



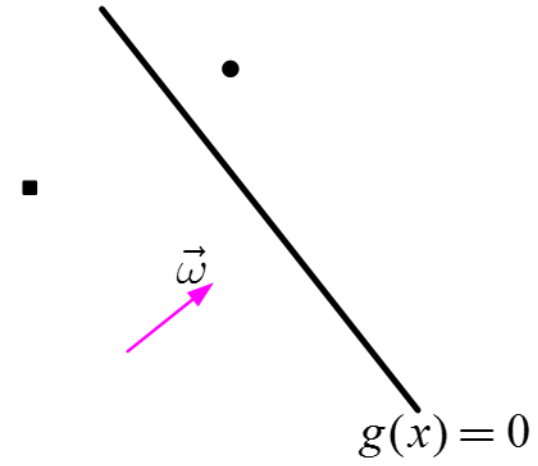
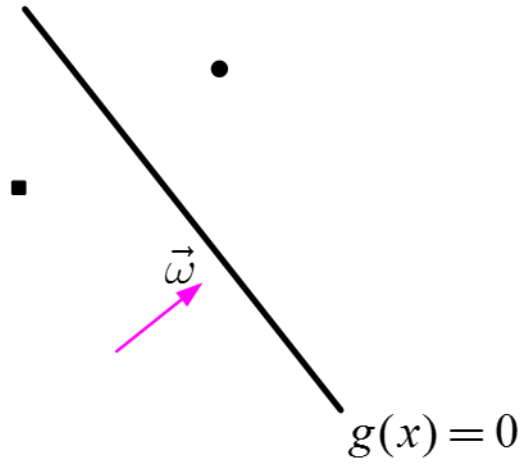
# Logistic Regression

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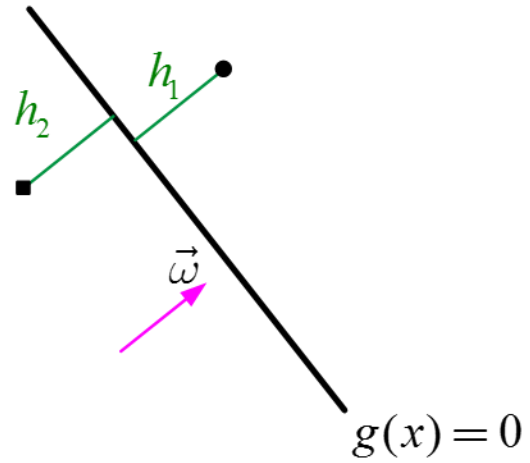
# Linear Classification: Logistic Regression

- Logistic regression is a classification algorithm
  - don't be confused
- Perceptron: make use of sign of data
- SVM: make use of margin (minimum distance)
  - Distance from two closest data points
- We want to use distance information of **all** data points
  - logistic regression

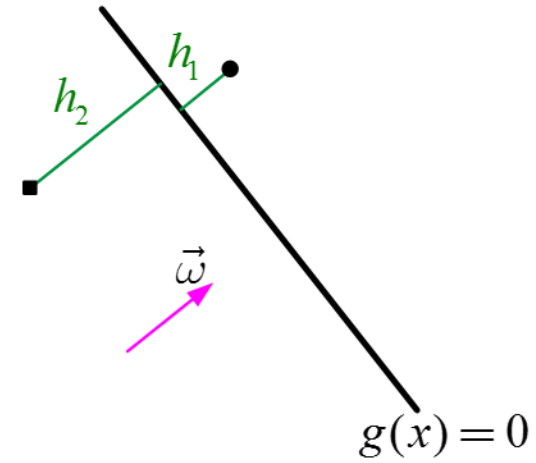
# Using Distances



# Using Distances

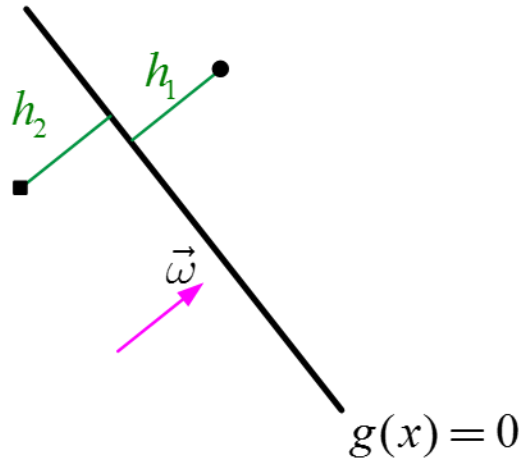


$$|h_1| + |h_2|$$



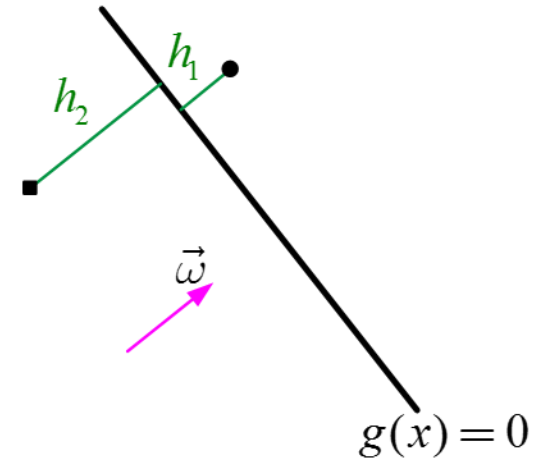
$$|h_1| + |h_2|$$

# Using Distances



$$|h_1| + |h_2|$$

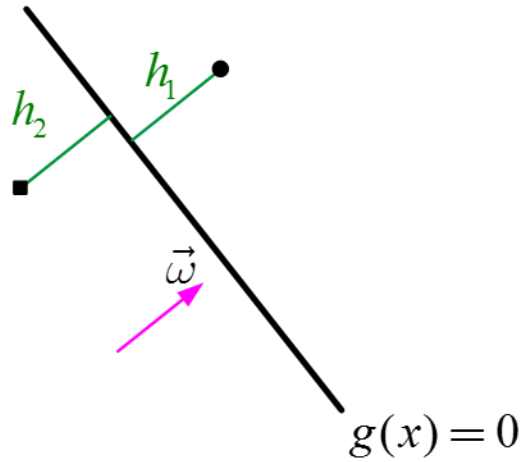
$$|h_1| \cdot |h_2|$$



$$|h_1| + |h_2|$$

$$|h_1| \cdot |h_2|$$

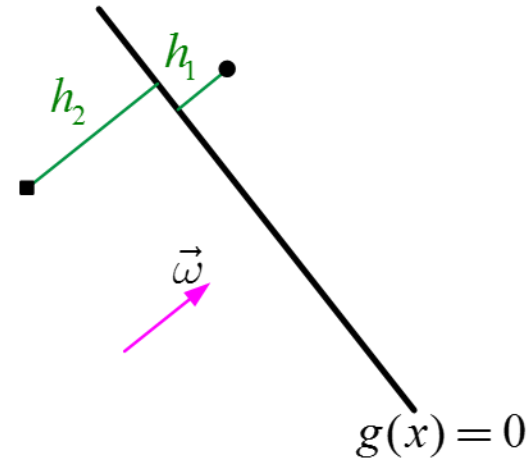
# Using Distances



$$|h_1| + |h_2|$$

$$|h_1| \cdot |h_2|$$

$$\frac{|h_1| + |h_2|}{2} \geq \sqrt{|h_1| \cdot |h_2|}$$



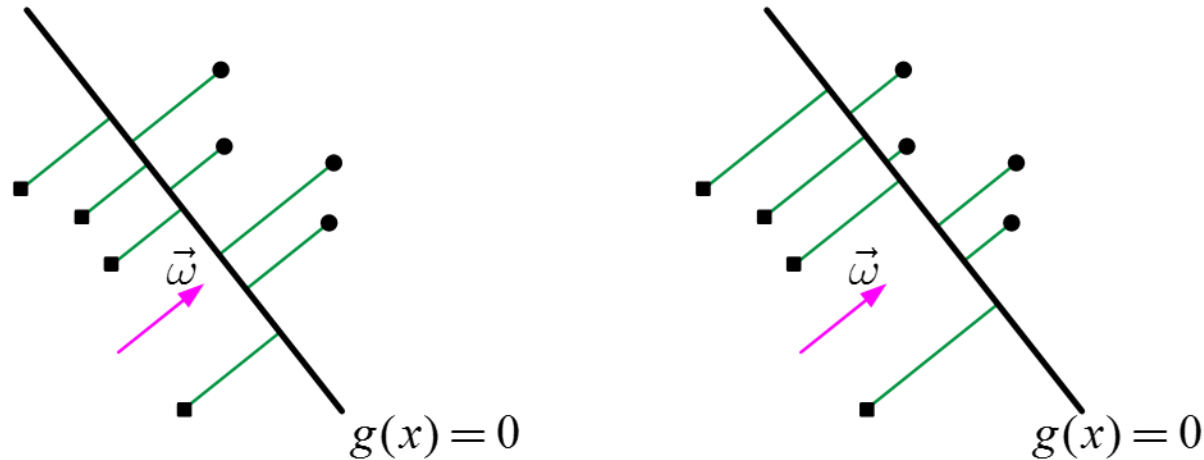
$$|h_1| + |h_2|$$

$$|h_1| \cdot |h_2|$$

$$\text{equal iff } |h_1| = |h_2|$$

# Using all Distances

- basic idea: to find the decision boundary (hyperplane) of  $g(x) = \omega^T x = 0$  such that maximizes  $\prod_i |h_i| \rightarrow$  **optimization**

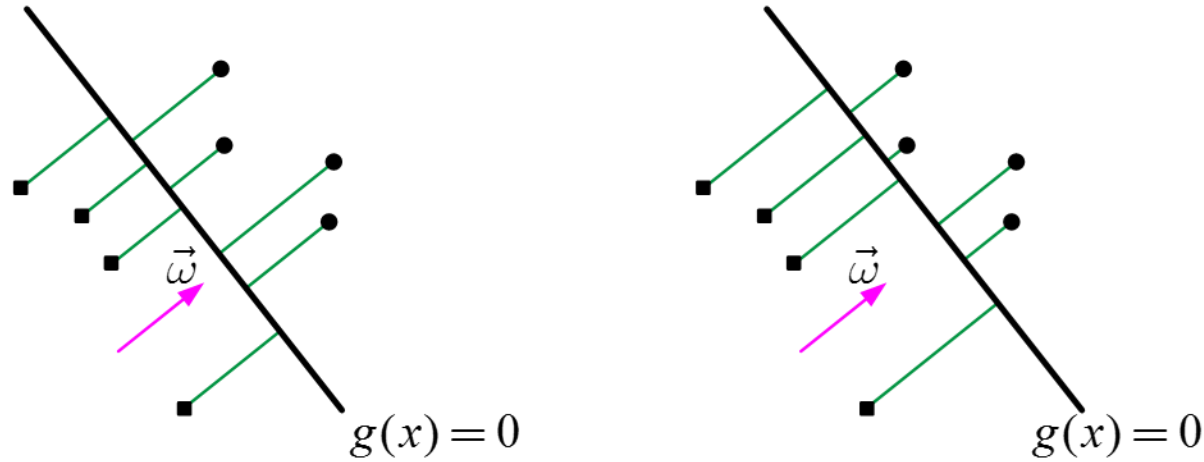


- Inequality of arithmetic and geometric means

$$\frac{x_1 + x_2 + \dots + x_m}{m} \geq \sqrt[m]{x_1 \cdot x_2 \cdot \dots \cdot x_m}$$

and that equality holds if and only if  $x_1 = x_2 = \dots = x_m$

# Using all Distances



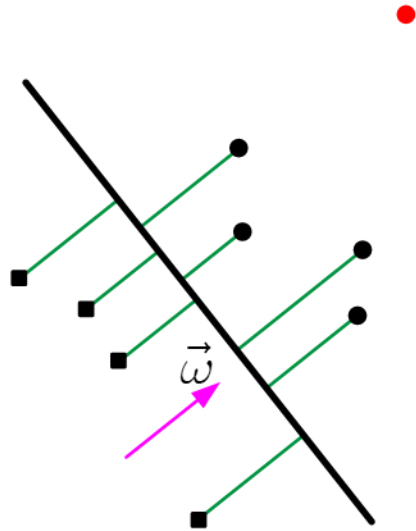
- Roughly speaking, this optimization of  $\max \prod_i |h_i|$  tends to position a **hyperplane in the middle of two classes**

$$h = \frac{g(x)}{\|\omega\|} = \frac{\omega^T x}{\|\omega\|} \sim \omega^T x$$

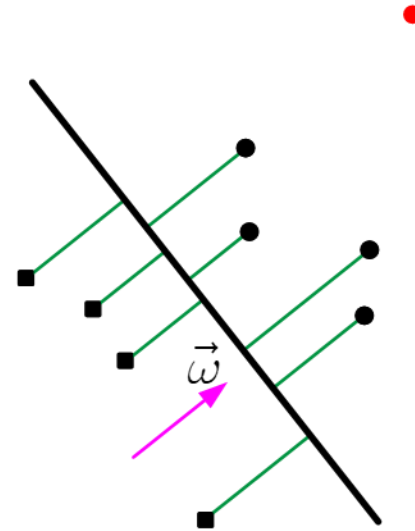


# Using all Distances with Outliers

- SVM vs. Logistic Regression



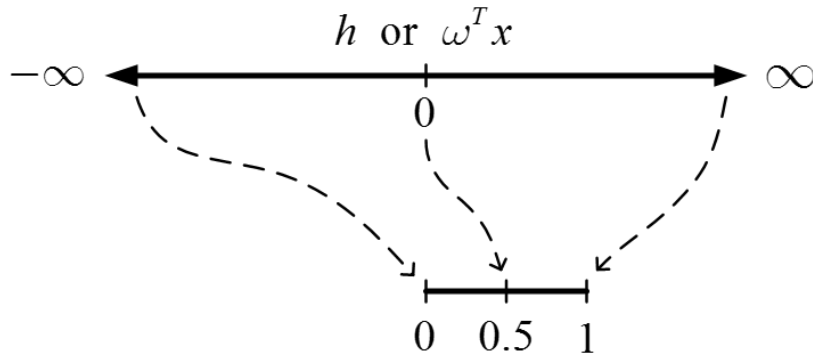
SVM



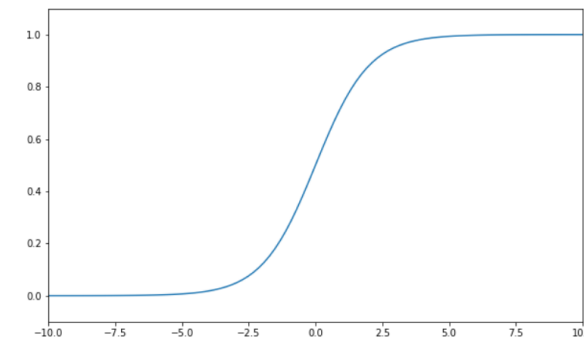
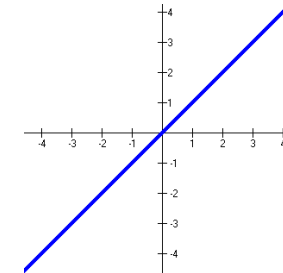
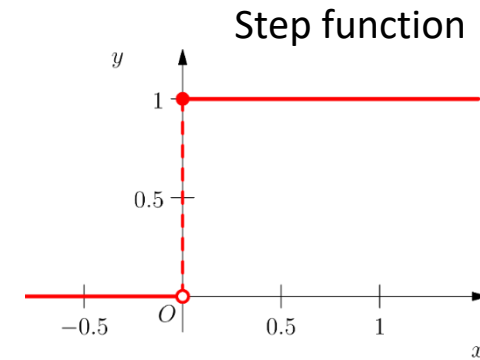
Logistic Regression

# Sigmoid Function

- We link or squeeze  $(-\infty, +\infty)$  to  $(0, 1)$  for several reasons:



$$\sigma(z) = \frac{1}{1 + e^{-z}} \implies \sigma(\omega^T x) = \frac{1}{1 + e^{-\omega^T x}}$$



# Sigmoid Function

- $\sigma(z)$  is the sigmoid function, or the logistic function
  - Logistic function always generates a value between 0 and 1
  - Crosses 0.5 at the origin, then flattens out

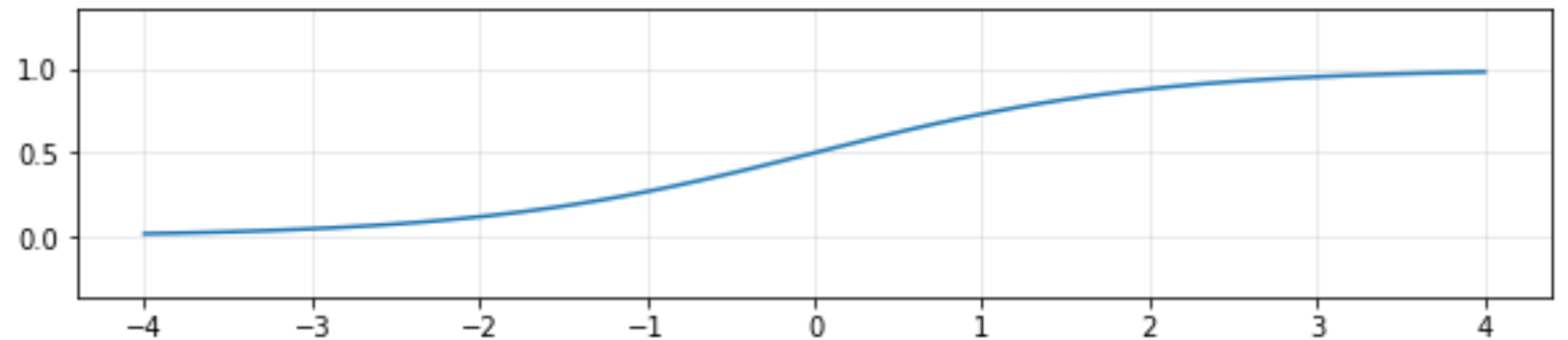
$$\sigma(z) = \frac{1}{1 + e^{-z}} \implies \sigma(\omega^T x) = \frac{1}{1 + e^{-\omega^T x}}$$

```
# plot a sigmoid function

import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

z = np.linspace(-4,4,100)
s = 1/(1 + np.exp(-z))

plt.figure(figsize=(10,2))
plt.plot(z, s)
plt.xlim([-4, 4])
plt.axis('equal')
plt.grid(alpha = 0.3)
plt.show()
```



# Sigmoid Function

- Benefit of mapping via the logistic function
  - Monotonic: same or similar optimization solution
  - Continuous and differentiable: good for gradient descent optimization
  - Probability or confidence: can be considered as probability

$$P(y = +1 \mid x, \omega) = \frac{1}{1 + e^{-\omega^T x}} \in [0, 1]$$

- Probability that the label is +1

$$P(y = +1 \mid x; \omega)$$

- Probability that the label is 0

$$P(y = 0 \mid x; \omega) = 1 - P(y = +1 \mid x; \omega)$$

# Goal: We Need to Fit $\omega$ to Data

- For a single data point  $(x, y)$  with parameters  $\omega$

$$P(y = +1 \mid x; \omega) = h_{\omega}(x) = \sigma(\omega^T x)$$

$$P(y = 0 \mid x; \omega) = 1 - h_{\omega}(x) = 1 - \sigma(\omega^T x)$$

- It can be compactly written as

$$P(y \mid x; \omega) = (h_{\omega}(x))^y (1 - h_{\omega}(x))^{1-y}$$

- For  $m$  training data points, the likelihood function of the parameters:

$$\begin{aligned} \mathcal{L}(\omega) &= P(y^{(1)}, \dots, y^{(m)} \mid x^{(1)}, \dots, x^{(m)}; \omega) \\ &= \prod_{i=1}^m P(y^{(i)} \mid x^{(i)}; \omega) \\ &= \prod_{i=1}^m \left( h_{\omega}(x^{(i)}) \right)^{y^{(i)}} \left( 1 - h_{\omega}(x^{(i)}) \right)^{1-y^{(i)}} \quad \left( \sim \prod_i |h_i| \right) \end{aligned}$$

## Goal: We Need to Fit $\omega$ to Data

$$\begin{aligned}\mathcal{L}(\omega) &= P\left(y^{(1)}, \dots, y^{(m)} \mid x^{(1)}, \dots, x^{(m)}; \omega\right) \\ &= \prod_{i=1}^m P\left(y^{(i)} \mid x^{(i)}; \omega\right) \\ &= \prod_{i=1}^m \left(h_{\omega}\left(x^{(i)}\right)\right)^{y^{(i)}} \left(1 - h_{\omega}\left(x^{(i)}\right)\right)^{1-y^{(i)}} \quad \left(\sim \prod_i |h_i|\right)\end{aligned}$$

- It would be easier to work on the log likelihood.

$$\ell(\omega) = \log \mathcal{L}(\omega) = \sum_{i=1}^m y^{(i)} \log h_{\omega}\left(x^{(i)}\right) + \left(1 - y^{(i)}\right) \log\left(1 - h_{\omega}\left(x^{(i)}\right)\right)$$

- The logistic regression problem can be solved as a (convex) optimization problem:

$$\hat{\omega} = \arg \max_{\omega} \ell(\omega)$$

- Again, it is an optimization problem

# Logistic Regression using GD

# Gradient Descent

- To use the gradient descent method, we need to find the derivative of it

$$\nabla \ell(\omega) = \begin{bmatrix} \frac{\partial \ell(\omega)}{\partial \omega_1} \\ \vdots \\ \frac{\partial \ell(\omega)}{\partial \omega_n} \end{bmatrix}$$

- We need to compute  $\frac{\partial \ell(\omega)}{\partial \omega_j}$

$$\ell(\omega) = \log \mathcal{L}(\omega) = \sum_{i=1}^m y^{(i)} \log h_{\omega} \left( x^{(i)} \right) + \left( 1 - y^{(i)} \right) \log \left( 1 - h_{\omega} \left( x^{(i)} \right) \right)$$



# Gradient Descent

$$\ell(\omega) = \log \mathcal{L}(\omega) = \sum_{i=1}^m y^{(i)} \log h_{\omega}(x^{(i)}) + (1 - y^{(i)}) \log(1 - h_{\omega}(x^{(i)}))$$

- Think about a single data point with a single parameter  $\omega$  for the simplicity.

$$\begin{aligned} & \frac{\partial}{\partial \omega} [y \log(\sigma) + (1 - y) \log(1 - \sigma)] \\ &= y \frac{\sigma'}{\sigma} + (1 - y) \frac{-\sigma'}{1 - \sigma} \\ &= \left( \frac{y}{\sigma} - \frac{1 - y}{1 - \sigma} \right) \sigma' \\ &= \frac{y - \sigma}{\sigma(1 - \sigma)} \sigma' \\ &= \frac{y - \sigma}{\sigma(1 - \sigma)} \sigma(1 - \sigma)x \\ &= (y - \sigma)x \end{aligned}$$

- For  $m$  training data points with parameters  $\omega$

$$\frac{\partial \ell(\omega)}{\partial \omega_j} = \sum_{i=1}^m \left( y^{(i)} - h_{\omega}(x^{(i)}) \right) x_j^{(i)} \quad \stackrel{\text{vectorization}}{=} \quad (y - h_{\omega}(x))^T x_j = x_j^T (y - h_{\omega}(x))$$

# Gradient Descent for Logistic Regression

$$\omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$$

$$X = \begin{bmatrix} (x^{(1)})^T \\ (x^{(2)})^T \\ (x^{(3)})^T \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} \\ 1 & x_1^{(3)} & x_2^{(3)} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ \vdots \end{bmatrix}$$

$$\frac{\partial \ell(\omega)}{\partial \omega_j} = \sum_{i=1}^m \left( y^{(i)} - h_{\omega}(x^{(i)}) \right) x_j^{(i)}$$

$$\stackrel{\text{vectorization}}{=} (y - h_{\omega}(x))^T x_j = x_j^T (y - h_{\omega}(x))$$

$$\nabla \ell(\omega) = \begin{bmatrix} \frac{\partial \ell(\omega)}{\partial \omega_0} \\ \frac{\partial \ell(\omega)}{\partial \omega_1} \\ \frac{\partial \ell(\omega)}{\partial \omega_2} \end{bmatrix} = X^T (y - h_{\omega}(x)) = X^T (y - \sigma(X\omega))$$

- Maximization problem
- Be careful on matrix shape

$$\omega \leftarrow \omega - \eta (-\nabla \ell(\omega))$$

# Python Implementation

```
# data generation

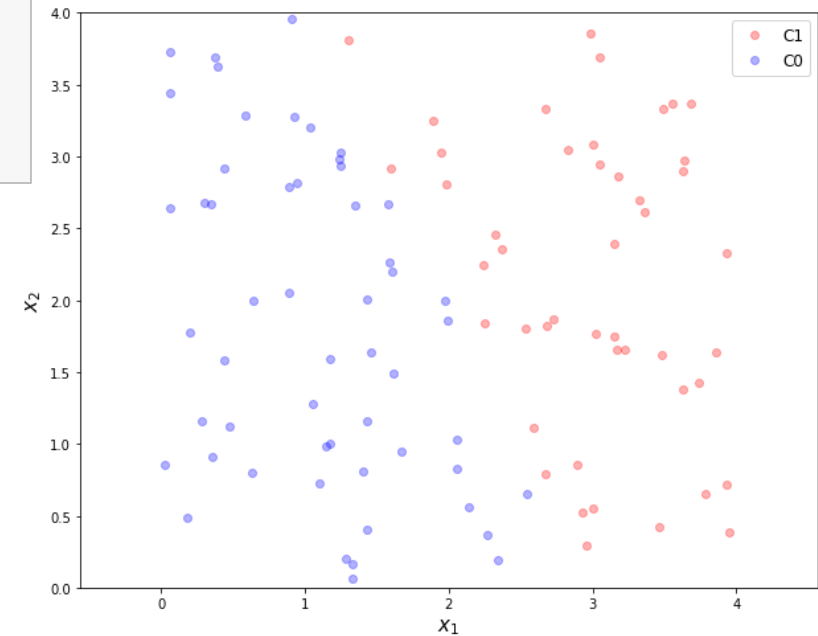
m = 100
w = np.array([[ -6], [ 2], [ 1]])
X = np.hstack([np.ones([m,1]), 4*np.random.rand(m,1), 4*np.random.rand(m,1)])

w = np.asmatrix(w)
X = np.asmatrix(X)

y = 1/(1 + np.exp(-X*w)) > 0.5

C1 = np.where(y == True)[0]
C0 = np.where(y == False)[0]

y = np.empty([m,1])
y[C1] = 1
y[C0] = 0
```



# Python Implementation

*# be careful with matrix shape*

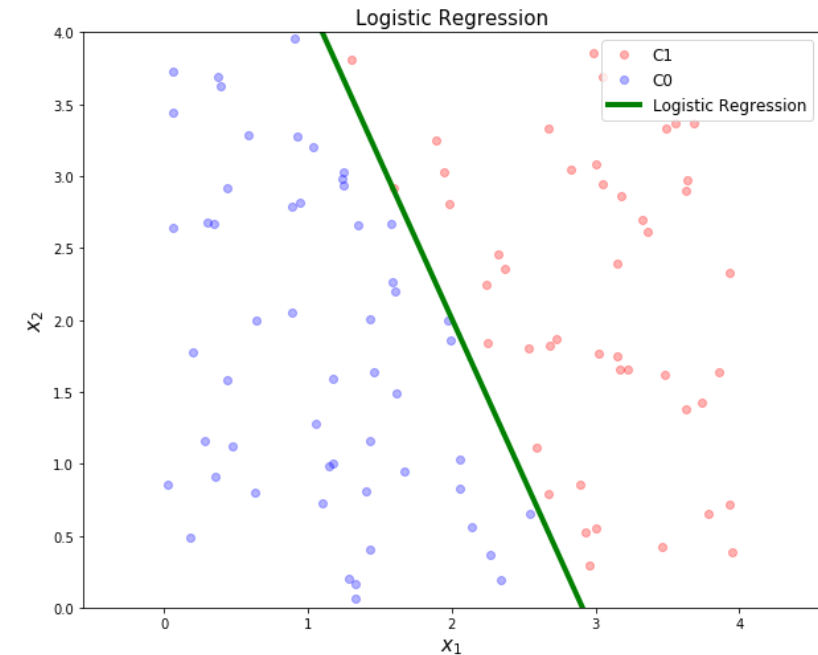
```
def h(x,w):  
    return 1/(1 + np.exp(-x*w))
```

```
alpha = 0.0001  
w = np.zeros([3,1])  
  
for i in range(1000):  
    df = -X.T*(y - h(X,w))  
    w = w - alpha*df  
  
print(w)
```

$$h_{\omega}(x) = h(x; \omega) = \sigma(\omega^T x) = \frac{1}{1 + e^{-\omega^T x}}$$

$$\nabla \ell(\omega) = \begin{bmatrix} \frac{\partial \ell(\omega)}{\partial \omega_0} \\ \frac{\partial \ell(\omega)}{\partial \omega_1} \\ \frac{\partial \ell(\omega)}{\partial \omega_2} \end{bmatrix} = X^T (y - h_{\omega}(x)) = X^T (y - \sigma(X\omega))$$

$$\omega \leftarrow \omega - \eta(-\nabla \ell(\omega))$$



# Logistic Regression using CVXPY

## Probabilistic Approach (or MLE)

- Consider a random variable  $y \in \{0, 1\}$

$$P(y = +1) = p, \quad P(y = 0) = 1 - p$$

where  $p \in [0, 1]$ , and is assumed to depend on a vector of explanatory variables  $x \in \mathbb{R}^n$

- Then, the logistic model has the form

$$p = \frac{1}{1 + e^{-\omega^T x}} = \frac{e^{\omega^T x}}{e^{\omega^T x} + 1}$$
$$1 - p = \frac{1}{e^{\omega^T x} + 1}$$

- We can re-order the training data so
  - for  $x_1, \dots, x_q$ , the outcome is  $y = +1$ , and
  - for  $x_{q+1}, \dots, x_m$ , the outcome is  $y = 0$

# Probabilistic Approach (or MLE)

- Likelihood function

$$\mathcal{L} = \prod_{i=1}^q p_i \prod_{i=q+1}^m (1 - p_i) \quad \left( \sim \prod_i |h_i| \right)$$

- Log likelihood function

$$\begin{aligned} \ell(\omega) &= \log \mathcal{L} = \sum_{i=1}^q \log p_i + \sum_{i=q+1}^m \log(1 - p_i) \\ &= \sum_{i=1}^q \log \frac{\exp(\omega^T x_i)}{1 + \exp(\omega^T x_i)} + \sum_{i=q+1}^m \log \frac{1}{1 + \exp(\omega^T x_i)} \\ &= \sum_{i=1}^q (\omega^T x_i) - \sum_{i=1}^m \log(1 + \exp(\omega^T x_i)) \end{aligned}$$

- Since  $\ell$  is a concave function of  $\omega$ , the logistic regression problem can be solved as a convex optimization problem

$$\hat{\omega} = \arg \max_{\omega} \ell(\omega)$$

# CVXPY Implementation

$$\begin{aligned}\ell(\omega) &= \log \mathcal{L} = \sum_{i=1}^q \log p_i + \sum_{i=q+1}^m \log(1 - p_i) \\ &= \sum_{i=1}^q \log \frac{\exp(\omega^T x_i)}{1 + \exp(\omega^T x_i)} + \sum_{i=q+1}^m \log \frac{1}{1 + \exp(\omega^T x_i)} \\ &= \sum_{i=1}^q (\omega^T x_i) - \sum_{i=1}^m \log(1 + \exp(\omega^T x_i))\end{aligned}$$

$$\omega = \begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix}$$
$$X = \begin{bmatrix} (x^{(1)})^T \\ (x^{(2)})^T \\ (x^{(3)})^T \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & x_1^{(1)} & x_2^{(1)} \\ 1 & x_1^{(2)} & x_2^{(2)} \\ 1 & x_1^{(3)} & x_2^{(3)} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

```
w = cvx.Variable([3, 1])

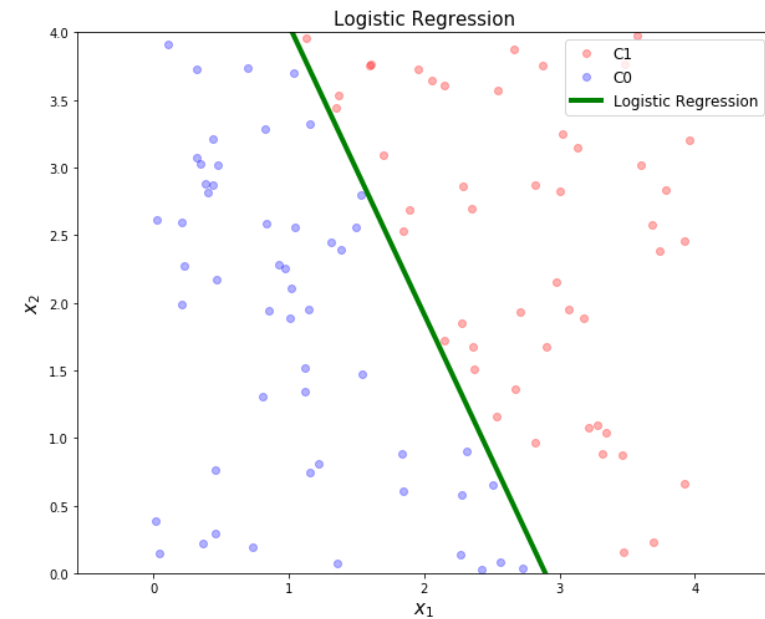
obj = cvx.Maximize(y.T*X*w - cvx.sum(cvx.logistic(X*w)))
prob = cvx.Problem(obj).solve()

w = w.value

xp = np.linspace(0,4,100).reshape(-1,1)
yp = - w[1,0]/w[2,0]*xp - w[0,0]/w[2,0]
```

$$\text{cvx.sum}(x) = \sum_{ij} x_{ij}$$

$$\text{cvx.logistic}(x) = \log(1 + e^x)$$





## In a More Compact Form

- Change  $y \in \{0, +1\} \rightarrow y \in \{-1, +1\}$  for computational convenience
  - Consider the following function

$$P(y = +1) = p = \sigma(\omega^T x), \quad P(y = -1) = 1 - p = 1 - \sigma(\omega^T x) = \sigma(-\omega^T x)$$
$$P(y \mid x, \omega) = \sigma(y\omega^T x) = \frac{1}{1 + \exp(-y\omega^T x)} \in [0, 1]$$

- Log-likelihood

$$\begin{aligned} \ell(\omega) = \log \mathcal{L} = \log P(y \mid x, \omega) &= \log \prod_{n=1}^m P(y_n \mid x_n, \omega) \\ &= \sum_{n=1}^m \log P(y_n \mid x_n, \omega) \\ &= \sum_{n=1}^m \log \frac{1}{1 + \exp(-y_n \omega^T x_n)} \\ &= \sum_{n=1}^m -\log(1 + \exp(-y_n \omega^T x_n)) \end{aligned}$$

# CVXPY Implementation

$$\begin{aligned}\hat{\omega} &= \arg \max_{\omega} \sum_{n=1}^m -\log(1 + \exp(-y_n \omega^T x_n)) \\ &= \arg \min_{\omega} \sum_{n=1}^m \log(1 + \exp(-y_n \omega^T x_n))\end{aligned}$$

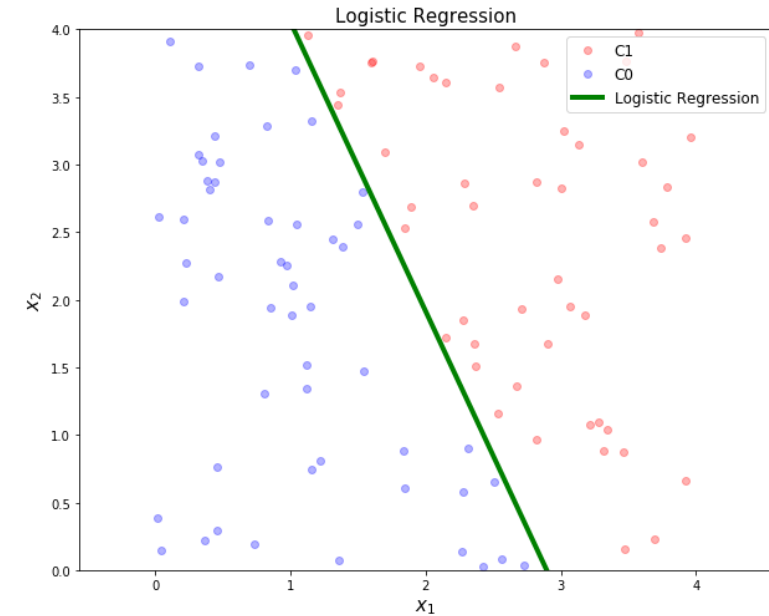
```
y = np.empty([m,1])
y[C1] = 1
y[C0] = -1
y = np.asmatrix(y)

w = cvx.Variable([3, 1])

obj = cvx.Minimize(cvx.sum(cvx.logistic(-cvx.multiply(y,X*w))))
prob = cvx.Problem(obj).solve()

w = w.value

xp = np.linspace(0,4,100).reshape(-1,1)
yp = - w[1,0]/w[2,0]*xp - w[0,0]/w[2,0]
```



$$\text{cvx.sum}(x) = \sum_{ij} x_{ij}$$

$$\text{cvx.logistic}(x) = \log(1 + e^x)$$

# Logistic Regression using Scikit-Learn

# Logistic Regression using Scikit-Learn

```
X = X[:,1:3]
```

```
X.shape
```

```
from sklearn import linear_model
```

```
clf = linear_model.LogisticRegression(solver='lbfgs')  
clf.fit(X,np.ravel(y))
```

```
w0 = clf.intercept_[0]
```

```
w1 = clf.coef_[0,0]
```

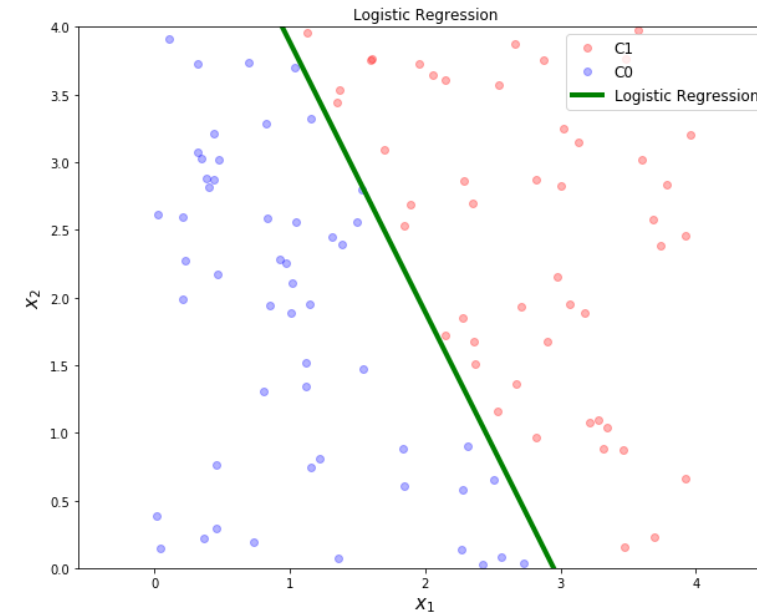
```
w2 = clf.coef_[0,1]
```

```
xp = np.linspace(0,4,100).reshape(-1,1)
```

```
yp = - w1/w2*xp - w0/w2
```

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \quad \omega_0, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

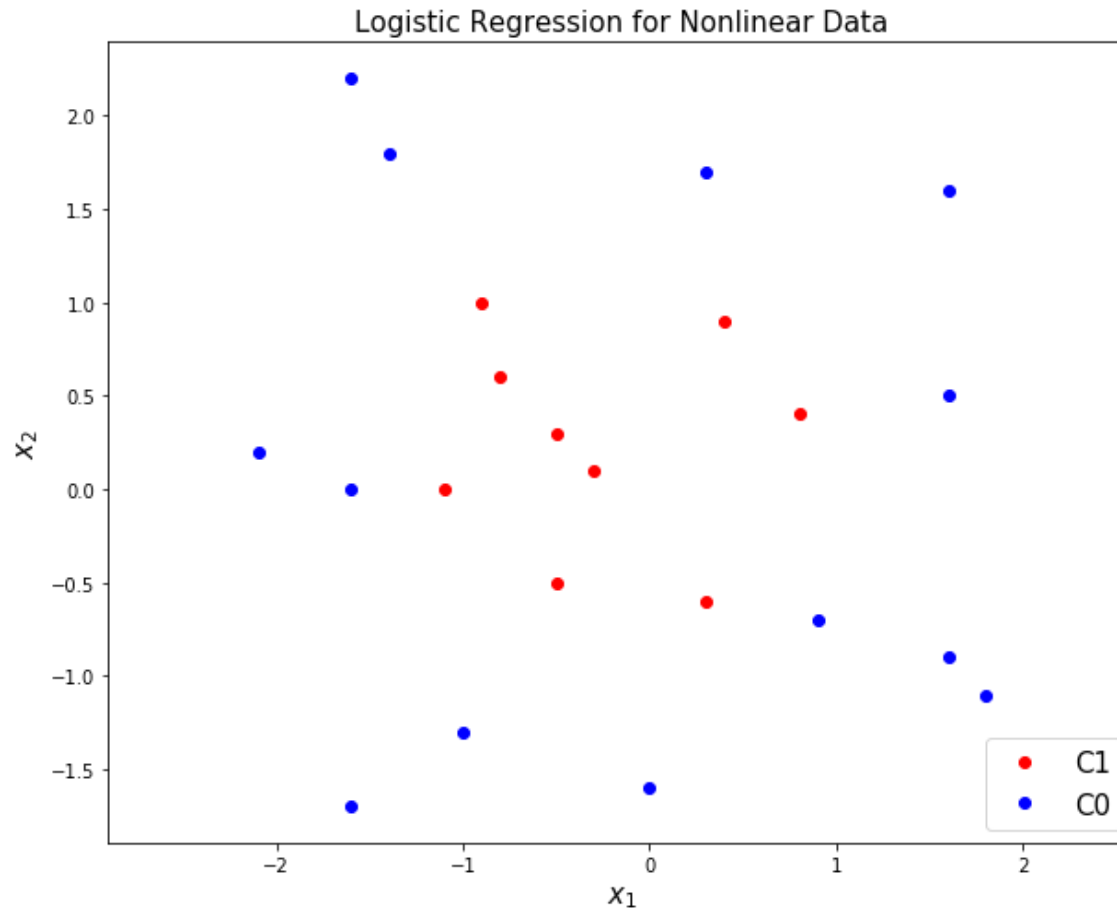
$$X = \begin{bmatrix} (x^{(1)})^T \\ (x^{(2)})^T \\ (x^{(3)})^T \\ \vdots \end{bmatrix} = \begin{bmatrix} x_1^{(1)} & x_2^{(1)} \\ x_1^{(2)} & x_2^{(2)} \\ x_1^{(3)} & x_2^{(3)} \\ \vdots & \vdots \end{bmatrix}, \quad y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ \vdots \end{bmatrix}$$



# Non-linear Classification

# Non-linear Classification

- Same idea as non-linear regression: non-linear features
  - Explicit or implicit Kernel



# Explicit Kernel

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies z = \phi(x) = \begin{bmatrix} 1 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \\ x_1^2 \\ \sqrt{2}x_1x_2 \\ x_2^2 \end{bmatrix}$$

```
N = X1.shape[0]
M = X0.shape[0]

X = np.vstack([X1, X0])
y = np.vstack([np.ones([N,1]), -np.ones([M,1])])

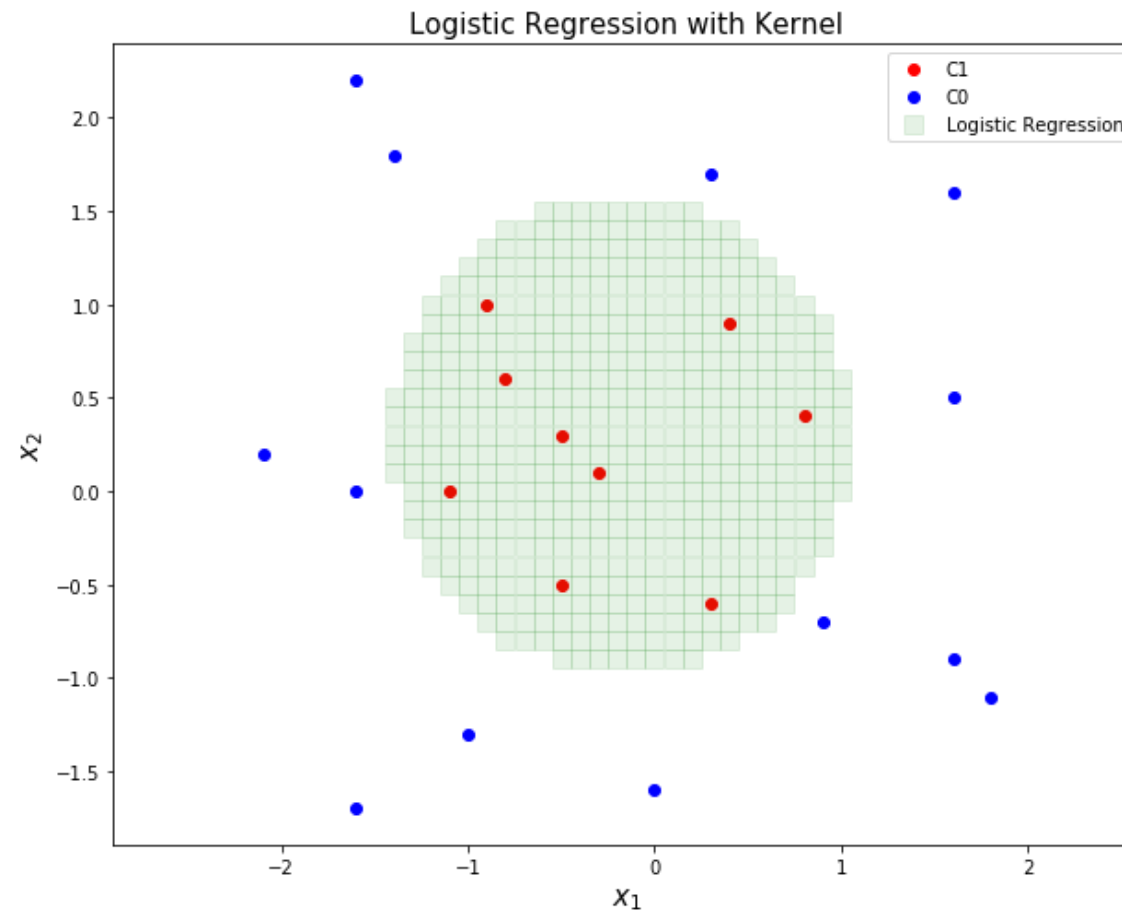
X = np.asmatrix(X)
y = np.asmatrix(y)

m = N + M
Z = np.hstack([np.ones([m,1]), np.sqrt(2)*X[:,0], np.sqrt(2)*X[:,1], np.square(X[:,0]),
               np.sqrt(2)*np.multiply(X[:,0], X[:,1]), np.square(X[:,1])])

w = cvx.Variable([6, 1])
obj = cvx.Minimize(cvx.sum(cvx.logistic(-cvx.multiply(y,Z*w))))
prob = cvx.Problem(obj).solve()

w = w.value
```

# Non-linear Classification





# Multiclass Classification

# Multiclass Classification

- Generalization to more than 2 classes is straightforward
  - one vs. all (one vs. rest)
  - one vs. one
- Using the softmax function instead of the logistic function
  - (refer to [UFLDL Tutorial](#))
  - see them as probability

$$P(y = k \mid x, \omega) = \frac{\exp(\omega_k^T x)}{\sum_k \exp(\omega_k^T x)} \in [0, 1]$$

- We maintain a separator weight vector  $\omega_k$  for each class  $k$