Recap interaction picture. $H = H_0 + H'$. Define time evolution operator with respect to H_0 as

$$\frac{\partial U_0(t,t_0)}{\partial t} = -\frac{i}{\hbar} H_0(t) U_0(t,t_0)$$

Integrating both sides from t_0 to t with $U_0(t_0,t_0) = 1$

$$\begin{aligned}
U_0(t,t_0) &= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_0(\tau) U_0(\tau,t_0) d\tau \\
&= 1 - \frac{i}{\hbar}\int_{t_0}^{t} H_0(\tau) d\tau + \left(\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} H_0(\tau) \int_{t_0}^{\tau} H_0(\tau') U_0(\tau',t_0) d\tau' d\tau \\
&= 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t}\int_{t_0}^{\tau_2}\cdots\int_{t_0}^{\tau_n} H_0(\tau) H_0(\tau_1)\cdots H_0(\tau_n)\, d\tau_n \cdots d\tau_1 d\tau \\
&\equiv \exp_+\left[-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H(\tau)\right]
\end{aligned}$$

which is a positive time ordered exponential. In interaction picture,

$$|\psi_S(t)\rangle \equiv U_0(t,t_0)\, |\psi_I(t)\rangle$$

Plug back to TDSE,

$$\begin{aligned}
\frac{\partial U_0(t,t_0)\, |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H U_0(t,t_0)\, |\psi_I(t)\rangle \\
\left(\frac{\partial U_0(t,t_0)}{\partial t}\right) |\psi_I(t)\rangle + U_0(t,t_0)\frac{\partial\, |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H U_0(t,t_0)\, |\psi_I(t)\rangle \\
-\frac{i}{\hbar} H_0(t) U_0(t,t_0)\, |\psi_I(t)\rangle + U_0(t,t_0)\frac{\partial\, |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H U_0(t,t_0)\, |\psi_I(t)\rangle \\
U_0(t,t_0)\frac{\partial\, |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H'(t) U_0(t,t_0)\, |\psi_I(t)\rangle \\
\frac{\partial\, |\psi_I(t)\rangle}{\partial t} &= -\frac{i}{\hbar} H_I'(t)\, |\psi_I(t)\rangle
\end{aligned}$$

where $H_I'(t) = U_0^\dagger(t,t_0) H'(t) U_0(t,t_0)$. We got a Schr$\ddot{o}$dinger-like equation.