Lecture 2 9/12/16

Birthday Problem - Group of $r \geq 2$ people. Assume all 365 (non 2/29) birthdays are equally likely, and that the birthdays of different people in the group are independent of one another. Find the smallest value of r such that

$$p(r) = \Pr(\text{at least 2 people have the same birthday}) > 1/2.$$

Solution $q(r) = 1 - p(r) = \Pr(\text{all } r \text{ birthdays are different})$. We want the smallest r such than $q(r) < 1/2$.

$$q(r) = \frac{364}{365} \cdots \frac{365-(r-1)}{365} = \prod_1^{r-1}\left(\frac{365-j}{365}\right)$$

$$= \prod_1^{r-1}\left(1 - \frac{j}{365}\right)$$

$q(22) = .5243$

$\boxed{q(23) = .4977 \quad p(23) = .5023}$

Note: $q(r) = \prod_1^{r-1}\left(1 - \frac{k}{365}\right) = e^{\sum_1^{r-1} \log\left(1 - \frac{k}{365}\right)} \approx e^{-\sum_1^{r-1} \frac{k}{365}}$

$$= e^{-\frac{r(r-1)}{730}}$$

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WANT, $e^{-\frac{r(r-1)}{730}} < \frac{1}{2} \iff \frac{r(r-1)}{730} > \log 2$

$$r^2 - r - 730 \log 2 > 0$$

$$r > \frac{1+\sqrt{1+2920 \log 2}}{2} = 22.9999$$

Suggests $r \geq 23$ is the answer

Poker Pr(Full House)

13 denominations A, 2, .. K

4 suits D, C, H, S

choose 5 cards without replacement

$$\# \text{ poker hands} = \frac{52(51)(50)(49)(48)}{120}$$

$$= 2,598,960$$

$\#$ full houses: $13(12)(4)(6) = 3744$

$$\Pr(\text{Full house}) = \frac{3744}{2,598,960} \approx .00144$$

$$\approx \frac{1}{694}$$

Gambler's Ruin Play series of games. On each win \$1 with probability p, lose \$1 with probability $q = 1-p$. Start with r dollars. Play until either you reach 0 or n $(1 \le r \le n-1)$, whichever comes first.

Find, $P(r) = \Pr(0 \text{ before } n \mid \text{start with } r)$

Solution $Q(r) = 1 - P(r)$, $Q(0) = 0$, $Q(n) = 1$

$$1 \le j \le n-1, \quad \underset{\displaystyle pQ(j) + qQ(j)}{\underset{\shortparallel}{Q(j)}} = p\,Q(j+1) + q\,Q(j-1)$$

$$p\big(Q(j+1) - Q(j)\big) = q\big(Q(j) - Q(j-1)\big)$$

$$\Delta(j) = Q(j+1) - Q(j); \quad \Delta(j) = \frac{q}{p}\Delta(j-1)$$

$$\Delta(j) = \left(\frac{q}{p}\right)^j \Delta(0) = \left(\frac{q}{p}\right)^j Q(1)$$

$$\Delta(j) = \gamma^j Q(1)$$

$$1 = \sum_0^{n-1} \Delta(j) = Q(1)\sum_0^{n-1}\gamma^j = \begin{cases} n\,Q(1) & \gamma = 1 \Leftrightarrow p = \frac{1}{2} \\ Q(1)\dfrac{1-\gamma^n}{1-\gamma}, & \gamma \ne 1 \end{cases}$$

$$\therefore\; Q(1) = \begin{cases} \dfrac{1}{n} & \gamma = 1 \Leftrightarrow p = 1/2 \\ \dfrac{1-\gamma}{1-\gamma^n}, & \gamma \ne 1 \end{cases}$$

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$$q(v) = \sum_0^{v-1} \Delta(i) = q(1) \sum_0^{v-1} \gamma^i$$

(i) $\gamma = 1;\ q(1) = \frac{1}{n},\ \sum_0^{v-1} \gamma^i = v$

$$q(v) = \frac{v}{n} \quad p(v) = 1 - \frac{v}{n}$$

(ii) $\gamma \neq 1;\ q(1) = \frac{1-\gamma}{1-\gamma^n},\ \sum_0^{v-1} \gamma^i = \frac{1-\gamma^v}{1-\gamma}$

$$q(v) = q(1) \sum_0^{v-1} \gamma^i = \frac{1-\gamma}{1-\gamma^n} \frac{1-\gamma^v}{1-\gamma} = \frac{1-\gamma^v}{1-\gamma^n}$$

$$p(v) = \frac{\gamma^v - \gamma^n}{1-\gamma^n}$$

<u>Ex</u> $p = .4,\ \gamma = \frac{.6}{.4} = 1.5,\ v = 5,\ n = 8$

$$q(5) = \frac{1-\gamma^5}{1-\gamma^8} = \frac{(1.5)^5 - 1}{(1.5)^8 - 1} = \frac{1688}{6305} \approx .2677$$

$$p(5) = 1 - q(5) = \frac{4617}{6305} \approx .7323$$

Craps 7, 11 Player wins

2, 3, 12 player loses

otherwise (4, 5, 6, 8, 9, 10), if player rolls, for example 8, then he/she continues to roll until the 1st time an 8 or 7 appears If 8 player wins, if 7 house wins.

Find Pr(player wins)

Outcome	Pr(Outcome)	P(Win/Outcome)	P(Outcome)Pr(W/Outc
4, 10	each $\frac{3}{36}$	1/3	$2 \times \frac{1}{36} = \frac{1}{18}$
5, 9	each $\frac{4}{36}$	2/5	$2 \times \frac{2}{45} = \frac{4}{45}$
6, 8	each $\frac{5}{36}$	5/11	$2 \times \frac{25}{36(11)} = \frac{25}{198}$
7	1/6	1	1/6
11	1/18	1	1/18

$$Pr(Win) = \sum_K Pr(Win/Outcome\,K)\,Pr(K) = \frac{244}{495} \approx .4929$$