1 The Magnus Expansion

1.1 The Cumulant Expansion

First, consider the cumulant expansion. Expand the operator or wavefunction perturbatively in some small parameter λ, $A = A_0 \sum_{n=0} \lambda^n A_n$. Personally, I think $A = \sum_{n=0} \frac{\lambda^n}{n!}\frac{\partial^n A}{\partial \lambda^n} = A_0 \sum \lambda^n \frac{\partial^n A}{A_0 n! \partial \lambda^n}$.

Given ansatz: $A = A_0 \exp(F) = A_0 \exp(\sum_{n=1} \lambda^n F_n)$. Finding the same order term, one gets,

$$\begin{aligned}
\sum_{n'=0}^{N'} \lambda^{n'} A_{n'} = \exp\left(\sum_{n=1} \lambda^n F_n\right) &= \sum_{n=0}^{N} \frac{1}{n!}\left(\sum_{m=1}^{M} \lambda^m F_m\right)^n \\
&= \sum_{n=0}^{N} \frac{1}{n!} \sum_{k_1+\cdots+k_M=n} \prod_{j}^{m} \left(\lambda^j F_j\right)^{k_j} \frac{n!}{k_1!\ldots k_M!} \\
&= \sum_{n=0}^{N} \sum_{k_1+\cdots+k_M=n} \frac{1}{k_1!\ldots k_M!} \prod_{j}^{m} \left(\lambda^j F_j\right)^{k_j}
\end{aligned}$$

where N' and N represent the order of $\sum_{n'=0} \lambda^{n'} A_{n'}$ and the order of the expansion of $\exp(F)$ to truncate. Note that even in the case $N = 1$, the right side still contains every order of λ, N does not reflect the order of λ on the right side. Take $N = 3 = N'$ as an example,

$$\begin{aligned}
&1 + \lambda A_1 + \lambda^2 A_2 + \lambda^3 A_3 \\
=&1 + \lambda F_1 + \lambda^2 F_2 + \lambda^3 F_3 + \frac{1}{2}\left(\lambda^2 F_1^2 + \lambda^3 F_1 F_2 + \lambda^3 F_2 F_1\right) + \frac{1}{3!}\lambda^3 F_1^3
\end{aligned}$$

Equate the terms with the same order,

$$
\begin{aligned}
F_1 &= A_1 \\
F_2 &= A_2 - \frac{1}{2}A_1^2 \\
F_3 &= A_3 - \frac{1}{2}(F_2F_1 + F_1F_2) - \frac{1}{6}F_1^3 \\
&= A_3 - \frac{1}{2}\left[(A_2 - \frac{1}{2}A_1^2)A_1 + A_1(A_2 - \frac{1}{2}A_1^2)\right] - \frac{1}{6}A_1^3 \\
&= A_3 - \frac{1}{2}\left(A_2A_1 + A_1A_2\right) - \frac{2}{3}A_1^3
\end{aligned}
$$

1.2 Time-Ordered Exponential Operator

The Magnus expansion for the time-ordered exponential operator:

$$
\exp_+\left[\lambda\int_{t_0}^{t} d\tau A(\tau)\right] \equiv \exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}\lambda^n F_n(t,t_0)\right]
$$

As one will see the factorial added here is to cancel out the prefactor in F_n, which simplifies the expression.

$$
\begin{aligned}
\exp_+\left[\lambda\int_{t_0}^{t} d\tau A(\tau)\right] &= 1 + \sum_n \lambda^n \int_{t_0}^{t} d\tau_n \cdots \int_{t_0}^{\tau_2} d\tau_1 A(\tau_n)\ldots A(\tau_1) \\
&\equiv 1 + \sum_n \lambda^n A_n
\end{aligned}
$$

which is not too complicated. Now, consider the expansion of F_n

$$
\begin{aligned}
\exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}\lambda^n F_n(t,t_0)\right] &= \sum_{m=0}\frac{1}{m!}\left(\sum_{n=1}^{\infty}\frac{1}{n!}\lambda^n F_n(t,t_0)\right)^m \\
&= 1 + \sum_{n=1}\frac{1}{n!}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\lambda^m F_m(t,t_0)\right)^n \\
&= 1 + \sum_{m=1}^{\infty}\frac{1}{m!}\lambda^m F_m(t,t_0) + \frac{1}{2!}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\lambda^m F_m(t,t_0)\right)^2 \\
&+ \sum_{n=3}\frac{1}{n!}\left(\sum_{m=1}^{\infty}\frac{1}{m!}\lambda^m F_m(t,t_0)\right)^n
\end{aligned}
$$

As one can see the lowest order of λ in each term equals n. For simplicity, the verification ends at the 2nd-order term, but the expression of the 3rd-order term will be given.

$$\begin{aligned}
&1+\lambda A_1+\lambda^2 A_2+\lambda^3 A_3\\
=&1+\left(\lambda F_1+\frac{1}{2!}\lambda^2 F_2+\frac{1}{3!}\lambda^3 F_3\right)+\frac{1}{2!}\left(\lambda^2 F_1^2+\frac{1}{2!}\lambda^3(F_2F_1+F_1F_2)\right)+\frac{1}{3!}\lambda^3 F_1^3
\end{aligned}$$

Again, equate terms in the same order,

$$\begin{aligned}
F_1 &= A_1\\
\frac{1}{2!}F_2 &= A_2-\frac{1}{2!}F_1^2 = A_2-\frac{1}{2}A_1^2\\
\frac{1}{3!}F_3 &= A_3-\frac{1}{2!}\frac{1}{2!}(F_2F_1+F_1F_2)-\frac{1}{3!}F_1^3
\end{aligned}$$

To get F_2, one needs to calculate A_1^2,

$$\begin{aligned}
A_1^2 &= \int_{t_0}^{t} d\tau_2 \int_{t_0}^{t} d\tau_1 A(\tau_2)A(\tau_1)\\
&= \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 A(\tau_2)A(\tau_1) + \int_{t_0}^{t} d\tau_2 \int_{\tau_2}^{t} d\tau_1 A(\tau_2)A(\tau_1)\\
&= A_2 + \int_{t_0}^{t} d\tau_2 \int_{\tau_2}^{t} d\tau_1 A(\tau_2)A(\tau_1)\\
&= A_2 + \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 A(\tau_2)A(\tau_1)\\
&= A_2 + \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 A(\tau_1)A(\tau_2)
\end{aligned}$$

where the order of integral is changed in the 2nd last step, and labels of dummy variables are changed in the last step. One has

$$\begin{aligned}
F_2(t,t_0) &= \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \left[A(\tau_2)A(\tau_1)-A(\tau_1)A(\tau_2)\right]\\
&= \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \left[A(\tau_2),A(\tau_1)\right]
\end{aligned}$$

Similarly,

$$F_3(t,t_0) = \int_{t_0}^{t} d\tau_3 \int_{t_0}^{\tau_3} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \left\{A(\tau_3)\left[A(\tau_2),A(\tau_1)\right]+\left[A(\tau_3),A(\tau_2)\right]A(\tau_1)\right\}$$

1.3 Time Evolution Operator in Interaction Picture

For the time evolution operator in the interaction picture,

$$U_I(t,t_0) = \exp_+\left[\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_I'(\tau)\right] \equiv \exp\left[\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n F_n(t,t_0)\right]$$

Substitute the corresponding terms back into the relation above,

$$\begin{aligned}
F_1 &= \int_{t_0}^{t} d\tau H_I'(\tau) \\
F_2(t,t_0) &= \int_{t_0}^{t} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \left[H_I'(\tau_2), H_I'(\tau_1)\right] \\
F_3(t,t_0) &= \int_{t_0}^{t} d\tau_3 \int_{t_0}^{\tau_3} d\tau_2 \int_{t_0}^{\tau_2} d\tau_1 \\
&\quad \left\{H_I'(\tau_3)\left[H_I'(\tau_2), H_I'(\tau_1)\right] + \left[H_I'(\tau_3), H_I'(\tau_2)\right] H_I'(\tau_1)\right\}
\end{aligned}$$