1.1 (1) $\frac{(1+i)^n}{(1-i)^{n+2}} = \frac{\sqrt{2}^n e^{i\frac{\pi}{4}n}}{\sqrt{2}^{n+2} e^{-i\frac{\pi}{4}(n+2)}} = \frac{1}{2} e^{i\frac{\pi}{4}(2n+2)}$

$= \frac{1}{2}(e^{i\frac{\pi}{2}})^{n+1}$

$= \frac{i^{n+1}}{2}$

(2) $\sqrt{i} = i^{\frac{1}{2}} = e^{i(\frac{\pi}{2}+2n\pi)\frac{1}{2}} = e^{i(\frac{\pi}{4}+n\pi)} = \cos(\frac{\pi}{4}+n\pi) + i\sin(\frac{\pi}{4}+n\pi)$

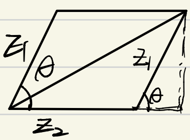
$\mathrm{Re}\sqrt{i} \begin{cases} = \frac{\sqrt{2}}{2}, & n \text{ is even} \\ = -\frac{\sqrt{2}}{2}, & n \text{ is odd} \end{cases}$

$\mathrm{Im}\sqrt{i} = \mathrm{Re}\sqrt{i}$

$\arg\sqrt{i} = \frac{\pi}{4} + n\pi$

1.2

(1) Let θ be the angle between z_1 and z_2



$|z_1 \pm z_2| = (|z_1|^2 + |z_2|^2 \pm 2|z_1||z_2|\cos(\theta))^{\frac{1}{2}}$

$\Rightarrow |z_1 + z_2| \geq |z_1 - z_2|$

$|z_1 - z_2| \geq (|z_1|^2 + |z_2|^2 - 2|z_1||z_2|)^{\frac{1}{2}} = \sqrt{(|z_1| - |z_2|)^2}$

Since $|z_1 - z_2| \geq 0$

$\Rightarrow |z_1 - z_2| \geq |z_1| - |z_2|$

$|z_1 + z_2| \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2| = |z_1| + |z_2|$

$\Rightarrow |z_1| - |z_2| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$

(2) $|z| = \sqrt{x^2+y^2} \leq \sqrt{(|x|+|y|)^2} = |x| + |y|$

$2(x^2+y^2) \geq x^2+y^2+2|x||y|$

$(|x|-|y|)^2 \geq 0$

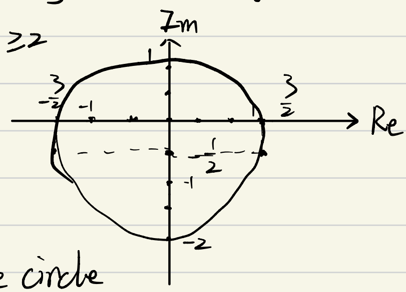
$\Rightarrow \frac{1}{\sqrt{2}}(|x|+|y|) \leq |z|$

1.3

(1) Let $z = x+iy$, $\mathrm{Im}\, z = y$, $|z| = x^2+y^2$

$\mathrm{Im}\, z + |z| = x^2+y^2+y \geq 2$

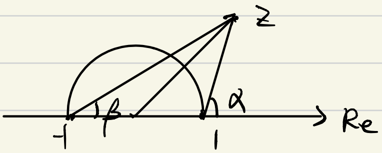
$x^2+(y+\frac{1}{2})^2 \geq (\frac{3}{2})^2$



$\{z : x+iy \mid x^2+(y+\frac{1}{2})^2 \geq \frac{9}{4}\}$

The area outside of the circle and the points on the circle

(2)



$\arg(z+1) = \alpha$, $\arg(z-1) = \beta$

$$\frac{z-1}{z+1} = \frac{x^2+y^2-1+i2y}{(x+1)^2+y^2} \Rightarrow 0 \leq \arg \frac{x^2+y^2-1+i2y}{(x+1)^2+y^2} < \frac{\pi}{2}$$

$\Rightarrow x^2+y^2-1 > 0$, $2y \geq 0$

1.4.

(1) de Moivre formula:

$$(\cos\theta + i\sin\theta)^n = e^{i\theta n} = e^{i(n\theta)} = \cos n\theta + i\sin n\theta$$

If θ is imaginary,

$$e^{i\theta} = e^{i(\theta_R + i\theta_I)} = e^{i\theta_R} e^{-\theta_I}$$

$$e^{i\theta n} = e^{i\theta_R n} e^{-\theta_I n}$$

$$= (\cos n\theta_R + i\sin n\theta_R)(\cos i n\theta_I + i\sin i n\theta_I)$$

$$= \cos n\theta_R \cos i n\theta_I + i\cos n\theta_R \sin i n\theta_I + i\sin n\theta_R \cos i n\theta_I$$

$$- \sin n\theta_R \sin i n\theta_I$$

$$= \cos(n\theta_R + ni\theta_I) + i\sin(n\theta_R + ni\theta_I)$$

$$= \cos n\theta + i\sin n\theta$$

Still holds

(2) $(z^n)^* = (z^*)^n$

Proof: $z^n = (x+iy)^n = \sum_{k=0} \binom{k}{n} x^k y^{n-k} i^{n-k}$

$$\Rightarrow (z^n)^* = \sum_{k=0} \binom{k}{n} x^k y^{n-k} (i^{n-k})^*$$

Note that $(i^n)^* = (-1)^n i^n$, since when n is even the i^n is real

$$(-1)^n i^n = (-i)^n$$

$$\Rightarrow (z^n)^* = \sum_{k=0} \binom{k}{n} x^k y^{n-k} (-i)^{n-k}$$

$$= \sum_{k=0} \binom{k}{n} x^k (-iy)^{n-k}$$

$$= (z^*)^n$$

1.5 (1) $\sin\phi + \sin 3\phi + \sin 5\phi + \cdots + \sin(2n-1)\phi$

$$= \frac{1}{2i}\left(e^{i\phi} - e^{-i\phi} + e^{3i\phi} - e^{-3i\phi} + \cdots + e^{(2n-1)i\phi} - e^{-(2n-1)i\phi}\right)$$

$$= \frac{1}{2i}\left[\frac{(1-e^{2ni\phi})}{1-e^{2i\phi}} e^{i\phi} - \frac{1-e^{-2ni\phi}}{1-e^{-2i\phi}} e^{-i\phi}\right]$$

$$= \frac{1}{2i} \frac{1-e^{2ni\phi} + 1 - e^{-2ni\phi}}{e^{-i\phi} - e^{i\phi}}$$

$$= \frac{1}{2i} \frac{2-2\cos 2n\phi}{-2i\sin\phi}$$

$$= \frac{1}{2} \frac{1-\cos 2n\phi}{\sin\phi}$$

(2) Similarly

$$\sum_{k=1}^{n} \cos(2k-1)\phi = \frac{1}{2} \frac{1-e^{2ni\phi} - 1 + e^{2ni\phi}}{e^{-i\phi} - e^{i\phi}}$$

$$= \frac{1}{2} \frac{\sin 2n\phi}{\sin\phi}$$

1.6. $\lim_{n\to\infty} \frac{2n+5i}{3n} \cos\frac{n\pi}{4} = \lim_{n\to\infty} \frac{2}{3}\cos\frac{n\pi}{4} + \frac{5}{3n} i \cos\frac{n\pi}{4}$

$$= \lim_{n\to\infty} \frac{2}{3}\cos\frac{n\pi}{4}$$

$\Rightarrow$ limit doesn't exist.