1 Ex 1.4

$$
\begin{aligned}
\hat{n}_n^2 &= \hat{n}_n \hat{d}_n^\dagger \hat{d}_n = (\hat{d}_n^\dagger \hat{n}_n + \delta_{nn} \hat{d}_n^\dagger)\hat{d}_n = \hat{d}_n^\dagger \hat{n}_n \hat{d}_n + \hat{d}_n^\dagger \hat{d}_n \\
&= \hat{d}_n^\dagger \hat{d}_n^\dagger \hat{d}_n \hat{d}_n + \hat{d}_n^\dagger \hat{d}_n \\
&= \hat{d}_n^\dagger \hat{d}_n
\end{aligned}
$$

where $\hat{d}_n^\dagger \hat{d}_n^\dagger = \hat{d}_n \hat{d}_n = 0$ for fermions.

2 Ex 1.5

$$
\begin{aligned}
\hat{N} &= \int dx\, \hat{\psi}^{\dagger}(x)\hat{\psi}(x) \\
&= \sum_{nm} \int dx\ \langle n|x\rangle\, \hat{d}_n^{\dagger}\, \langle x|m\rangle\, \hat{d}_m \\
&= \sum_{nm} \int dx\ \langle n|x\rangle\, \langle x|m\rangle\, \hat{d}_n^{\dagger}\hat{d}_m \\
&= \sum_{nm} \langle n|m\rangle\, \hat{d}_n^{\dagger}\hat{d}_m = \sum_{nm} \delta_{nm}\hat{d}_n^{\dagger}\hat{d}_m \\
&= \sum_{n} \hat{d}_n^{\dagger}\hat{d}_n
\end{aligned}
$$

Not sure what he means by a generic ket, assume it is (anti)symmetrized. $\varepsilon = \pm 1$, $+1$ for symmetry basis. Consider an arbitrary N-particle basis:

$$
\hat{N}\, |\lambda_a\rangle_1\, |\lambda_b\rangle_2 \cdots |\lambda_z\rangle_N = \sum_j N_j\, |\lambda_a\rangle_1\, |\lambda_b\rangle_2 \cdots |\lambda_z\rangle_N = N\, |\lambda_a\rangle_1\, |\lambda_b\rangle_2 \cdots |\lambda_z\rangle_N
$$

Then,

$$
\begin{aligned}
|\Psi_N\rangle &= \frac{1}{\sqrt{N!N_a!N_b!\cdots N_z!}} \sum_s P_s \varepsilon^{p_s}\, |\lambda_a\rangle_1\, |\lambda_b\rangle_2 \cdots |\lambda_z\rangle_N \\
\hat{N}\, |\Psi_N\rangle &= \frac{1}{\sqrt{N!N_a!N_b!\cdots N_z!}} \sum_s P_s \varepsilon^{p_s} \hat{N}\, |\lambda_a\rangle_1\, |\lambda_b\rangle_2 \cdots |\lambda_z\rangle_N \\
&= \frac{N}{\sqrt{N!N_a!N_b!\cdots N_z!}} \sum_s P_s \varepsilon^{p_s}\, |\lambda_a\rangle_1\, |\lambda_b\rangle_2 \cdots |\lambda_z\rangle_N \\
&= N\, |\Psi_N\rangle
\end{aligned}
$$

3 Ex 1.6

The second quantization form of $\hat{H}_0$ and $\hat{H}_{\mathsf{int}}$:

$$\hat{H}_0 = \int dx\, dx'\, \hat{\psi}^\dagger(x) \langle x | \hat{h} | x' \rangle \hat{\psi}(x')$$
$$\hat{H}_{\mathsf{int}} = \frac{1}{2} \int dx\, dx'\, v(x,x') \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') \hat{\psi}(x') \hat{\psi}(x)$$

Then,

$$\begin{aligned}
[\hat{N}, \hat{H}_0]_- &= [\int dx\, \hat{\psi}^\dagger(x)\hat{\psi}(x), \int dx'\, dx''\, \hat{\psi}^\dagger(x') \langle x' | \hat{h} | x'' \rangle \hat{\psi}(x'')] \\
&= \int dx\, dx'\, dx''\ \langle x' | \hat{h} | x'' \rangle [\hat{\psi}^\dagger(x)\hat{\psi}(x), \hat{\psi}^\dagger(x')\hat{\psi}(x'')] \\
&= \int dx\, dx'\, dx''\ \langle x' | \hat{h} | x'' \rangle [\hat{n}(x), \hat{\psi}^\dagger(x')\hat{\psi}(x'')] \\
&= \int dx\, dx'\, dx''\ \langle x' | \hat{h} | x'' \rangle \Big([\hat{n}(x), \hat{\psi}^\dagger(x')]\hat{\psi}(x'') + \hat{\psi}^\dagger(x')[\hat{n}(x), \hat{\psi}(x'')] \Big) \\
&= \int dx\, dx'\, dx''\ \langle x' | \hat{h} | x'' \rangle \Big(\delta(x - x')\hat{\psi}^\dagger(x')\hat{\psi}(x'') - \delta(x - x'')\hat{\psi}^\dagger(x')\hat{\psi}(x'') \Big) \\
&= \int dx'\, dx''\ \langle x' | \hat{h} | x'' \rangle \Big(\hat{\psi}^\dagger(x')\hat{\psi}(x'') - \hat{\psi}^\dagger(x')\hat{\psi}(x'') \Big) \\
&= 0
\end{aligned}$$

Then, for the interacting Hamiltonian,

$$
\begin{aligned}
&[\hat{N}, \hat{H}_{\mathsf{int}}]_- \\
=&[\int dx\, \hat{\psi}^\dagger(x)\hat{\psi}(x), \frac{1}{2}\int dx'\, dx''\, v(x',x'')\hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x')] \\
=&\frac{1}{2}\int dx\, dx'\, dx''\, v(x',x'')[\hat{\psi}^\dagger(x)\hat{\psi}(x), \hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x')] \\
=&\frac{1}{2}\int dx\, dx'\, dx''\, v(x',x'')[\hat{n}(x), \hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x')] \\
=&\frac{1}{2}\int dx\, dx'\, dx''\, v(x',x'')([\hat{n}(x), \hat{\psi}^\dagger(x')]\hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x') \\
&+ \hat{\psi}^\dagger(x')[\hat{n}(x), \hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x')]) \\
=&\frac{1}{2}\int dx\, dx'\, dx''\, v(x',x'')(\delta(x-x')\hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x') \\
&+ \hat{\psi}^\dagger(x')[\hat{n}(x), \hat{n}(x'')\hat{\psi}(x')]) \\
=&\frac{1}{2}\int dx\, dx'\, dx''\, v(x',x'')(\delta(x-x')\hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x') \\
&+ \hat{\psi}^\dagger(x')\left([\hat{n}(x), \hat{n}(x'')]\hat{\psi}(x') + \hat{n}(x'')[\hat{n}(x), \hat{\psi}(x')]\right)) \\
=&\frac{1}{2}\int dx\, dx'\, dx''\, v(x',x'')(\delta(x-x')\hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x') \\
&- \delta(x-x')\hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x')\hat{\psi}(x')) \\
=&\frac{1}{2}\int dx'\, dx''\, v(x',x'')\left(\hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x'')\hat{\psi}(x') - \hat{\psi}^\dagger(x')\hat{\psi}^\dagger(x'')\hat{\psi}(x')\hat{\psi}(x')\right) \\
=&0
\end{aligned}
$$

4 Ex 1.7

Not sure how the operators work. I guess $\hat{n}_{s\uparrow} \equiv \hat{d}^{\dagger}_{s\uparrow}\hat{d}_{s\uparrow}$. Clearly, $\hat{n}_{s\uparrow}|0\rangle = 0$ and $\hat{n}_{s\downarrow}|0\rangle = 0$. Then,

$$\begin{aligned}
\hat{S}^z_s |s\uparrow\rangle &= \frac{1}{2}\left(\hat{n}_{s\uparrow} - \hat{n}_{s\downarrow}\right)\hat{d}^{\dagger}_{s\uparrow}|0\rangle = \frac{1}{2}\left(\hat{n}_{s\uparrow}\hat{d}^{\dagger}_{s\uparrow} - \hat{n}_{s\downarrow}\hat{d}^{\dagger}_{s\uparrow}\right)|0\rangle \\
&= \frac{1}{2}\left(\delta_{\uparrow\uparrow}\hat{d}^{\dagger}_{s\uparrow} + \hat{d}^{\dagger}_{s\uparrow}\hat{n}_{s\uparrow} - \delta_{\downarrow\uparrow}\hat{d}^{\dagger}_{s\uparrow} - \hat{d}^{\dagger}_{s\uparrow}\hat{n}_{s\downarrow}\right)|0\rangle \\
&= \frac{1}{2}\hat{d}^{\dagger}_{s\uparrow}|0\rangle = \frac{1}{2}|s\uparrow\rangle \\
\hat{S}^+_s |s\uparrow\rangle &= \hat{d}^{\dagger}_{s\uparrow}\hat{d}_{s\downarrow}\hat{d}^{\dagger}_{s\uparrow}|0\rangle = \hat{d}^{\dagger}_{s\uparrow}\left(\delta_{\downarrow\uparrow} - \hat{d}^{\dagger}_{s\uparrow}\hat{d}_{s\downarrow}\right)|0\rangle \\
&= \hat{d}^{\dagger}_{s\uparrow}\hat{d}^{\dagger}_{s\uparrow}|\emptyset\rangle = |\emptyset\rangle \\
\hat{S}^-_s |s\uparrow\rangle &= \hat{d}^{\dagger}_{s\downarrow}\hat{d}_{s\uparrow}\hat{d}^{\dagger}_{s\uparrow}|0\rangle = \hat{d}^{\dagger}_{s\downarrow}\left(\delta_{\uparrow\uparrow} - \hat{d}^{\dagger}_{s\uparrow}\hat{d}_{s\uparrow}\right)|0\rangle \\
&= \hat{d}^{\dagger}_{s\downarrow}|0\rangle = |s\downarrow\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{S}^z_s |s\downarrow\rangle &= \frac{1}{2}\left(\hat{n}_{s\uparrow} - \hat{n}_{s\downarrow}\right)\hat{d}^{\dagger}_{s\downarrow}|0\rangle = \frac{1}{2}\left(\hat{n}_{s\uparrow}\hat{d}^{\dagger}_{s\downarrow} - \hat{n}_{s\downarrow}\hat{d}^{\dagger}_{s\downarrow}\right)|0\rangle \\
&= \frac{1}{2}\left(\delta_{\uparrow\downarrow}\hat{d}^{\dagger}_{s\downarrow} + \hat{d}^{\dagger}_{s\downarrow}\hat{n}_{s\uparrow} - \delta_{\downarrow\downarrow}\hat{d}^{\dagger}_{s\downarrow} - \hat{d}^{\dagger}_{s\downarrow}\hat{n}_{s\downarrow}\right)|0\rangle \\
&= -\frac{1}{2}\hat{d}^{\dagger}_{s\downarrow}|0\rangle = -\frac{1}{2}|s\downarrow\rangle
\end{aligned}$$

Using the results above,

$$\begin{aligned}
\hat{S}^+_s |s\downarrow\rangle &= \left(\langle s\downarrow|\,\hat{S}^-_s\right)^{\dagger} \\
\Rightarrow \langle s\uparrow|\,\hat{S}^+_s |s\downarrow\rangle &= \left(\langle s\downarrow|\,\hat{S}^-_s |s\uparrow\rangle\right)^{\dagger} = \left(\langle s\downarrow|s\downarrow\rangle\right)^{\dagger} = 1 \\
\Rightarrow \hat{S}^+_s |s\downarrow\rangle &= |s\uparrow\rangle
\end{aligned}$$

Then,

$$\begin{aligned}
\hat{S}^-_s |s\downarrow\rangle &= \left(\langle s\downarrow|\,\hat{S}^+_s\right)^{\dagger} \\
\Rightarrow \langle s\uparrow|\,\hat{S}^-_s |s\downarrow\rangle &= \left(\langle s\downarrow|\,\hat{S}^+_s |s\uparrow\rangle\right)^{\dagger} = \left(\langle s\downarrow|\emptyset\rangle\right)^{\dagger} = 0 \\
\Rightarrow \langle s\downarrow|\,\hat{S}^-_s |s\downarrow\rangle &= \left(\langle s\downarrow|\,\hat{S}^+_s |s\downarrow\rangle\right)^{\dagger} = \left(\langle s\downarrow|s\uparrow\rangle\right)^{\dagger} = 0
\end{aligned}$$

It indicates that $\hat{S}_s^- \, |s\downarrow\rangle$ is orthogonal to both $|s\uparrow\rangle$ and $|s\downarrow\rangle$. Thus, $\hat{S}_s^- \, |s\downarrow\rangle = |\emptyset\rangle$.

$\hat{S}_s^z$ corresponds to σ^z, $\hat{S}_s^+$ and $\hat{S}_s^-$ correspond to the ladder operators for spin projection.

5 Ex 1.8

Let σ_- be the reverse of σ

$$\begin{aligned}
\frac{1}{2}\sum_{\sigma\sigma'} \hat{d}^{\dagger}_{s\sigma}\sigma^x_{\sigma\sigma'}\hat{d}^{\dagger}_{s\sigma'} &= \frac{1}{2}\sum_{\sigma\sigma'} \hat{d}^{\dagger}_{s\sigma}\delta_{\sigma\sigma'_-}\hat{d}^{\dagger}_{s\sigma'} = \frac{1}{2}\sum_{\sigma} \hat{d}^{\dagger}_{s\sigma}\hat{d}^{\dagger}_{s\sigma_-} \\
&= \frac{1}{2}\left(\hat{d}^{\dagger}_{s\uparrow}\hat{d}^{\dagger}_{s\downarrow} + \hat{d}^{\dagger}_{s\downarrow}\hat{d}^{\dagger}_{s\uparrow}\right) \\
&= \frac{1}{2}\left(\hat{S}^{+}_{s} + \hat{S}^{-}_{s}\right) = \hat{S}^{x}_{s}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\sum_{\sigma\sigma'} \hat{d}^{\dagger}_{s\sigma}\sigma^y_{\sigma\sigma'}\hat{d}^{\dagger}_{s\sigma'} &= \frac{1}{2}\sum_{\sigma}\left(\hat{d}^{\dagger}_{s\sigma}\left\langle\sigma\right|\sigma^y\left|\uparrow\right\rangle\hat{d}^{\dagger}_{s\uparrow} + \hat{d}^{\dagger}_{s\sigma}\left\langle\sigma\right|\sigma^y\left|\downarrow\right\rangle\hat{d}^{\dagger}_{s\downarrow}\right) \\
&= \frac{1}{2}\sum_{\sigma} i\left(\hat{d}^{\dagger}_{s\sigma}\left\langle\sigma|\downarrow\right\rangle\hat{d}^{\dagger}_{s\uparrow} - \hat{d}^{\dagger}_{s\sigma}\left\langle\sigma|\uparrow\right\rangle\hat{d}^{\dagger}_{s\downarrow}\right) \\
&= \frac{i}{2}\left(\hat{d}^{\dagger}_{s\downarrow}\hat{d}^{\dagger}_{s\uparrow} - \hat{d}^{\dagger}_{s\uparrow}\hat{d}^{\dagger}_{s\downarrow}\right) = \frac{1}{2i}\left(\hat{d}^{\dagger}_{s\uparrow}\hat{d}^{\dagger}_{s\downarrow} - \hat{d}^{\dagger}_{s\downarrow}\hat{d}^{\dagger}_{s\uparrow}\right) \\
&= \frac{1}{2i}\left(\hat{S}^{+}_{s} - \hat{S}^{-}_{s}\right) = \hat{S}^{y}_{s}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\sum_{\sigma\sigma'} \hat{d}^{\dagger}_{s\sigma}\sigma^z_{\sigma\sigma'}\hat{d}^{\dagger}_{s\sigma'} &= \frac{1}{2}\sum_{\sigma}\left(\hat{d}^{\dagger}_{s\sigma}\left\langle\sigma\right|\sigma^z\left|\uparrow\right\rangle\hat{d}^{\dagger}_{s\uparrow} + \hat{d}^{\dagger}_{s\sigma}\left\langle\sigma\right|\sigma^z\left|\downarrow\right\rangle\hat{d}^{\dagger}_{s\downarrow}\right) \\
&= \frac{1}{2}\sum_{\sigma}\left(\hat{d}^{\dagger}_{s\sigma}\left\langle\sigma|\uparrow\right\rangle\hat{d}^{\dagger}_{s\uparrow} - \hat{d}^{\dagger}_{s\sigma}\left\langle\sigma|\downarrow\right\rangle\hat{d}^{\dagger}_{s\downarrow}\right) \\
&= \frac{1}{2}\left(\hat{d}^{\dagger}_{s\uparrow}\hat{d}^{\dagger}_{s\uparrow} - \hat{d}^{\dagger}_{s\downarrow}\hat{d}^{\dagger}_{s\downarrow}\right) \\
&= \frac{1}{2}\left(\hat{n}_{s\uparrow} - \hat{n}_{s\downarrow}\right) = \hat{S}^{z}_{s}
\end{aligned}$$

Then,

$$\begin{aligned}
[\hat{S}^i_s, \hat{S}^j_{s'}]_- &= \frac{1}{4}\sum_{\sigma\sigma'}\sum_{\tau\tau'}\left[\hat{d}^{\dagger}_{s\sigma}\sigma^i_{\sigma\sigma'}\hat{d}_{s\sigma'}, \hat{d}^{\dagger}_{s'\tau}\sigma^j_{\tau\tau'}\hat{d}_{s'\tau'}\right] \\
&= \frac{1}{4}\sum_{\sigma\sigma'}\sum_{\tau\tau'}\sigma^i_{\sigma\sigma'}\sigma^j_{\tau\tau'}\left[\hat{d}^{\dagger}_{s\sigma}\hat{d}_{s\sigma'}, \hat{d}^{\dagger}_{s'\tau}\hat{d}_{s'\tau'}\right] \\
&= \frac{1}{4}\sum_{\sigma\sigma'}\sum_{\tau\tau'}\sigma^i_{\sigma\sigma'}\sigma^j_{\tau\tau'}\left(\left[\hat{d}^{\dagger}_{s\sigma}\hat{d}_{s\sigma'}, \hat{d}^{\dagger}_{s'\tau}\right]\hat{d}_{s'\tau'} + \hat{d}^{\dagger}_{s'\tau}\left[\hat{d}^{\dagger}_{s\sigma}\hat{d}_{s\sigma'}, \hat{d}_{s'\tau'}\right]\right)
\end{aligned}$$

$$
\begin{aligned}
\left[\hat{d}^\dagger_{s\sigma}\hat{d}_{s\sigma'}, \hat{d}^\dagger_{s'\tau}\right]\hat{d}_{s'\tau'} &= \hat{d}^\dagger_{s\sigma}\left[\hat{d}_{s\sigma'}, \hat{d}^\dagger_{s'\tau}\right]\hat{d}_{s'\tau'} + \left[\hat{d}^\dagger_{s\sigma}, \hat{d}^\dagger_{s'\tau}\right]\hat{d}_{s\sigma'}\hat{d}_{s'\tau'} \\
&= \delta_{ss'}\delta_{\sigma'\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau'} + 2\delta_{ss'}\hat{d}^\dagger_{s\sigma}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'}\hat{d}_{s'\tau'}
\end{aligned}
$$

$$
\begin{aligned}
\hat{d}^\dagger_{s'\tau}\left[\hat{d}^\dagger_{s\sigma}\hat{d}_{s\sigma'}, \hat{d}_{s'\tau'}\right] &= \hat{d}^\dagger_{s'\tau}\hat{d}^\dagger_{s\sigma}\left[\hat{d}_{s\sigma'}, \hat{d}_{s'\tau'}\right] + \hat{d}^\dagger_{s'\tau}\left[\hat{d}^\dagger_{s\sigma}, \hat{d}_{s'\tau'}\right]\hat{d}_{s\sigma'} \\
&= 2\delta_{ss'}\hat{d}^\dagger_{s'\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s\sigma'}\hat{d}_{s'\tau'} - \delta_{ss'}\delta_{\sigma\tau'}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'} \\
&= -\delta_{ss'}\delta_{\sigma\tau'}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'} - 2\delta_{ss'}\hat{d}^\dagger_{s\sigma}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'}\hat{d}_{s'\tau'}
\end{aligned}
$$

Plugging them back,

$$
\begin{aligned}
&[\hat{S}^i_s, \hat{S}^j_{s'}]_- \\
=&\frac{1}{4}\delta_{ss'}\sum_{\sigma\sigma'}\sum_{\tau\tau'}\sigma^i_{\sigma\sigma'}\sigma^j_{\tau\tau'}\left(\delta_{ss'}\delta_{\sigma'\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau'} - \delta_{ss'}\delta_{\sigma\tau'}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'}\right) \\
=&\frac{1}{4}\delta_{ss'}\sum_{\sigma\sigma'}\sigma^i_{\sigma\sigma'}\left[\sum_{\tau\tau'}\sigma^j_{\tau\tau'}\left(\delta_{\sigma'\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau'} - \delta_{\sigma\tau'}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'}\right)\right] \\
=&\frac{1}{4}\delta_{ss'}\sum_{\sigma\sigma'}\sigma^i_{\sigma\sigma'}\left(\sum_{\tau'}\sigma^j_{\sigma'\tau'}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau'} - \sum_{\tau}\sigma^j_{\tau\sigma}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'}\right) \\
=&\frac{1}{4}\delta_{ss'}\sum_{\sigma\sigma'}\sigma^i_{\sigma\sigma'}\left(\sum_{\tau}\sigma^j_{\sigma'\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau} - \sum_{\tau}\sigma^j_{\tau\sigma}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'}\right) \\
=&\frac{1}{4}\delta_{ss'}\sum_{\tau}\sum_{\sigma\sigma'}\left(\sigma^i_{\sigma\sigma'}\sigma^j_{\sigma'\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau} - \sigma^j_{\tau\sigma}\sigma^i_{\sigma\sigma'}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'}\right) \\
=&\frac{1}{4}\delta_{ss'}\sum_{\tau}\left[\sum_{\sigma}(\sigma^i\sigma^j)_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau} - \sum_{\sigma'}(\sigma^j\sigma^i)_{\tau\sigma'}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma'}\right] \\
=&\frac{1}{4}\delta_{ss'}\sum_{\tau}\left[\sum_{\sigma}(\sigma^i\sigma^j)_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau} - \sum_{\sigma}(\sigma^j\sigma^i)_{\tau\sigma}\hat{d}^\dagger_{s'\tau}\hat{d}_{s\sigma}\right] \\
=&\frac{1}{4}\delta_{ss'}\left[\sum_{\tau\sigma}(\sigma^i\sigma^j)_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s'\tau} - \sum_{\tau\sigma}(\sigma^j\sigma^i)_{\sigma\tau}\hat{d}^\dagger_{s'\sigma}\hat{d}_{s\tau}\right]
\end{aligned}
$$

Note that $\delta_{ss'}\hat{d}^\dagger_{s'\sigma'} = \delta_{ss'}\hat{d}^\dagger_{s\sigma'}$. Then,

$$
\begin{aligned}
&[\hat{S}^i_s, \hat{S}^j_{s'}]_- \\
=&\frac{1}{4}\delta_{ss'}\left[\sum_{\tau\sigma}(\sigma^i\sigma^j)_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s\tau} - \sum_{\tau\sigma}(\sigma^j\sigma^i)_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s\tau}\right] \\
=&\frac{1}{4}\delta_{ss'}\sum_{\tau\sigma}(\sigma^i\sigma^j - \sigma^j\sigma^i)_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s\tau} \\
=&\frac{1}{4}\delta_{ss'}\sum_{\tau\sigma}\left(\left[\sigma^i, \sigma^j\right]\right)_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s\tau} \\
=&\frac{1}{4}\delta_{ss'}\sum_{\tau\sigma}\left(2i\varepsilon_{ijk}\sigma^k\right)_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s\tau} \\
=&i\delta_{ss'}\varepsilon_{ijk}\frac{1}{2}\sum_{\tau\sigma}\sigma^k_{\sigma\tau}\hat{d}^\dagger_{s\sigma}\hat{d}_{s\tau} \\
=&i\delta_{ss'}\varepsilon_{ijk}\hat{S}^k_s = i\delta_{ss'}\sum_{k=x,y,z}\varepsilon_{ijk}\hat{S}^k_s
\end{aligned}
$$