Relativity and Cosmology Assignments

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1 Einstein's velocity addition

In this problem, you will derive the velocity addition rule in relativity using Lorentz transformations. Consider a particle A in the frame B. Its velocity w.r.t frame B is V_{AB}. In another frame C moving w.r.t B with a velocity V_{CB}. Show that the velocity of A w.r.t to the frame C using **Lorentz transformation** is:

$$V_{AC} = \frac{V_{AB} + V_{BC}}{1 + (V_{AB}V_{BC}/c^2)} \tag{1}$$

(***Hint*** *: You can use consider the velocity to be* $u = \frac{dx}{dt}$ *in the frame B and then write the Lorentz transformation equations for* dx *and* dt *to the frame C where these become* $\bar{dx}$ *and* $\bar{dt}$)

2 Superluminal Motion

Astronomers observed radio galaxies moving with velocities exceeding the velocity of c! M87 is an example of such a galaxy in the Virgo cluster. The distance to this galaxy, M87, is about D = 62 million light years. One can use this distance to convert angular separations into linear separations across the line of sight.

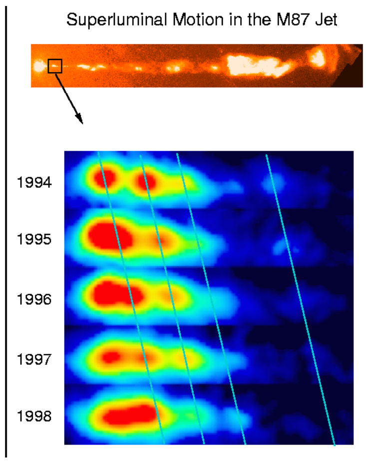

Figure 1: Blobs

1. Now, look at one of the blobs: the innermost one, which appears most clearly in 1996 and 1997. Approximately calculate the velocity of the blob. Hold on a minute! Is this against special relativity postulate 2?

2. This calculation is not correct. The clouds are actually moving straight towards us with a velocity below c. As the cloud rapidly approaches, the distance light has to travel reduces and hence it would appear to reach us sooner. This effect can be understood from the second diagram. The cloud stars at N and moves at an angle θ w.r.t the line of sight of the observer O. Let $\vec{V}$ be the velocity of the cloud and let t_{obs} be the time the observer sees the light emitted from the cloud at time t. From the distance travelled by light calculate the relation between t_{obs} and t.

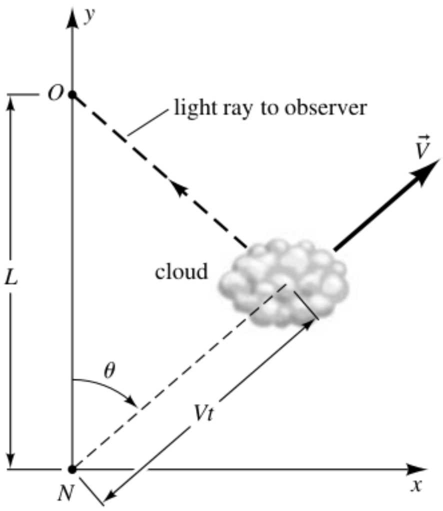

Figure 2: Apparent motion of the cloud

3. Now we using the t_{obs} relation, figure out the transverse speed V_T observed by the observer using $V_T = dx/dt_{obs}$.

4. Can you now plot V_T for different θ and see at what angles you see an apparent motion greater than c ?

3 Relativistic Doppler Effect

In this exercise you will derive the Doppler effect using Lorentz transformations.

Consider the source to be at rest and the receiver is moving away with some velocity v and $v > 0$. Use subscript s for source and r for receiver.

1. Consider the first pulse received by the receiver at $t_s = 0$ and $x_s = 0$. λ_s is the wavelength of the wave. Now is time t_{rs} the receiver receives the second pulse. Then show that

$$ct_{rs} = \lambda_s + vt_{rs}$$

2. We now need to make a Lorentz transformation to receiver frame and change t_{rs} to t_r. Show that after doing Lorentz transformation

$$t_r = \frac{t_{rs}}{\gamma}$$

3. Now substitute this in the previous relation and obtain the famous relation between the frequencies.

$$f_r = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}} f_s$$

4 The Friedmann equations

In this exercise, you will derive the Friedmann equations from Newtonian mechanics. Understand that the Friedmann equations essentially imply conservation of energy.

Consider a galaxy of mass m to be at the surface of a sphere of radius R. Since you know that the universe is homogeneous and isotropic, you can essentially treat the sphere to have a mass M uniformly distributed with a density ρ.

1. Arrive at an expression for the interaction energy (E) between the galaxy and the mass in the sphere.

   (***Hint:*** *The galaxy is moving away due to the expanding universe (Hubble's law) as well as being pulled by gravity to the centre of the sphere.*)

2. Shift to a comoving coordinate system by plugging $R = a(t)\chi$, where χ is the comoving distance. Define $k = \frac{-2E}{m\chi^2 c^2}$ to arrive at the first Friedmann equation (This equation represents a Universe with no dark energy):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} \tag{2}$$

   Observe that there is an extra c^2 factor in the RHS as against the expression shown in the presentation. This is due to switching between SI units and **cosmological units**. More on this can be found here.

3. Consider the 1st law of thermodynamics (this is where the conservation of energy part comes in):

$$dE = TdS - PdV \tag{3}$$

   Assume a reversible expansion. Using everything that you have at your disposal, arrive at the fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \tag{4}$$

4. Using the fluid equation and the 1st Friedmann equation, arrive at the 2nd Friedmann equation (also called the acceleration equation):

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \tag{5}$$

5. *(Brownie points)* In a cosmological context, what exactly is the 1st law of thermodynamics telling us?