2021年期中考试卷答案

I. Fill in the space underlined (50% in total)

1. In 1909, Robert A. Millikan measured first the charge of an electron. As shown in Fig. 1, a spherical oil droplet (球形油粒) of radius R and effective mass density (有效密度) ρ_m carries a total charge q in a gravity field g. The electric field $E =$ ___________ will suspend (悬浮) the charged droplet.

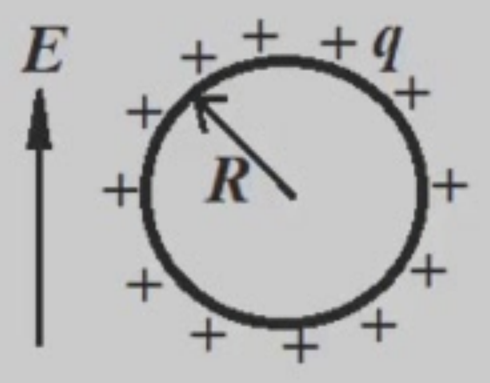


Figure 1

$$\left(\frac{4}{3}\pi R^3 \rho_m\right) g = qE$$

$$E = \frac{4\pi R^3 \rho_m g}{3q}$$

2. As shown in Fig.2, charges Q, Q, and q lie on the corners of an equilateral triangle (等边三角形) with sides of length a. What must q be ______ for electrical field to be zero at P point half-way up the altitude (一半高度).

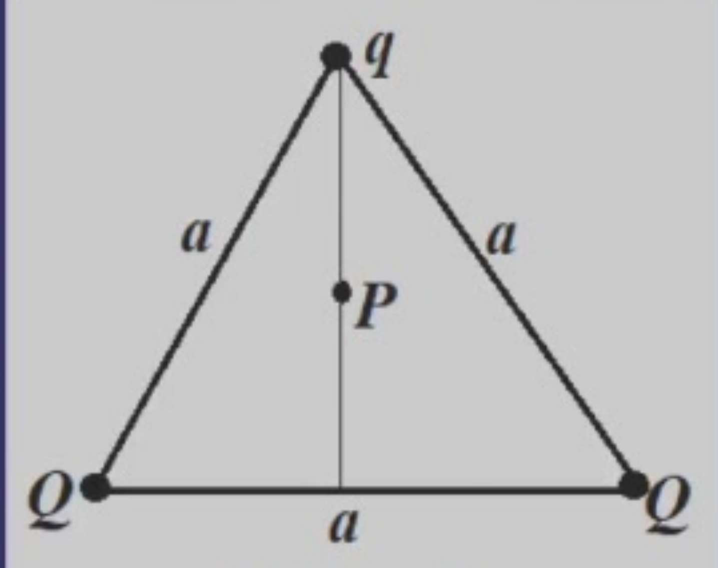
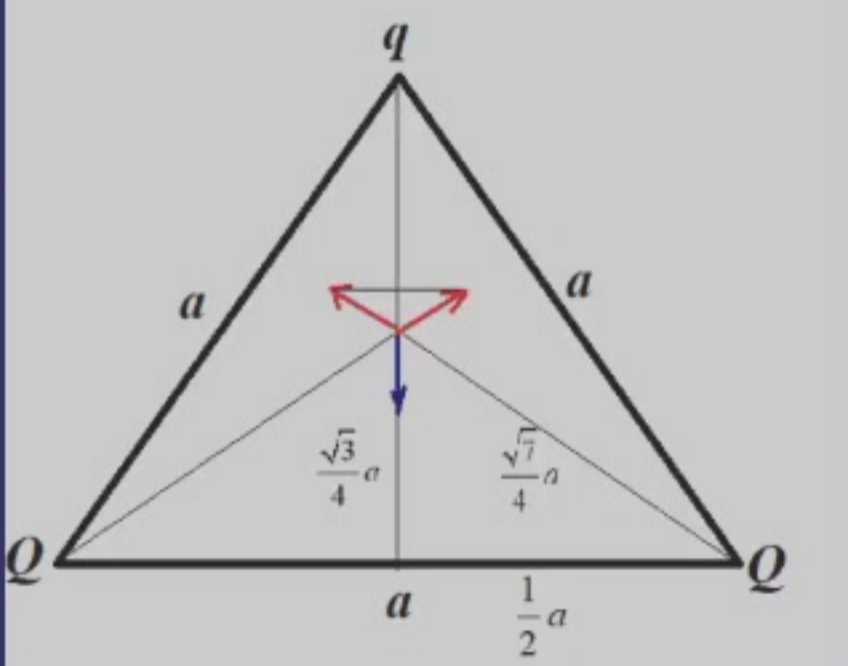


Figure 2



$$2\cdot\frac{1}{4\pi\varepsilon_0}\frac{Q}{(\frac{\sqrt{7}}{4}a)^2}\cdot\frac{\frac{\sqrt{3}}{4}a}{\frac{\sqrt{7}}{4}a}=\frac{1}{4\pi\varepsilon_0}\frac{q}{(\frac{\sqrt{3}}{4}a)^2}$$

$$2\cdot\frac{16}{7}Q\cdot\sqrt{\frac{3}{7}}=\frac{16}{3}q$$

$$q=\frac{6}{7}\sqrt{\frac{3}{7}}Q$$

3. As shown in Fig. 3, there is a rectangle loop (矩形电流环) with the size of $a \times b$ carrying a current I. The magnetic induction strength B = __________ at the central point P.

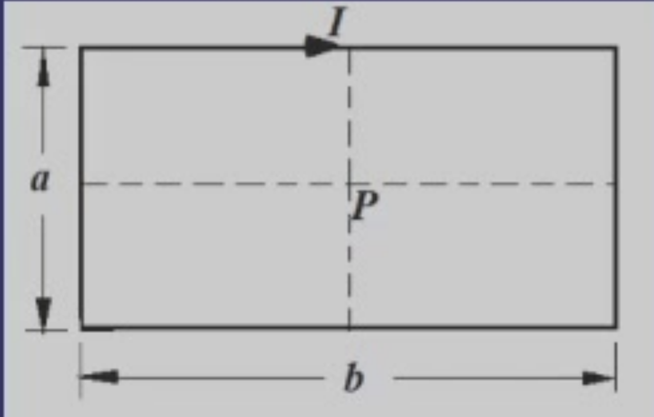
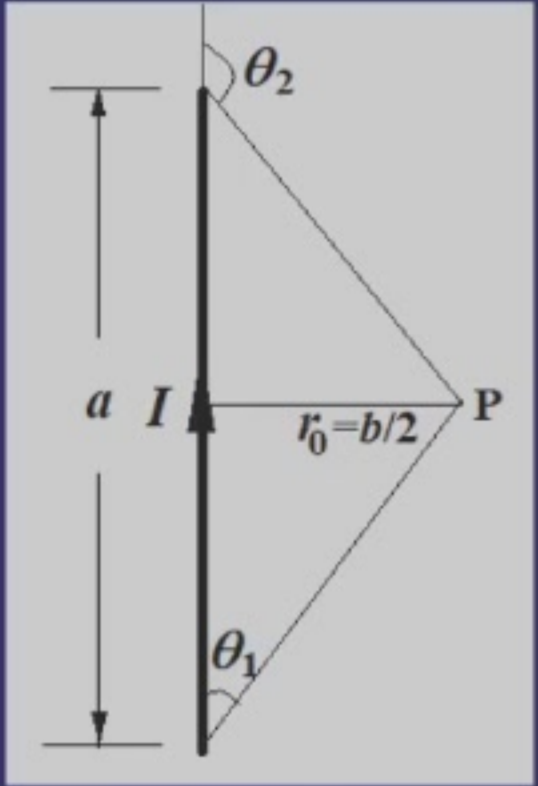


Figure 3



$$B_a = \frac{\mu_0 I}{4\pi r_0}(\cos\theta_1 - \cos\theta_2)$$
$$= \frac{\mu_0 I}{4\pi \frac{b}{2}}[\cos\theta_1 - \cos(\pi - \theta_1)]$$
$$= \frac{\mu_0 I}{2\pi b}[2\frac{a/2}{\sqrt{(\frac{a}{2})^2 + (\frac{b}{2})^2}}]$$
$$= \frac{\mu_0 I}{\pi}\frac{a}{b\sqrt{a^2+b^2}}$$

$$B = \frac{\mu_0 I}{\pi}[\frac{2a}{b\sqrt{a^2+b^2}} + \frac{2b}{a\sqrt{a^2+b^2}}]$$
$$= \frac{2\mu_0 I}{\pi ab}\frac{a^2+b^2}{\sqrt{a^2+b^2}}$$
$$= \frac{2\mu_0 I\sqrt{a^2+b^2}}{\pi ab}$$

4. As shown in Fig. 4, a long cable (电缆) is composed of an inner conducting wire of radius a and an outer conducting barrel (筒) of radius b. The inner conducting wire carries a current with density (电流密度) J_0 flowing out paper, the outer conducting barrel carries a current with density J_0 flowing into paper. The magnetic field $B =$ ________ for $r \le a$; $B =$ ________ for $a < r \le b$; $B =$ ________ for $r > b$, respectively.

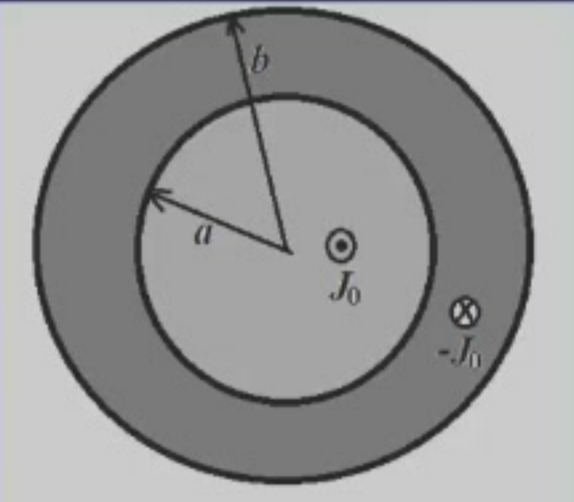
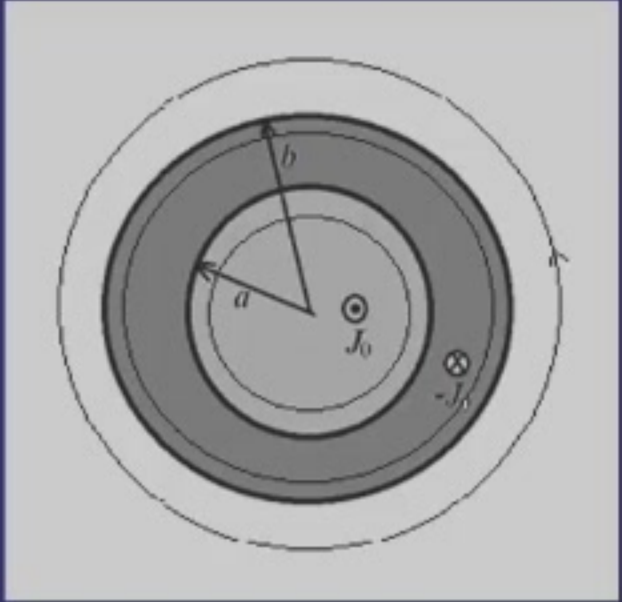


Figure 4



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 i$$

$$r \le a, \quad B \cdot 2\pi r = \mu_0 J_0 \pi r^2, \; B = \frac{1}{2}\mu_0 J_0 r$$

$$a < r \le b, \; B \cdot 2\pi r = \mu_0 [J_0 \pi a^2 - J_0(\pi r^2 - \pi a^2)],$$

$$B = \frac{1}{2}\mu_0 J_0 [\frac{2a^2}{r} - r]$$

$$r > b, \quad B \cdot 2\pi r = \mu_0 [J_0 \pi a^2 - J_0(\pi b^2 - \pi a^2)],$$

$$B = \frac{1}{2}\mu_0 J_0 \frac{2a^2 - b^2}{r}$$

5. **As shown in Fig. 5, a rectangular (矩形) loop of wire with length *a*, width *b*, and resistance *R* is placed near an infinitely long wire carrying current *i*. The distance from the long wire to the loop is *D*. The current in the loop is ____________ as it moves away from the long wire with speed *v*.**

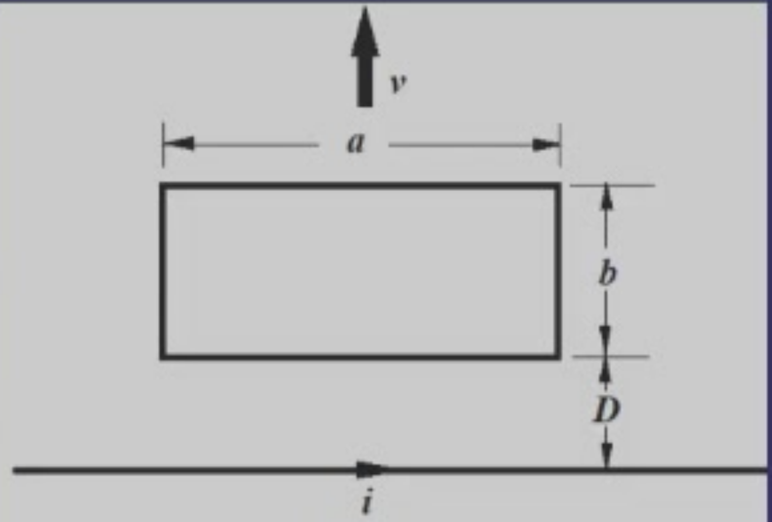


Figure 5

$$B = \frac{\mu_0 i}{2\pi r}$$

$$\Phi = \int_D^{D+b} \frac{\mu_0 i}{2\pi r} a dr = \frac{\mu_0 a i}{2\pi}[\ln(D+b) - \ln D]$$

$$\varepsilon = -\frac{d\Phi}{dt} = -\frac{\mu_0 a i}{2\pi}[\frac{1}{D+b} - \frac{1}{D}]\frac{dD}{dt}$$

$$= \frac{\mu_0 a i}{2\pi}\frac{b}{D(D+b)} v = \frac{\mu_0 i}{2\pi}\frac{abv}{D(D+b)}$$

$$I = \frac{\varepsilon}{R} = \frac{\mu_0 i}{2\pi R}\frac{abv}{D(D+b)}$$

6. A capacitor has square (正方形) plates, each of side a, making an angle θ with each other as shown in Fig. 6. For small θ, the capacitance is given by

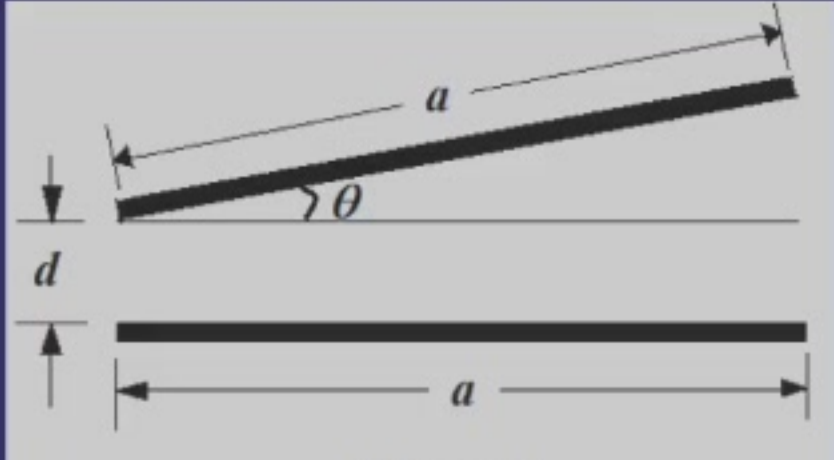
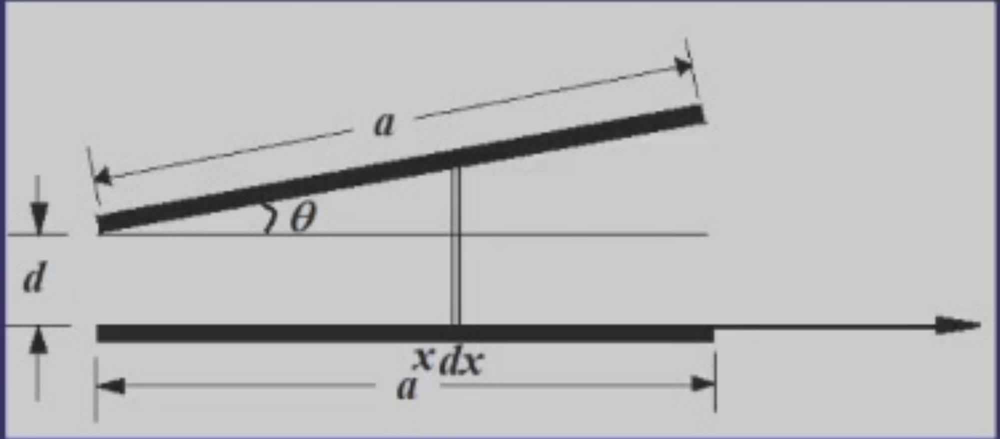


Figure 6

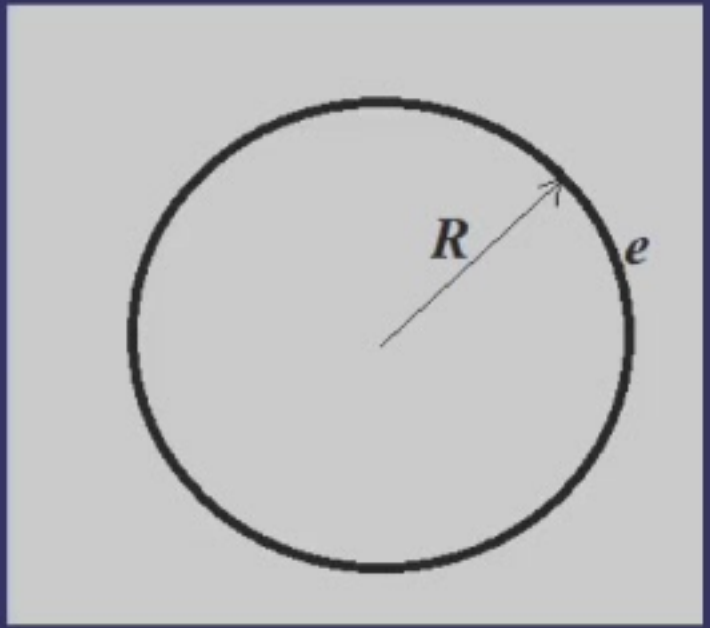


$$dC = \frac{\varepsilon_0 a dx}{d + x \cdot \theta}$$

$$C = \int_0^a \frac{\varepsilon_0 a dx}{d + x \cdot \theta} = \frac{\varepsilon_0 a}{\theta} \ln(d + x\theta)\Big|_0^a$$

$$= \frac{\varepsilon_0 a}{\theta} \ln \frac{d + \theta a}{d}$$

7. Assume that the electron is a sphere of radius *R* over whose surface the electron charge is uniformly distributed. The energy associated with the external electric field in vacuum of the electron as a function of *R* is ______________.



$$E = -\frac{e}{4\pi\varepsilon_0 r^2}$$

$$U = \int_R^\infty \frac{1}{2}\varepsilon_0 E^2 \cdot 4\pi r^2 dr$$

$$= \int_R^\infty \frac{1}{2}\varepsilon_0 \left(\frac{e}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 dr$$

$$= \frac{e^2}{8\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2} = \frac{e^2}{8\pi\varepsilon_0 R}$$

8. If we know the potential distribution (电势分布) in the spherical coordinate (球坐标系) given as

$$V(r,\theta,\varphi) = Ar^2 \sin\theta\cos\varphi$$

, then the electric field $\vec{E} = \_\_\_\_\_\_\hat{r} + \_\_\_\_\_\_\hat{\theta} + \_\_\_\_\_\_\hat{\varphi}$

, and the volume charge density (体电荷密度) distribution:

$$\rho(r,\theta,\varphi) = \_\_\_\_\_\_\_\_$$

$$\vec{E} = -\nabla V = -[\frac{\partial V}{\partial r}\hat{r} + \frac{\partial V}{r\partial\theta}\hat{\theta} + \frac{\partial V}{r\sin\theta\partial\varphi}\hat{\varphi}]$$

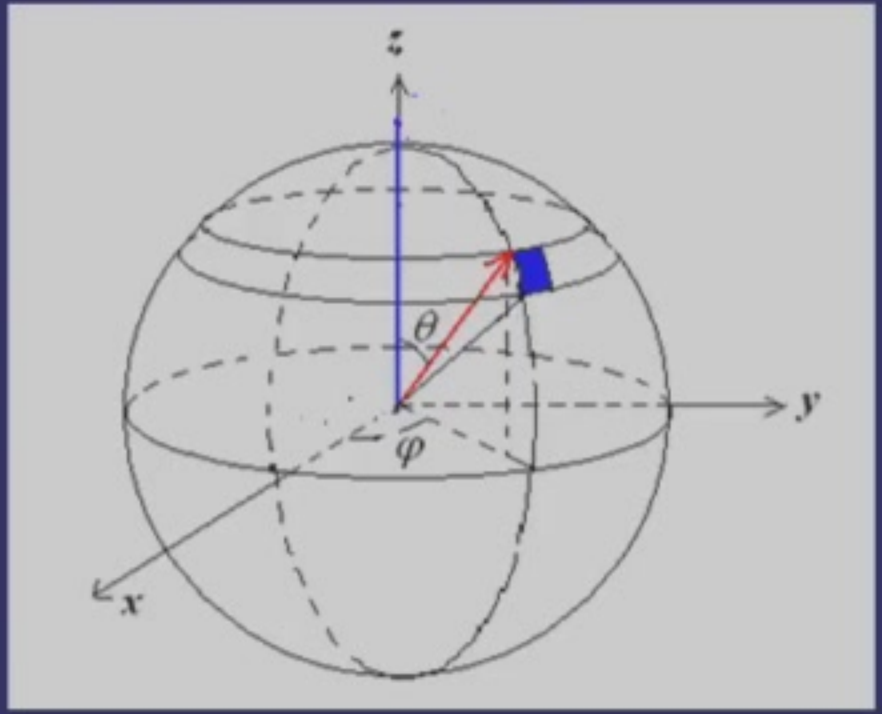
$$= -Ar[2\sin\theta\cos\varphi\hat{r} + \cos\theta\cos\varphi\hat{\theta} - \sin\varphi\hat{\varphi}]$$

$$\rho_e = \varepsilon_0 \nabla\cdot\vec{E}$$

$$= \varepsilon_0[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta E_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}]$$

$$= -A\varepsilon_0[6\sin\theta\cos\varphi + \frac{(\cos^2\theta - \sin^2\theta)\cos\varphi}{\sin\theta} - \frac{\cos\varphi}{\sin\theta}]$$

$$= -4A\varepsilon_0 \sin\theta\cos\varphi$$



II. Problems (Present the necessary equations in solution) (50%)

1. (20%) (a). As shown in Fig. 7(a), an infinitely long hollow semi-cylinder (半圆柱壳) of radius R carries a uniform surface charge density (面电荷密度) σ_0. What is the electric field along the axis of the cylinder?

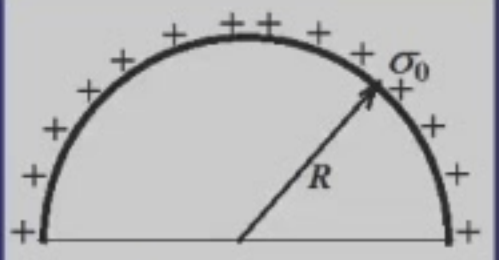
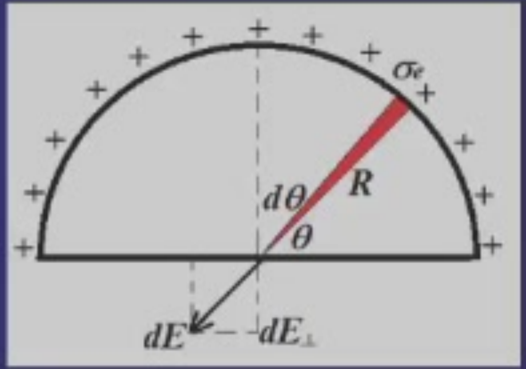
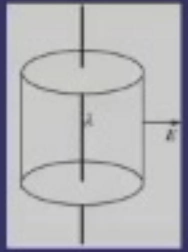


Figure 7(a)





$$\oiint \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}, \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

$$dE = \frac{\sigma_e R d\theta}{2\pi\varepsilon_0 R} = \frac{\sigma_e d\theta}{2\pi\varepsilon_0}$$

$$dE_\perp = dE\sin\theta,\ E_\perp = \int_0^\pi \frac{\sigma_e \sin\theta d\theta}{2\pi\varepsilon_0} = -\frac{\sigma_e}{2\pi\varepsilon_0}\cos\theta\Big|_0^\pi = \frac{\sigma_e}{\pi\varepsilon_0}$$

(b). As shown in Fig. 7(b), an infinitely long semi-cylinder (半圆柱体) of radius R carries a uniform volume charge density (体电荷密度) ρ_0. What is the electric field along the axis of the cylinder?

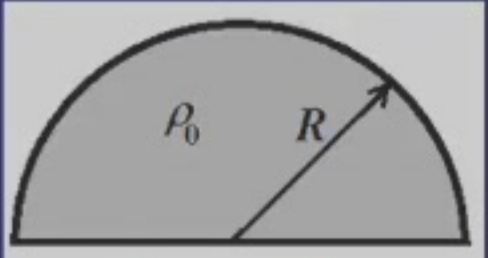
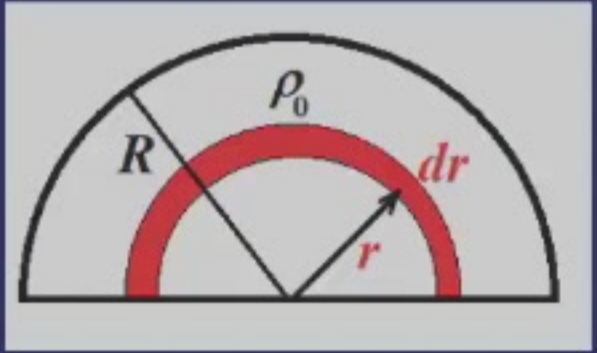


Figure 7(b)



$$dq = \rho_0 \cdot 2\pi r L dr, \quad d\sigma = \frac{dq}{2\pi r L} = \rho_0 dr$$

$$dE_\perp = \frac{\rho_0 dr}{\pi\varepsilon_0}, \quad E_\perp = \int_0^R \frac{\rho_0 dr}{\pi\varepsilon_0} = \frac{\rho_0 R}{\pi\varepsilon_0}$$

2. (15%) A cylinder of radius a and height L as in Fig. 8 has polarization (极化强度):

$$\vec{P} = \frac{P_0 z}{L}\hat{z}$$

(a) What is the induced charge distribution (束缚电荷分布) in the cylinder?

(b) Find the electric field E and the displacement field D (电位移矢量) everywhere along the z axis.

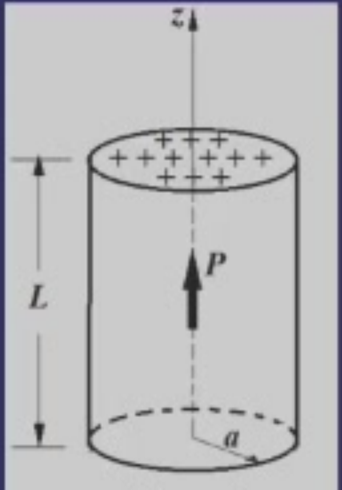


Figure 8

$$\vec{P} = \frac{P_0 z}{L}\hat{z}$$

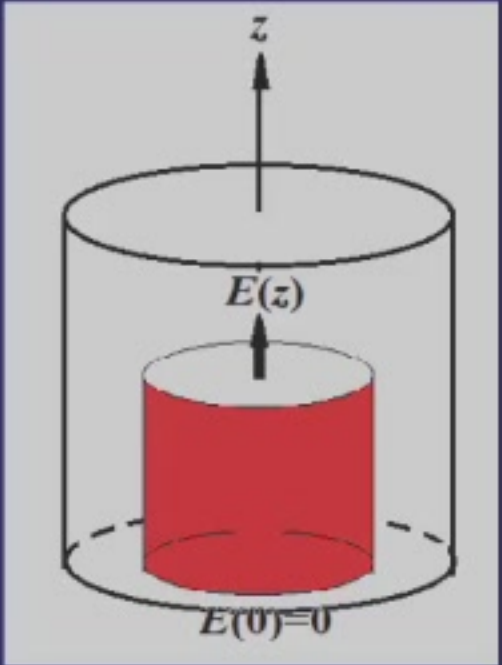
$$\oiint \vec{P}\cdot d\vec{A} = -q'$$

$$\nabla\cdot\vec{P} = -\rho_e'$$

(a) 在柱坐标系中：

$$\nabla\cdot\vec{P} = \frac{1}{r}\frac{\partial}{\partial r}(rP_r) + \frac{1}{r}\frac{\partial}{\partial \varphi}P_\varphi + \frac{\partial P_z}{\partial z}$$

$$\rho_e' = -\frac{\partial P_z}{\partial z} = -\frac{P_0}{L}$$



(b) $$\oiint \vec{E}\cdot d\vec{A} = \frac{q'}{\varepsilon_0}$$

$$r \le a,\quad E\cdot\pi r^2 - 0 = \frac{\rho_e'\cdot\pi r^2 z}{\varepsilon_0}$$

$$E(z) = \frac{\rho_e'}{\varepsilon_0}z = -\frac{P_0 z}{\varepsilon_0 L},\ E(0) = 0,\ E(L) = -\frac{P_0}{\varepsilon_0}$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = 0$$

3. (15%) As shown in Fig. 9, a pair of parallel plate electrodes (平行板电极) with length l, width d, and distance s, at voltage difference V_0 enclose (填充) an Ohmic material (欧姆材料) whose conductivity (电导率) varies linearly (线性变化) as $\sigma(x) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}$**, from σ_1 at lower electrode to σ_2 at the upper electrode. The dielectric constant (介电常数) κ_e of the material is a constant.**

(a) Find the electric fields and the resistance (电阻) between electrodes

(b) What are the volume and surface charge distributions (体和面电荷分布)?

(c) What is the total volume charge in the system and how is it related to the surface charge on the electrodes?

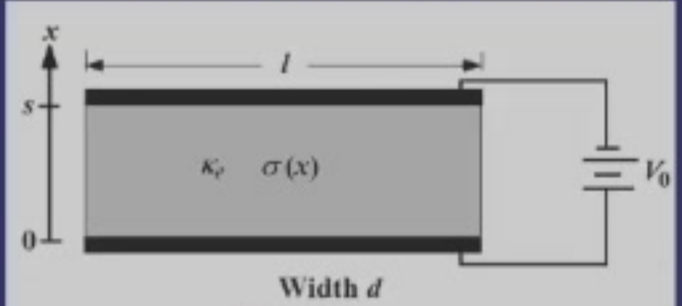
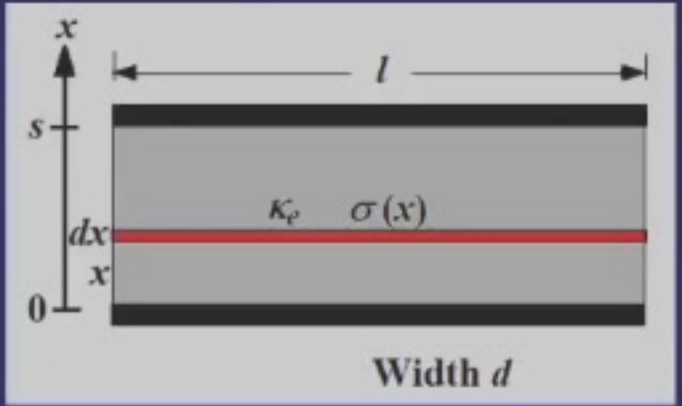


Figure 9



(a) $$dR = \rho\frac{dx}{A} = \frac{1}{\sigma}\cdot\frac{dx}{ld} = \frac{dx}{[\sigma_1 + (\sigma_2 - \sigma_1)\frac{x}{s}]ld}$$

$$= \frac{sdx}{[\sigma_1 s + (\sigma_2 - \sigma_1)x]ld}$$

$$R = \int_0^s \frac{sdx}{[\sigma_1 s + (\sigma_2 - \sigma_1)x]ld}$$

$$= \frac{s}{ld(\sigma_2 - \sigma_1)}\ln[\sigma_1 s + (\sigma_2 - \sigma_1)x]\Big|_0^s$$

$$= \frac{s}{ld(\sigma_2 - \sigma_1)}\ln\frac{\sigma_2}{\sigma_1}$$

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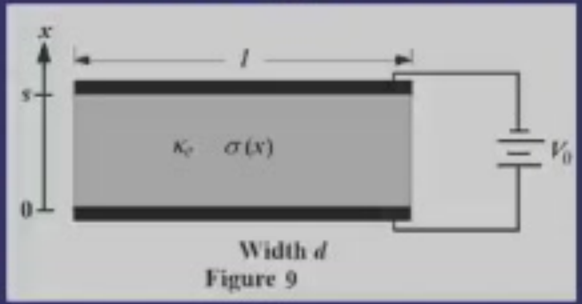

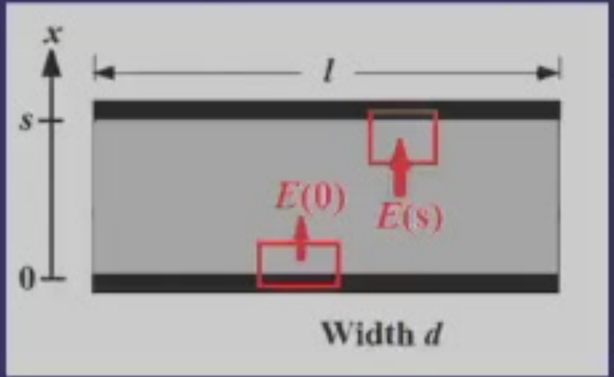

$$R=\frac{s}{ld(\sigma_2-\sigma_1)}\ln\frac{\sigma_2}{\sigma_1}$$

(a) $I=j\cdot ld=\dfrac{V_0}{R}=\dfrac{V_0 ld(\sigma_2-\sigma_1)}{s\ln\dfrac{\sigma_2}{\sigma_1}}$

$$j=\frac{V_0(\sigma_2-\sigma_1)}{s\ln\dfrac{\sigma_2}{\sigma_1}}$$

$$\vec{j}=\sigma\vec{E}$$

$$\therefore E=\frac{j}{\sigma}=\frac{V_0(\sigma_2-\sigma_1)}{[\sigma_1+(\sigma_2-\sigma_1)\frac{x}{s}]s\ln\dfrac{\sigma_2}{\sigma_1}}$$

$$=\frac{V_0(\sigma_2-\sigma_1)}{[\sigma_1 s+(\sigma_2-\sigma_1)x]\ln\dfrac{\sigma_2}{\sigma_1}}$$

(b) The total charge density:

$$\rho_e=\varepsilon_0\nabla\cdot\vec{E}=\varepsilon_0\frac{\partial E_x}{\partial x}$$

$$=-\frac{\varepsilon_0 V_0(\sigma_2-\sigma_1)^2}{[\sigma_1 s+(\sigma_2-\sigma_1)x]^2\ln\dfrac{\sigma_2}{\sigma_1}}$$

The total surface charge density at $x=0$

$$E(0)=\frac{V_0(\sigma_2-\sigma_1)}{\sigma_1 s\ln\dfrac{\sigma_2}{\sigma_1}}$$

$$\oiint \vec{E}\cdot d\vec{A}=\frac{q}{\varepsilon_0}$$

$$E\cdot\Delta A=\frac{1}{\varepsilon_0}\sigma_e(0)\Delta A$$

$$\sigma_e(0)=\varepsilon_0 E(0)=\frac{\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_1 s\ln\dfrac{\sigma_2}{\sigma_1}}$$

The total surface charge density at $x=s$

$$E(s)=\frac{V_0(\sigma_2-\sigma_1)}{\sigma_2 s\ln\dfrac{\sigma_2}{\sigma_1}}$$

$$\oiint \vec{E}\cdot d\vec{A}=\frac{q}{\varepsilon_0}$$

$$-E\cdot\Delta A=\frac{1}{\varepsilon_0}\sigma_e(s)\Delta A$$

$$\sigma_e(s)=-\varepsilon_0 E(s)=-\frac{\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_2 s\ln\dfrac{\sigma_2}{\sigma_1}}$$

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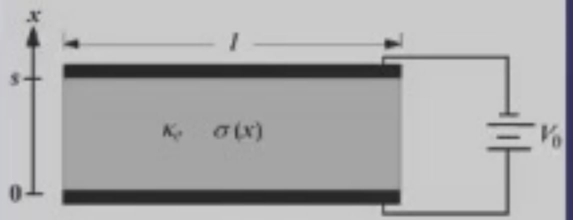
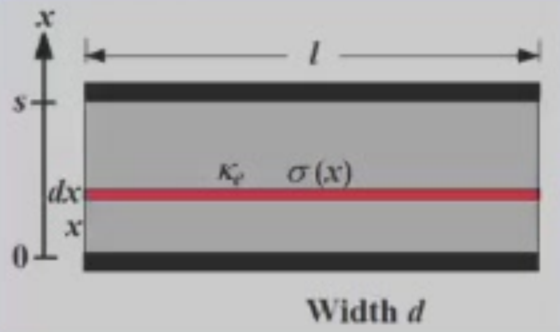


Width d
Figure 9



$$R = \frac{s}{ld(\sigma_2 - \sigma_1)}\ln\frac{\sigma_2}{\sigma_1}$$

(c) $$\rho_e = -\frac{\varepsilon_0 V_0(\sigma_2-\sigma_1)^2}{[\sigma_1 s+(\sigma_2-\sigma_1)x]^2 \ln\frac{\sigma_2}{\sigma_1}}$$

$$Q = -ld\int_0^s \rho_e dx$$

$$= -ld\int_0^s \frac{\varepsilon_0 V_0(\sigma_2-\sigma_1)^2}{[\sigma_1 s+(\sigma_2-\sigma_1)x]^2 \ln\frac{\sigma_2}{\sigma_1}} dx$$

$$= -\frac{ld\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\ln\frac{\sigma_2}{\sigma_1}}\int_0^s \frac{1}{[\sigma_1 s+(\sigma_2-\sigma_1)x]^2} d(\sigma_2-\sigma_1)x$$

$$= \frac{ld\varepsilon_0 V_0(\sigma_2-\sigma_1)}{3\ln\frac{\sigma_2}{\sigma_1}} \frac{1}{[\sigma_1 s+(\sigma_2-\sigma_1)x]^3}\Bigg|_0^s$$

$$= \frac{ld\varepsilon_0 V_0(\sigma_2-\sigma_1)}{3s^3\ln\frac{\sigma_2}{\sigma_1}}\left[\frac{1}{\sigma_2^3}-\frac{1}{\sigma_1^3}\right]$$

$$E = \frac{V_0(\sigma_2-\sigma_1)}{[\sigma_1 s+(\sigma_2-\sigma_1)x]\ln\frac{\sigma_2}{\sigma_1}}$$

$$D = \kappa_e\varepsilon_0 E = \frac{\kappa_e\varepsilon_0 V_0(\sigma_2-\sigma_1)}{[\sigma_1 s+(\sigma_2-\sigma_1)x]\ln\frac{\sigma_2}{\sigma_1}}$$

$$D(0) = \frac{\kappa_e\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_1 s\ln\frac{\sigma_2}{\sigma_1}}$$

$$D(0)\cdot\Delta A = \sigma_{e0}(0)\cdot\Delta A$$

$$\sigma_{e0}(0) = D(0) = \frac{\kappa_e\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_1 s\ln\frac{\sigma_2}{\sigma_1}}$$

$$D(s) = \frac{\kappa_e\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_2 s\ln\frac{\sigma_2}{\sigma_1}}$$

$$-D(s)\cdot\Delta A = \sigma_{e0}(s)\cdot\Delta A$$

$$\sigma_{e0}(s) = -D(s) = -\frac{\kappa_e\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_2 s\ln\frac{\sigma_2}{\sigma_1}}$$

$$P_x = \frac{(\kappa_e-1)\varepsilon_0 V_0(\sigma_2-\sigma_1)}{[\sigma_1 s+(\sigma_2-\sigma_1)x]\ln\frac{\sigma_2}{\sigma_1}}$$

$$P_x(0) = \frac{(\kappa_e-1)\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_1 s\ln\frac{\sigma_2}{\sigma_1}}$$

$$\sigma_e'(0) = \vec{P}\cdot\hat{n} = -P_x(0)$$

$$= -\frac{(\kappa_e-1)\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_1 s\ln\frac{\sigma_2}{\sigma_1}}$$

$$P_x(s) = \frac{(\kappa_e-1)\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_2 s\ln\frac{\sigma_2}{\sigma_1}}$$

$$\sigma_e'(s) = \vec{P}\cdot\hat{n} = P_x(s)$$

$$= \frac{(\kappa_e-1)\varepsilon_0 V_0(\sigma_2-\sigma_1)}{\sigma_2 s\ln\frac{\sigma_2}{\sigma_1}}$$