1 Ex 1.2

Expand field operator in general basis,

$$
\begin{aligned}
|x\rangle &= \\
&= \sum_n \langle n|x\rangle |n\rangle = \sum_n \varphi_n^*(x)\hat{d}_n^\dagger |0\rangle
\end{aligned}
$$

Thus, $\hat{\psi}^\dagger(x) = \sum_n \varphi_n^*(x)\hat{d}_n^\dagger$, where $\langle n|x\rangle \equiv \varphi_n^*(x)$ is defined in the transformation from space-spin basis to general basis. Take the adjoint:

$$
\langle 0| \hat{\psi}(x) = \langle 0| \sum_n \varphi_n(x)\hat{d}_n
$$

which implies $\hat{\psi}(x) = \sum_n \varphi_n(x)\hat{d}_n$

2 Ex 1.3

$|n\rangle = |\mathbf{p}\tau\rangle$, and $\langle \mathbf{x}|\mathbf{p}\tau\rangle = e^{i\mathbf{p}\cdot\mathbf{r}}\delta_{\sigma\tau}$

$$
\begin{aligned}
|\mathbf{p}\tau\rangle &= \hat{d}_{\mathbf{p}\tau}^\dagger |0\rangle \\
&= \int d\mathbf{x}\, |\mathbf{x}\rangle \langle \mathbf{x}|\mathbf{p}\tau\rangle = \int d\mathbf{x}\, |\mathbf{x}\rangle\, e^{i\mathbf{p}\cdot\mathbf{r}}\delta_{\sigma\tau} \\
&= \int d\mathbf{x} e^{i\mathbf{p}\cdot\mathbf{r}}\delta_{\sigma\tau}\hat{\psi}^\dagger(x) |0\rangle
\end{aligned}
$$

$\Rightarrow \hat{d}_{\mathbf{p}\tau}^\dagger = \int e^{i\mathbf{p}\cdot\mathbf{r}}\delta_{\sigma\tau}\hat{\psi}^\dagger(x)d\mathbf{x}$ and $\hat{d}_{\mathbf{p}\tau} = \int e^{-i\mathbf{p}\cdot\mathbf{r}}\delta_{\sigma\tau}\hat{\psi}(x)d\mathbf{x}$. The (anti)commutation relation:

$$
\begin{aligned}
&[\hat{d}_{\mathbf{p}\tau}, \hat{d}_{\mathbf{p}'\tau'}^\dagger]_\mp \\
=& \int d\mathbf{x}\, d\mathbf{x}'\, e^{i(\mathbf{p}'\cdot\mathbf{r}'-\mathbf{p}\cdot\mathbf{r})}\delta_{\sigma\tau}\delta_{\sigma'\tau'}[\hat{\psi}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{x}')]_\mp \\
=& \int d\mathbf{x}\, d\mathbf{x}'\, e^{i(\mathbf{p}'\cdot\mathbf{r}'-\mathbf{p}\cdot\mathbf{r})}\delta_{\sigma\tau}\delta_{\sigma'\tau'}\delta(\mathbf{x}-\mathbf{x}') \\
=& \int d\mathbf{x}\, e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}\delta_{\sigma\tau}\delta_{\sigma\tau'} = \delta_{\tau\tau'}\int d\mathbf{r}\, e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}} \\
=&(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')\delta_{\tau\tau'}
\end{aligned}
$$

The transformation from field operator to momentum $\hat{d}$-operators

$$\begin{aligned}
|\mathbf{x}\rangle &= \hat{\psi}^\dagger(\mathbf{x})\,|0\rangle \\
&= \frac{1}{(2\pi)^3}\sum_\tau \int d\mathbf{p}\ |\mathbf{p}\tau\rangle\,\langle\mathbf{p}\tau|\mathbf{x}\rangle \\
&= \frac{1}{(2\pi)^3}\sum_\tau \int d\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{r}}\delta_{\sigma\tau}\hat{d}^\dagger_{\mathbf{p}\tau}\,|0\rangle
\end{aligned}$$

$$\Rightarrow \hat{\psi}^\dagger(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{r}}\hat{d}^\dagger_{\mathbf{p}\sigma} \quad \hat{\psi}(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}}\hat{d}_{\mathbf{p}\sigma}$$

where the completeness relation of $\{|\mathbf{p}\tau\rangle\}$ is used.

Proof for the completeness relation of $\{|\mathbf{p}\tau\rangle\}$:

$$\langle\mathbf{p}\tau|\mathbf{p}'\tau'\rangle = \int d\mathbf{x}\ \langle\mathbf{p}\tau|\mathbf{x}\rangle\,\langle\mathbf{x}|\mathbf{p}'\tau'\rangle = \sum_\sigma \delta_{\tau\sigma}\delta_{\sigma\tau'}\int d\mathbf{r}\, e^{i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}} = (2\pi)^3\delta_{\tau\tau'}\delta(\mathbf{p}-\mathbf{p}')$$

which implies $\mathbb{1} = \sum_\tau \int \dfrac{d\mathbf{p}}{(2\pi)^3}\,|\mathbf{p}\tau\rangle\,\langle\mathbf{p}\tau|$