

(2)  $L$  守恒, 有

$$Rv \cos \alpha = (R+h)v_0 = rv_\theta = r \frac{d\theta}{dt} \rightarrow \begin{cases} \frac{d\theta}{dt} = \frac{(R+h)v_0}{r} \\ v = \frac{R+h}{R \cos \alpha} v_0 \end{cases}$$

$E$  守恒, 有

$$\frac{1}{2}mv_0^2 - \frac{GmM}{R+h} = \frac{1}{2}m(v(r))^2 - \frac{GmM}{r} = \frac{1}{2}mv^2 - \frac{GmM}{R}$$

解得  $v = \frac{\sqrt{2GhM + hRv_0^2 + R^2v_0^2}}{\sqrt{hR + R^2}}$  与  $\cos \alpha = \frac{v_0(h+R)^2}{\sqrt{R(h+R)}\sqrt{2GhM + Rv_0^2(h+R)}}$ , 同时由于  
 $v(r) = v_\theta^2 + v_r^2 = \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2$

$$\frac{1}{2}mv_0^2 - \frac{GmM}{R+h} = \frac{1}{2}m \left( \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 \right)^2 - \frac{GmM}{r}$$

解得

$$\begin{aligned} \frac{dr}{dt} &= -\sqrt{-\frac{2GM}{h+R} - v_0^2 + \frac{2GM}{r} - \frac{v_0^2(h+R)^2}{r^2}} \\ dr &= dt \left( -\sqrt{-\frac{2GM}{h+R} - v_0^2 + \frac{2GM}{r} - \frac{v_0^2(h+R)^2}{r^2}} \right) \\ r dr &= dt \left( -\sqrt{-\frac{2GMr^2}{h+R} - r^2v_0^2 + 2GMr - v_0^2(h+R)^2} \right) \\ \frac{r dr}{-\sqrt{\left(-\frac{2GM}{h+R} - v_0^2\right)r^2 + 2GMr - v_0^2(h+R)^2}} &= dt \end{aligned}$$

同时积分, 初状态  $t = 0, r = R + h$ , 末状态  $t = t, r = R$ 。

$$\begin{aligned} \int_0^t dt &= \int_{R+h}^R \frac{r dr}{-\sqrt{\left(-\frac{2GM}{h+R} - v_0^2\right)r^2 + 2GMr - v_0^2(h+R)^2}} \\ \int_0^t dt &= \int_R^{R+h} \frac{r dr}{\sqrt{\left(-\frac{2GM}{h+R} - v_0^2\right)r^2 + 2GMr - v_0^2(h+R)^2}} \end{aligned}$$

利用

$$\int \frac{x dx}{\sqrt{a+bx+cx^2}} = \frac{\sqrt{a+bx+cx^2}}{c} - \frac{b}{2(-c)^{3/2}} \arcsin \frac{2cx+b}{\sqrt{\Delta}} + C (c < 0, C \text{ 为积分常数})$$

代入数据  $c = -\frac{2GM}{h+R} - v_0^2 < 0, b = 2GM, a = -v_0^2(h+R)^2$

$$t = \frac{R+h}{2GM - v_0^2(R+h)} \sqrt{\frac{2GMRh}{R+h} - v_0^2(2R+h)h} \\ + GM \left( \frac{R+h}{2GM - v_0^2(R+h)} \right)^{3/2} \left( \frac{\pi}{2} + \arcsin \frac{v_0^2 R (R+h) - GM (R-h)}{GM(R+h) - v_0^2(R+h)^2} \right)$$