Proof: Sufficiency of 5 Doublings of String x

Problem Statement

Let x be a string of length n and s be a string of length m, where both strings consist of lowercase Latin letters. You are allowed to repeatedly append the current string x to itself (i.e., $x \leftarrow x + x$). After each operation, the length of x doubles.

We want to determine the minimum number of operations after which s appears as a substring of x, or conclude that this is impossible. It is given that:

$$n \cdot m \leq 25$$

Goal

We aim to prove the following:

If s can appear in x as a substring after some number of self-appending operations, then it will necessarily appear within 5 operations (i.e., when $|x| = 32n$).

Operation Growth

Each operation doubles the length of x. After k operations, the length of x becomes:

$$|x_k| = n \cdot 2^k$$

Thus, the growth of x over the first few operations is:

Operations	Length of x
0	n
1	$2n$
2	$4n$
3	$8n$
4	$16n$
5	$32n$

Key Insight: Repetition Structure

Let x_0 be the original string. Each subsequent x_k is a repetition of x_0:

$$x_k = \underbrace{x_0\ x_0\ \ldots\ x_0}_{2^k \text{ times}}$$

Therefore, all versions of x are made solely by concatenating copies of the original x_0, and any substring that appears in x_k must consist of parts (or overlaps) of these repeated x_0 segments.

What Must Be Guaranteed

To ensure that s appears as a substring of x_k:

1. x_k must be at least as long as s (i.e., $|x_k| \geq m$).
2. x_k must be long enough to allow all possible **alignments** of s to be tried — especially those where s spans across boundaries between adjacent repetitions of x_0.

Worst-Case Overlap Alignment

Suppose s starts near the end of one repetition of x_0 and ends at the beginning of the next. In such a case, s overlaps multiple repetitions of x_0. To handle this, x_k must be long enough to simulate a sliding window of size m over all possible positions.

If k is the number of repetitions of x_0, then the total length of x is $k \cdot n$. The number of m-length substrings available in such a string is:

$$k \cdot n - m + 1$$

To ensure that s is **guaranteed** to appear (if possible), a sufficient condition is:

$$k \cdot n \geq m + n - 1$$

Even more conservatively, we can demand:

$$k \cdot n \geq 2m$$

which ensures enough overlap between repetitions to cover all alignment positions.

Using the Problem Constraint

We are given:

$$n \cdot m \leq 25 \Rightarrow m \leq \frac{25}{n}$$

We require:

$$32n \geq 2m \Rightarrow m \leq 16n$$

This always holds because:

$$\frac{25}{n} \leq 16n \quad \text{for all } n \in [1, 25]$$

Proof of this inequality:
Multiply both sides:

$$25 \leq 16n^2 \Rightarrow n^2 \geq \frac{25}{16} = 1.5625 \Rightarrow n \geq \sqrt{1.5625} \approx 1.25$$

Since n is a positive integer and $n \geq 1$, the inequality always holds.

Conclusion

After 5 operations, the string x has length $32n$, which satisfies:

- $32n \geq 2m$
- Covers all possible alignments of s of length m
- Adheres to the global constraint $n \cdot m \leq 25$

Thus, if s does not appear as a substring of x after 5 operations, it will never appear in any longer repetition of x.

$$\boxed{\text{If } s \text{ can appear in } x, \text{ it will appear within 5 operations (i.e., length } 32n\text{).}}$$