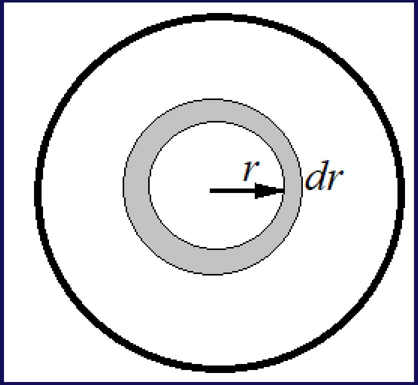
2023 年期中考试答案

I. Fill in the space underlined (50% in total)

1. The current density (电流密度) across a cylindrical (圆柱形) conductor of radius R varies according to the equation: $j = j_0(1 - r/R)$, where r is the distance from the axis $r = 0$ and decreases linearly to zero at the surface $r = R$. The current I in the conductor is of __________.



$$I = \int_0^R j_0\left(1 - \frac{r}{R}\right) \cdot 2\pi r dr$$

$$= 2\pi j_0\left[\frac{1}{2}R^2 - \frac{1}{3}R^2\right]$$

$$= \frac{1}{3} j_0 \pi R^2$$

2. There is a thin disk (圆盘) of radius *a*, with total charge *Q* uniformly distributed (均匀分布) on its surface, rotating at a constant angular speed (角速度) *ω*, as shown in Fig. 1. What is the magnetic dipole moment (磁偶极矩)? ________________

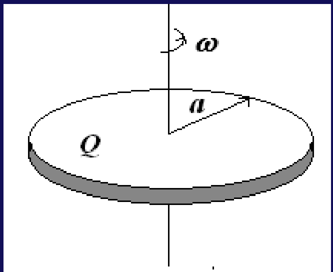
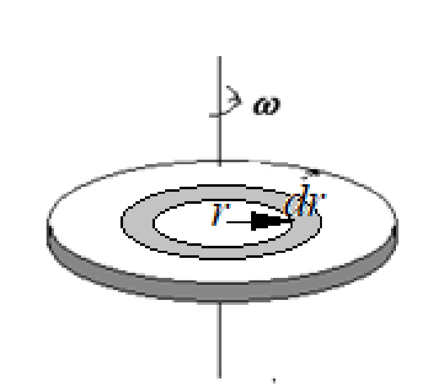


Figure 1



$$\sigma = \frac{Q}{\pi a^2}$$

$$di = \frac{\sigma dA}{T} = \frac{\frac{Q}{\pi a^2} 2\pi r dr}{2\pi / \omega} = \frac{\omega Q}{\pi a^2} r dr$$

$$d\mu = \pi r^2 di = \frac{\omega Q}{a^2} r^3 dr$$

$$\mu = \int_0^a \frac{\omega Q}{a^2} r^3 dr = \frac{1}{4} Q \omega a^2$$

3. As shown in Fig. 2, the spherical region $a < r < b$ carries a charge per unit volume of $\rho = A/r$, where A is a constant. At the center ($r = 0$) of the enclosed cavity is a point charge q. When $A =$ ________, so that the electric field in the region $a < r < b$ has constant magnitude?

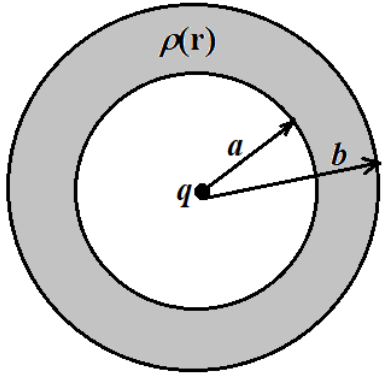


Figure 2

$$\oiint \vec{E} \cdot d\vec{A} = \frac{\sum q}{\varepsilon_0}$$

$$E \cdot 4\pi r^2 = \frac{1}{\varepsilon_0}\left[q + \int_a^r \frac{A}{r} 4\pi r^2 dr\right]$$

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q + 4\pi A \frac{1}{2}(r^2 - a^2)}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[2\pi A + \frac{q - 2\pi A a^2}{r^2}\right]$$

$$q - 2\pi A a^2 = 0$$

$$A = \frac{q}{2\pi a^2}$$

4. As shown in Fig. 3, on a thin rod (细棒) of length L lying along the x axis with one end at the origin (x = 0), there is distributed a charge per unit length given by $\lambda = Ax$, where A is a constant and x is the distance from the origin. Taking the electrostatic potential at infinity (无穷远) to be zero, find the potential V = _______ at the point P on the y axis. Determine the vertical component, E_y = _______ of the electric field at P.

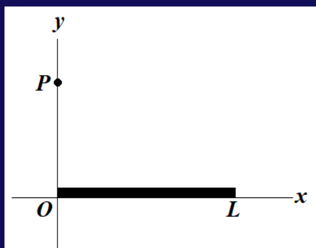
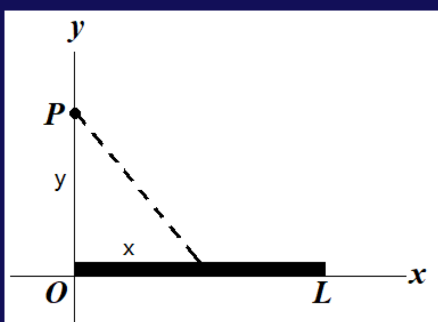


Figure 3



$$dV(y) = \frac{\lambda dx}{4\pi\varepsilon_0\sqrt{x^2+y^2}} = \frac{Axdx}{4\pi\varepsilon_0\sqrt{x^2+y^2}}$$

$$V(y) = \int_0^L \frac{Axdx}{4\pi\varepsilon_0\sqrt{x^2+y^2}} = \frac{A}{4\pi\varepsilon_0}(\sqrt{L^2+y^2} - y)$$

$$E(y) = -\frac{\partial V}{\partial y} = \frac{A}{4\pi\varepsilon_0}(1 - \frac{1}{\sqrt{L^2+y^2}})$$

5. A long wire is bent into the shape shown in Fig. 4, without cross contact (无交叉接触) at P. The radius of the circular section is *R*. Determine the magnitude ______ and direction ______ of *B* at the center C of the circular portion when the current *i* is as indicated.

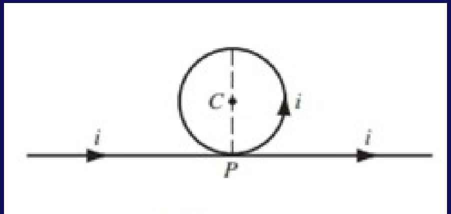


Figure 4

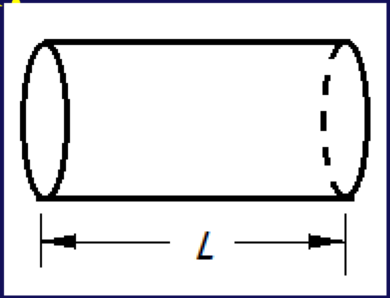
$$B=\frac{\mu_0 i}{2R}+\frac{\mu_0 i}{2\pi R}=\frac{\mu_0 i}{2R}(1+\frac{1}{\pi})$$

6. A length *L* of wire carries a current *i*. If the wire is formed into a circular coil (one turn only), the maximum torque (最大力矩) in a given magnetic field, *B*, is of _________.

$$L=2\pi R,\ R=\frac{L}{2\pi},\ \mu=i\pi R^2=\frac{L^2}{4\pi}i$$

$$\tau_{\max}=\mu B=\frac{L^2 iB}{4\pi}$$

7. A steady (稳定) beam of alpha (α) particles (q=2e) traveling with kinetic energy 22.4 MeV carries a current of 250 nA. At any instant (任何时刻), how many alpha particles are there in a given 18.0-cm length of the beam?



$$j = nqu = 2neu$$

$$I = jA = 2neuA = \frac{nAL}{L}2eu = \frac{N}{L}2eu, \; N = \frac{IL}{2eu}$$

$$m_{He} = \frac{4\times10^{-3}}{6.02\times10^{23}} = 6.64\times10^{-27}\,\text{kg}$$

$$u = \sqrt{\frac{2E_k}{m_{He}}} = \sqrt{\frac{2\times22.4\times10^{6}\times1.6\times10^{-19}}{6.64\times10^{-27}}} = 3.29\times10^{7}\,\text{m/s}$$

$$N = \frac{IL}{2eu} = \frac{250\times10^{-9}\times0.18}{2\times1.6\times10^{-19}\times3.29\times10^{7}} = 4282$$

8. As shown in Fig. 5, a resistance coil (绕线电阻), wired to an external battery (与外电源相连接), is placed inside an adiabatic cylinder (绝热的圆筒) fitted with a frictionless piston (无摩擦的活塞) and containing an ideal gas. A current $i = 240$ mA flows through the coil, which has a resistance $R = 550\ \Omega$. The piston, mass $m = 11.8$ kg, moves upward at speed $v =$ ________, so that the temperature of the gas remains unchanged.

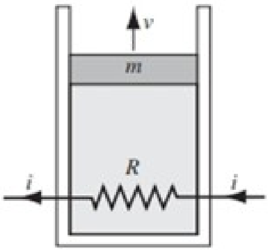


Figure 5

$$P = i^2 R = 0.24^2 \times 550 = 31.68$$

$$P = mgv,$$

$$v = \frac{P}{mg} = \frac{31.68}{11.8 \times 10} = 0.268 \text{ m/s}$$

9. A capacitor has square plates, each of side a, making an angle θ with each other as shown in Fig. 6. The capacitance is given by ________

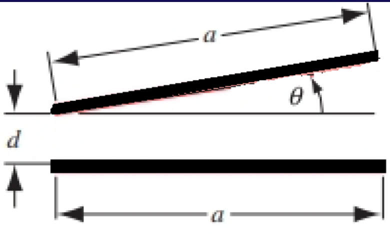
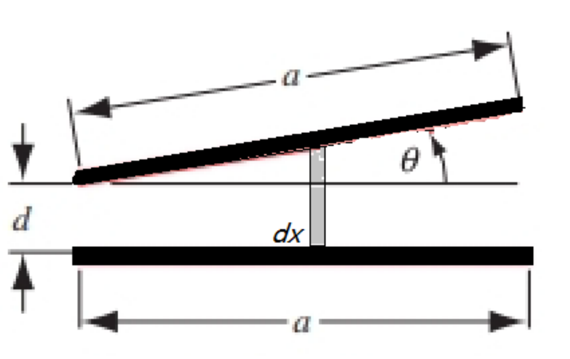


Figure 6



$$dC = \frac{\varepsilon_0 a dx}{d + x tg\theta}$$

$$C = \int_0^a \frac{\varepsilon_0 a dx}{d + x tg\theta} = \frac{\varepsilon_0 a}{tg\theta} \ln\left(\frac{d + a tg\theta}{d}\right)$$

$$\text{for small } \theta,\ C = \frac{\varepsilon_0 a^2}{d}\left(1 - \frac{a\theta}{2d}\right)$$

10. **A wire carrying current *i* has the configuration shown in Fig. 7. Two semi-infinite straight sections, each tangent to the same circle, are connected by a circular arc, of angle θ, along the circumference of the circle, with all sections lying in the same plane. What must θ be in order to for B to be zero at the center of the circle? _________.**

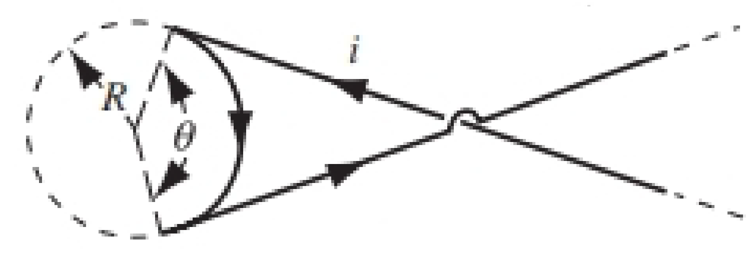


Figure 7

$$B = \frac{\mu_0 i}{2\pi R} - \frac{\mu_0 i}{2R} \cdot \frac{\theta}{2\pi} = \frac{\mu_0 i}{2\pi R}(1 - \frac{\theta}{2})$$

II. Problems (Present the necessary equations in solution) (50%)

1. (10%) There are two coaxial cylinders (圆筒) with radius of r_1 and r_2, respectively, and the same length of l. Two kind conductors with conductivity (电导率) σ_1 and σ_2, respectively, are filled between them as shown in Fig. 8. Find the resistance (电阻) between two coaxial cylinders.

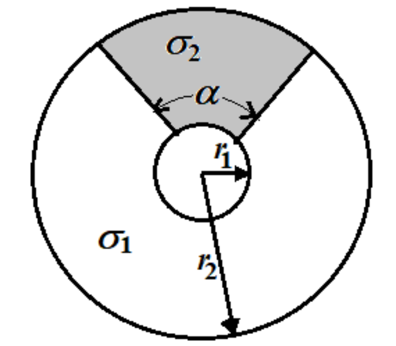
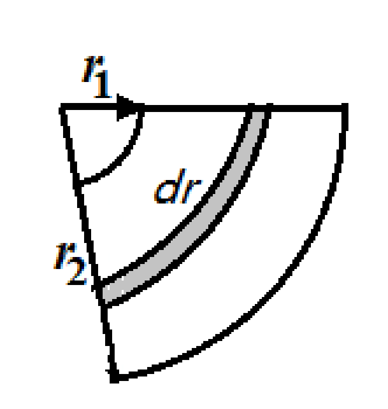


Figure 8



$$dR = \rho\frac{dr}{A} = \frac{1}{\sigma_1}\square\frac{dr}{l\cdot r\alpha}$$

$$R_1 = \frac{q}{\sigma_1 l\alpha}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{q}{\sigma_1 l\alpha}\ln\frac{r_2}{r_1}$$

$$R_2 = \frac{q}{\sigma_1 l(2\pi-\alpha)}\ln\frac{r_2}{r_1}$$

$$\frac{1}{R} = \frac{1}{R_1}+\frac{1}{R_2} = \frac{l}{\ln\frac{r_2}{r_1}}[\sigma_1\alpha+\sigma_2(2\pi-\alpha)]$$

$$R = \frac{\ln\frac{r_2}{r_1}}{l}\frac{1}{\sigma_1\alpha+\sigma_2(2\pi-\alpha)}$$

2. (10%) As shown in Fig. 9, the surface charge distribution (面电荷密度分布) on a dielectric sphere (电介质球) with radius R is

$$\sigma_f = \sigma_0(3\cos\theta - 1)$$

What is the electric field E at the center point O of the sphere?

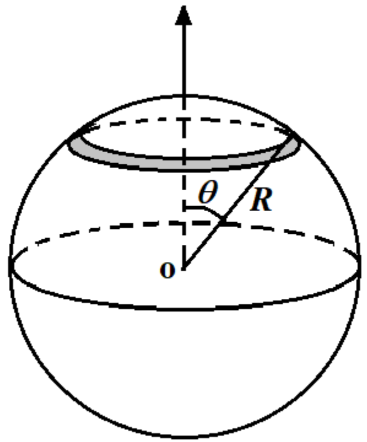


Figure 9

$$dE_z = \frac{R\cos\theta dq}{4\pi\varepsilon_0[R^2\cos^2\theta + R^2\sin^2\theta]^{3/2}} = \frac{\cos\theta dq}{4\pi\varepsilon_0 R^2}$$

$$dq = \sigma dA = \sigma_0(3\cos\theta - 1)\cdot 2\pi R\sin\theta \cdot R d\theta$$

$$= 2\pi R^2\sigma_0(3\cos\theta - 1)\sin\theta d\theta$$

$$dE_z = \frac{\cos\theta dq}{4\pi\varepsilon_0 R^2} = \frac{\cos\theta}{4\pi\varepsilon_0 R^2} 2\pi R^2\sigma_0(3\cos\theta - 1)\sin\theta d\theta$$

$$= \frac{\sigma_0}{2\varepsilon_0}(3\cos^2\theta - \cos\theta)\sin\theta d\theta$$

$$E_z = -\int_0^{\pi}\frac{\sigma_0}{2\varepsilon_0}(3\cos^2\theta - \cos\theta)d\cos\theta$$

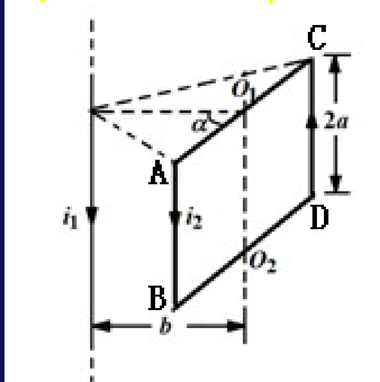
$$= -\frac{\sigma_0}{\varepsilon_0}$$

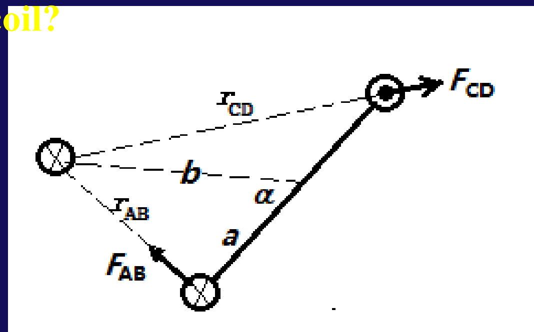
2. (15%) As shown in Fig. 10, there is a square coil (正方形线圈) with a side length (边长) of $2a$, carrying a current of i_2 near a long wire carrying a current of i_1. The distance between the symmetric axis O_1O_2 of the coil and the long wire is of b. The coil can rotate around its axis O_1O_2 which is parallel to the long wire.

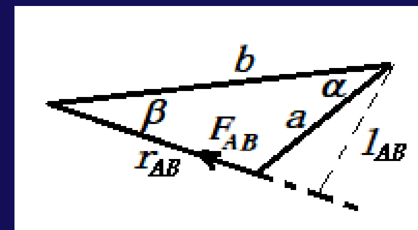
(a) Find the force F and torque (力矩) τ on the coil as it rotates at the α angle.

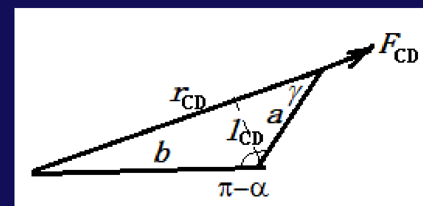
(b) What value is the α angle as the coil is located at equilibrium position (平衡位置)?

(c) When the coil is rotated from the equilibrium position to $\alpha = \pi/2$, how much work (功) does the force by the current i_1 on the coil?









$$(a):\ B_{AB} = \frac{\mu_0 i_1}{2\pi r_{AB}},\ r_{AB} = \sqrt{a^2+b^2-2ab\cos\alpha}$$

$$F_{AB} = 2ai_2\frac{\mu_0 i_1}{2\pi\sqrt{a^2+b^2-2ab\cos\alpha}} = \frac{\mu_0 i_1 i_2 a}{\pi\sqrt{a^2+b^2-2ab\cos\alpha}}$$

$$B_{CD} = \frac{\mu_0 i_1}{2\pi r_{CD}},\ r_{CD} = \sqrt{a^2+b^2-2ab\cos(\pi-\alpha)}$$

$$F_{CD} = \frac{\mu_0 i_1 i_2 a}{\pi\sqrt{a^2+b^2+2ab\cos\alpha}}$$

$$\frac{a}{\sin\beta} = \frac{\sqrt{a^2+b^2-2ab\cos\alpha}}{\sin\alpha},\ l_{AB} = b\sin\beta = \frac{ab\sin\alpha}{\sqrt{a^2+b^2-2ab\cos\alpha}}$$

$$\frac{b}{\sin\beta} = \frac{\sqrt{a^2+b^2+2ab\cos\alpha}}{\sin(\pi-\alpha)},\ l_{CD} = a\sin\gamma = \frac{ab\sin\alpha}{\sqrt{a^2+b^2+2ab\cos\alpha}}$$

$$\tau = F_{AB}\cdot l_{AB} + F_{CD}\cdot l_{CD} = \frac{\mu_0 i_1 i_2 a^2 b\sin\alpha}{\pi}\left[\frac{1}{a^2+b^2-2ab\cos\alpha} + \frac{1}{a^2+b^2+2ab\cos\alpha}\right]$$

$$= \frac{2\mu_0 i_1 i_2 a^2 b\sin\alpha}{\pi}\frac{a^2+b^2}{(a^2+b^2)^2-4a^2b^2\cos^2\alpha}$$

$$(b):\ \tau = 0,\quad \sin\alpha = 0,\ \alpha = 0 \text{ or } \pi$$

2. (15%) As shown in Fig. 10, there is a square coil (正方形线圈) with a side length (边长) of $2a$, carrying a current of i_2 near a long wire carrying a current of i_1. The distance between the symmetric axis O_1O_2 of the coil and the long wire is of b. The coil can rotate around its axis O_1O_2 which is parallel to the long wire.

(a) Find the force F and torque (力矩) τ on the coil as it rotates at the α angle.

(b) What value is the α angle as the coil is located at equilibrium position (平衡位置)?

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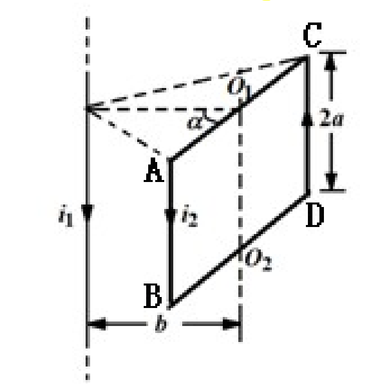


(c). $$\tau = \frac{2\mu_0 i_1 i_2 a^2 b \sin\alpha}{\pi}\frac{a^2+b^2}{(a^2+b^2)^2-4a^2b^2\cos^2\alpha}$$

$$W=\int_0^{\pi/2}\tau d\alpha=\int_0^{\pi/2}\frac{2\mu_0 i_1 i_2 a^2 b \sin\alpha}{\pi}\frac{a^2+b^2}{(a^2+b^2)^2-4a^2b^2\cos^2\alpha}d\alpha$$

$$=-\frac{2\mu_0 i_1 i_2 a^2 b}{\pi}\int_0^{\pi/2}\frac{a^2+b^2}{(a^2+b^2)^2-4a^2b^2\cos^2\alpha}d\cos\alpha$$

$$=-\frac{2\mu_0 i_1 i_2 a^2 b}{\pi}\frac{1}{2ab}\int_0^{\pi/2}\frac{1}{1-(\frac{2ab}{a^2+b^2}\cos\alpha)^2}d(\frac{2ab}{a^2+b^2}\cos\alpha)$$

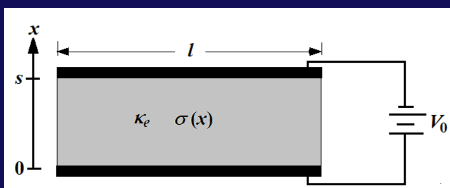
$$=\frac{\mu_0 i_1 i_2 a}{\pi}\ln\frac{b-a}{b+a}$$

3. (15%) As shown in Fig. 9, a pair of parallel plate electrodes (平行板电极) with length l, width d, and distance s, at voltage difference V_0 enclose (填充) an Ohmic material (欧姆材料) whose conductivity (电导率) varies linearly (线性变化) as $\sigma(x) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}$, from σ_1 at lower electrode to σ_2 at the upper electrode. The dielectric constant (介电常数) κ_e of the material is a constant.

(a) Find the electric fields and the resistance (电阻) between electrodes

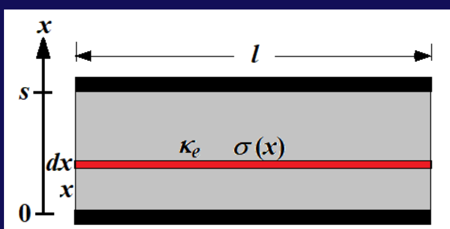
(b) What are the volume and surface charge distributions (体和面电荷分布)?

(c) What is the total volume charge in the system and how is it related to the surface charge on the electrodes?



Width d

Figure 9



Width d

(a) $$dR = \rho\frac{dx}{A} = \frac{1}{\sigma}\cdot\frac{dx}{ld} = \frac{dx}{[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]ld}$$

$$= \frac{s\,dx}{[\sigma_1 s + (\sigma_2 - \sigma_1)x]ld}$$

$$R = \int_0^s \frac{s\,dx}{[\sigma_1 s + (\sigma_2 - \sigma_1)x]ld}$$

$$= \frac{s}{ld(\sigma_2 - \sigma_1)}\ln[\sigma_1 s + (\sigma_2 - \sigma_1)x]\Big|_0^s$$

$$= \frac{s}{ld(\sigma_2 - \sigma_1)}\ln\frac{\sigma_2}{\sigma_1}$$

$$I = \frac{V_0}{R} = j\cdot ld = \frac{V_0 ld(\sigma_2 - \sigma_1)}{s\ln\frac{\sigma_2}{\sigma_1}}$$

$$\therefore j = \frac{V_0(\sigma_2 - \sigma_1)}{s\ln\frac{\sigma_2}{\sigma_1}}$$

$$\vec{j} = \sigma\vec{E}$$

$$\therefore E = \frac{j}{\sigma} = \frac{V_0(\sigma_2 - \sigma_1)}{[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]s\ln\frac{\sigma_2}{\sigma_1}}$$

$$= \frac{V_0(\sigma_2 - \sigma_1)}{[\sigma_1 s + (\sigma_2 - \sigma_1)x]\ln\frac{\sigma_2}{\sigma_1}}$$

3. (15%) As shown in Fig. 9, a pair of parallel plate electrodes (平行板电极) with length l, width d, and distance s, at voltage difference V_0 enclose (填充) an Ohmic material (欧姆材料) whose conductivity (电导率) varies linearly (线性变化) a $\sigma(x) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}$, from σ_1 at lower electrode to σ_2 at the upper electrode. The dielectric constant (介电常数) κ_e of the material is a constant.

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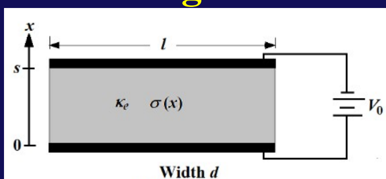
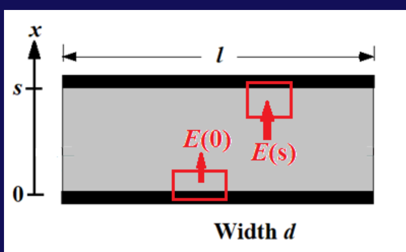


Figure 9



$$R = \frac{s}{ld(\sigma_2 - \sigma_1)} \ln\frac{\sigma_2}{\sigma_1}$$

$$\text{(b)}\quad E = \frac{j}{\sigma} = \frac{V_0(\sigma_2 - \sigma_1)}{[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]s\ln\frac{\sigma_2}{\sigma_1}}$$

$$= \frac{V_0(\sigma_2 - \sigma_1)}{[\sigma_1 s + (\sigma_2 - \sigma_1)x]\ln\frac{\sigma_2}{\sigma_1}}$$

The volume charge density:

$$\rho_e = \varepsilon_0 \nabla \cdot \vec{E} = \varepsilon_0 \frac{\partial E_x}{\partial x}$$

$$= -\frac{\varepsilon_0 V_0(\sigma_2 - \sigma_1)^2}{[\sigma_1 s + (\sigma_2 = \sigma_1)x]^2 \ln\frac{\sigma_2}{\sigma_1}}$$

The total surface charge density at $x = 0$

$$E(0) = \frac{V_0(\sigma_2 - \sigma_1)}{\sigma_1 s \ln\frac{\sigma_2}{\sigma_1}}$$

$$\oiint \vec{E} \cdot dA = \frac{q}{\varepsilon_0}$$

$$E \cdot \Delta A = \frac{1}{\varepsilon_0}\sigma_e(0)\Delta A$$

$$\sigma_e(0) = \varepsilon_0 E(0) = \frac{\varepsilon_0 V_0(\sigma_2 - \sigma_1)}{\sigma_1 s \ln\frac{\sigma_2}{\sigma_1}}$$

The total surface charge density at $x = s$

$$E(s) = \frac{V_0(\sigma_2 - \sigma_1)}{\sigma_2 s \ln\frac{\sigma_2}{\sigma_1}}$$

$$\oiint \vec{E} \cdot dA = \frac{q}{\varepsilon_0}$$

$$-E \cdot \Delta A = \frac{1}{\varepsilon_0}\sigma_e(s)\Delta A$$

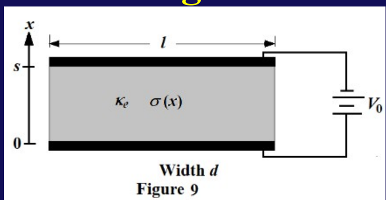
$$\sigma_e(s) = -\varepsilon_0 E(s) = -\frac{\varepsilon_0 V_0(\sigma_2 - \sigma_1)}{\sigma_2 s \ln\frac{\sigma_2}{\sigma_1}}$$

3. (15%) As shown in Fig. 9, a pair of parallel plate electrodes (平行板电极) with length l, width d, and distance s, at voltage difference V_0 enclose (填充) an Ohmic material (欧姆材料) whose conductivity (电导率) varies linearly (线性变化) as $\sigma(x)=\sigma_1+(\sigma_2-\sigma_1)\frac{x}{s}$, from σ_1 at lower electrode to σ_2 at the upper electrode. The dielectric constant (介电常数) κ_e of the material is a constant.

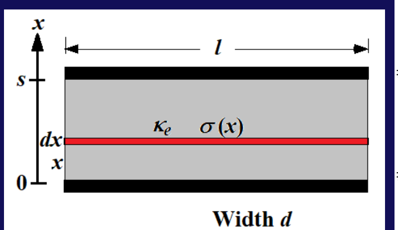
(a) Find the electric fields and the resistance (电阻) between electrodes

(b) What are the volume and surface charge distributions (体和面电荷分布)?

(c) What is the total volume charge in the system and how is it related to the surface charge on the electrodes?



Width d
Figure 9



Width d

$$R=\frac{s}{ld(\sigma_2-\sigma_1)}\ln\frac{\sigma_2}{\sigma_1}$$

(c) $$\rho_e=-\frac{\varepsilon_0V_0(\sigma_2-\sigma_1)^2}{[\sigma_1s+(\sigma_2-\sigma_1)x]^2\ln\frac{\sigma_2}{\sigma_1}}$$

$$Q=-ld\int_0^s\rho_e dx$$

$$=-ld\int_0^s\frac{\varepsilon_0V_0(\sigma_2-\sigma_1)^2}{[\sigma_1s+(\sigma_2-\sigma_1)x]^2\ln\frac{\sigma_2}{\sigma_1}}dx$$

$$=-\frac{ld\varepsilon_0V_0(\sigma_2-\sigma_1)}{\ln\frac{\sigma_2}{\sigma_1}}\int_0^s\frac{1}{[\sigma_1s+(\sigma_2-\sigma_1)x]^2}d((\sigma_2-\sigma_1)x)$$

$$=\frac{ld\varepsilon_0V_0(\sigma_2-\sigma_1)}{3\ln\frac{\sigma_2}{\sigma_1}}\frac{1}{[\sigma_1s+(\sigma_2-\sigma_1)x]^3}\Bigg|_0^s$$

$$=\frac{ld\varepsilon_0V_0(\sigma_2-\sigma_1)}{3s^3\ln\frac{\sigma_2}{\sigma_1}}\left[\frac{1}{\sigma_2^3}-\frac{1}{\sigma_1^3}\right]$$

$$E=\frac{V_0(\sigma_2-\sigma_1)}{[\sigma_1s+(\sigma_2-\sigma_1)x]\ln\frac{\sigma_2}{\sigma_1}}$$

$$D=\kappa_e\varepsilon_0E=\frac{\kappa_e\varepsilon_0V_0(\sigma_2-\sigma_1)}{[\sigma_1s+(\sigma_2-\sigma_1)x]\ln\frac{\sigma_2}{\sigma_1}}$$

$$D(0)=\frac{\kappa_e\varepsilon_0V_0(\sigma_2-\sigma_1)}{\sigma_1s\ln\frac{\sigma_2}{\sigma_1}}$$

$$D(0)\cdot\Delta A=\sigma_{e0}(0)\cdot\Delta A$$

$$\sigma_{e0}(0)=D(0)=\frac{\kappa_e\varepsilon_0V_0(\sigma_2-\sigma_1)}{\sigma_1s\ln\frac{\sigma_2}{\sigma_1}}$$

$$D(s)=\frac{\kappa_e\varepsilon_0V_0(\sigma_2-\sigma_1)}{\sigma_2s\ln\frac{\sigma_2}{\sigma_1}}$$

$$-D(s)\cdot\Delta A=\sigma_{e0}(s)\cdot\Delta A$$

$$\sigma_{e0}(s)=-D(s)=-\frac{\kappa_e\varepsilon_0V_0(\sigma_2-\sigma_1)}{\sigma_2s\ln\frac{\sigma_2}{\sigma_1}}$$

$$P_x=\frac{(\kappa_e-1)\varepsilon_0V_0(\sigma_2-\sigma_1)}{[\sigma_1s+(\sigma_2-\sigma_1)x]\ln\frac{\sigma_2}{\sigma_1}}$$

$$P_x(0)=\frac{(\kappa_e-1)\varepsilon_0V_0(\sigma_2-\sigma_1)}{\sigma_1s\ln\frac{\sigma_2}{\sigma_1}}$$

$$\sigma_e'(0)=\vec{P}\cdot\hat{n}=-P_x(0)$$

$$=-\frac{(\kappa_e-1)\varepsilon_0V_0(\sigma_2-\sigma_1)}{\sigma_1s\ln\frac{\sigma_2}{\sigma_1}}$$

$$P_x(s)=\frac{(\kappa_e-1)\varepsilon_0V_0(\sigma_2-\sigma_1)}{\sigma_2s\ln\frac{\sigma_2}{\sigma_1}}$$

$$\sigma_e'(s)=\vec{P}\cdot\hat{n}=P_x(s)$$

$$=\frac{(\kappa_e-1)\varepsilon_0V_0(\sigma_2-\sigma_1)}{\sigma_2s\ln\frac{\sigma_2}{\sigma_1}}$$