# GR and Black Holes Assignment

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## 1 Clocks, Gravity and Gravitational Redshift

We will now consider a *Gedanken experiment.* Alice, in the nose of the rocket, emits signals at equal intervals ($\Delta\tau_A$) on a clock at her height ($h$). Bob, in the tail, measures the time interval ($\Delta\tau_B$) between receipt of the signals on an identical clock at his location. As shown in the left figure 1, the rocket is stationary on Earth's surface where bodies fall with acceleration $g$. We will derive the relation between $\Delta\tau_A$ and $\Delta\tau_B$.

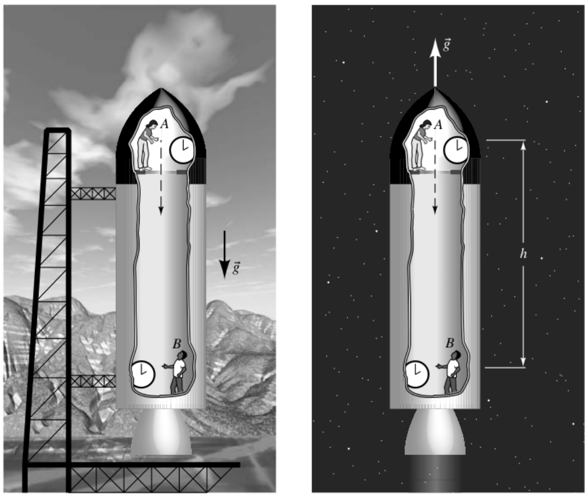


Figure 1: The thought experiment showing equivalence between a stationary ship on earth where there is uniform acceleration due to gravity $g$. The second figure is an equivalent frame where the rocket is accelerating.

1. Consider the rocket accelerating along the z-axis. Suppose that t = 0 is the time the first pulse is emitted, $t_1$ is the time it is received. $\Delta\tau_A$ is the time second pulse is emitted, and $t_1 + \Delta\tau_B$ is the time the second pulse is received as shown in the figure 2. Write down the distance travelled by the **first** and **second** pulse in terms of the times defined.

2. Now keep only the **linear** terms in the equations after substituting $z(t)$

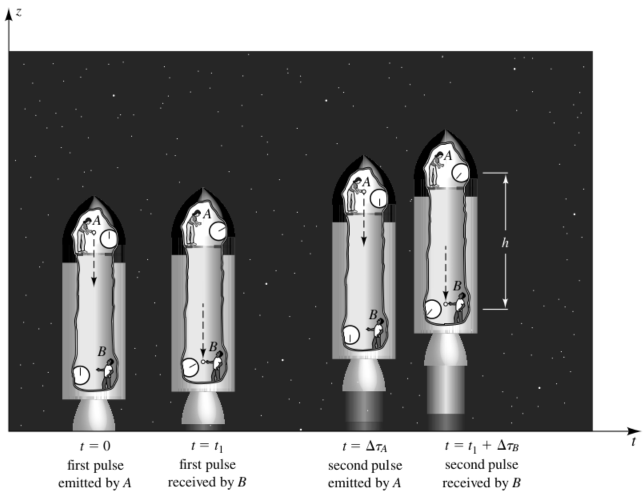


Figure 2: The pulse sent at different times

in terms of the uniform acceleration $g$, show that

$$\Delta\tau_B = \Delta\tau_B\left(1 - \frac{gh}{c^2}\right) \tag{1}$$

3. According to *Equivalence Principle*, the same should apply to unifrom gravitational field $g$ on Earth. Relate (1) to the gravitational potenmtials at Alice ($\Phi_A$) and Bob's($\Phi_B$) location. We arrive at a very general GR result for any general potential $\Phi$.

$$\Delta\tau_B = \left(1 + \frac{\Phi_A - \Phi_B}{c^2}\right)\Delta\tau_A \tag{2}$$

When the receiver is at a higher gravitational potential than the emitter, the signals will be received more slowly than they were emitted. When the receiver is at a lower potential than the emitter, the signals will be received more quickly.

4. Let's have some fun now. Say you are working on the top floor of your department building. Your heart is like a clock, we wil use it for our measurements! Using (1) figure out how fast a heart will beat more times on the top floor in a given interval of time than the heart of a your friend on the ground floor.Assume a heighr of 30m.

5. We can apply (1) to photons with frequency. Consider light emited from the surface of a star with frequency $\omega_*$ which is recieved at a very farway from the star with frequency $\omega_\infty$. This will be lesser than $\omega_*$ and this phenomena is called **gravitational redshift**. Since frequency is *redshifted*. Arrive at this relation.

## 2 Schwarschild Metric properties

You will now derive some properties of Schwarschild Metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi) \tag{3}$$

1. Firstly, figure out the singularities in the metric i.e, find r where the metric coefficient(s) blow up.
2. Do a coordinate transformation to *Edington Finkelstein* coordinates $(v, r)$

   $$t = v - r - 2Mlog\left|\frac{r}{2M} - 1\right| \tag{4}$$

   Show that in this coordinates the metric becomes

   $$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + sin^2\theta d\phi) \tag{5}$$

   Notice that the singularity at $r = 2M$ is removed here. This singularity was just a *coordinate singularity.*
3. If $|M/r| << 1$ then expand the metric to arrive at the static, weak field metric. Now using the newtonian potential,

   $$\Phi = -\frac{GM}{r}$$

   show that the weak field metric becomes:

   $$ds^2 = -\left(1 + 2\Phi\right)dt^2 + \left(1 - 2\Phi\right)(dx^2 + dy^2 + dz^2)$$

## 3 Simple calculation of Chandrashekhar Mass Limit

You will now do a back-of-the-envelope calculation of the Chandreshakar Mass limit.

White dwarfs support against their gravity by the pressure of electrons from Pauli exclusion principle. This pressure is called *Fermi Pressure.*

1. Consider a spherical configuration of radius R consisting of A electrons and A protons. We can think of each of the electron to occupy a volume of characteristic size $\lambda$ such that there is a notat of A electrons occupying a spherical volume of radius R. Then,

   $$\lambda \sim \frac{R}{A^{1/3}}.$$

   Now uising the deBroglie relatiopn $p = 2\pi\hbar/\lambda$, calculate the approx. Fermi momentum, $p_F$.

2. When the sphere is compressed, $p_F$ rises and has to compete the gravitational pressure. Using the relativistic energy $E = [(p_F c)^2 + (m_e c^2)^2]^{1/2} \approx p_F c$, show that the total Fermi energy is

$$E_F \sim A^{4/3} \frac{\hbar c}{R} \tag{6}$$

3. The protons supply most of the mass. The total mass of protons is $m_p A$. Calculate the gravitational energy $E_G$

$$E_G \sim -\frac{G(m_p A)^2}{R} \tag{7}$$

4. Now equate (6) and (7) to arrive at the critical A to balance out the graviutational and fermi pressure.Plugging in the constants show that $A_{crit} \sim 10^{57}$.

   If A is sufficiently large, the total energy $E = E_F + E_G$ will be negative and the white dwarf will collapse

5. Further show that the critical Mass associated with $A_{crit}$ is of the order of $M_{crit} \sim 1 M_\odot$. If you follow the careful calculation that requires a bit of Quantum and Statistical Mechanics, you will arrive at the number Chandra found:

$$M_{chandra} = 1.4 M_\odot \tag{8}$$