Black holes and Gravitational waves Assignment Solutions

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1 Clocks, Gravity and Gravitational Redshift

Solution:

1.1

If position of Bob at $t = 0$ is taken to be origin of z-axis, Then position of Bob

$$z_B = \frac{gt^2}{2}$$

Then position of Alice will be

$$z_A = h + \frac{gt^2}{2}$$

Therefore, the distance travelled by the first pulse

$$z_A(0) - z_B(t_1) = ct_1$$

$$h - \frac{gt_1^2}{2} = ct_1$$

And the distance travelled by the second pulse

$$z_A(\delta\tau_A) - z_B(t_1 + \delta\tau_B) = c(t_1\delta\tau_B - \delta\tau_A)$$

$$h - \frac{gt_1^2}{2} - gt_1\delta\tau_B = c(t_1\delta\tau_B - \delta\tau_A)$$

Subtracting equation 2 from equation 1

$$gt_1\delta\tau_B = c\delta\tau_A - c\delta\tau_B$$

1.2

Approximating

$$t_1 = \frac{h}{c}$$

$$\delta\tau_B = \delta\tau_A\left(1 - \frac{gh}{c^2}\right)$$

1.3

Received pulses are $\left(1-\frac{gh}{c^2}\right)$ times shorter than the the pulse emitted.

According to Equivalence Principle,

$$\phi_A - \phi_B = gh$$

$$\delta\tau_B = \delta\tau_A \left(1 - \frac{\phi_A - \phi_B}{c^2}\right)$$

1.4

Use equation 3 and substituting $h = 30$, and $\delta\tau_A = 72/60 = 1.2$ Then take the inverse of $\delta\tau_B$

1.5

Use equation 3 and then take the inverse of $\delta\tau_B$

$$\omega_{shifted} = \omega_* \left(1 - \frac{GM}{rc^2}\right)$$

2 Schwarschild Metric properties

Solution:

2.1

From the equation

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 r^2 (d\theta^2 + \sin^2\theta d\phi)$$

We can see that the metric coefficient(s) blows up when $r = 2M$ and $r = 0$

2.2

Using the Edington Finkelstein transformation,

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right|$$

Differentiating the equation,

$$dt = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr$$

Squaring both sides,

$$dt^2 = dv^2 + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 - 2dvdr$$

2.3

Using

$$\phi = \frac{-GM}{r}$$

and the transformation from spherical coordinates to cartesian coordinate

$$ds^2 = -(1+2\phi)\,dt^2 - (1-2\phi)(dx^2+dy^2+dz^2)$$

3 Simple Calculation of Chandrashekhar Mass limit

Solution:

3.1

White Dwarfs support themselves against gravity by the pressure of electrons arising from the Pauli's Exclusion principle. This Fermi pressure has an energy associated with it, called the Fermi energy. We can roughly estimate this energy by considering a spherical configuration of radius R , consisting of A electrons and A protons (to maintain electrical neutrality).

We can approximate that as the protons supply most of the mass, they account for most of the gravitational energy whereas the electrons account for the Fermi energy.

We've been given the relation $\lambda \sim \frac{R}{A^{\frac{1}{3}}}$. We can use this to find the Fermi momentum p_F.

$$p_F = \frac{2\pi\hbar}{\lambda} \sim \frac{\hbar}{\lambda} \sim \frac{A^{\frac{1}{3}}\hbar}{R}$$

3.2

The electrons primarily account for Fermi energy and we've been given energy for one electron. Hence, using the relativistic energy $E = [(p_F c)^2 + (m_e c^2)^2]^{1/2} \approx p_F c$, the total Fermi energy for A electrons is

$$E_G \sim AE \sim Ap_F c \sim A^{4/3}\frac{\hbar c}{R}$$

3.3

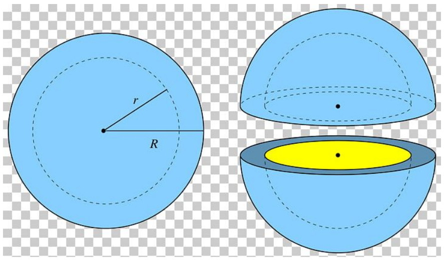
The protons supply most of the mass hence calculating gravitational energy for a spherical configuration of radius R and mass of A protons:

The gravitational potential due to a uniformly filled sphere (density $= \rho$) at a point inside the sphere r distance away from the center:

$$V = -\frac{GM}{2R^3}(3R^2 - a^2)$$

Consider a small spherical shell of radius r inside the sphere, its potential energy will be

$$dE_G = V dm = -\frac{GM}{2R^3}(3R^2 - a^2)(\rho 4\pi a^2 da)$$



The total gravitational energy will be

$$E_G = \int_0^R -\frac{GM}{2R^3}(3R^2 - a^2)\rho 4\pi a^2 da$$

On integrating,

$$E_G = -\frac{6GM^2}{5R} \sim -\frac{GM^2}{R} = -\frac{G(Am_p)^2}{R}$$

3.4

Equating the Fermi energy and gravitational energy magnitudes:

$$A^{4/3}\frac{\hbar c}{R} = -\frac{G(Am_p)^2}{R}$$

Simplifying

$$A_{crit} = (\frac{\hbar c}{Gm_p^2})^{\frac{3}{2}} = (\frac{6.626X10^{-34}X3X10^8}{4x3.14X6.67X10^{-11}(1.67X10^{-27})^2})^{\frac{3}{2}} \sim 10^{57}$$

3.5

Mass associated with A_{crit},

$$M_{crit} \sim m_p A_{crit} \sim 1.67X10^{-27}X10^{57}$$

$$M_{crit} \sim 10^{30} \sim 1M_{\odot}$$