In order to prove the Hammersley Clifford Theorem, it is sufficient, like mentioned in the script, to show the converse of Corollary 1, that is:

Theorem 1. *If* $\{i,j\} \in E$*, then* x_i *and* x_j *are dependent conditioned on* $x_K, K = [n] \backslash \{i,j\}$*.*

Proof. Per definition, independency means that for all $x \in \mathcal{X}_{[n]}$ we have

$$p(x_i, x_j | x_{-\{i,j\}}) = p(x_i | x_{-\{i,j\}}) \cdot p(x_j | x_{-\{i,j\}})$$

and hence

$$\begin{aligned} p(x_i | x_{-\{i,j\}}) &= \frac{p(x_i, x_j | x_{-\{i,j\}})}{p(x_j | x_{-\{i,j\}})} \\ &= \frac{\frac{p(x)}{p(x_{-\{i,j\}})}}{\frac{p(x_{-i})}{p(x_{-\{i,j\}})}} \\ &= \frac{p(x)}{p(x_{-i})} \\ &= p(x_i | x_{-i}) \quad . \end{aligned}$$

Thus, we only have to find one $x \in \mathcal{X}_{[n]}$ s.t. this equality breaks.

Note that $p(x_i | x_{-\{i,j\}})$ does not depend on $x_j \in \mathcal{X}_j$ as we marginalize over it. Hence, we would like to find a pair $(x, x') \in \mathcal{X}_{[n]}^2$ s.t. they only differ in x_j and $p(x_i | x_{-i}) \neq p(x'_i | x'_{-i})$, as this would imply that either $p(x_i | x_{-\{i,j\}}) \neq p(x_i | x_{-i})$ or $p(x'_i | x'_{-\{i,j\}}) \neq p(x'_i | x'_{-i})$.

To this end, let us analyze $p(x_i | x_{-i})$ with regard to the influence of x_j. We have

$$
\begin{aligned}
p(x_i|x_{-i}) &= \frac{p(x)}{\sum\limits_{x_i \in \mathcal{X}_i} p(x)} \\
&= \frac{\left(\prod\limits_{I \in \mathcal{I}: j \in I} a_I(x_I)\right) \cdot \left(\prod\limits_{I \in \mathcal{I}: j \notin I} a_I(x_I)\right)}{\sum\limits_{x_i \in \mathcal{X}_i} \left[\left(\prod\limits_{I \in \mathcal{I}: i \notin I} a_I(x_I)\right) \cdot \left(\prod\limits_{I \in \mathcal{I}: i \in I} a_I(x_I)\right)\right]} \\
&= \frac{\left(\prod\limits_{I \in \mathcal{I}: j \in I} a_I(x_I)\right) \cdot \left(\prod\limits_{I \in \mathcal{I}: j \notin I} a_I(x_I)\right)}{\left(\prod\limits_{I \in \mathcal{I}: i \notin I} a_I(x_I)\right) \cdot \left(\sum\limits_{x_i \in \mathcal{X}_i} \prod\limits_{I \in \mathcal{I}: i \in I} a_I(x_I)\right)} \\
&= \frac{\left(\prod\limits_{I \in \mathcal{I}: \{i,j\} \subseteq I} a_I(x_I)\right) \cdot \left(\prod\limits_{I \in \mathcal{I}: i \notin I, j \in I} a_I(x_I)\right)}{\left(\prod\limits_{I \in \mathcal{I}: i \notin I} a_I(x_I)\right) \cdot \left(\sum\limits_{x_i \in \mathcal{X}_i} \prod\limits_{I \in \mathcal{I}: i \in I} a_I(x_I)\right)} \cdot \left(\prod_{I \in \mathcal{I}: j \notin I} a_I(x_I)\right) \\
&= \frac{\left(\prod\limits_{I \in \mathcal{I}: \{i,j\} \subseteq I} a_I(x_I)\right) \cdot \left(\prod\limits_{I \in \mathcal{I}: i \notin I, j \in I} a_I(x_I)\right)}{\left(\prod\limits_{I \in \mathcal{I}: i \notin I, j \in I} a_I(x_I)\right) \cdot \left(\prod\limits_{I \in \mathcal{I}: i \notin I, j \notin I} a_I(x_I)\right) \cdot \left(\sum\limits_{x_i \in \mathcal{X}_i} \prod\limits_{I \in \mathcal{I}: i \in I} a_I(x_I)\right)} \cdot \left(\prod_{I \in \mathcal{I}: j \notin I} a_I(x_I)\right) \\
&= \frac{\left(\prod\limits_{I \in \mathcal{I}: \{i,j\} \subseteq I} a_I(x_I)\right)}{\left(\sum\limits_{x_i \in \mathcal{X}_i} \prod\limits_{I \in \mathcal{I}: i \in I} a_I(x_I)\right)} \cdot \frac{\left(\prod\limits_{I \in \mathcal{I}: j \notin I} a_I(x_I)\right)}{\left(\prod\limits_{I \in \mathcal{I}: i \notin I, j \notin I} a_I(x_I)\right)} \\
&=: \frac{\left(\prod\limits_{I \in \mathcal{I}: \{i,j\} \subseteq I} a_I(x_I)\right)}{\left(\sum\limits_{x_i \in \mathcal{X}_i} \prod\limits_{I \in \mathcal{I}: i \in I} a_I(x_I)\right)} \cdot c(x_{-j}) \quad .
\end{aligned}
$$

Note that $c(x_{-j})$ does not depend on x_j, so we can fully focus on

$$
\frac{\left(\prod\limits_{I \in \mathcal{I}: \{i,j\} \subseteq I} a_I(x_I)\right)}{\left(\sum\limits_{x_i \in \mathcal{X}_i} \prod\limits_{I \in \mathcal{I}: i \in I} a_I(x_I)\right)} =: \frac{n(x)}{d(x)} \quad . \tag{1}
$$

Since we have $\{i,j\} \in E$, there must be an interval $I \in \mathcal{I}$ and $x_I^* \in \mathcal{X}_I$ s.t. $\{i,j\} \subseteq I$ and

$$q_I(x_I^*) \neq 0 \iff a_I(x_I^*) \neq 1 \quad .$$

Among all those intervals $\mathscr{I} \subseteq \mathcal{I}$ that have these properties, we pick a minimal interval I, i.e. there is no other interval I' in $\mathscr{I}$ except for I s.t. $I' \subseteq I$. This is done in order to have only one factor in $n(x)$ as we will see in a moment. We also fix the associated $x_I^* \in \mathcal{X}_I$. Note that both I and x_I^* are not necessarily uniquely determined, but that is not an issue.

Now, we define our x s.t. it equals x_I^* for all $x_k, k \in I \backslash \{i\}$. For all other indices $k \notin I$ we set $x_k \coloneqq 1$. For now, we let the value of x_i undefined. For x', we set $x'_k \coloneqq x_k$ for all $k \in [n] \backslash \{j\}$ like already discussed, and for index j we set $x'_j \coloneqq 1$. Note that $x_j \neq 1$.

By doing so, we achieved that the numerator $n(x)$ of (1) evaluates to

$$a_I(x_I)$$

for x, since for all index sets $I' \in \mathcal{I}, \{i,j\} \subseteq I'$ other than I we have that $I' \nsubseteq I$ per construction of I and hence it will contain an index $k \in [n]$ s.t. $k \in I', k \notin I$. Thus, $x_k = 1$, and therefore all $a_{I'}(x_{I'})$ will evaluate to 1. Similarly, $n(x')$ evaluates to

$$a_I(x'_I) = 1 \quad ,$$

because we have set $x'_j = 1$.

Now, note that divisor $d(x)$ of our expression (1) does not depend on x_i, which is why we haven't defined it yet. Now there are two cases:

Case 1: $d(x) = d(x')$
Then we set $x_i \coloneqq x'_i \coloneqq x_i^*$. Thus, $n(x) = a_I(x_I) \neq 1 = a_I(x'_i) = n(x')$ per definition of I, and hence ultimately $p(x_i|x_{-i}) \neq p(x'_i|x'_{-i})$.

Case 2: $d(x) \neq d(x')$
Then we set $x_i \coloneqq x'_i \coloneqq 1$. Thus, $n(x) = a_I(x_I) = 1 = a_I(x'_i) = n(x')$, and hence ultimately $p(x_i|x_{-i}) \neq p(x'_i|x'_{-i})$.

Either way, we get $p(x_i|x_{-i}) \neq p(x'_i|x'_{-i})$ as desired, which concludes the proof. $\square$