n] = e^{j((\omega+2\pi)n+\phi)} = e^{j(\omega n+\phi)} e^{j2\pi n} \end{aligned}$$

- But note that

$$x_2[n] = x_1[n]$$

- The signal x_1 and x_2 have different frequencies but are identical
- We say that x_1 and x_2 are aliases
- This phenomenon is called aliasing

```
N = 15;
n = 0:N-1;

x1 = cos(2*pi/N*n);
x2 = cos((2*pi/N + 2*pi)*n);
```

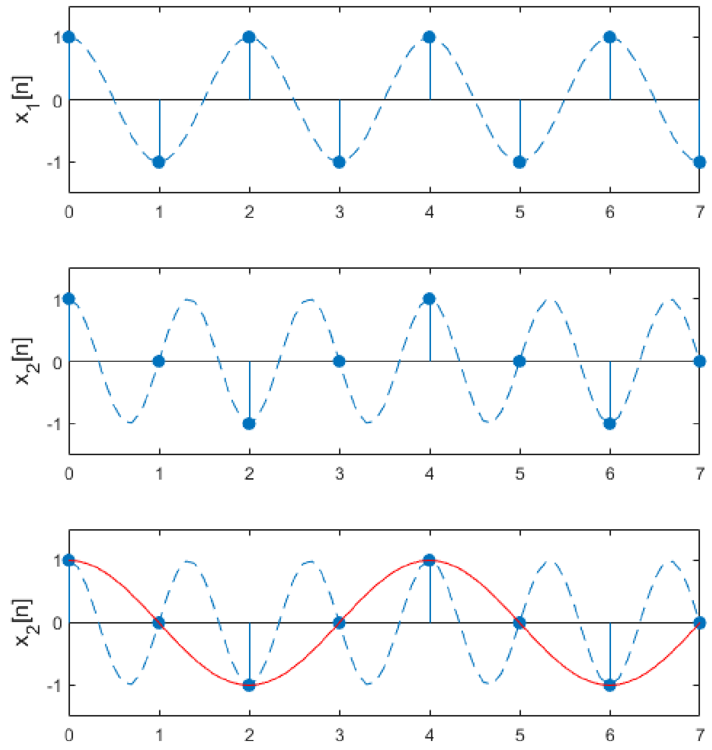


Alias-free Frequencies in Discrete Sinusoids

- Alias-free frequencies
  - The only frequencies that lead to unique (distinct) sinusoids lie in an interval of length 2π
  - Two intervals are typically used in the signal processing

$$0 \leq \omega < 2\pi$$
$$-\pi < \omega \leq \pi$$

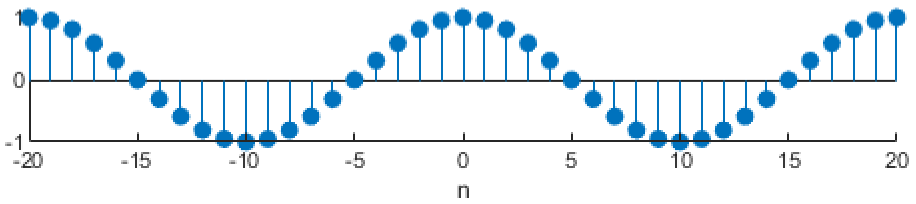
```
N = 8;
n = 0:N-1;

xn1 = cos(pi*n);
xn2 = cos(3/2*pi*n);
```



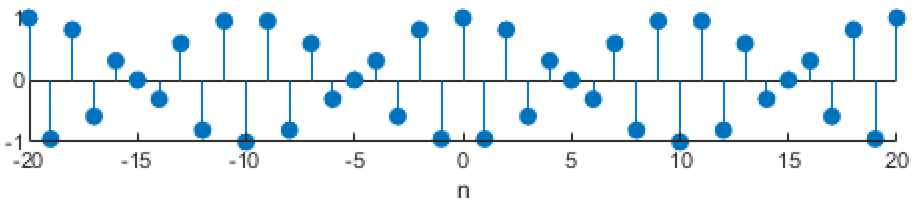
Low and High Frequencies in Discrete Sinusoids

- Low frequencies: ω closed to 0 and 2π
- High frequencies: ω closed to π and $-\pi$

- $\cos\left(\frac{2\pi}{20}n\right)$



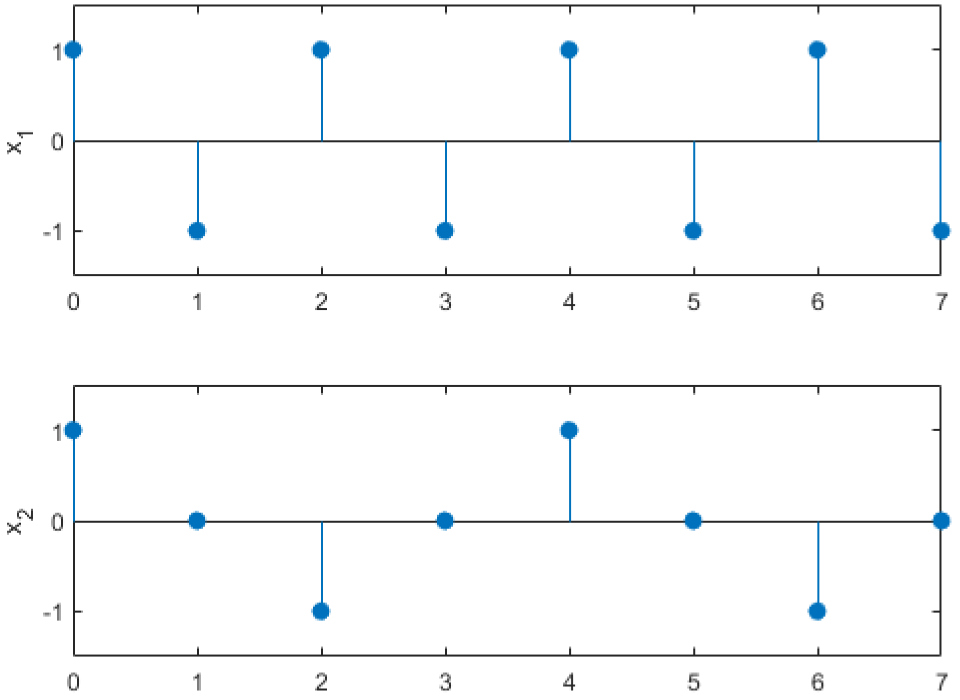
- $\cos\left(9 \times \frac{2\pi}{20}n\right) = \cos\left(\frac{18\pi}{20}n\right)$



Low and High Frequencies in Discrete Sinusoids

- Which one is a higher frequency? $\omega_0 = \pi \quad \text{or} \quad \omega_0 = \dfrac{3\pi}{2}$

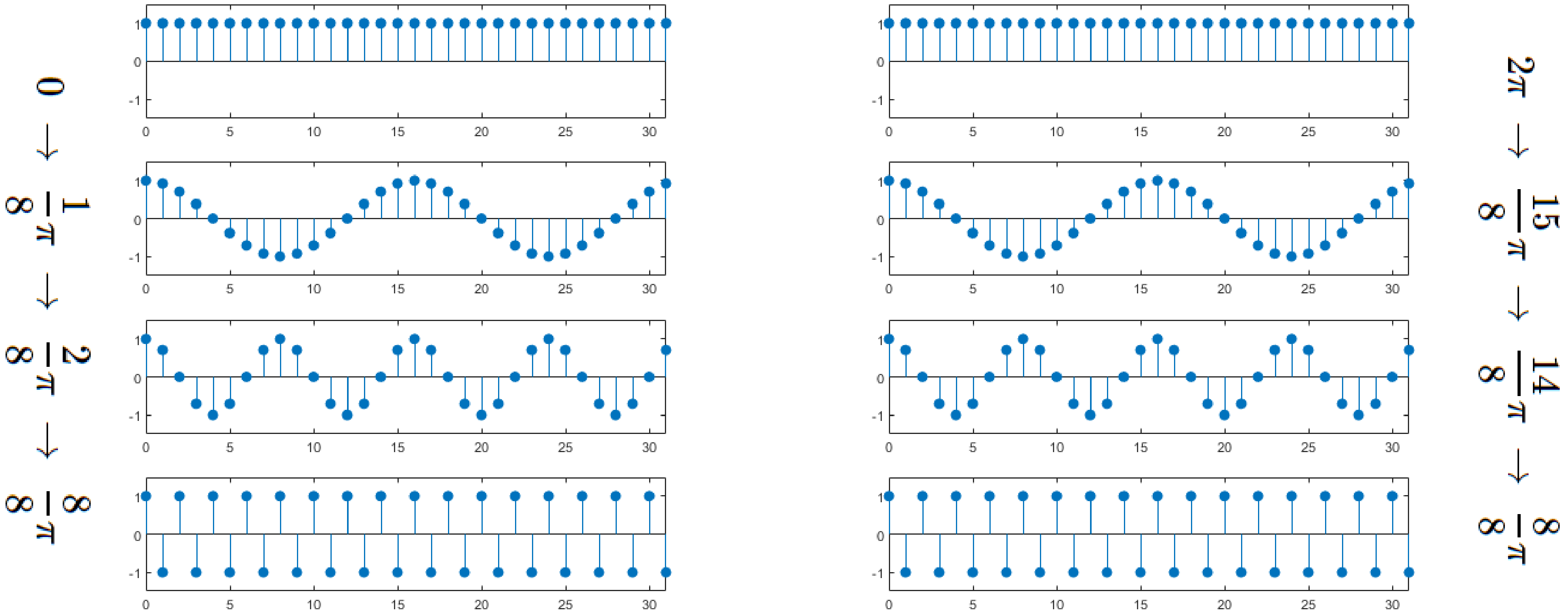
```
N = 8;
n = 0:N-1;

x1 = cos(pi*n);
x2 = cos(3/2*pi*n);
```



Frequency in Discrete Sinusoids

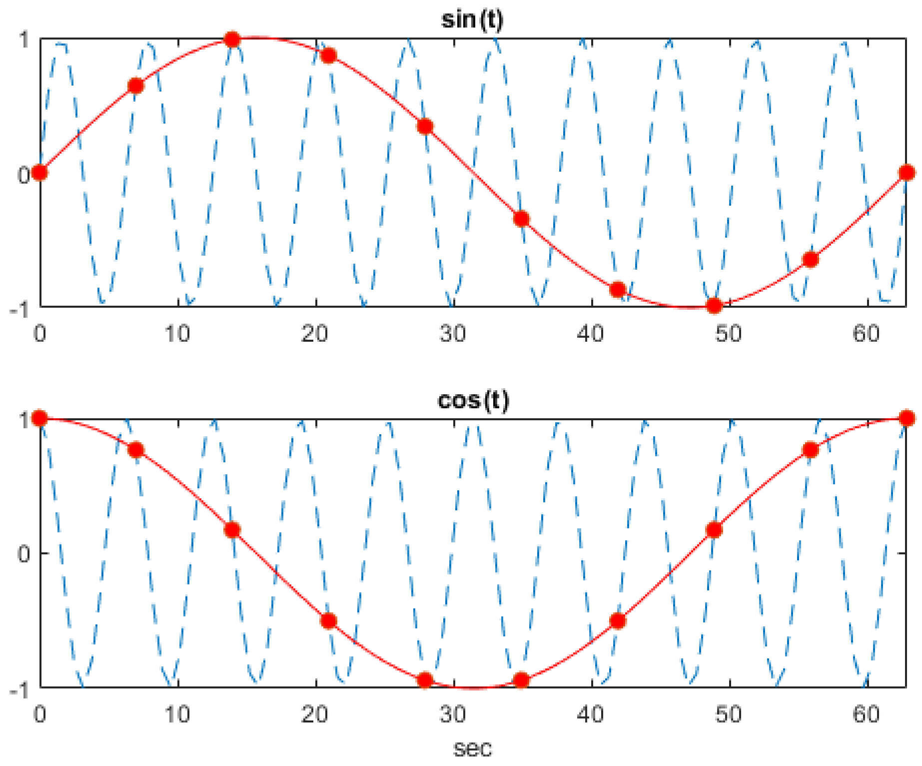
```
N = 32;
n = 0:N-1;
x1 = cos(0*pi*n);
x2 = cos(1/8*pi*n);
x3 = cos(2/8*pi*n);
x4 = cos(1*pi*n);
```

```
N = 32;
n = 0:N-1;
x5 = cos(2*pi*n);
x6 = cos(15/8*pi*n);
x7 = cos(14/8*pi*n);
x8 = cos(1*pi*n);
```



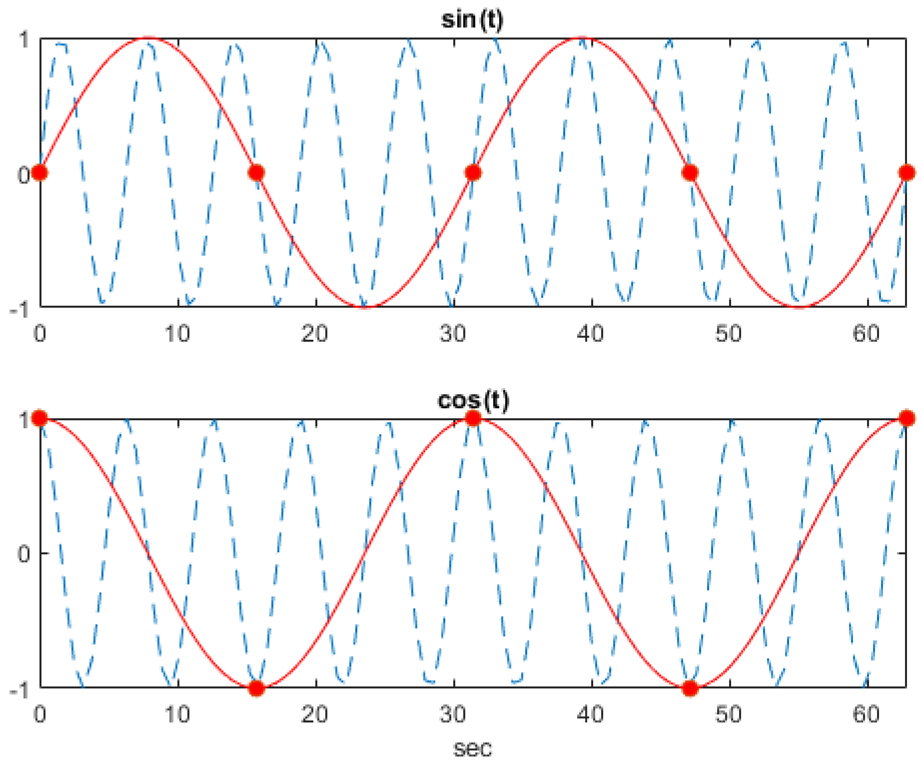
Aliasing

```
t = linspace(0,10*2*pi,100);
x = sin(t);
y = cos(t);

ts = linspace(0,10*2*pi,10);
xs = sin(ts);
ys = cos(ts);
```

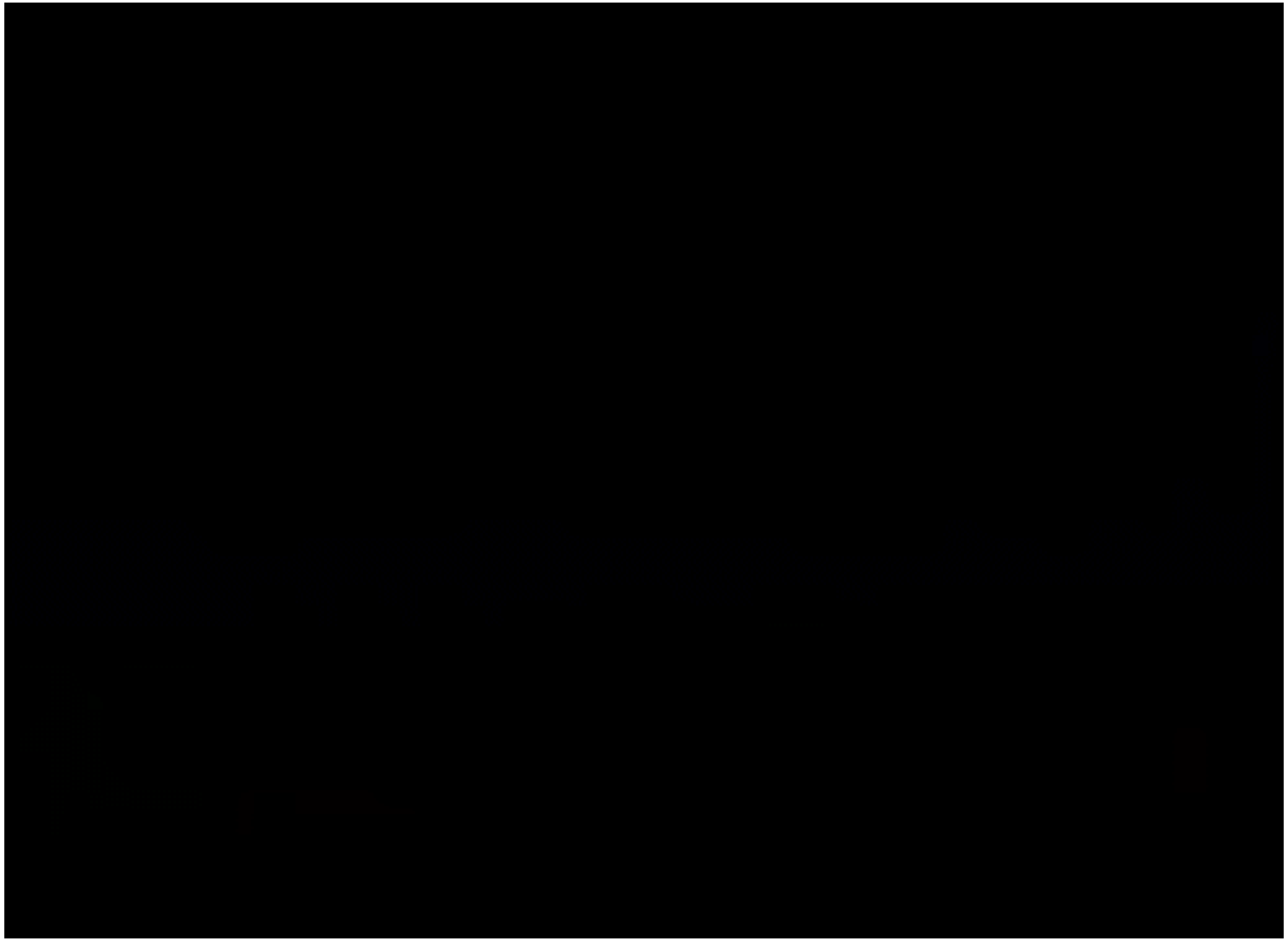


Aliasing

```
t = linspace(0,10*2*pi,100);
x = sin(t);
y = cos(t);

ts = linspace(0,10*2*pi,5);
xs = sin(ts);
ys = cos(ts);
```



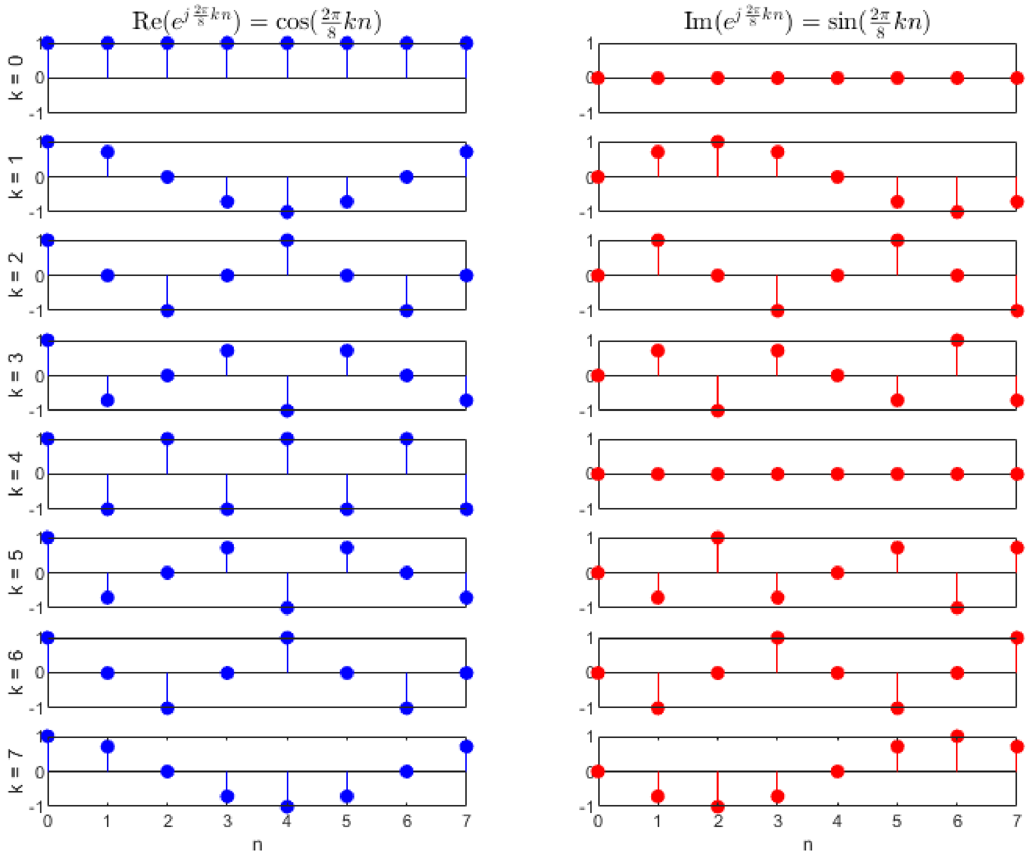
Aliasing: Wheel

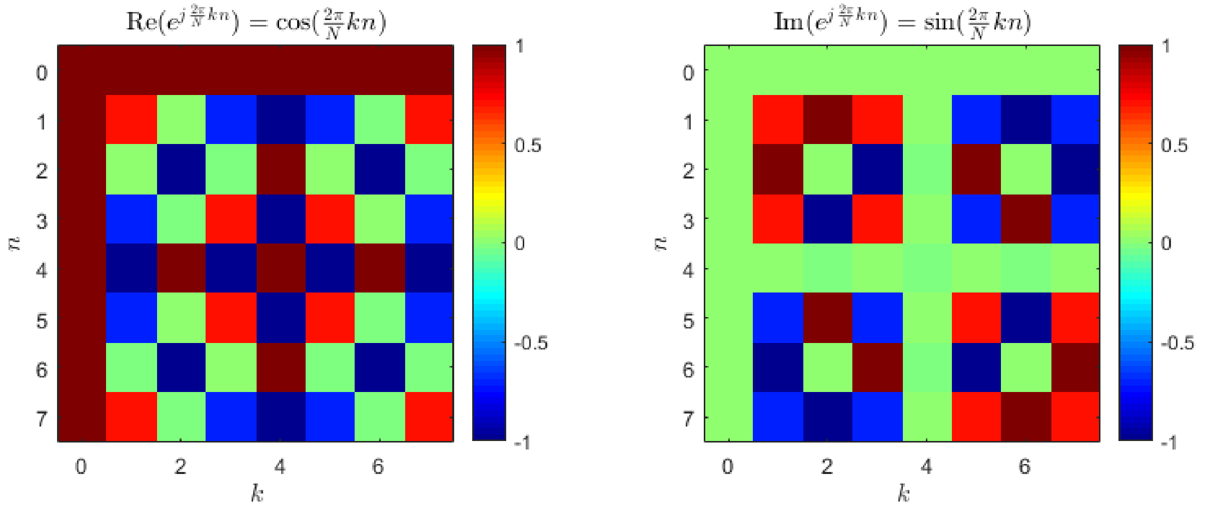


Visual Matrix of Discrete Sinusoids

Visual Matrix of Discrete Sinusoids

$$x[n] = e^{j\omega n}, \qquad \omega = \frac{2\pi}{N}k$$





Complex Exponential Signals with Damping

Complex Exponential Signals

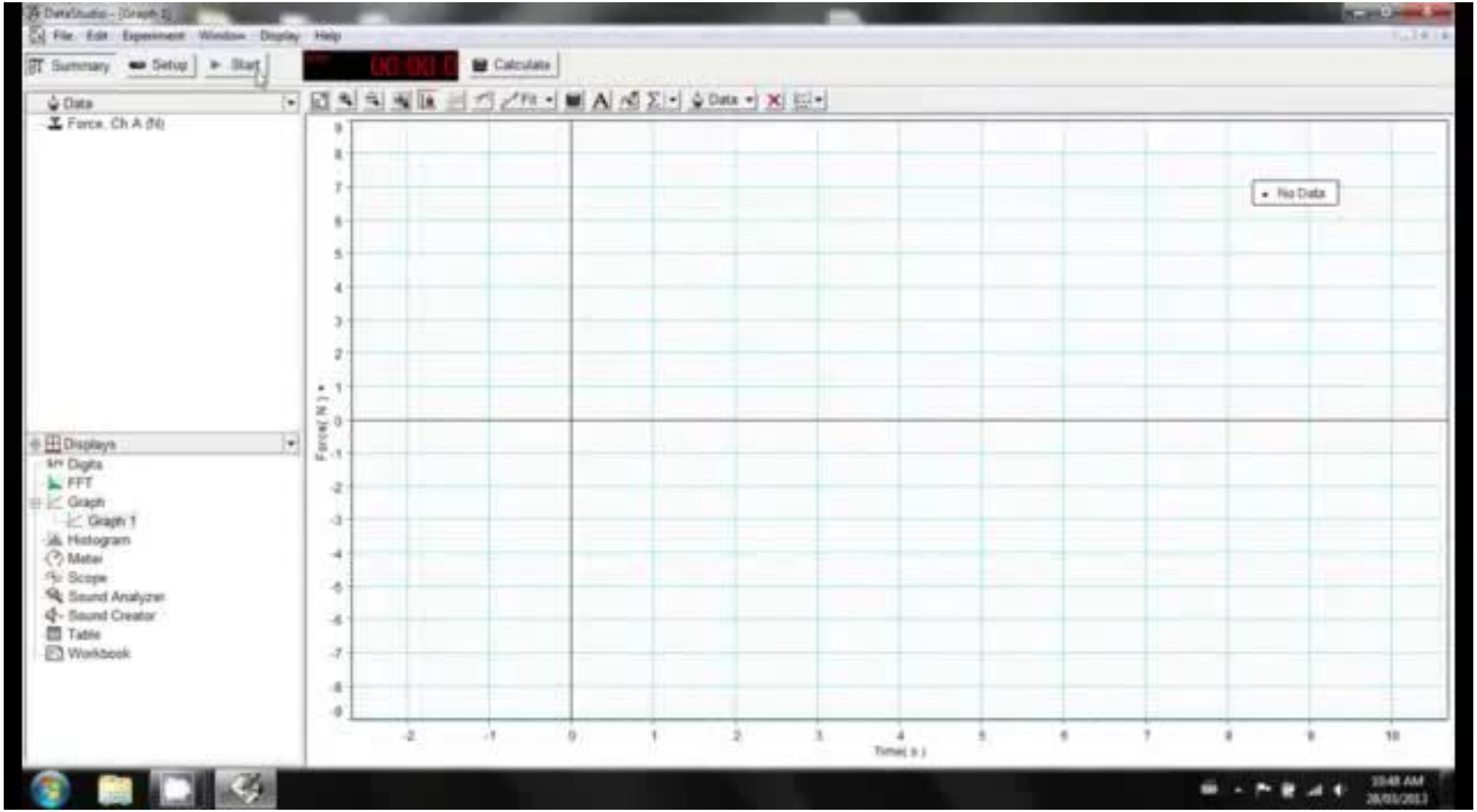
- Consider the general complex number $z = |z|e^{j\angle z}, \ z \in \mathbb{C}$
  - $|z|$ = magnitude of z
  - $\angle z$ = phase angle of z
  - Can visualize $z \in \mathbb{C}$ as a point in the complex plane

- Complex exponential is a spiral

$$z^n = \left(|z|e^{j\omega}\right)^n = |z|^n e^{j\omega n}$$

  - $|z|^n$ is a real exponential envelope
  - $e^{j\omega n}$ is a complex sinusoid
  - z^n is a helix

Damped Free Oscillation

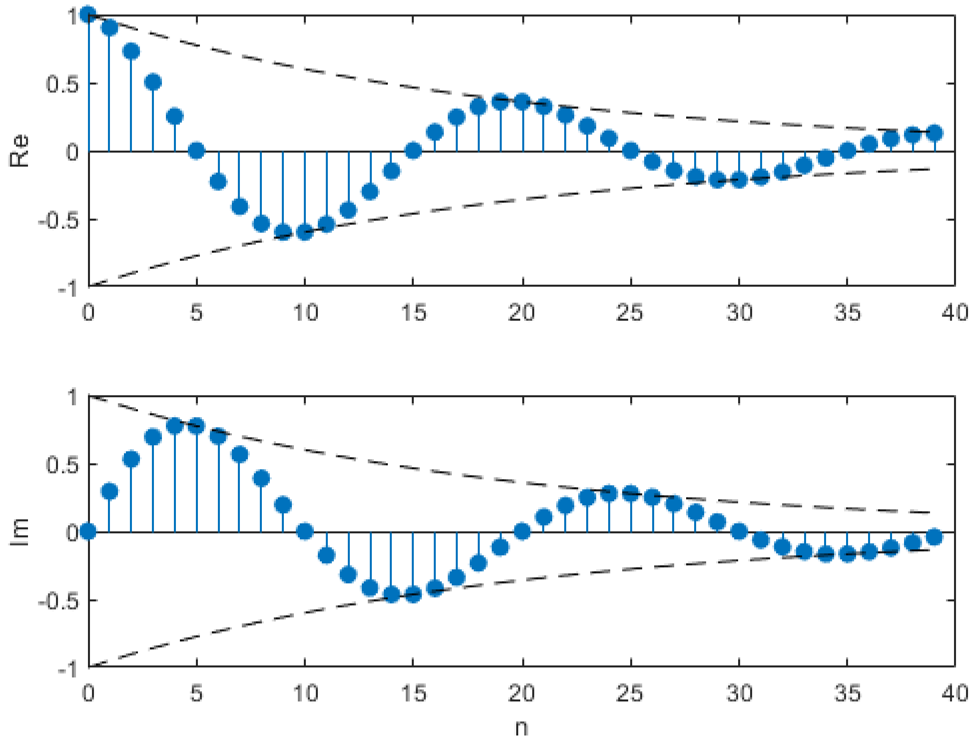


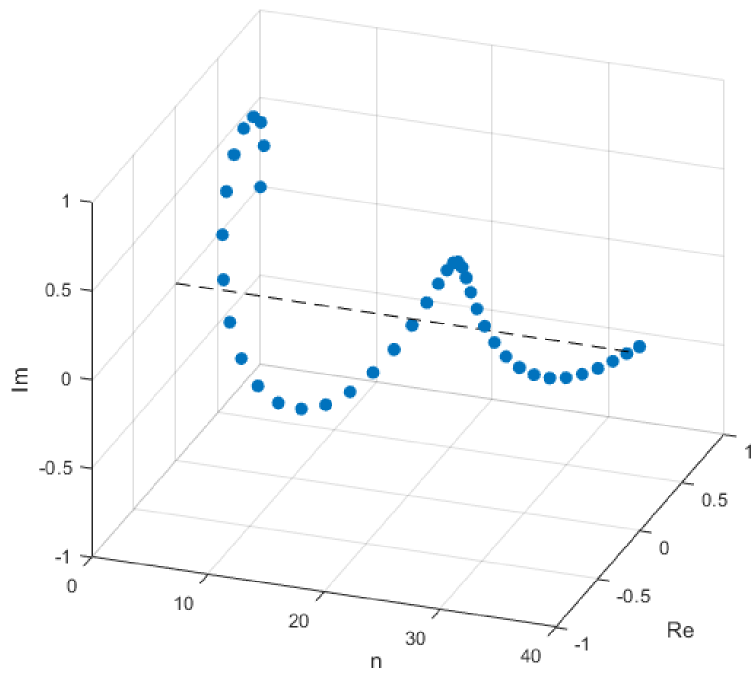
PHY245: Damped Mass On A Spring, https://www.youtube.com/watch?v=ZqedDWEAUN4

Plot Complex Signals

- Rectangular form

$$x[n] = \gamma^n e^{j\frac{2\pi}{N}n}$$

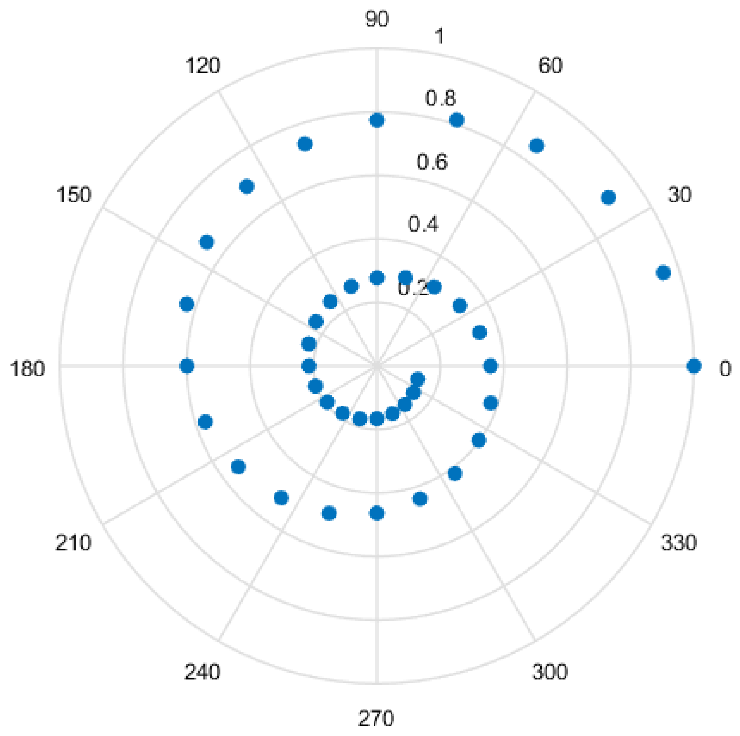


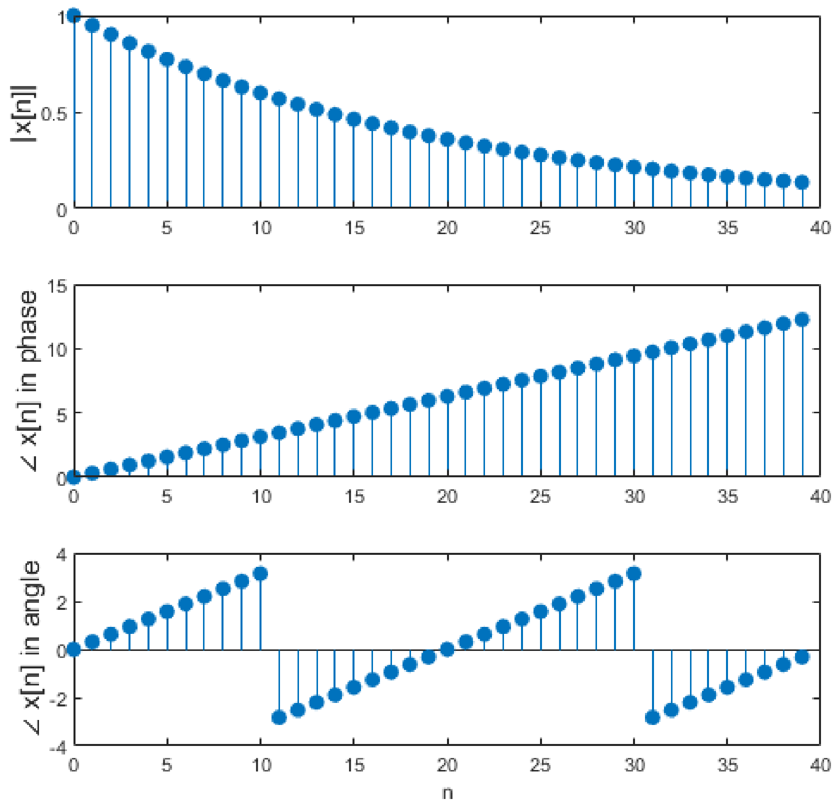


Plot Complex Signals

- Polar form

$$x[n] = \gamma^n e^{j\frac{2\pi}{N}n}$$
$$= |x[n]|\, e^{j\angle x[n]}$$





Signals are Vectors

Signals are Vectors

- Vectors in $\mathbb{R}^N$ or $\mathbb{C}^N$

$$x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Transpose of a Vector

- the transpose operation T converts a column vector to a row vector (and vice versa)

$$x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}^T = \begin{bmatrix} x[0] & x[1] & \cdots & x[N-1] \end{bmatrix}$$

- In addition to transpose, the conjugate transpose (aka Hermitian transpose) operation H takes the complex conjugate

$$x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}^H = \begin{bmatrix} x[0]^* & x[1]^* & \cdots & x[N-1]^* \end{bmatrix}$$

Transpose in MATLAB

- Be careful

```
a = [2 + 1j, 1 - 2j];
a'
a.'
```

```
ans =

   2.0000 - 1.0000i
   1.0000 + 2.0000i


ans =

   2.0000 + 1.0000i
   1.0000 - 2.0000i
```

Matrix Multiplication as Linear Combination

- Linear Combination = Matrix Multiplication

- Given a collection of M vectors $x_{0,} x_{1,} \cdots x_{M-1}$ and scalars $\alpha_{0,} \alpha_{1,} \cdots \alpha_{M-1}$, the linear combination of the vectors is given by

$$y = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{M-1} x_{M-1} = \sum_{m=0}^{M-1} \alpha_m x_m$$

Matrix Multiplication as Linear Combination

- Step 1: stack the vectors x_m as column vectors into an $N \times M$ matrix

$$X = [x_0 \mid x_1 \mid \cdots \mid x_{M-1}]$$

- Step 2: stack the scalars α_m into an $M \times 1$ column vector

$$\alpha = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix}$$

- Step 3: we can now write a linear combination as the matrix/vector product

$$y = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{M-1} x_{M-1} = \sum_{m=0}^{M-1} \alpha_m x_m = [x_0 \mid x_1 \mid \cdots \mid x_{M-1}] \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = X\alpha$$

- Note: the row-n , column-m element of the matrix $[X]_{n,m} = x_m[n]$

Inner Product

- The inner product (or dot product) between two vectors $x, y \in \mathbb{C}^N$ is given by

$$\langle x, y \rangle = y^H x = \sum_{n=0}^{N-1} x[n] y[n]^*$$

- The inner product takes two signals (vectors in $\mathbb{C}^N$) and produces a single (complex) number
- Inner product of a signal with itself

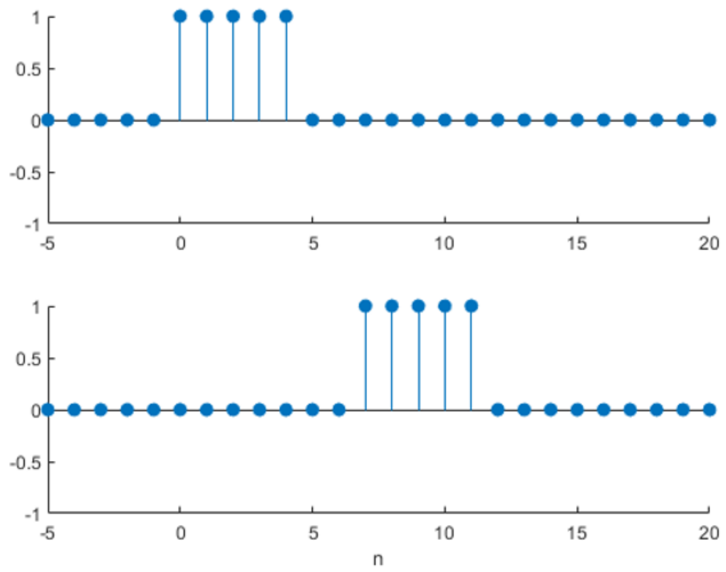
$$\langle x, x \rangle = \sum_{n=0}^{N-1} x[n]\, x[n]^* = \sum_{n=0}^{N-1} |x[n]|^2 = \|x\|_2^2$$

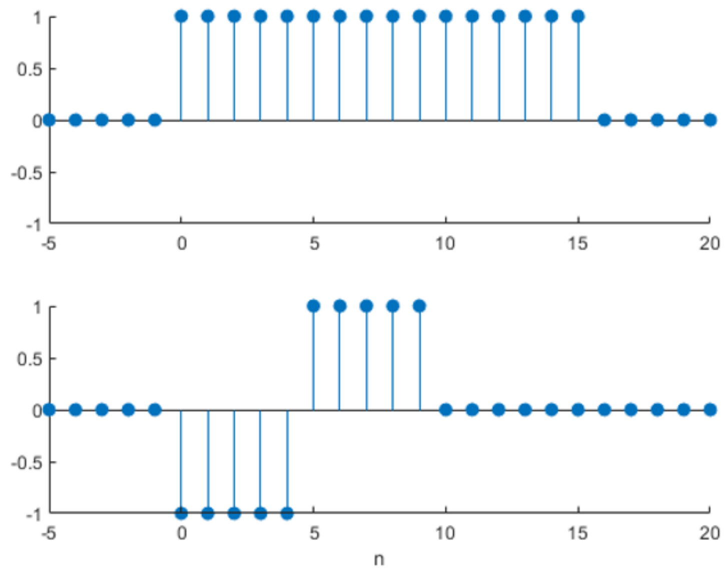
- Two vectors $x, y \in \mathbb{C}^N$ are orthogonal if

$$\langle x, y \rangle = 0$$

Orthogonal Signals

- Two sets of orthogonal signals





Harmonic Sinusoids are Orthogonal

$$d_k[n] = e^{j\frac{2\pi k}{N}n}, \quad n, k, N \in \mathbb{Z}, 0 \le n \le N-1, 0 \le k \le N-1$$

- Claim: $\langle d_k, d_l \rangle = 0,\ k \neq l$

$$\langle d_k, d_l \rangle = \sum_{n=0}^{N-1} d_k[n] d_l^*[n] = \sum_{n=0}^{N-1} e^{j\frac{2\pi k}{N}n} \left(e^{j\frac{2\pi l}{N}n}\right)^* = \sum_{n=0}^{N-1} e^{j\frac{2\pi k}{N}n} e^{-j\frac{2\pi l}{N}n}$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n} \quad \text{let } r = k - l \in \mathbb{Z}, r \neq 0$$

$$= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}rn} = \sum_{n=0}^{N-1} a^n \quad \text{with } \ a = e^{j\frac{2\pi}{N}r}, \text{then use} \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

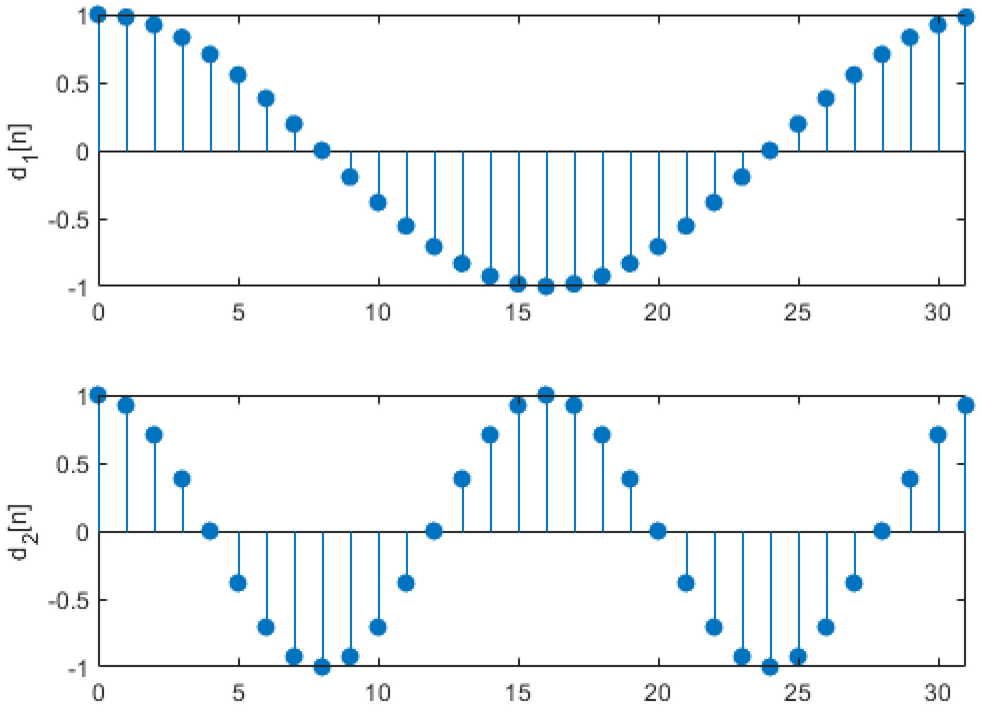
$$= \frac{1 - e^{j\frac{2\pi r N}{N}}}{1 - e^{j\frac{2\pi r}{N}}} = 0$$

POSTECH

Harmonic Sinusoids are Orthogonal

```
N = 32;
n = 0:N-1;

k = 1;
d1 = cos(2*pi/N*k*n)';

k = 2;
d2 = cos(2*pi/N*k*n)';
```

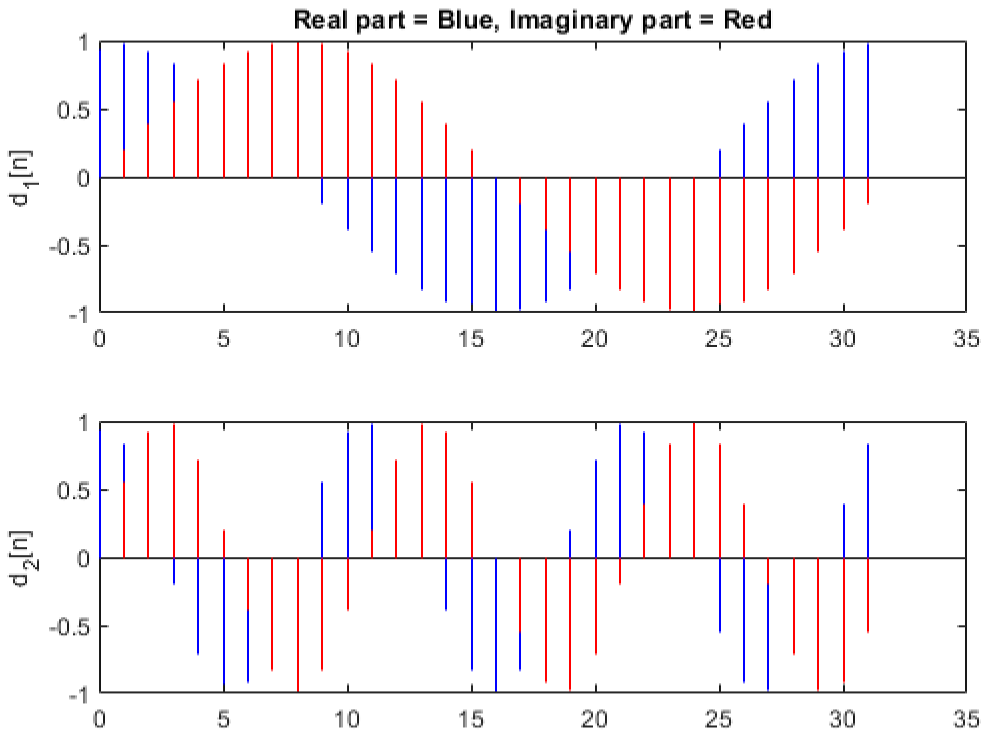
```
innerproduct =

  -2.2572e-15
```



Harmonic Sinusoids are Orthogonal

```
N = 32;
n = 0:N-1;

k = 1;
d1 = exp(1j*2*pi/N*k*n).';

k = 3;
d2 = exp(1j*2*pi/N*k*n).';
```

```
innerproduct =

  -1.8965e-15 + 8.7512e-16i
```



Normalized Harmonic Sinusoids

$$d_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N}n}$$

```
N = 16;
n = 0:N-1;

k = 3;
s_3 = 1/sqrt(N)*exp(1j*2*pi/N*k*n).';

k = 5;
s_5 = 1/sqrt(N)*exp(1j*2*pi/N*k*n).';

% ': complex conjugate transpose
s_3'*s_5    % to see they are orthogonal
s_3'*s_3    % to see it is normalized
s_5'*s_5    % to see it is normalized
```

```
ans =

  -4.1633e-16 + 8.1780e-17i


ans =

     1


ans =

     1
```

Matrix Multiplication as a Sequence of Inner Products of Rows

- Consider the matrix multiplication $y = X\alpha$
- The row-n , column-m element of the matrix $[X]_{n,m} = x_m[n]$
- We can compute each element $y[n]$ in y as the inner product of the n-th row of X with the vector α

$$y = \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ x_0[n] & x_1[n] & \cdots & x_{M-1}[n] \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = X\alpha$$

- Can write $y[n]$

$$y[n] = \sum_{m=0}^{M-1} \alpha_m x_m[n] = \langle \text{row } n \text{ of } X, \alpha \rangle$$