Proof for the left-to-reader exercise:

To avoid the confusion,

$$\hat{h} \to \hat{h}_j$$

$\hat{h}_j$ is the noninteracting Hamiltonian acting on jth particle.

$$\begin{aligned}
\hat{H}_0 \left|n_1 \dots n_N\right\rangle =& \frac{1}{\sqrt{N!}}\Big(\sum_j \hat{h}_j \left|n_1\right\rangle \left|n_2\right\rangle \dots \left|n_N\right\rangle + \sum_j \hat{h}_j \left|n_2\right\rangle \left|n_1\right\rangle \dots \left|n_N\right\rangle \\
& + \dots + \sum_j \hat{h}_j \left|n_N\right\rangle \left|n_{N-1}\right\rangle \dots \left|n_1\right\rangle\Big) \\
=& \frac{1}{\sqrt{N!}}(\epsilon_{n_1} + \epsilon_{n_2} + \dots + \epsilon_{n_N})(\left|n_1\right\rangle \left|n_2\right\rangle \dots \left|n_N\right\rangle + \left|n_2\right\rangle \left|n_1\right\rangle \dots \left|n_N\right\rangle \\
& + \dots + \left|n_N\right\rangle \left|n_{N-1}\right\rangle \dots \left|n_1\right\rangle) \\
& = (\epsilon_{n_1} + \epsilon_{n_2} + \dots + \epsilon_{n_N}) \left|n_1 \dots n_N\right\rangle
\end{aligned}$$