(Einstein summation convention)

Proof for $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}$. For 3-dimensional fundamental basis set, $e_1 = (1\,0\,0)^T$, $e_2 = (0\,1\,0)^T$, and $e_3 = (0\,0\,1)^T$. So, for an arbitrary unit vector $e_i = (\delta_{i1}\,\delta_{i2}\,\delta_{i3})^T$. Then,

$$\begin{aligned} e_i \cdot (e_j \times e_k) =& (\delta_{j2}\delta_{k3} - \delta_{j3}\delta_{k2})\delta_{i1} + (\delta_{j3}\delta_{k1} - \delta_{j1}\delta_{k3})\delta_{i2} + (\delta_{j1}\delta_{k2} - \delta_{j2}\delta_{k1})\delta_{i3} \\ =& \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \end{aligned}$$

And by definition $e_j \times e_k = \varepsilon_{jki}e_i = \varepsilon_{ijk}e_i$, then,

$$\varepsilon_{ijk} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix}$$

So,

$$\begin{aligned} \varepsilon_{ijk}\varepsilon_{lmn} =& \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \begin{vmatrix} \delta_{l1} & \delta_{l2} & \delta_{l3} \\ \delta_{m1} & \delta_{m2} & \delta_{m3} \\ \delta_{n1} & \delta_{n2} & \delta_{n3} \end{vmatrix} = \begin{vmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{vmatrix} \begin{vmatrix} \delta_{l1} & \delta_{m1} & \delta_{n1} \\ \delta_{l2} & \delta_{m2} & \delta_{n2} \\ \delta_{l3} & \delta_{m3} & \delta_{n3} \end{vmatrix} \\ =& \left| \begin{pmatrix} \delta_{i1} & \delta_{i2} & \delta_{i3} \\ \delta_{j1} & \delta_{j2} & \delta_{j3} \\ \delta_{k1} & \delta_{k2} & \delta_{k3} \end{pmatrix} \begin{pmatrix} \delta_{l1} & \delta_{m1} & \delta_{n1} \\ \delta_{l2} & \delta_{m2} & \delta_{n2} \\ \delta_{l3} & \delta_{m3} & \delta_{n3} \end{pmatrix} \right| \\ =& \left| \begin{pmatrix} \delta_{ip}\delta_{pl} & \delta_{ip}\delta_{pm} & \delta_{ip}\delta_{pn} \\ \delta_{jp}\delta_{pl} & \delta_{jp}\delta_{pm} & \delta_{jp}\delta_{pn} \\ \delta_{kp}\delta_{pl} & \delta_{kp}\delta_{pm} & \delta_{kp}\delta_{pn} \end{pmatrix} \right| \end{aligned}$$

Recall that $\delta_{ip}\delta_{pl} = \delta_{il}$. The above can be further written as,

$$\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

For $\varepsilon_{ijk}\varepsilon_{klm}$, one only needs to set $n \to k$, then,

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{lmk} =& \varepsilon_{ijk}\varepsilon_{klm} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{ik} \\ \delta_{jl} & \delta_{jm} & \delta_{jk} \\ \delta_{kl} & \delta_{km} & 1 \end{vmatrix} \\
=& \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} - (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})\delta_{km} + (\delta_{im}\delta_{jk} - \delta_{ik}\delta_{jm})\delta_{kl} \\
=& \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} - \delta_{il}\delta_{jk}\delta_{km} + \delta_{ik}\delta_{jl}\delta_{km} + \delta_{im}\delta_{jk}\delta_{kl} - \delta_{ik}\delta_{jm}\delta_{kl} \\
=& 3(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) - \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} + \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} \\
=& \delta_{il}\delta_{jm} - \delta_{jl}\delta_{im}
\end{aligned}
$$

Note that for the second last step, factor 3 comes from the summation over k. Always remember that Einstein summation convention is implemented throughout the proof. The $\varepsilon_{ijk}\varepsilon_{klm}$ here, is the a summation over k.