Recap interaction picture. $H = H_0 + H'$. Define time evolution operator with respect to H_0 as

$$\frac{\partial U_0(t,t_0)}{\partial t} = -\frac{i}{\hbar} H_0(t) U_0(t,t_0)$$

Integrating both sides from t_0 to t with $U_0(t_0,t_0) = 1$

$$\begin{aligned}
U_0(t,t_0) &= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_0(\tau) U_0(\tau,t_0) d\tau \\
&= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_0(\tau) d\tau + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} H_0(\tau) \int_{t_0}^{\tau} H_0(\tau') U_0(\tau',t_0) d\tau' d\tau \\
&= 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} d\tau_n \int_{t_0}^{\tau_n} d\tau_{n-1} \cdots \int_{t_0}^{\tau_2} d\tau_1 \, H_0(\tau_n) H_0(\tau_{n-1}) \cdots H_0(\tau_1) \\
&\equiv \exp_+ \left[-\frac{i}{\hbar} \int_{t_0}^{t} d\tau H_0(\tau) \right]
\end{aligned}$$

which is a positive time ordered exponential. In interaction picture,

$$|\psi_S(t)\rangle \equiv U_0(t,t_0) |\psi_I(t)\rangle$$

Plug back to TDSE,

$$\begin{aligned}
\frac{\partial U_0(t,t_0) |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H U_0(t,t_0) |\psi_I(t)\rangle \\
\left(\frac{\partial U_0(t,t_0)}{\partial t}\right) |\psi_I(t)\rangle + U_0(t,t_0) \frac{\partial |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H U_0(t,t_0) |\psi_I(t)\rangle \\
-\frac{i}{\hbar} H_0(t) U_0(t,t_0) |\psi_I(t)\rangle + U_0(t,t_0) \frac{\partial |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H U_0(t,t_0) |\psi_I(t)\rangle \\
U_0(t,t_0) \frac{\partial |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H'(t) U_0(t,t_0) |\psi_I(t)\rangle \\
\frac{\partial |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H'_I(t) |\psi_I(t)\rangle
\end{aligned}$$

where $H'_I(t) = U_0^\dagger(t,t_0) H'(t) U_0(t,t_0)$. We got a Schrödinger-like equation. Define interaction time evolution operator, $|\psi_I(t)\rangle = U_I(t,t_0) |\psi_I(t_0)\rangle$. One has,

$$
\begin{aligned}
|\psi_S(t)\rangle &= U_0(t,t_0)U_I(t,t_0)\,|\psi_I(t_0)\rangle \\
&= U_0(t,t_0)U_I(t,t_0)\,|\psi_S(t_0)\rangle \\
&= U(t,t_0)\,|\psi_S(t_0)\rangle
\end{aligned}
$$

where $|\psi_S(t_0)\rangle = U_0(t_0,t_0)\,|\psi_I(t_0)\rangle = |\psi_I(t_0)\rangle$ by definition. And $U(t,t_0)$ is the time evolution operator under Schr$\ddot{o}$dinger picture, it evolves the system with total Hamiltonian. Plug this back to the equation above, one gets the infinite series expansion of $U_I(t,t_0)$

$$
U_I(t,t_0) = \exp_+\left[-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_I'(\tau)\right]
$$

With this, one can write $U(t,t_0)$ in another way,

$$
\begin{aligned}
&U(t,t_0) \\
=&U_0(t,t_0)\exp_+\left[-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_I'(\tau)\right] \\
=&U_0(t,t_0)+U_0(t,t_0)\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t}d\tau_n\cdots\int_{t_0}^{\tau_2}d\tau_1\,H_I'(\tau_n)\cdots H_I'(\tau_1) \\
=&U_0(t,t_0)+\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t}d\tau_n\cdots\int_{t_0}^{\tau_2}d\tau_1\,U_0(t,t_0)U_0^\dagger(\tau_n,t_0)H'(\tau_n)U_0(\tau_n,t_0)\times \\
&U_0^\dagger(\tau_{n-1},t_0)H'(\tau_{n-1})U_0(\tau_{n-1},t_0)\cdots U_0^\dagger(\tau_1,t_0)H'(\tau_1)U_0(\tau_1,t_0) \\
=&U_0(t,t_0)+\sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n\int_{t_0}^{t}d\tau_n\cdots\int_{t_0}^{\tau_2}d\tau_1\,U_0(t,\tau_n)H'(\tau_n)U_0(\tau_n,\tau_{n-1})H'(\tau_{n-1})\times \\
&\cdots U_0(\tau_2,\tau_1)H'(\tau_1)U_0(\tau_1,t_0)
\end{aligned}
$$