Proof for the left-to-reader exercise:

$$
\begin{aligned}
[\hat{c}_\alpha, \hat{c}^\dagger_\beta]_\mp &= \sum_{nm} [U_{\alpha n}\hat{d}_n, U^\dagger_{\beta m}\hat{d}^\dagger_m]_\mp \\
&= \sum_{nm} U_{\alpha n}U^\dagger_{\beta m}[\hat{d}_n, \hat{d}^\dagger_m]_\mp \\
&= \sum_{nm} U_{\alpha n}U^\dagger_{\beta m}\delta_{nm} \\
&= \sum_{n} U_{\alpha n}U^\dagger_{\beta n} = \sum_{n} \langle\alpha|n\rangle \langle n|\beta\rangle \\
&= \langle\alpha|\beta\rangle = \delta_{\alpha\beta}
\end{aligned}
$$