

1 Information Theory

1.1 Entropy

Definition 1.1 (Entropy). Let X be a discrete random variable taking values in a finite set \mathcal{X} with probability mass function $p(x) = P(X = x)$. The *entropy* of X , denoted $H(X)$, is defined as:

$$H(X) := - \sum_{x \in \mathcal{X}} p(x) \log p(x),$$

where the logarithm is typically taken base 2 (bits) or base e (nats).

Remark 1.1. If $p(x) = 0$, we set $p(x) \log p(x) := 0$. This ensures that $p(x) \log p(x)$ is continuous on $[0, 1]$.

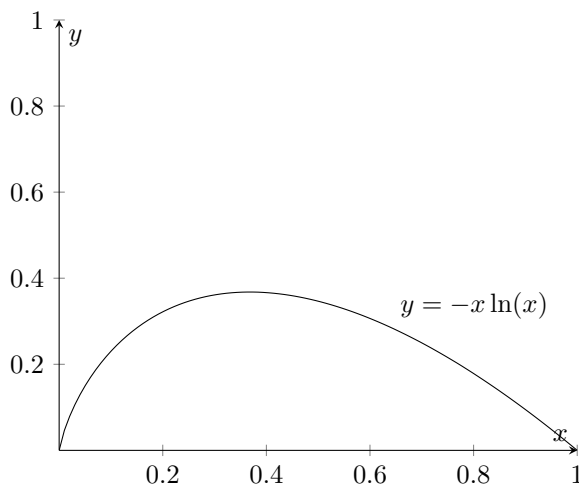


Figure 1: Plot of the function $y = -x \ln(x)$.

Remark 1.2. Entropy measures the uncertainty or information content of a random variable. Higher entropy indicates more unpredictability.

Proposition 1.1 (Non-Negativity of Entropy). *For any discrete random variable X , we have $H(X) \geq 0$.*

Proof. Since $0 \leq p(x) \leq 1$ and $-\log p(x) \geq 0$, each term in the sum is non-negative, so their total sum is non-negative. \square

Lemma 1.1 (Jensen's Inequality). *Let $X \in \mathcal{X}$ be a random variable over a finite set \mathcal{X} , and let ϕ be a convex function defined for all X . Then:*

$$\phi(E[X]) \leq E[\phi(X)] \quad .$$

Proof. We use induction over $n = |\mathcal{X}|$. The base case $n = 1$ is trivial. Hence, assume that the claim holds for some n . We now prove the claim for $n + 1$. Clearly, for $n > 1$, we must have $P(X = x_k) < 1$ for some $x_k \in \mathcal{X}$. Without loss of generality, we assume $k = n + 1$. Hence:

$$\begin{aligned} \phi(E[X]) &= \phi\left(\sum_{i=1}^{n+1} p(x_i)x_i\right) \\ &= \phi\left(\left[(1 - p(x_{n+1}))\sum_{i=1}^n \frac{p(x_i)}{1 - p(x_{n+1})}x_i\right] + p(x_{n+1})x_{n+1}\right) \\ &\stackrel{\text{convexity}}{\leq} (1 - p(x_{n+1}))\phi\left(\sum_{i=1}^n \frac{p(x_i)}{1 - p(x_{n+1})}x_i\right) + p(x_{n+1})\phi(x_{n+1}) \\ &\stackrel{\text{inductive hypothesis}}{\leq} (1 - p(x_{n+1}))\sum_{i=1}^n \frac{p(x_i)}{1 - p(x_{n+1})}\phi(x_i) + p(x_{n+1})\phi(x_{n+1}) \\ &= \sum_{i=1}^{n+1} p(x_i)\phi(x_i) = E[\phi(X)] \quad . \end{aligned}$$

□

Remark 1.3. For strictly convex ϕ , it can be shown that

$\phi(E[X]) = E[\phi(X)]$ is maximized $\iff X$ is sampled from a uniform distribution .

Proposition 1.2 (Maximum Entropy). *For a discrete random variable X over n outcomes, entropy is maximized when X is uniform:*

$$H(X) \leq \log n \quad .$$

Proof. We have:

$$\begin{aligned} -H(X) &= -E[-\log(p(X))] \\ &= E\left[-\log\left(\frac{1}{p(X)}\right)\right] \\ &\stackrel{\text{Jensen's Inequality}}{\geq} -\log\left(E\left[\frac{1}{p(X)}\right]\right) \\ &= -\log n \quad , \end{aligned}$$

where we assumed $p(X) > 0$. Of course, the cases where $p(X) = 0$ follow directly, since $p(X) \log p(X) = 0$.

$H(X) \leq \log n$ follows directly. Note that we have equality iff X has uniform distribution (since $-\log(x)$ is strictly convex). \square

1.1.1 Joint, Conditional, and Cross Entropy

Definition 1.2 (Joint Entropy). For a pair of discrete random variables X and Y , the joint entropy is:

$$H(X, Y) := - \sum_{x, y} p(x, y) \log p(x, y) \quad .$$

Definition 1.3 (Conditional Entropy). The conditional entropy of Y given X is defined as:

$$H(Y | X) := \sum_x p(x) H(Y | X = x) = - \sum_{x, y} p(x, y) \log p(y | x).$$

Corollary 1.1. *We immediately see from the first equation that $H(Y | X) \geq 0$.*

Theorem 1.1 (Chain Rule for Entropy).

$$H(X, Y) = H(X) + H(Y | X) \quad .$$

Proof. We have:

$$\begin{aligned} H(X, Y) &= - \sum_{x,y} p(x, y) \log p(x, y) \\ &= - \sum_{x,y} p(x, y) \log (p(x)p(y | x)) \\ &= - \sum_{x,y} p(x, y) \log p(x) - \sum_{x,y} p(x, y) \log p(y | x) \\ &= H(X) + H(Y | X) \quad . \end{aligned}$$

□

Corollary 1.2. $H(X, Y) \geq 0$ follows directly.

Definition 1.4 (Cross-Entropy). Let p and q be two probability distributions over a finite set \mathcal{X} , with $p(x) > 0 \Rightarrow q(x) > 0$. The *cross-entropy* of p relative to q is defined as:

$$H_q(p) := - \sum_{x \in \mathcal{X}} p(x) \log q(x) \quad .$$

Remark 1.4. Cross-entropy measures the expected number of bits required to encode samples from p using a code optimized for the distribution q .

Remark 1.5. Cross-entropy is non-negative (see section 1.2).

1.1.2 Properties of Entropy

Proposition 1.3. *Conditional entropy satisfies:*

$$H(Y | X) \leq H(Y) \quad ,$$

with equality if and only if X and Y are independent.

Proof. From the chain rule:

$$H(X, Y) = H(Y) + H(X | Y) = H(X) + H(Y | X) \quad ,$$

which implies:

$$H(Y | X) = H(Y) + H(X | Y) - H(X) = H(Y) - I(X; Y) \quad ,$$

with mutual information $I(X; Y) \geq 0$ (see section 1.3). Equality holds if and only if $I(X; Y) = 0$, i.e., X and Y are independent. □

Corollary 1.3 (Subadditivity of Entropy). *For any two random variables X and Y ,*

$$H(X, Y) \leq H(X) + H(Y) \quad ,$$

with equality if and only if X and Y are independent.

Proof. From the chain rule:

$$H(X, Y) = H(X) + H(Y \mid X) \leq H(X) + H(Y) \quad ,$$

since $H(Y \mid X) \leq H(Y)$ based on proposition 1.3. Equality holds if and only if $H(Y \mid X) = H(Y)$, i.e., X and Y are independent. \square

Theorem 1.2 (Concavity of Entropy). *The entropy function $H(p)$, where $p \in \Delta$ is a probability vector, is concave on the probability simplex Δ .*

Proof. This follows from the fact that $f(x) = -x \log x$ is concave for $x \in [0, 1]$, and entropy is the sum of such terms. Therefore, for every convex combination $p = \lambda p_1 + (1 - \lambda)p_2$:

$$H(p) \geq \lambda H(p_1) + (1 - \lambda)H(p_2) \quad .$$

\square

Summary of Key Properties

- Non-negativity: $H(X) \geq 0$
- Maximum entropy: $H(X) \leq \log |\mathcal{X}|$
- Chain rule: $H(X, Y) = H(X) + H(Y \mid X)$
- Subadditivity: $H(X, Y) \leq H(X) + H(Y)$
- Conditioning reduces entropy: $H(Y \mid X) \leq H(Y)$
- Concavity: $H(p)$ is concave in the distribution p

1.2 Kullback-Leibler Divergence

Definition 1.5 (KL Divergence). Let P and Q be two discrete probability distributions over the same finite set \mathcal{X} , with $P(x) > 0 \Rightarrow Q(x) > 0$. The Kullback-Leibler divergence (or relative entropy) from P to Q is defined as:

$$\begin{aligned} D_{\text{KL}}(P\|Q) &:= \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)} \\ &= - \sum_x P(x) \log Q(x) + \sum_x P(x) \log P(x) \\ &= H_Q(P) - H(P) \quad . \end{aligned}$$

Remark 1.6. If $P(x) = Q(x) = 0$, we set $P(x) \log \frac{P(x)}{Q(x)} := 0$.

Remark 1.7. KL divergence measures the inefficiency of assuming that the distribution is Q when the true distribution is P . It is not a metric: it is not symmetric and does not satisfy the triangle inequality.

Lemma 1.2 (Gibb's Inequality). *Suppose that $P = \{p_1, \dots, p_n\}$ and $Q = \{q_1, \dots, q_n\}$ are discrete probability distributions. Then:*

$$- \sum_{i=1}^n p_i \log p_i \leq - \sum_{i=1}^n p_i \log q_i \quad .$$

Proof. The claim is equivalent to $\sum_{i=1}^n p_i \log p_i - \sum_{i=1}^n p_i \log q_i \geq 0$. We have:

$$\begin{aligned} \sum_{i=1}^n p_i \log p_i - \sum_{i=1}^n p_i \log q_i &= \sum_{i=1}^n p_i \log \frac{p_i}{q_i} \\ &= \sum_{i=1}^n p_i \left(- \log \frac{q_i}{p_i} \right) \\ &\stackrel{\text{Jensen's Inequality}}{\geq} - \log \left(\sum_{i=1}^n p_i \frac{q_i}{p_i} \right) \\ &= - \log(1) = 0 \quad . \end{aligned}$$

□

Corollary 1.4. *It directly follows from the proof that $D_{\text{KL}}(P\|Q) \geq 0$ and $0 \leq H(P) \leq H_Q(P)$.*

Proposition 1.4 (Additivity). *Let $P = P_1 \times P_2$, $Q = Q_1 \times Q_2$. Then:*

$$D_{\text{KL}}(P\|Q) = D_{\text{KL}}(P_1\|Q_1) + D_{\text{KL}}(P_2\|Q_2) \quad .$$

Proof.

$$\begin{aligned} D_{\text{KL}}(P_1 \times P_2\|Q_1 \times Q_2) &= \sum_{x,y} P_1(x)P_2(y) \log \frac{P_1(x)P_2(y)}{Q_1(x)Q_2(y)} \\ &= \sum_{x,y} P_1(x)P_2(y) \left(\log \frac{P_1(x)}{Q_1(x)} + \log \frac{P_2(y)}{Q_2(y)} \right) \\ &= \sum_x P_1(x) \log \frac{P_1(x)}{Q_1(x)} + \sum_y P_2(y) \log \frac{P_2(y)}{Q_2(y)} \\ &= D_{\text{KL}}(P_1\|Q_1) + D_{\text{KL}}(P_2\|Q_2) \quad . \end{aligned}$$

□

Proposition 1.5 (Entropy Representation via KL Divergence). *Let U be the uniform distribution over \mathcal{X} , where $|\mathcal{X}| = n$. Then for any distribution P ,*

$$H(P) = \log n - D_{\text{KL}}(P\|U) \quad .$$

Proof.

$$\begin{aligned} D_{\text{KL}}(P\|U) &= \sum_x P(x) \log \frac{P(x)}{1/n} = \sum_x P(x) \log P(x) + \sum_x P(x) \log n \\ &= -H(P) + \log n \quad . \end{aligned}$$

□

Summary of Key Properties

- $D_{\text{KL}}(P\|Q) \geq 0$
- $D_{\text{KL}}(P\|Q) = 0 \iff P = Q$
- Asymmetric: $D_{\text{KL}}(P\|Q) \neq D_{\text{KL}}(Q\|P)$
- Additive over independent distributions
- Connection to entropy: $H(P) = \log n - D_{\text{KL}}(P\|U)$

1.3 Mutual Information

Definition 1.6 (Mutual Information). Let X and Y be discrete random variables with joint distribution $p(x, y)$ and marginals $p(x)$, $p(y)$. The *mutual information* between X and Y is defined as:

$$I(X; Y) := \sum_{x, y} p(x, y) \log \left(\frac{p(x, y)}{p(x)p(y)} \right) .$$

Remark 1.8. Mutual information quantifies how much knowing X reduces uncertainty about Y , and vice versa. Per definition, it is symmetric: $I(X; Y) = I(Y; X)$.

Proposition 1.6 (Equivalent Expressions). *Mutual information can also be expressed as:*

$$\begin{aligned} I(X; Y) &= D_{\text{KL}}(p(x, y) \| p(x)p(y)) \\ &= H_{p(x)p(y)}(p(x, y)) - H(X, Y) \\ &= \left[- \sum_{x, y} p(x, y) \log(p(x)p(y)) \right] - H(X, Y) \\ &= H(X) + H(Y) - H(X, Y) \\ &= H(X) - H(X | Y) \\ &= H(Y) - H(Y | X) \end{aligned}$$

Proof. Each follows from basic entropy identities and the definition of KL divergence. \square

Corollary 1.5. $I(X; Y) \geq 0$, since $I(X; Y) = D_{\text{KL}}(p(x, y) \| p(x)p(y))$ and KL divergence is always non-negative.

Definition 1.7 (Conditional Mutual Information). Let X, Y, Z be discrete random variables. The *conditional mutual information* of X and Y given Z is defined as:

$$I(X; Y | Z) := \sum_{x, y, z} p(x, y, z) \log \frac{p(x, y | z)}{p(x | z)p(y | z)} .$$

Equivalently, in terms of entropy:

$$I(X; Y | Z) = H(X | Z) - H(X | (Y, Z)) .$$

Proof.

$$\begin{aligned}
& H(X | Z) - H(X | (Y, Z)) \\
&= \sum_z p(z) H(X | Z = z) - \sum_{y,z} p(y, z) H(X | Y = y, Z = z) \\
&= - \sum_z p(z) \sum_x p(x | z) \log p(x | z) + \sum_{y,z} p(y, z) \sum_x p(x | y, z) \log p(x | y, z) \\
&= \sum_{x,y,z} p(x, y, z) \log \frac{p(x | y, z)}{p(x | z)} \\
&= \sum_{x,y,z} p(x, y, z) \log \frac{p(x, y | z)}{p(x | z)p(y | z)} \\
&= I(X; Y | Z) \quad .
\end{aligned}$$

□

Remark 1.9. Conditional mutual information measures how much knowing Y reduces the uncertainty of X , *given* that we already know Z .

Proposition 1.7 (Chain Rule for Mutual Information). *Let X , Y , and Z be random variables. Then:*

$$I(X; Y, Z) = I(X; Z) + I(X; Y | Z) \quad .$$

Proof. We use entropy-based expressions for mutual information:

$$\begin{aligned}
I(X; Y, Z) &= H(X) - H(X | (Y, Z)) \\
&= I(X; Z) + H(X | Z) - H(X | (Y, Z)) \\
&= I(X; Z) + H(X | Z) - (H(X | Z) - I(X; Y | Z)) \\
&= I(X; Z) + I(X; Y | Z) \quad .
\end{aligned}$$

□

Proposition 1.8 (Non-Negativity of Conditional Mutual Information). *It holds true that*

$$I(X; Y | Z) \geq 0 \quad .$$

Proof. We have:

$$\begin{aligned}
I(X; Y | Z) &= \sum_{x,y,z} p(x, y, z) \log \frac{p(x, y | z)}{p(x | z)p(y | z)} \\
&= \sum_z p(z) \sum_{x,y} p(x, y | z) \log \frac{p(x, y | z)}{p(x | z)p(y | z)} \\
&= \sum_z p(z) D_{\text{KL}}(p(x, y | z) \| p(x | z)p(y | z)) \geq 0 \quad .
\end{aligned}$$

□

Corollary 1.6. *As a direct consequence, we have*

$$I(X; Z) \leq I(X; Y, Z) \quad .$$

Definition 1.8 (Conditional Independence). Let X, Y, Z be discrete random variables. We say that X is *conditionally independent* of Z given Y , and write:

$$X \perp Z \mid Y$$

if and only if

$$p(z \mid x, y) = p(z \mid y) \quad \text{for all } x, y, z \quad .$$

Equivalently:

$$p(x, z \mid y) = p(x \mid y)p(z \mid y) \quad .$$

Proposition 1.9. *If $X \perp Z \mid Y$, then the conditional mutual information between X and Z given Y is zero:*

$$I(X; Z \mid Y) = 0 \quad .$$

Proof. By definition of conditional mutual information:

$$I(X; Z \mid Y) = \sum_{x, z, y} p(x, z, y) \log \frac{p(x, z \mid y)}{p(x \mid y)p(z \mid y)} \quad .$$

If $X \perp Z \mid Y$, then:

$$p(x, z \mid y) = p(x \mid y)p(z \mid y) \quad ,$$

so the logarithm becomes:

$$\log \frac{p(x \mid y)p(z \mid y)}{p(x \mid y)p(z \mid y)} = \log 1 = 0 \quad .$$

Hence, each term in the sum is zero, and:

$$I(X; Z \mid Y) = 0 \quad .$$

□

1.3.1 Data Processing Inequality

Lemma 1.3 (Markov Chain). *Let X, Y, Z be random discrete random variables forming the Markov chain $X \rightarrow Y \rightarrow Z$. Then:*

$$X \perp Z \mid Y \quad .$$

Proof. Per definition from Markov chains, we have:

$$p(z \mid x, y) = p(z \mid y) \quad ,$$

and hence $X \perp Z \mid Y$. □

Theorem 1.3 (Data Processing Inequality). *If $X \rightarrow Y \rightarrow Z$ is a Markov chain, then:*

$$I(X; Z) \leq I(X; Y) \quad .$$

Proof. We use the chain rule and conditional independence:

$$\begin{aligned} I(X; Z) &= I(X; Z, Y) - I(X; Y \mid Z) \\ &= I(X; Y) + I(X; Z \mid Y) - I(X; Y \mid Z) \quad . \end{aligned}$$

Since $X \rightarrow Y \rightarrow Z$, we have $I(X; Z \mid Y) = 0$. Thus:

$$I(X; Z) = I(X; Y) - I(X; Y \mid Z) \leq I(X; Y) \quad ,$$

because $I(X; Y \mid Z) \geq 0$. □

Corollary 1.7 (No Gain in Processing). *Any function $f(Y)$ of Y cannot increase information about X :*

$$I(X; f(Y)) \leq I(X; Y) \quad .$$

Proof. This follows by applying the DPI to the chain $X \rightarrow Y \rightarrow f(Y)$. □

Summary of Key Properties

- $I(X; Y) \geq 0$
- $I(X; Y) = 0$ if and only if $X \perp Y$
- $I(X; Y) = D_{\text{KL}}(p(x, y) \parallel p(x)p(y))$
- Chain rule: $I(X; Y, Z) = I(X; Z) + I(X; Y \mid Z)$
- Data Processing Inequality: $X \rightarrow Y \rightarrow Z \Rightarrow I(X; Z) \leq I(X; Y)$

1.4 Bounding Mutual Information via Matrix Rank of the Joint Distribution

Theorem 1.4. *Let X, Y be random variables from finite sets \mathcal{X}, \mathcal{Y} , and let matrix \mathbf{P} denote their joint probability distribution, i.e. $\mathbf{P}_{ij} = p(x_i, y_j)$. Let $r := \text{rank } \mathbf{P}$ denote the rank of matrix \mathbf{P} . Then we have*

$$I(X; Y) \leq \log r \quad .$$

Proof. Let $n := |\mathcal{X}|$ and $m := |\mathcal{Y}|$. If \mathbf{P} has rank r , then so must the transition matrix $\mathbf{P}_{Y|X} \in \mathbb{R}^{m \times n}$ defined as $(\mathbf{P}_{Y|X})_{ij} := p(y_i | x_j) = \frac{p(x_j, y_i)}{\sum_k p(x_k, y_i)}$, since $\mathbf{P}_{Y|X}$ is created from \mathbf{P} by transposing and column scaling. If one column consisted of only zeros, i.e. $\sum_k p(x_k, y_i) = 0$, we may just copy a different scaled column vector to this column.

Now, let's analyze matrix $\mathbf{P}_{Y|X}$. First, note that it is a Markov chain transition matrix, and hence all its columns lie in the m -dimensional unit simplex. Consider the convex hull of the column vectors, it is a r -dimensional convex polytope in the m -dimensional unit simplex. Thus, we can find a r -dimensional simplex with corners collected by matrix \mathbf{U} s.t. it is a superset of this polytope and still a subset of the (potentially) higher dimensional unit simplex.

Thus, every column vector in $\mathbf{P}_{Y|X}$ can be written as a convex combination of the column vectors in \mathbf{U} . It follows that $\mathbf{P}_{Y|X}$ can be decomposed as

$$\mathbf{P}_{Y|X} = \mathbf{U}\mathbf{V} \quad , \quad \mathbf{U} \in \mathbb{R}^{m \times r}, \mathbf{V} \in \mathbb{R}^{r \times n} \quad ,$$

where both \mathbf{U} and \mathbf{V} are Markov chain transition matrices as well.

Hence, we can introduce a latent variable $Z \in \{1, \dots, r\}$, which forms the Markov chain

$$X \xrightarrow{\mathbf{V}} Z \xrightarrow{\mathbf{U}} Y \quad .$$

Finally, based on theorem 1.3 it follows that

$$I(X; Y) \leq I(X; Z) = H(Z) - H(Z | X) \leq H(Z) \leq \log r \quad .$$

□

1.5 Convergence of Mutual Information

Theorem 1.5 (Element-Wise Exponential Convergence Implies Exponential Convergence). *Let $f : \mathcal{D} \rightarrow \mathbb{R}^m$ be a function defined on a convex domain $\mathcal{D} \subseteq \mathbb{R}^n$ that is a Cartesian product of real intervals, i.e., $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \times \dots \times \mathcal{D}_n$ where each $\mathcal{D}_i \subseteq \mathbb{R}$ is an interval. Let $\{\mathbf{x}_k\}_{k=1}^\infty \subset \mathcal{D}$ be a sequence converging exponentially fast to $\mathbf{x}_0 \in \mathcal{D}$.*

Let \mathbf{e}_j denote the j -th standard basis vector in \mathbb{R}^n . Suppose that for each input coordinate $j \in \{1, 2, \dots, n\}$ there exist functions $K_j(C', \rho)$, $C_j(C', \rho)$, $P_j(C', \rho)$ s.t. for every sequence $\{\mathbf{u}_k\}_{k=1}^\infty \subset \mathcal{D}$ converging to \mathbf{u}_0 where the difference $\mathbf{u}_\ell - \mathbf{u}_{\ell'}$ is parallel to \mathbf{e}_j (i.e., they only differ in the j -th coordinate) that satisfies $|\mathbf{u}_0 - \mathbf{u}_k| \leq C' \rho^k$ for all k and some $\rho \in [0, 1)$, $C' > 0$, we have for all $k \geq K_j(C', \rho)$:

$$\|f(\mathbf{u}_0) - f(\mathbf{u}_k)\| \leq C_j(C', \rho) \rho^k k^{P_j(C', \rho)} .$$

Then, there exist constants $C > 0$ and $\sigma \in [0, 1)$ such that for all sufficiently large k :

$$\|f(\mathbf{x}_0) - f(\mathbf{x}_k)\| \leq C \sigma^k .$$

Proof. Let the sequence $\{\mathbf{x}_k\}_{k=1}^\infty \subset \mathcal{D}$ converge exponentially to $\mathbf{x}_0 \in \mathcal{D}$. By definition, there exist constants $C_x > 0$ and $\rho \in [0, 1)$ such that for all k ,

$$\|\mathbf{x}_k - \mathbf{x}_0\| \leq C_x \rho^k .$$

Let $\mathbf{x}_k = (x_{k,1}, \dots, x_{k,n})^T$ and $\mathbf{x}_0 = (x_{0,1}, \dots, x_{0,n})^T$. An immediate consequence is that each coordinate also converges exponentially, i.e., for each $j \in \{1, \dots, n\}$:

$$|x_{k,j} - x_{0,j}| \leq \|\mathbf{x}_k - \mathbf{x}_0\|_\infty \leq \|\mathbf{x}_k - \mathbf{x}_0\| \leq C_x \rho^k ,$$

where we use the equivalence of norms in \mathbb{R}^n .

To bound $\|f(\mathbf{x}_0) - f(\mathbf{x}_k)\|$, we define a sequence of $n + 1$ intermediate points that form a path from \mathbf{x}_k to \mathbf{x}_0 by changing one coordinate at a time. For each k , let:

$$\begin{aligned} \mathbf{z}_{k,0} &:= \mathbf{x}_k = (x_{k,1}, x_{k,2}, \dots, x_{k,n}) \\ \mathbf{z}_{k,1} &:= (x_{0,1}, x_{k,2}, \dots, x_{k,n}) \\ &\vdots \\ \mathbf{z}_{k,j} &:= (x_{0,1}, \dots, x_{0,j}, x_{k,j+1}, \dots, x_{k,n}) \\ &\vdots \\ \mathbf{z}_{k,n} &:= (x_{0,1}, \dots, x_{0,n}) = \mathbf{x}_0 . \end{aligned}$$

Since \mathcal{D} is a cartesian product intervals and both \mathbf{x}_k and \mathbf{x}_0 are in \mathcal{D} , all intermediate points $\mathbf{z}_{k,j}$ are also contained in \mathcal{D} . We can express the total difference $f(\mathbf{x}_0) - f(\mathbf{x}_k)$ as a telescoping sum:

$$f(\mathbf{x}_0) - f(\mathbf{x}_k) = f(\mathbf{z}_{k,n}) - f(\mathbf{z}_{k,0}) = \sum_{j=1}^n (f(\mathbf{z}_{k,j}) - f(\mathbf{z}_{k,j-1})) \quad .$$

By the triangle inequality, we have:

$$\|f(\mathbf{x}_0) - f(\mathbf{x}_k)\| \leq \sum_{j=1}^n \|f(\mathbf{z}_{k,j}) - f(\mathbf{z}_{k,j-1})\| \quad .$$

Now, we analyze each term $\|f(\mathbf{z}_{k,j}) - f(\mathbf{z}_{k,j-1})\|$ for a fixed $j \in \{1, \dots, n\}$. The points $\mathbf{z}_{k,j}$ and $\mathbf{z}_{k,j-1}$ differ only in their j -th coordinate.

Let us define a sequence $\{\mathbf{u}_m\}_{m=1}^\infty$ and a limit point \mathbf{u}_0 that fit the condition in the theorem's hypothesis. For the given j and k , let

$$\begin{aligned} \mathbf{u}_m &:= (x_{0,1}, \dots, x_{0,j-1}, x_{m,j}, x_{k,j+1}, \dots, x_{k,n}) \\ \mathbf{u}_0 &:= (x_{0,1}, \dots, x_{0,j-1}, x_{0,j}, x_{k,j+1}, \dots, x_{k,n}) \quad . \end{aligned}$$

Note that $\mathbf{u}_0 = \mathbf{z}_{k,j}$ and by setting $m = k$, we get $\mathbf{u}_k = \mathbf{z}_{k,j-1}$. The sequence $\{\mathbf{u}_m\}$ lies on a line parallel to the j -th coordinate axis. As $m \rightarrow \infty$, $\mathbf{u}_m \rightarrow \mathbf{u}_0$ because $x_{m,j} \rightarrow x_{0,j}$. The convergence is exponential:

$$\|\mathbf{u}_m - \mathbf{u}_0\| = |x_{m,j} - x_{0,j}| \leq C_x \rho^m \quad .$$

The hypothesis states that for any such sequence, there exist constants $K_j(C_x, \rho)$, $C_j(C_x, \rho)$, $P_j(C_x, \rho)$ which are independent of the specific line, such that for all $k \geq K_j(C_x, \rho)$ we have $\|f(\mathbf{u}_0) - f(\mathbf{u}_m)\| \leq C_j(C_x, \rho) \rho^m m^{P_j(C_x, \rho)}$. Applying this for $m = k$:

$$\|f(\mathbf{z}_{k,j}) - f(\mathbf{z}_{k,j-1})\| = \|f(\mathbf{u}_0) - f(\mathbf{u}_k)\| \leq C_j(C_x, \rho) \rho^k k^{P_j(C_x, \rho)} \quad .$$

This inequality holds for each $j = 1, \dots, n$. Substituting these bounds back into the sum:

$$\|f(\mathbf{x}_0) - f(\mathbf{x}_k)\| \leq \sum_{j=1}^n C_j(C_x, \rho) \rho^k k^{P_j(C_x, \rho)} \quad .$$

Let $K := \max_{j \in \{1, \dots, n\}} \{K_j(C_x, \rho)\}$, $C := \sum_{j=1}^n C_j(C_x, \rho)$ and $P := \max_{j \in \{1, \dots, n\}} \{P_j(C_x, \rho)\}$. Hence, for all $k \geq K$:

$$\|f(\mathbf{x}_0) - f(\mathbf{x}_k)\| \leq \sum_{j=1}^n C_j(C_x, \rho) \rho^k k^P = \left(\sum_{j=1}^n C_j(C_x, \rho) \right) \rho^k k^P = C \rho^k k^P \quad .$$

This shows that $\{f(\mathbf{x}_k)\}$ converges exponentially to $f(\mathbf{x}_0)$ with a rate of σ s.t. $\rho < \sigma < 1$ (like $\sigma := \sqrt{\rho}$). This completes the proof. \square

Lemma 1.4. *Let the function $f : [0, \infty) \rightarrow \mathbb{R}$ be defined as $f(x) = x \log x$ for $x \in (0, 1)$, and $f(x) = 0$ everywhere else. If a sequence $\{x_k\}_{k=1}^\infty \subset [0, 1]$ converging to a limit $x_\infty \in [0, 1]$ satisfies $|x_k - x_\infty| \leq C\rho^k$ for some $C \in \mathbb{R}_{>0}$, $\rho \in [0, 1)$, then the sequence $\{f(x_k)\}$ converges to $f(x_\infty)$ with $|f(x_k) - f(x_\infty)| \leq C\rho^k |\log C + k \log \rho|$ for $k \geq \log_\rho \left(\frac{e^{-1}}{C} \right) =: K$.*

Proof. We consider two cases for the limit x_∞ :

Case 1: $x_\infty = 0$

In this case, $|x_k - 0| = x_k \leq C\rho^k$. We want to bound the difference $|f(x_k) - f(0)| = |x_k \log x_k|$. Note that $|x \log x|$ is monotonically increasing on $[0, e^{-1}]$. For $k \geq K$ we have $x_k \leq e^{-1}$. Thus, for all $k \geq K$:

$$|x_k \log x_k| \leq |C\rho^k \log(C\rho^k)| = C\rho^k |\log C + k \log \rho| \quad .$$

Case 2: $x_\infty > 0$

Similarly, for $k \geq K$ we have $|x_k - x_\infty| \leq C\rho^k \leq e^{-1}$. Based on the function graph, it follows that for $k \geq K$ we have:

$$|f(x_k) - f(x_\infty)| \leq |f(C\rho^k) - f(0)| = |f(C\rho^k)| = C\rho^k |\log C + k \log \rho| \quad .$$

□

Theorem 1.6 (Element-Wise Exponential Convergence Property of Mutual Information). *Let the function $f : [0, \infty) \rightarrow \mathbb{R}$ be defined as $f(x) = x \log x$ for $x \in (0, 1)$, and $f(x) = 0$ everywhere else. Define the function $I : [0, 1]^{mn} \rightarrow \mathbb{R}$ for a matrix \mathbf{M} as:*

$$I(\mathbf{M}) = \sum_{i=1}^m \sum_{j=1}^n f(M_{ij}) - \sum_{i=1}^m f\left(\sum_{j=1}^n M_{ij}\right) - \sum_{j=1}^n f\left(\sum_{i=1}^m M_{ij}\right) \quad .$$

This function exhibits element-wise exponential convergence. That is, for any single component (i_0, j_0) , if a sequence of matrices $\{\mathbf{U}_k\}_{k=1}^\infty \subset [0, 1]^{mn}$ converges to a limit \mathbf{U}_∞ , varies only in the (i_0, j_0) -th component and satisfies $\|\mathbf{U}_k - \mathbf{U}_\infty\| \leq C\rho^k$ for some $C > 0$, $\rho \in [0, 1)$ and all k , then the sequence of values $\{I(\mathbf{U}_k)\}$ converges to $I(\mathbf{U}_\infty)$ with $|I(\mathbf{U}_k) - I(\mathbf{U}_\infty)| \leq C'(C, \rho)\rho^k k^{P(C, \rho)}$ for all $k \geq K(C, \rho)$.

Proof. The function $I(\mathbf{M})$ is a sum of terms involving f applied to the matrix entries and their row and column sums. Let $m_i(\mathbf{M}) = \sum_j M_{ij}$ and $m'_j(\mathbf{M}) = \sum_i M_{ij}$. We have:

$$I(\mathbf{M}) = \sum_{i,j} f(M_{ij}) - \sum_i f(m_i(\mathbf{M})) - \sum_j f(m'_j(\mathbf{M})) \quad .$$

We are given a sequence $\{\mathbf{U}_k\}$ that varies only in the (i_0, j_0) -th component, $u_k = U_k(i_0, j_0)$. All other components are constant. The exponential convergence of $\{\mathbf{U}_k\}$ means $|u_k - u_\infty| \leq C\rho^k$.

The difference $I(\mathbf{U}_k) - I(\mathbf{U}_\infty)$ consists only of terms whose arguments change with k . These are:

1. The entry term: $f(u_k)$.
2. The row-sum term: $f(m_{i_0}(\mathbf{U}_k))$, where $m_{i_0}(\mathbf{U}_k) = u_k + \text{const.}$
3. The column-sum term: $f(m'_{j_0}(\mathbf{U}_k))$, where $m'_{j_0}(\mathbf{U}_k) = u_k + \text{const.}$

By the triangle inequality, the total error is bounded by the sum of the absolute errors of these three terms:

$$\begin{aligned} |I(\mathbf{U}_k) - I(\mathbf{U}_\infty)| &\leq |f(u_k) - f(u_\infty)| \\ &\quad + |f(m_{i_0}(\mathbf{U}_k)) - f(m_{i_0}(\mathbf{U}_\infty))| \\ &\quad + |f(m'_{j_0}(\mathbf{U}_k)) - f(m'_{j_0}(\mathbf{U}_\infty))| \quad . \end{aligned}$$

The arguments to the function f in each of these three terms converge exponentially to their limits with rate ρ and constant C , since $|m_{i_0}(\mathbf{U}_k) - m_{i_0}(\mathbf{U}_\infty)| = |u_k - u_\infty|$ and $|m'_{j_0}(\mathbf{U}_k) - m'_{j_0}(\mathbf{U}_\infty)| = |u_k - u_\infty|$.

Hence, by lemma 1.4, we have:

$$\begin{aligned} |I(\mathbf{U}_k) - I(\mathbf{U}_\infty)| &\leq 3C\rho^k |\log C + k \log \rho| \\ &\leq 3C\rho^k k (|\log C| + |\log \rho|) \\ &= C'\rho^k k \quad , \end{aligned}$$

with $C' := 3C(|\log C| + |\log \rho|)$ and for all $k \geq K(C, \rho)$. Note that K , C' and P only depend on C and ρ . \square

Corollary 1.8. *Using theorem 1.5, we see that if a sequence $\{\mathbf{P}_k\}$ of joint probability distributions converges exponentially fast, then $\{I(\mathbf{P}_k)\}$ converges exponentially fast as well.*

Corollary 1.9. *A joint probability matrix $\mathbf{P}_{X,Y}$ can be calculated from the conditional probability matrix $\mathbf{P}_{Y|X}$ and the diagonal matrix \mathbf{P}_X with the probabilities for X on its diagonal using $\mathbf{P}_{X,Y} = \mathbf{P}_{Y|X}\mathbf{P}_X$. Hence, if $\mathbf{P}_{Y|X}$ converges exponentially fast while \mathbf{P}_X stays constant, the mutual information $I(X; Y)$ will converge exponentially fast as well.*