Relativity Assignment Solutions

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1 Einstein's velocity addition

Solution:

Particle A moves with velocity V_{AB} as observed by frame B and V_{AC} as observed by frame C. Frame C is moving with velocity V_{CB} as observed by B. For ease of calculations, let

u = velocity of A wrt B = $\frac{dx}{dt}$

In a different frame C, velocity will be $\frac{dx'}{dt'}$

Lorentz transformations give us the relations

$$x' = \gamma(x - \beta ct) \tag{1}$$

$$ct' = \gamma(ct - \beta x) \tag{2}$$

$$\beta = \frac{v}{c}$$

A change in x' and t' can be written as

$$dx' = \gamma(dx - \beta cdt)$$

$$cdt' = \gamma(cdt - \beta dx)$$

Dividing,

$$\frac{dx'}{cdt'} = \frac{dx - \beta cdt}{cdt - \beta dx}$$

Simplifying,

$$\frac{dx'}{cdt'} = \frac{\frac{dx}{dt} - \beta c}{c - \beta \frac{dx}{dt}}$$

$$\frac{dx'}{cdt'} = \frac{u - \beta c}{c - \beta u}$$

$$\frac{dx'}{dt'} = \frac{u - v}{1 - \frac{uv}{c^2}}$$

Substitute

u = V_{AB}
v = $V_{CB} = -V_{BC}$
$\frac{dx'}{dt'} = V_{AC}$
And we get

$$V_{AC} = \frac{V_{AB} + V_{BC}}{1 + (V_{AB}V_{BC}/c^2)}$$

2 Superluminal Motion

Solution:

2.1

Let us say that one of the blobs is moving with velocity v in $+x$ direction. The angle subtended by the blob's path at the observer (Earth) is (angle) . Also it's given that the blob is at a distance $D = 62$ million light years from the Earth.

Therefore,

$$v = D\omega$$

$$v = (putvalues)$$

Which is way beyond c!

So first of all, this calculation is wrong XP. This is because of our wrong perception of the blob's motion. This is a type of optical illusion arising from the fact that space is 3-dimensional, while our images are not. We think that it is moving horizontally as we see in this 2-D diagram but it is in fact headed almost directly towards us and that too at speeds closer to that of light!

This apparent superluminal motion works because of the limited speed of light. If a jet of hot gas is moving towards us at speeds close to light, the light that's travelling towards us is coming from a place much closer than we think it is and arrives much sooner than we expect it to.

2.2

From the figure

$$ct_{obs} = \sqrt{(L - Vtcos\theta)^2 + (Vtsin\theta)^2}$$

$$ct_{obs} = L - Vtsin\theta, Vt << c$$

$$t_{obs} = t\left(1 - \frac{v}{c}cos\theta\right) + \frac{L}{C}$$

2.3

Transverse velocity is V_T

$$V_T = \frac{dx}{dt}$$

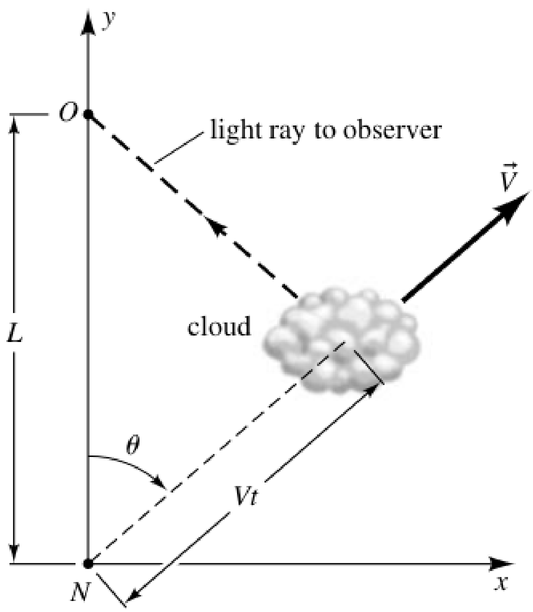

Figure 1: Actual superluminal motion

now,

$$\frac{dx}{dt} = \frac{dx}{dt_{obs}}\frac{dt_{obs}}{dt}$$

$$\frac{dx}{dt_{obs}} = V sin\theta$$

and

$$\frac{dt_{obs}}{dt} = 1 - \frac{v}{c}cos\theta$$

Therefore,

$$V_T = \frac{V sin\theta}{1 - \frac{v}{c}cos\theta}$$

2.4

According to question $V_T > c$

$$\frac{V sin\theta}{1 - \frac{v}{c}cos\theta} > c$$

$$V sin\theta > c - V cos\theta$$

$$sin\theta + cos\theta > \frac{c}{v}$$

Squaring both sides

$$1 + 2sin\theta cos\theta > \left(\frac{c}{v}\right)^2$$

$$= \frac{1}{2}(sin)^{-1}\left(\frac{1-\frac{v}{c}^2}{\frac{v}{c}^2}\right)$$

3 Relativistic Doppler Effect

Solution:

3.1

The source (S) is at rest and the receiver (R) is moving away from S with velocity v and $v > 0$. The first pulse received by the receiver at $t_s = 0$ and $x_s = 0$. λ_s is the wavelength of the wave. Now at time t_{rs} the receiver receives the second pulse. Let us consider two situations.

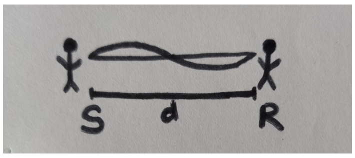

Figure 2: Case-1

CASE 1 - Both S and R are at rest.

Suppose the distance between the two is d and N pulses passed the receiver since $t_s = 0$ till t_{rs}. Then

$$d = \lambda_s N$$

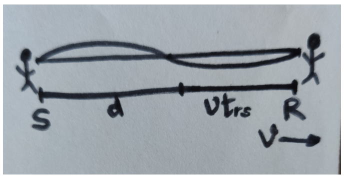

Figure 3: Case-2

CASE 2 - R starts moving with velocity v.

Now it's given that the pulse takes t_{rs} time to reach R. Also 2nd postulate of Special Relativity states that speed of light is constant in vacuum irrespective of any inertial reference frame. Hence

$$ct_{rs} = d + vt_{rs}$$

$$ct_{rs} = \lambda_s N + vt_{rs}$$

Rearranging and putting N = 1;

$$\lambda_s = ct_{rs} - vt_{rs}$$

Hence, proved.

3.2

Now let us change the reference frame from S to R. Perform Lorentz transformation and we get

$$ct_{rs} = \gamma \left(ct_r - \frac{v}{c} x \right)$$

for $x = 0$

$$t_{rs} = \gamma t_r$$

3.3

$$\lambda_s = \frac{c}{f_s}$$

Also, using $ct_{rs} = \lambda_s N + vt_{rs}$ from 3.1

$$t_{rs} = \frac{1}{f_s \left(1 - \frac{v}{c}\right)}$$

$$f_{rs} = \frac{1}{t_{rs}}$$

$$f_{rs} = f_s \left(1 - \frac{v}{c}\right)$$

$$f_r = f_{rs} \gamma$$

$$f_r = f_s \frac{(1 - \frac{v}{c})}{\sqrt{1 - \frac{v}{c}^2}}$$

$$\frac{f_r}{f_s} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

4 The Friedman equations

Solution:

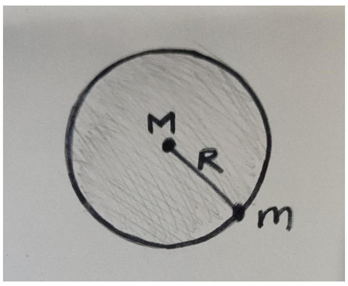

Figure 4: Imagine the galaxy present on a sphere of radius 'R'

4.1

So let's consider a certain spherical section of the universe of radius 'R'. Since the universe is assumed to be homogeneous and isotropic, it has uniform density ρ (however it may not be constant). This situation can be simplified such that the contents of the sphere are concentrated at the centre as a point mass 'M'.

according to Hubble's law:

$$\frac{dR}{dt} = HR$$

Now calculating the energy interaction of m:

Potential energy (due to gravity):

$$U = \frac{-GMm}{R} = \frac{-4\pi G\rho m R^2}{3}$$

Kinetic energy (due to universe expansion):

$$K = \frac{m\left(\frac{dR}{dt}\right)^2}{2}$$

Total energy:

$$E = U + K = \frac{-4\pi G\rho m R^2}{3} + \frac{mHR^2}{2}$$

Therefore,

$$\frac{2E}{mR^2} = H - \frac{8}{3}\pi G\rho$$

4.2

Now we shift to co-moving coordinate system:

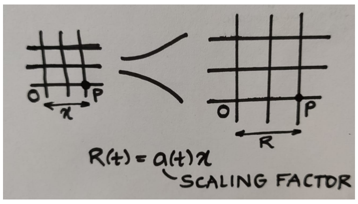

Figure 5: Co-moving coordinates

$$R(t) = a(t)x$$
$$\dot{R}(t) = \dot{a}(t)x + a(t)\dot{x} = \dot{a}(t)x$$

Total energy:

$$E = \frac{-4\pi G\rho m R^2}{3} + \frac{m\dot{R}^2}{2} = \frac{-4\pi G\rho m a^2 x^2}{3} + \frac{m\dot{a}^2\dot{x}^2}{2}$$

Divide both sides by $\frac{ma^2x^2}{2}$,

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} + \frac{2E}{mx^2a^2}$$

Substitute $k = \frac{-2E}{mx^2c^2}$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$$

4.3

According to first law of thermodynamics:

$$dE = TdS - PdV$$

For a reversible process, $dS = 0$
Therefore;

$$\frac{dE}{dV} = -P = \frac{\dot{E}}{\dot{V}}$$

Volume is given by:

$$V = \frac{4\pi a^3x^3}{3}$$

$$\dot{V} = \frac{4\pi}{3}x^3(3a^2)\dot{a} = 3V\left(\frac{\dot{a}}{a}\right)$$

Substituting $\dot{V}$,

$$\dot{E} = -P\dot{V} = -P\left(3V\left(\frac{\dot{a}}{a}\right)\right)$$

By mass energy equivalence;

$$E = \rho V c^2$$

Differentiating wrt time;

$$\dot{E} = \dot{\rho}Vc^2 + \rho\dot{V}c^2$$

$$-P\left(3V\left(\frac{\dot{a}}{a}\right)\right) = \dot{\rho}Vc^2 + \rho\dot{V}c^2$$

Dividing by $-V$

$$3P\left(\frac{\dot{a}}{a}\right) = \left(\dot{\rho}V + \rho\dot{V}\right)\frac{c^2}{-V} = \left(\dot{\rho} + \rho\frac{\dot{V}}{-V}\right)c^2$$

$$3P\left(\frac{\dot{a}}{a}\right) = \left(\dot{\rho} - \rho\frac{3\dot{a}}{a}\right)c^2$$

Simplifying

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\frac{P}{c^2} + \rho\right) = 0$$

4.4

Using the result from 4.2

$$(\dot{a})^2 = \frac{8\pi G\rho a^2}{3} - kc^2$$

$$2a\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + \rho 2a\dot{a}) + 0$$

Dividing by $2a^2$ and substituting $\dot{\rho} = -3\frac{\dot{a}}{a}\left(\frac{P}{c^2} + \rho\right)$ (from 4.3)

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(-3\frac{\dot{a}}{a}\left(\frac{P}{c^2} + \rho\right) + 2\rho\frac{\dot{a}}{a}\right)$$

Simplifying,

$$\frac{\ddot{a}}{\dot{a}} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

4.5

in context to cosmology, First Law of Thermodynamics tells us that, as volume V of universe expands by dV, the pressure P in volume V does the work PdV, which decreases the total energy by that amount.

$$d\left(\rho\frac{4}{3}\pi R^3\right) = -Pd\left(\frac{4}{3}\pi R^3\right)$$