Applications of QM (Tutorial)

Horizon Summer School

July 2021

Question 1

Prove $[\hat{a}, \hat{a}^\dagger] = 1$ using $[\hat{x}, \hat{p}] = i\hbar$

Question 2

If we consider $\hbar = \omega = m = 1$, the Hamiltonian reduces to,

$$\mathcal{H} = \frac{1}{2}(\hat{X}^2 + \hat{P}^2)$$

$\hat{X}$ and $\hat{P}$ can be written in terms of $\hat{a}$ and $\hat{a}^\dagger$ as,

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \quad \hat{P} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

For the number state $|n\rangle$, calculate $(\Delta\hat{X})^2$ and $(\Delta\hat{P})^2$ and verify the Uncertainty relation.

Note: $(\Delta\hat{X})^2 = \left\langle \hat{X}^2 \right\rangle - \left\langle \hat{X} \right\rangle^2$

Question 3

Prove the Baker-Hausdorf lemma, i.e. for two operators $\hat{A}$ and $\hat{B}$

$$e^{\lambda\hat{A}}\hat{B}e^{-\lambda A} = \hat{B} + \lambda[\hat{A}, \hat{B}] + \frac{\lambda^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + ...$$

Question 4

For the case when $[\hat{A}, \hat{B}] \neq 0$ and $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, prove the disentanglement theorem,

$$e^{\hat{A}+\hat{B}} = \exp\left\{\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right)\right\}e^{\hat{A}}e^{\hat{B}}$$

Question 5

Show that $|\xi\rangle$ is a squeezed state by calculating ΔX and ΔP in this state. Find the values of θ for which it is squeezed in the X quadrature and the values for which it is squeezed in the P quadrature.

Hint: You need to calculate $\langle\xi| X |\xi\rangle$ and $\langle\xi| X^2 |\xi\rangle$. (Similarly for P)

$$\langle\xi| X |\xi\rangle = \frac{1}{2} \langle 0| S^\dagger(\hat{a} + \hat{a}^\dagger)S |0\rangle = \frac{1}{2}(\langle 0| S^\dagger \hat{a} S |0\rangle + \langle 0| S^\dagger \hat{a}^\dagger S |0\rangle)$$

You need to calculate the quantities $S^\dagger \hat{a} S$ and $S^\dagger \hat{a}^\dagger S$ (Use Baker-Hausdorf formula.)

$$\langle\xi| X^2 |\xi\rangle = \frac{1}{4} \langle 0| S^\dagger(\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger \hat{a} + 1)S |0\rangle$$

You have terms like $S^\dagger \hat{a}^2 S = S^\dagger \hat{a} S S^\dagger \hat{a} S$

Submission

Link: `https://forms.gle/4yoKZ1NrAd5Gpgiq5`

Submit it on or before Sunday, 11:59 pm