Introductory Quantum Optics and some applications

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CFI Summer School 2021:
A random walk through physics and astronomy

Horizon: The Physics and Astronomy Club of IIT Madras

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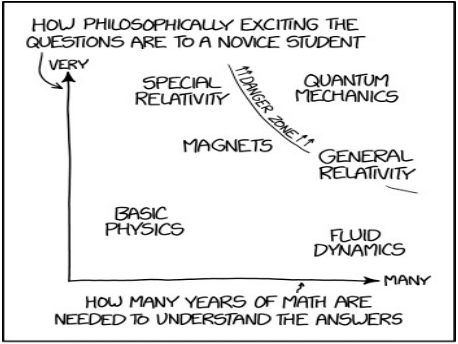
Contents

Before we start!



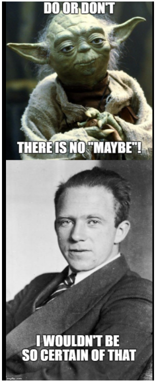
WHY SO MANY PEOPLE HAVE WEIRD IDEAS ABOUT QUANTUM MECHANICS

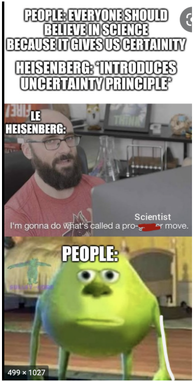
But trust me, I will not do any difficult math! There will be a lot of linear algebra.

Prelude

1. The canonical commutation relation: $[\hat{x}, \hat{p}] = i\hbar$
2. Any two operators (observables) that do not commute cannot be measured simultaneously.
3. Any observalble (O) that commutes with the Hamiltonian (H) is a constant of motion and the system can be described by the Hilbert space spanned by that observable, i.e. H and O are simultaneously diagonalisable.
4. A hermitian operator has real eigenvalues and orthogonal eigenkets. $\langle n|m\rangle = \delta_{nm}$
5. Any state can be represented as a superposition of a complete basis state. Completeness.

The Uncertainty Principle





Uncertainty Principle: You cannot measure two things at the same time.
Random physicist: I can measure L^2 and L_z at the same time.
Uncertainty Principle: Aah yeah. You got me there!

The Uncertainty Principle

For an observable $\hat{A}$, define the operator, $\Delta\hat{A} = \hat{A} - \left\langle \hat{A} \right\rangle$ We have,

$$\left\langle (\Delta A)^2 \right\rangle = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2$$

The generalized Uncertainty Principle states that,

$$\left\langle (\Delta A)^2 \right\rangle \left\langle (\Delta B)^2 \right\rangle \geq \frac{1}{4} | \left\langle [\hat{A}, \hat{B}] \right\rangle |^2$$

Three important lemma to prove this:

1. Schwartz inequality.
2. The expectation value of a Hermitian Operator is real.
3. The expectation value of an anti-Hermitian Operator is purely imaginary.

Schwartz inequality,

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$$

This can be proved by noting that,

$$(\langle\alpha| + \lambda^* \langle\beta|)(|\alpha\rangle + \lambda|\beta\rangle) \geq 0$$

Then, the Schwartz inequality can be proved by taking $\lambda = -\langle\beta|\alpha\rangle\langle\beta|\beta\rangle$.
Consider,

$$|\alpha\rangle = (\Delta\hat{A})\,|\rangle$$

$$|\beta\rangle = (\Delta\hat{B})\,|\rangle$$

$|\rangle$ denotes any ket.

Using Lemma 1 and Hermiticity of $\Delta\hat{A}$ and $\Delta\hat{B}$, we can get,

$$\left\langle(\Delta A)^2\right\rangle\left\langle(\Delta B)^2\right\rangle \geq |\left\langle(\Delta A)(\Delta B)\right\rangle|^2$$

The RHS is,

$$(\Delta A)(\Delta B) = \frac{1}{2}[(\Delta A),(\Delta B)] + \frac{1}{2}\{(\Delta A),(\Delta B)\}$$

It is easy to check that,

$$[(\Delta A),(\Delta B)] = [\hat{A},\hat{B}]$$

is anti-Hermitian.

$$[\hat{A},\hat{B}]^\dagger = -[\hat{A},\hat{B}]$$

The Uncertainty Principle

Whereas, $\{(\Delta A), (\Delta B)\}$ is Hermitian. Now take the expectation value.

$$\langle(\Delta A)(\Delta B)\rangle = \frac{1}{2}\langle[A, B]\rangle + \frac{1}{2}\langle\{(\Delta A), (\Delta B)\}\rangle$$

The first term is purely imaginary and the second is purely real.

$$|\langle(\Delta A)(\Delta B)\rangle|^2 = \frac{1}{4}|\langle[A, B]\rangle|^2 + \frac{1}{4}|\langle\{(\Delta A), (\Delta B)\}\rangle|^2$$

In most cases, one can ignore the second term which is always positive. Hence, we get,

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}\left|\left\langle[\hat{A}, \hat{B}]\right\rangle\right|^2$$

Some remarks

1. From the generalised Uncertainty Principle, we can see that any two observables which do not commute, cannot be measured simultaneously.
2. But if they commute, there is no uncertainty between them and they can be measured (diagonalised) simultaneously using the same basis states.

There are two ways of solving this. First is to directly solve the Schrodinger equation and use Hermite polynomials (usually taught first in QM classes), second is to use an algebraic method. I will teach the algebraic method. This is how the wavefunctions look like when one solves it using the analytical method.

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}.$$

TABLE 2.1: The first few Hermite polynomials, $H_n(\xi)$.

$H_0 = 1,$
$H_1 = 2\xi,$
$H_2 = 4\xi^2 - 2,$
$H_3 = 8\xi^3 - 12\xi,$
$H_4 = 16\xi^4 - 48\xi^2 + 12,$
$H_5 = 32\xi^5 - 160\xi^3 + 120\xi.$

Do those wave functions look intimidating?

The Quantum Harmonic Oscillator

The well known Hamiltonian,

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$$

Note: x and p are operators
Consider the operators:

$$a = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + i\frac{\hat{p}}{m\omega}) \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - i\frac{\hat{p}}{m\omega})$$

$\hat{a}$: Annihilation operator
$\hat{a}^\dagger$: Creation operator
They satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1$$

Energy eigenvalues and eigenkets

Define the Number operator,

$$\hat{N} = \hat{a}^\dagger \hat{a}$$

Using simple algebra,

$$\hat{a}^\dagger \hat{a} = \left(\frac{m\omega}{2\hbar}\right)\left(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}\right) + \left(\frac{i}{2\hbar}\right)[\hat{x}, \hat{p}]$$

$$\hat{N} = \frac{\mathcal{H}}{m\omega} - \frac{1}{2}$$

One can write,

$$\mathcal{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

Note: $[\mathcal{H}, \hat{N}] = 0$

Energy eigenvalues and eigenkets

Let the eigenkets of the $\hat{N}$ be,

$$\hat{N}\left|n\right\rangle = n\left|n\right\rangle$$

Since, $\mathcal{H}$ and $\hat{N}$ commute, the system can be spanned by the set of states $\{\left|n\right\rangle\}$.
Hence, the energy eigenvalues in this basis are,

$$\mathcal{H}\left|n\right\rangle = E_n\left|n\right\rangle$$

where,

$$E_n = \hbar\omega(n + \frac{1}{2})$$

I will show later that n is a non-negative integer.

The creation and annihilation operators

What exactly are $\hat{a}$ and $\hat{a}^\dagger$?

$$\text{First note:} \quad [\hat{N}, \hat{a}] = [\hat{a}^\dagger \hat{a}, \hat{a}] = \hat{a}^\dagger [\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} = -\hat{a}$$

$$\text{And:} \quad [\hat{N}, \hat{a}^\dagger] = [\hat{a}^\dagger \hat{a}, \hat{a}^\dagger] = \hat{a}^\dagger [\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} = \hat{a}^\dagger$$

Hence,

$$\hat{N}\hat{a}^\dagger |n\rangle = ([\hat{N}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{N}) |n\rangle = (\hat{a}^\dagger + \hat{a}^\dagger \hat{N}) |n\rangle = (n+1)\hat{a}^\dagger |n\rangle$$

Likewise,

$$\hat{N}\hat{a} |n\rangle = ([\hat{N}, \hat{a}] + \hat{a}\hat{N}) |n\rangle = (-\hat{a} + \hat{a}\hat{N}) |n\rangle = (n-1)\hat{a} |n\rangle$$

Hence the name creation and annihilation operators.

Note: [AB,C] = ABC - CAB = ABC + ACB - ACB -CAB = A[B,C] + [A,C]B

The creation and annihilation operator

$\hat{a}^\dagger |n\rangle$ and $\hat{a}|n\rangle$ are also eigenkets of $\hat{N}$ as well as the Hamiltonian $\mathcal{H}$.
One can conclude,

$$\hat{a}^\dagger |n\rangle = c\,|n+1\rangle \quad \text{and} \quad \hat{a}\,|n\rangle = c\,|n-1\rangle$$

Determine c (a complex number):

1. Complex conjugate of the second eqn: $\langle n|\,\hat{a}^\dagger = c^* \langle n-1|$.
2. Take inner product: $\langle n|\,\hat{a}^\dagger \hat{a}\,|n\rangle = |c|^2$
3. LHS is n. Hence, $c = \sqrt{n}$. We have, $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$.

Similarly, one can show that, $\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$

Concluding this portion

Keep on applying the annihilation operator to a state $|n\rangle$,

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$$\hat{a}^2|n\rangle = \sqrt{n(n-1)}|n-2\rangle$$

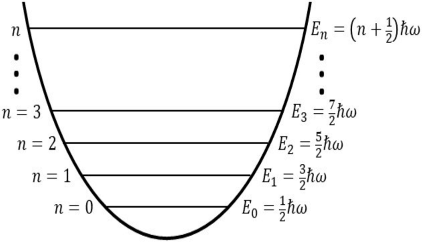
$$\hat{a}^3|n\rangle = \sqrt{n(n-1)(n-2)}|n-3\rangle$$

This can go on but has to terminate somewhere. Why?
Because norm of $\hat{a}|n\rangle = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = n \geq 0$.
The lowest value n can take is 0 and it changes by ± 1 under the application of $\hat{a}$ and $\hat{a}^\dagger$. Hence, n is a non-negative integer.
The ground (lowest energy) state is,

$$E_0 = \frac{1}{2}\hbar\omega; \qquad E_n = \hbar\omega(n+\frac{1}{2}), \quad n = 0, 1, 2, 3, ...$$

We successfully construct the simultaneous eigenkets of $\mathcal{H}$ and $\hat{N}$.

Concluding this portion



$$
\begin{aligned}
|1\rangle &= a^\dagger|0\rangle, \\
|2\rangle &= \left(\frac{a^\dagger}{\sqrt{2}}\right)|1\rangle = \left[\frac{(a^\dagger)^2}{\sqrt{2}}\right]|0\rangle, \\
|3\rangle &= \left(\frac{a^\dagger}{\sqrt{3}}\right)|2\rangle = \left[\frac{(a^\dagger)^3}{\sqrt{3!}}\right]|0\rangle, \\
&\vdots \\
|n\rangle &= \left[\frac{(a^\dagger)^n}{\sqrt{n!}}\right]|0\rangle.
\end{aligned}
$$

Figure: Different energy levels of a Quantum Harmonic Oscillator.

Exercise

1. Prove $[\hat{a}, \hat{a}^\dagger] = 1$ using $[\hat{x}, \hat{p}] = i\hbar$
2. If we consider $\hbar = \omega = m = 1$, the Hamiltonian reduces to,

$$\mathcal{H} = \frac{1}{2}(\hat{X}^2 + \hat{P}^2)$$

$\hat{X}$ and $\hat{P}$ can be written in terms of $\hat{a}$ and $\hat{a}^\dagger$ as,

$$\hat{X} = \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \quad \hat{P} = \frac{1}{2}(\hat{a} - \hat{a}^\dagger)$$

For the number state $|n\rangle$, calculate $(\Delta\hat{X})^2$ and $(\Delta\hat{P})^2$ and verify the Uncertainty relation.

Note: $(\Delta\hat{X})^2 = \left\langle \hat{X}^2 \right\rangle - \left\langle \hat{X} \right\rangle^2$

Why learn QHO?

Because it is everywhere. Areas like Quantum Field Theory, Condensed Matter Physics, Quantum Optics, start with Harmonic Oscillators.

Vacuum fluctuations

We already saw that the zero photon state $|0\rangle$ has an energy,

$$E_0 = \frac{1}{2}\hbar\omega$$

A state of nothingness (vacuum) has some energy! This phenomena is completely quantum mechanical. Let us verify the uncertainty relation. In the state $|0\rangle$

$$\langle X \rangle = \langle P \rangle = 0$$

$$\left\langle \hat{X}^2 \right\rangle = \frac{1}{4} \langle 0| \hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger \hat{a} + 1 |0\rangle = \frac{1}{4}$$

$$\left\langle \hat{P}^2 \right\rangle = -\frac{1}{4} \langle 0| \hat{a}^2 + \hat{a}^{\dagger 2} - 2\hat{a}^\dagger \hat{a} - 1 |0\rangle = \frac{1}{4}$$

Hence, $\Delta X \Delta P = \frac{1}{4}$

Quantization of a single mode field

Consider radiation field confined to a 1D cavity between $z = 0$ and $z = L$. Consider no sources and Electric field must vanish at the boundaries.

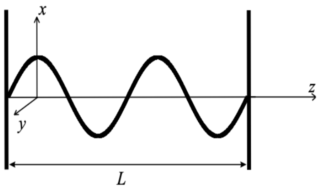


Fig. 2.1. Cavity with perfectly conducting walls located at $z = 0$ and $z = L$. The electric field is polarized along the x-direction.

Figure: One dimensional cavity

E is polarized along x.

The Maxwell equations for this case are,

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

$$\nabla \times \mathrm{B} = \mu_0 \epsilon_0 \frac{\partial \mathrm{E}}{\partial t}$$

$$\nabla . \mathrm{E} = 0$$

$$\nabla . \mathrm{B} = 0$$

With the desired boundary conditions, these equations can be solved to give,

$$E_x(z, t) = Kq(t) \sin kz$$

Here, K is a constant which depends upon ϵ_0, V (effective volume of the cavity), ω, etc.

Because of the boundary condition at $z = L$, k is quantized.

$$k = \frac{m\pi}{L}, \qquad m = 1, 2, ...$$

After solving the Maxwell equations, one can show that the dimension of $q(t)$ is that of length. $q(t)$ can hence be treated as a canonical position (just like position x). Single mode corresponds to only one m value.

From the expression of E, one can write,

$$B_y(z,t) = \left(\frac{\mu_0\epsilon_0}{k}\right)K\dot{q}(t)cos(kz)$$

In exercise, you will show $K = (2\omega^2/V\epsilon_0)^{1/2}$.

Energy density of a radiation,

$$\mathcal{E} = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

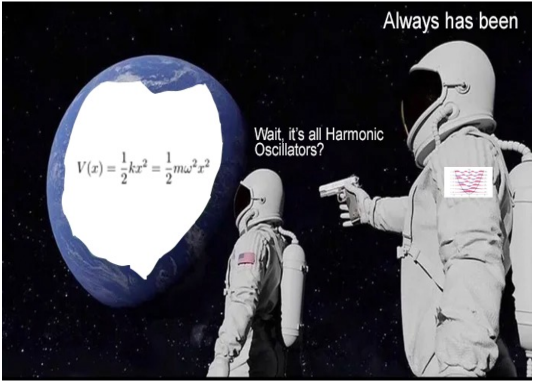
Hence, the Hamiltonian,

$$\mathcal{H} = \int dV \left[\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right]$$

It is easy to see that,

$$\mathcal{H} = \frac{1}{2}(p^2 + \omega^2 q^2)$$

This equation has the structure of a Harmonic Oscillator where $p = \dot{q}$ plays the role of canonical momentum and q plays the role of canonical position.


Always has been
Wait, it's all Harmonic Oscillators?
$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$

After quantizing, q and p can be treated as operators satisfying,

$$[\hat{q}, \hat{p}] = i\hbar$$

We can again define creation and annihilation operators,

$$\hat{a} = (2\hbar\omega)^{-1/2}(\omega\hat{q} + i\hat{p})$$

$$\hat{a}^{\dagger} = (2\hbar\omega)^{-1/2}(\omega\hat{q} - i\hat{p})$$

The Electric and Magnetic field operators can hence be defined as,

$$\hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^{\dagger})\sin kz$$

$$\hat{B}_y(z,t) = \mathcal{B}_0(\hat{a} - \hat{a}^{\dagger})\cos kz$$

After this, everything from the HO section follow. We can define the number operator as,

$$\hat{N} = \hat{a}^\dagger \hat{a}$$

and the Hamiltonian would be,

$$\mathcal{H} = \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)\hbar\omega$$

The quantized energies in the basis $\{|n\rangle\}$ (photon number basis) would be,

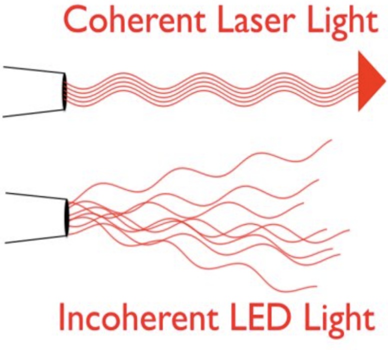
$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

Each of these light quanta are know as photons. If one adds a photon to a state, its energy increases by $\hbar\omega = h\nu$. (Remember $\hbar = h/(2\pi)$ and $\omega = 2\pi\nu$)

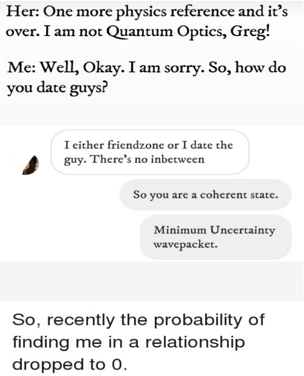
Exercise

1. Prove the Baker-Hausdorf lemma, i.e. for two operators $\hat{A}$ and $\hat{B}$
$$e^{\lambda\hat{A}}\hat{B}e^{-\lambda A} = \hat{B} + \lambda[\hat{A},\hat{B}] + \frac{\lambda^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$$
2. For the case when $[\hat{A},\hat{B}] \neq 0$ and $[\hat{A},[\hat{A},\hat{B}]] = 0$, prove the disentanglement theorem,
$$e^{\hat{A}+\hat{B}} = \exp\left\{\left(-\frac{1}{2}[\hat{A},\hat{B}]\right)\right\}e^{\hat{A}}e^{\hat{B}}$$

a laser source could be a coherent source that has a property that the wavelengths are in phase in space and time (constant relative phase).



They are the 'most classical' state of light. They are the states of minimum uncertainty.



Coherent state of light

They are the eigenstates of the annihilation operator and are the so called minimum uncertainty states.

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

$\hat{a}$ is not Hermitian and hence, α is a complex number.
Since, $\{|n\rangle\}$ is a complete basis, $|\alpha\rangle$ can be written as a superposition,

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle$$

Apply $\hat{a}$ on both sides,

$$\hat{a}\,|\alpha\rangle = \sum_{n=1}^{\infty} C_n\sqrt{n}\,|n-1\rangle = \alpha\sum_{n=0}^{\infty} C_n\,|n\rangle$$

Coherent state

We get,

$$C_n\sqrt{n} = \alpha C_{n-1}$$

Continuing,

$$C_n = \frac{\alpha}{\sqrt{n}} C_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}} C_{n-2} = \ldots$$

$$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0$$

Hence,

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Coherent State

C_0 can be computed by normalization,

$$\langle\alpha|\alpha\rangle = 1 = |C_0|^2 \sum_n \sum_m \frac{\alpha^{*n}\alpha^m}{\sqrt{m!n!}} \langle n|m\rangle$$

$$= |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |C_0|^2 e^{|\alpha|^2}$$

Hence, the coherent state of light in terms of the photon number basis is,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Note: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ implies $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$

Let us verify the uncertainty principle.

$$\hat{X}\frac{\hat{a}+\hat{a}^\dagger}{2}, \qquad \hat{P}=\frac{\hat{a}-\hat{a}^\dagger}{2i}$$

$$\left\langle \hat{X} \right\rangle = \langle\alpha| \frac{\hat{a}+\hat{a}^\dagger}{2} |\alpha\rangle = \frac{1}{2}(\alpha+\alpha^*)$$

$$\left\langle \hat{X}^2 \right\rangle = \frac{1}{4} \langle\alpha| \hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+1 |\alpha\rangle$$

$$= \frac{1}{4}(\alpha^2+\alpha^{*2}+2|\alpha|^2+1)$$

Hence,

$$(\Delta\hat{X})^2 = \left\langle \hat{X}^2 \right\rangle - \left\langle \hat{X} \right\rangle = \frac{1}{4}$$

$$\Delta\hat{X} = \frac{1}{2}$$

State of minimum uncertainty

Uncertainty in $\hat{P}$.

$$\left\langle \hat{P} \right\rangle = \langle\alpha| \frac{\hat{a} - \hat{a}^\dagger}{2i} |\alpha\rangle = \frac{1}{2i}(\alpha - \alpha^*)$$

$$\left\langle \hat{P}^2 \right\rangle = -\frac{1}{4} \langle\alpha| \hat{a}^2 + \hat{a}^{\dagger 2} - 2\hat{a}^\dagger \hat{a} - 1 |\alpha\rangle$$

$$= -\frac{1}{4}(\alpha^2 + \alpha^{*2} - 2|\alpha|^2 - 1)$$

Hence again,

$$(\Delta\hat{P})^2 = \left\langle \hat{P}^2 \right\rangle - \left\langle \hat{P} \right\rangle = \frac{1}{4}, \quad \Delta P = \frac{1}{2}$$

This gives,

$$\Delta X \Delta P = \frac{1}{4} \quad \text{(Minimum uncertainty)}$$

Classical state of light

1. Because of minimum uncertainty, coherent state is called classical light.
2. But note: $|0\rangle$ also has minimum uncertainty but it is not classical. In fact it is the most quantum state.
3. Hence, for classicality of light, minimum uncertainty is necessary but not sufficient.
4. Classicality can be proved by the so called Glauber-Sudarshan P function. (Beyond the scope of this session)

Two level quantum systems

Consider a system with only two levels $|g\rangle$ and $|e\rangle$. For it to be a physical system, it should satisfy the orthonormalisation condition and the completeness relation.

Orthonormal:

$$\langle g|g\rangle = \langle e|e\rangle = 1$$

$$\langle e|g\rangle = \langle g|e\rangle = 0$$

Completeness:

$$|e\rangle\langle e| + |g\rangle\langle g| = \mathcal{I}$$

Classical bits consist of 0 and 1. Qubits (quantum bits) used for Quantum computation are quantum two level systems.

An application of coherent light

Suppose you can create two coherent light,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,, \qquad |\beta\rangle = e^{-|\beta|^2/2} \sum_{m=0}^{\infty} \frac{\beta^m}{\sqrt{m!}} |m\rangle$$

such that, α and β are real and $\beta = -\alpha$. The two states are normalised.

$$\langle\alpha|\alpha\rangle = \langle\beta|\beta\rangle = 1$$

$$\langle\beta|\alpha\rangle = e^{-|\alpha|^2/2} e^{-|\beta|^2/2} \sum_{n,m=0}^{\infty} \frac{\beta^{*m}\alpha^n}{\sqrt{m!n!}} \delta_{mn}$$

$$= e^{-|\alpha|^2/2} e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha\beta^*)^n}{n!}$$

An application of coherent light

$$|\langle\beta|\alpha\rangle|^2 = e^{-|\alpha|^2}e^{-|\beta|^2}\left|\sum_{n=0}^{\infty}\frac{(\alpha\beta^*)^n}{n!}\right|$$

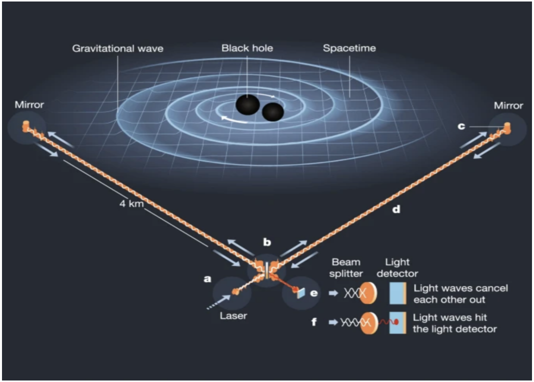
$$= e^{-|\alpha|^2}e^{-|\beta|^2}e^{2\alpha\beta}$$

$$= e^{-|\alpha-\beta|^2}$$

For large α, $|\langle\beta|\alpha\rangle|^2 \to 0$ This is how a two level system can be formed using coherent light.

Squeezed light

How are gravitational waves detected at LIGO, VIRGO, etc.?



They use squeezed light in order to measure only one of the quadratures precisely. Stay tuned for more information!

Squeezed state of light

Consider two observables $\hat{A}$ and $\hat{B}$ which do not commute. You need to measure only one of them and do not care about the other. Squeezed states come in handy in such situations.
For Coherent state, we saw,

$$\Delta x = \frac{1}{2}, \qquad \Delta p = \frac{1}{2}$$

For squeezed state, we will have,

$$\Delta x < \frac{1}{2}$$

(Squeezing in the $\hat{X}$ quadrature)
More precision in measurement of $\hat{X}$ than it is in $\hat{P}$

Squeezed state

Consider the state,

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

Let us calculate ΔX in this state.

$$\langle\psi|\hat{X}|\psi\rangle = \left\{\frac{\sqrt{3}}{2}\langle 0| + \frac{1}{2}\langle 1|\right\}\frac{1}{2}(\hat{a}+\hat{a}^\dagger)\left\{\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right\} = \frac{\sqrt{3}}{4}$$

$$\langle\psi|\hat{X}^2|\psi\rangle = \left\{\frac{\sqrt{3}}{2}\langle 0| + \frac{1}{2}\langle 1|\right\}\frac{1}{4}(2\hat{a}^\dagger\hat{a}+1)\left\{\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right\} = \frac{3}{8}$$

Hence,

$$\left\langle \hat{X}^2 \right\rangle - \left\langle \hat{X} \right\rangle = \frac{3}{8} - \frac{3}{16} = \frac{3}{16} < \frac{1}{4}$$

Calculate ΔP and verify the uncertainty proncipile.

Squeezed state

The above is a simple squeezed state of light, squeezed in the $\hat{X}$ quadrature. But it is difficult to realise in the lab.
A realisable state is the squeezed vacuum which is an infinite superposition of $\{|2n\rangle\}$ states.
The Squeezing operator,

$$S(\xi) = \exp\left\{\frac{1}{2}(\xi^* \hat{a}^2 - \xi \hat{a}^{\dagger 2})\right\}, \quad \xi = re^{i\theta}$$

such that,

$$S(\xi)\,|0\rangle = |\xi\rangle$$

Exercise

Show that $|\xi\rangle$ is a squeezed state by calculating ΔX and ΔP in this state. Find the values of θ for which it is squeezed in the X quadrature and the values for which it is squeezed in the P quadrature.

Hint: You need to calculate $\langle\xi| X |\xi\rangle$ and $\langle\xi| X |\xi\rangle$. (Similarly for P)

$$\langle\xi| X^2 |\xi\rangle = \frac{1}{2} \langle 0| S^\dagger(\hat{a} + \hat{a}^\dagger)S |0\rangle = \frac{1}{2}(\langle 0| S^\dagger \hat{a} S |0\rangle + \langle 0| S^\dagger \hat{a}^\dagger S |0\rangle)$$

You need to calculate the quantities $S^\dagger \hat{a} S$ and $S^\dagger \hat{a}^\dagger S$ (Use Baker-Hausdorf formula.)

$$\langle\xi| X^2 |\xi\rangle = \frac{1}{4} \langle 0| (S^\dagger \hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger \hat{a} + 1)S |0\rangle$$

You have terms like $S^\dagger \hat{a}^2 S = S^\dagger \hat{a} S S^\dagger \hat{a} S$

Van der Waals force

Basically a force of attraction between two uncharged materials. There are two pictures for studying this force.

1. Fluctuating dipoles - The dipoles in the two materials interact leading to an attractive force.
2. Casimir effect - Arises due to vacuum fluctuations.

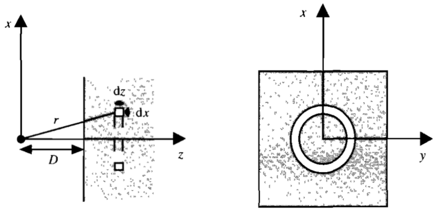
Using perturbation theory, one can show that the interaction between two uncharged atoms goes as $1/r^6$.

$$V(r) = -C/r^6$$

For the interaction between an atom and a semi-infinite sheet, one needs to integrate over all the interactions. The attraction now goes as,

$$V(D) \propto -1/D^3$$

Van der Waals force



Now, if you consider the interaction between two semi-infinite sheets (separated by a distance h), you have to do another integral and in that case, the interaction per unit area goes as,

$$U(h) = -\frac{A}{12\pi h^2}, \qquad A \text{ - Hamaker constant}$$

Van der Waals force

For large separations, effect of finite speed of propagation of the fields from the fluctuating dipoles become significant. This is called the retardation effect. In this regime, the interaction between two atoms goes as $1/r^7$ and that between two sheets goes as $1/h^3$.

Lifshitz theory - A more powerful approach in calculating this force. This is an involved topic. I will consider a simple case of interaction of two conductors at 0K. This is known as the Casimir effect which arises due to the zero point energy.

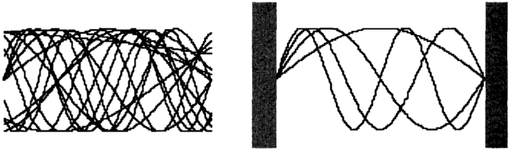


Fig. 4.2 Sketch in one dimension of some of the possible standing wave modes in free space (left), and in the space between perfectly conducting metal plates (right). The difference in zero-point energies between the two situations gives rise to the attractive Casimir force between the plates.

Situation 1 - Absolute vacuum. There are fluctuations with energy $(1/2)\hbar\omega$. There are infinite possible standing modes (multimode field). Hence the total energy is infinite.
Situation 2 - Put two parallel metallic conductors separated by some distance z. In this case, the number of allowed frequencies in the z direction will be quantuized. But there will be infinite possibilities in the x and y directions. The total force again would be infinite.

Van der Waals force

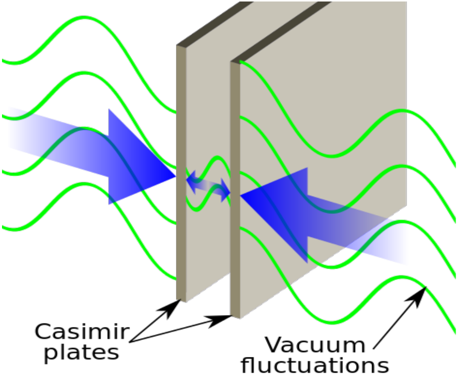
Though the total energies are infinite in both the cases, the difference between the two energies can be shown to be finite. If this difference is negative, then we can be sure that there is an attraction between the two conductors. In turns out that,

$$U_C(l) = E(l) - E_{free}(l) = -\frac{\hbar c \pi^2}{720 l^3}$$

The $1/l^3$ dependence is same as the previous case in the retarded regime.

Conclusion: This force can be considered as a direct manifestation of vacuum fluctuations in the space between two bodies.

Casimir effect



Cling films

Figure: Every time one wraps a sandwich with a cling film, one is relying on the subtleties of vacuum fluctuations and virtual photons to stop the crumbs falling out

The End.

Thank you!