

(2) L 守恒, 有

$$Rv \cos \alpha = (R + h)v_0 = rv_\theta = r \frac{d\theta}{dt} \rightarrow \begin{cases} \frac{d\theta}{dt} = \frac{(R + h)v_0}{r} \\ v = \frac{R + h}{R \cos \alpha} v_0 \end{cases}$$

E 守恒, 有

$$\frac{1}{2}mv_0^2 - \frac{GmM}{R + h} = \frac{1}{2}m(v(r))^2 - \frac{GmM}{r} = \frac{1}{2}mv^2 - \frac{GmM}{R}$$

解得 $v = \frac{\sqrt{2GhM + hRv_0^2 + R^2v_0^2}}{\sqrt{hR + R^2}}$ 与 $\cos \alpha = \frac{v_0(h + R)^2}{\sqrt{R(h + R)}\sqrt{2GhM + Rv_0^2(h + R)}}$, 同时由于

$$v(r)^2 = v_\theta^2 + v_r^2 = \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2$$

$$\frac{1}{2}mv_0^2 - \frac{GmM}{R + h} = \frac{1}{2}m\left(\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\right) - \frac{GmM}{r}$$

解得

$$\begin{aligned} \frac{dr}{dt} &= -\sqrt{-\frac{2GM}{h + R} - v_0^2 + \frac{2GM}{r} - \frac{v_0^2(h + R)^2}{r^2}} \\ dr &= dt \left(-\sqrt{-\frac{2GM}{h + R} - v_0^2 + \frac{2GM}{r} - \frac{v_0^2(h + R)^2}{r^2}} \right) \\ r dr &= dt \left(-\sqrt{-\frac{2GMr^2}{h + R} - r^2v_0^2 + 2GMr - v_0^2(h + R)^2} \right) \\ \frac{r dr}{-\sqrt{\left(-\frac{2GM}{h + R} - v_0^2\right)r^2 + 2GMr - v_0^2(h + R)^2}} &= dt \end{aligned}$$

同时积分, 初状态 $t = 0, r = R + h$, 末状态 $t = t, r = R$ 。

$$\begin{aligned} \int_0^t dt &= \int_{R+h}^R \frac{r dr}{-\sqrt{\left(-\frac{2GM}{h + R} - v_0^2\right)r^2 + 2GMr - v_0^2(h + R)^2}} \\ \int_0^t dt &= \int_R^{R+h} \frac{r dr}{\sqrt{\left(-\frac{2GM}{h + R} - v_0^2\right)r^2 + 2GMr - v_0^2(h + R)^2}} \end{aligned}$$

利用

$$\int \frac{x dx}{\sqrt{a + bx + cx^2}} = \frac{\sqrt{a + bx + cx^2}}{c} - \frac{b}{2(-c)^{3/2}} \arcsin \frac{2cx + b}{\sqrt{\Delta}} + C (c < 0, C \text{ 为积分常数})$$

代入数据 $c = -\frac{2GM}{h+R} - v_0^2 < 0, b = 2GM, a = -v_0^2(h+R)^2$

$$t = \frac{R+h}{2GM - v_0^2(R+h)} \sqrt{\frac{2GMRh}{R+h} - v_0^2(2R+h)h} \\ + GM \left(\frac{R+h}{2GM - v_0^2(R+h)} \right)^{3/2} \left(\frac{\pi}{2} + \arcsin \frac{v_0^2 R (R+h) - GM (R-h)}{GM(R+h) - v_0^2(R+h)^2} \right)$$