

See [Cis19] and the question¹ in mathoverflow. The main objective is Proposition 3.4, which implies [Cis19, Lemma5.5.16].

Notation 0.1. We employ the following notations.

- Given any category \mathbf{C} and two objects $A, B \in \mathbf{C}$, we write $\mathbf{C}(A, B)$ for the homset.
- Δ is the category of simplices. We write Δ^n for the n -simplex, which is also seen as a category.
- We write \mathbf{C}^{Δ^1} for the arrow category of \mathbf{C} . We write \square for $\Delta^1 \times \Delta^1$.
- We write \mathbf{Set} for the category of sets.
- \mathbf{sSet} is the category of simplicial sets; i.e., $\mathbf{sSet} = [\Delta^{\text{op}}, \mathbf{Set}]$. Every set is seen as a simplicial set by the diagonal functor $\mathbf{Set} \rightarrow \mathbf{sSet}$.
- \mathbf{bisSet} is the category of bisimplicial sets; i.e., $\mathbf{bisSet} = [\Delta^{\text{op}} \times \Delta^{\text{op}}, \mathbf{Set}]$
- For each small category \mathbb{A} , we write $\mathfrak{y}: \mathbb{A} \rightarrow [\mathbb{A}^{\text{op}}, \mathbf{Set}]$ for the yoneda embedding.
- We write $- \boxtimes -: \mathbf{sSet} \times \mathbf{sSet} \rightarrow \mathbf{bisSet}$ for the nerve functor of $\mathfrak{y} \times \mathfrak{y}: \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \mathbf{sSet} \times \mathbf{sSet}$.
- There are functors $- \multimap -: \mathbf{sSet}^{\text{op}} \times \mathbf{bisSet} \rightarrow \mathbf{sSet}$ and $- \circ -: \mathbf{bisSet} \times \mathbf{sSet}^{\text{op}} \rightarrow \mathbf{sSet}$ such that there exists the following family of bijections natural in $A, B \in \mathbf{sSet}^{\text{op}}$ and $X \in \mathbf{bisSet}$.

$$(1) \quad \mathbf{bisSet}(A \boxtimes B, X) \cong \mathbf{sSet}(A, X \circ B) \cong \mathbf{sSet}(B, A \multimap X)$$

- We write $- \boxtimes^L -: \mathbf{sSet}^{\Delta^1} \times \mathbf{sSet}^{\Delta^1} \rightarrow \mathbf{bisSet}^{\Delta^1}$ for the *pushout product* or *Leibniz product* with respect to \boxtimes ; given two morphisms $f: A \rightarrow B$ and $g: C \rightarrow D$ in \mathbf{sSet} , the morphism $f \boxtimes^L g$ is the canonical one of the form $A \boxtimes D \cup_{A \boxtimes C} B \boxtimes C \rightarrow B \boxtimes D$ whose domain is the fibred coproduct of $f \boxtimes \text{id}_C$ and $\text{id}_A \boxtimes g$.
- We write $- \overset{L}{\circ} -: \mathbf{sSet}^{\Delta^1, \text{op}} \times \mathbf{bisSet}^{\Delta^1} \rightarrow \mathbf{sSet}^{\Delta^1}$ and $- \overset{L}{\multimap} -: \mathbf{bisSet}^{\Delta^1} \times \mathbf{sSet}^{\Delta^1, \text{op}} \rightarrow \mathbf{sSet}^{\Delta^1}$ for the *pullback product* for \multimap and \circ respectively. Therefore, given a morphism $f: A \rightarrow B$ in \mathbf{sSet} and a morphism $h: X \rightarrow Y$ in \mathbf{bisSet} , $f \overset{L}{\circ} h$ is the canonical morphism towards the fibred product of $f \multimap \text{id}_Y$ and $\text{id}_A \multimap h$.
- Let i and k be morphisms. We write $i \pitchfork k$ if i has left lifting property to k , or equivalently, k has right lifting property to i .
- Given a class of morphisms \mathbf{l} , we write $\ell(\mathbf{l})$ for the class of morphisms that have left lifting property to those in \mathbf{l} . Dually, we write $r(\mathbf{l})$ for the class of morphisms that have right lifting property to those in \mathbf{l} .
- Given two classes of morphisms \mathbf{l} and \mathbf{K} in \mathbf{sSet} , we obtain a class of morphisms $\mathbf{l} \boxtimes^L \mathbf{K} = \{i \boxtimes^L k \mid i \in \mathbf{l}, k \in \mathbf{K}\}$.
- We write \mathbf{l}_{mono} for the canonical cellular model for \mathbf{sSet} ; i.e., the set of boundary inclusions.
- We write \mathbf{l}_{ft} for the set of generating left anodyne extensions; i.e. the set of horn inclusions lifting initial objects.
- We write \mathbf{l}_{ftbi} for $\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}} \cup \mathbf{l}_{\text{mono}} \boxtimes^L \mathbf{l}_{\text{ft}}$. The set \mathbf{l}_{ftbi} is the generator for left bi-anodyne extensions.
- For each simplicial set A , we write $(\mathbf{sSet}/A)_{\text{co}}$ for the slice category equipped with the homotopical structure defined for the covariant model structure.
- For each bisimplicial set X , we write $(\mathbf{bisSet}/X)_{\text{co}}$ for the slice category equipped with the homotopical structure for the bicovariant model structure.
- We write ω for the least infinite ordinal seen as a category by its order.
- We regard the horn Λ_2^2 as a subcategory of Δ^2 . ■

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¹<https://mathoverflow.net/questions/467753/on-lemma-5-5-16-of-cisinskis-higher-categories-and-homotopical-algebra>

1. ON CLASSES OF MORPHISMS IN **bisSet**

Lemma 1.1. *For each morphisms i, j in **sSet** and a morphism h in **bisSet**, the following are equivalent.*

- $i \boxtimes^L j \rhd h$
- $i \rhd h \circ^L j$
- $j \rhd i \circ^L h$

Proof. Follows from (1) and the universality of the pushout and the pullbacks defining $i \boxtimes^L j$, $h \circ^L j$, and $i \circ^L h$. \square

Lemma 1.2. *Suppose we are given two small classes \mathbf{I} and \mathbf{J} of morphisms in **sSet**, and a morphism f in **bisSet**. The following are equivalent.*

- i) $h \in r(\mathbf{I} \boxtimes^L \mathbf{J})$.
- ii) For each $j \in \mathbf{J}$, $h \circ^L j \in r(\mathbf{I})$.
- iii) For each $i \in \mathbf{I}$, $i \circ^L h \in r(\mathbf{J})$.
- iv) For each $j \in \ell r(\mathbf{J})$, $h \circ^L j \in r(\mathbf{I})$.
- v) For each $i \in \ell r(\mathbf{I})$, $i \circ^L h \in r(\mathbf{J})$.

Proof. By Lemma 1.1, the first three arguments are equivalent to the following.

$$j \rhd i \circ^L h \quad \text{for any } i \in \mathbf{I} \text{ and } j \in \mathbf{J}.$$

By the small object argument, this is also equivalent to

$$j \rhd i \circ^L h \quad \text{for any } i \in \mathbf{I} \text{ and } j \in \ell r(\mathbf{J}),$$

which is equivalent to iv). v) is also equivalent to them in the same way. \square

Corollary 1.3 ([Cis19, Lemma 5.5.6]). *A morphism q in **bisSet** is a trivial fibration if and only if $q \circ -$ is a trivial fibration for each monomorphism i in **sSet**.*

Proof. $\mathbf{I}_{\text{mono}} \boxtimes^L \mathbf{I}_{\text{mono}}$ is a cellular model for **bisSet** (see [Cis19, Example 1.3.4 and 2.4.5]), and this is a direct consequence of Lemma 1.2, where both \mathbf{I} and \mathbf{J} are \mathbf{I}_{mono} . \square

Lemma 1.4. $\text{dom}: (\mathbf{sSet}/A)_{\text{co}} \rightarrow (\mathbf{sSet})_{\text{co}}$ *preserves and lifts trivial fibrations and naive fibrations.*

Proof. Recall that dom preserves and lifts both monomorphisms and anodyne extensions.

Let $f: S \rightarrow T$ be a morphism in \mathbf{sSet}/A . We show that f is a trivial/naive fibration if and only if $\text{dom}(f)$ is a trivial/naive fibration. The if part is easily checked by considering that dom preserves monomorphisms/anodyne extensions.

Let i be a monomorphism/anodyne extension in $(\mathbf{sSet})_{\text{co}}$ and $(u, v): i \rightarrow \text{dom}(f)$ be a morphism in \mathbf{sSet}^{Δ^1} . It suffices to show there is a diagonal filler for this square. The morphisms towards A from f is followed by (u, v) , and this enables us to see (u, v) as a square in \mathbf{sSet}/A . Moreover, its domain is a monomorphism/anodyne extension. Therefore, desired diagonal filler exists if f is a trivial/naive fibration in \mathbf{sSet}/A . \square

Lemma 1.5. *A left fibration between trivial fibrations is a trivial fibration.*

Proof. Suppose we are given a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow h \quad \swarrow g & \\ & C & \end{array}$$

in **sSet** such that f is a left fibration (i.e., a naive fibration in $(\mathbf{sSet})_{\text{co}}$) and g and h are trivial fibrations. By Lemma 1.4, this lifts to another triangle

$$\begin{array}{ccc} (A, h) & \xrightarrow{f} & (B, g) \\ & \searrow h \quad \swarrow g & \\ & (C, \text{id}) & \end{array}$$

in $(\mathbf{sSet}/C)_{\text{co}}$ satisfying the same condition. Since (A, h) and (B, g) are fibrant, f is a fibration. Therefore, the 2 out of 3 property for weak equivalences ensures that f is a trivial fibration. \square

Proposition 1.6. *Suppose we are given a morphism $q: X \rightarrow Y$ in \mathbf{bisSet} that belongs to $r(\mathbf{lft} \boxtimes^L \mathbf{lmono})$ and a monomorphism $i: K \rightarrow L$ in \mathbf{sSet} . $q \circ^L i$ is a trivial fibration if $q \circ K$ and $q \circ L$ are trivial fibrations.*

Proof. By Lemma 1.2, $q \circ^L i$ is a left fibration filling the following diagram.

$$\begin{array}{ccccc}
 X \circ L & \xleftarrow{X \circ i} & X \circ K & & \\
 \uparrow q \circ L & & \uparrow x & \swarrow q \circ K & \\
 Y \circ L & \xleftarrow{\quad} & \cdot & & Y \circ K \\
 & & \nwarrow q \circ^L i & &
 \end{array}$$

The pullback square assures x is also a trivial fibration, and this follows from Lemma 1.5. \square

2. INSIGHTS FROM REEDY CATEGORY THEORY

Observation 2.1. Δ^1 can be seen as a lattice so that there are the meet $\wedge: \square \rightarrow \Delta^1$ and the join $\vee: \square \rightarrow \Delta^1$. Given a category \mathbf{C} with pushout and pullback, we obtain $L_\wedge: \mathbf{C}^\square \rightarrow \mathbf{C}^{\Delta^1}$ as the right Kan extension along the meet \wedge , while we have $L_\vee: \mathbf{C}^\square \rightarrow \mathbf{C}^{\Delta^1}$ as the left Kan extension along the join \vee .

On the other hand, suppose we are given a functor $\otimes: \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{C}$. Consider the composite

$$\Delta^1 \times \Delta^1 \times \mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \xrightarrow{\text{evaluation}} \mathbf{A} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{C},$$

where the former is the canonical evaluation functor that sends (i, j, X, Y) to $(X(i), Y(j))$. The currying of this composite gives a functor $\bar{\otimes}: \mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \rightarrow \mathbf{C}^\square$.

Now we have two functors $L_\wedge \circ \bar{\otimes}$ and $L_\vee \circ \bar{\otimes}$ of the form $\mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \rightarrow \mathbf{C}^{\Delta^1}$. The first one is the *pullback product* of \otimes , while the second one is the *pushout product* or the *Leibniz product* of \otimes . \blacksquare

Definition 2.2. Suppose that we are given a cocomplete category \mathbf{C} and a class S of objects in \mathbf{C} . We say S is **saturated by monomorphisms** if for each diagram $F: \mathbb{I}^{\text{op}} \rightarrow \mathbf{C}$ satisfying the following conditions, the colimit of F is in S ;

- \mathbb{I} is a discrete category, or either of ω^{op} or Λ_2^2 .
- If $u: i \rightarrow j$ in \mathbb{I} is of the form $n+1 \rightarrow n$ in ω^{op} or $1 \rightarrow 2$ in Λ_2^2 , then $F(u)$ is a monomorphism.
- The image of F is in S . \blacksquare

Fact 2.3 ([Cis19, Corollary 1.3.10]). *A class of simplicial sets contains all simplicial sets if it is saturated by monomorphisms and contains representables.*

Definition 2.4. Suppose that we are given

- a complete category \mathbf{C} ,
- a weak factorisation system $(\text{Cof}, \text{TFib})$ whose elements are called *cofibrations* and *trivial fibrations* respectively,
- a small category \mathbb{I} that is a discrete category or either of ω^{op} or Λ_2^2 , and
- two functors $F, G: \mathbb{I} \rightarrow \mathbf{C}$.

A natural transformation $\alpha: F \Rightarrow G$ is a **Reedy trivial fibration** if it satisfies the followings.

- For each object $i \in \mathbb{I}$, $\alpha_i: F(i) \rightarrow G(i)$ is a trivial fibration.
- Let $u: i \rightarrow j$ in \mathbb{I} be either of $n+1 \rightarrow n$ in ω^{op} or $1 \rightarrow 2$ in Λ_2^2 . Then the naturality square

$$\begin{array}{ccc}
 F(i) & \xrightarrow{F(u)} & F(j) \\
 \alpha_i \downarrow & & \downarrow \alpha_j \\
 G(i) & \xrightarrow{G(u)} & G(j)
 \end{array}$$

is sent by L_\wedge (see Observation 2.1) to a trivial fibration. \blacksquare

Proposition 2.5. *The limit of a Reedy trivial fibration is a trivial fibration.*

Proof. Let us write $\Delta_{\mathbb{I}}: \mathbf{C} \rightarrow [\mathbb{I}, \mathbf{C}]$ for the diagonal functor, and write $\lim_{\mathbb{I}}$ for its right adjoint. Suppose that we are given a cofibration $f: K \rightarrowtail L$ in \mathbf{C} and a Reedy trivial fibration $\alpha: F \rightarrow G$. We show $f \pitchfork \lim_{\mathbb{I}}(\alpha)$, which is equivalent to $\Delta_{\mathbb{I}}(f) \pitchfork \alpha$ in $[\mathbb{I}, \mathbf{C}]$. To this end, let $(u, v): \Delta_{\mathbb{I}}(f) \rightarrow \alpha$ be a morphism in $[\mathbb{I}, \mathbf{C}]^{\Delta^1}$. It suffices to find the diagonal filler $k: \Delta_{\mathbb{I}}(L) \rightarrow F$ for this square.

- When \mathbb{I} is a discrete category, take diagonal fillers $k_i: L \rightarrow G(i)$ for each squares $(u_i, v_i): f \rightarrow \alpha_i$. Such fillers exist because α_i is a trivial fibration for each $i \in \mathbb{I}$.
- Suppose $\mathbb{I} = \omega^{\text{op}}$. We construct the diagonal filler $(k_n)_{n \geq 0}$ by induction on n .
 (n=0) k_0 is the diagonal filler for the square $(u_0, v_0): f \rightarrow \alpha_0$.
 (n>0) Suppose we have constructed k_{n-1} that fits into the following commutative diagram.

$$\begin{array}{ccccc} K & \xrightarrow{u_n} & F(n) & \longrightarrow & F(n-1) \\ f \downarrow & & \downarrow & \nearrow k_{n-1} & \downarrow \alpha_{n-1} \\ L & \xrightarrow{v_n} & G(n) & \longrightarrow & G(n-1) \end{array}$$

Since the square on the right is sent by L_{\wedge} to a trivial fibration, we have another square whose right side is a trivial fibration;

$$\begin{array}{ccc} K & \xrightarrow{u_n} & F(n) \\ f \downarrow & & \downarrow \\ K & \xrightarrow{\bar{k}_{n-1}} & \cdot \end{array}$$

where the bottom side is the morphism induced from k_{n-1} and v_n by the universality of the pullback defining the right side. We define k_n as a filler for this square.

- Suppose $\mathbb{I} = \Lambda_2^2$. Define $(k_x)_{x=0,1,2}$ as follows.
 - k_0 is the filler for $(u_0, v_0): f \rightarrow \alpha_0$, and define k_2 as the composite

$$L \xrightarrow{k_0} F(0) \rightarrow F(2)$$

where the latter is the image under F of $0 \rightarrow 2$.

- Now we have the following commutative diagram.

$$\begin{array}{ccccc} K & \xrightarrow{u_1} & F(1) & \longrightarrow & F(2) \\ f \downarrow & & \downarrow & \nearrow k_2 & \downarrow \alpha_2 \\ L & \xrightarrow{v_1} & G(1) & \longrightarrow & G(2) \end{array}$$

We define k_1 in the same way as the definition of k_n above for $\mathbb{I} = \omega^{\text{op}}$. □

3. THE MAIN PROPOSITION

Definition 3.1. We say a morphism i in **bisSet** is a *levelwise left anodyne extension* if $i \circ \Delta^n$ is a left anodyne extension for each $n \geq 0$. ■

Lemma 3.2. *Morphisms in $\ell r(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}})$ are levelwise left anodyne extensions.*

Proof. Firstly, observe that the class of levelwise left anodyne extensions is described as

$$\bigcup_{n \geq 0} ((-) \circ \Delta^n)^{-1}(\ell r(\mathbf{l}_{\text{ft}})),$$

where $((-) \circ \Delta^n)^{-1}$ means the inverse image of the function on morphisms induced from the functor $(-) \circ \Delta^n: \mathbf{bisSet} \rightarrow \mathbf{sSet}$. Since $(-) \circ \Delta^n$ preserves colimits and saturated classes are closed under unions, this class is saturated. Therefore, it suffices to show that morphisms in $\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}}$ are levelwise left anodyne extensions.

Let $i: A \rightarrow B \in \mathbf{lft}$ and $j: C \rightarrow D \in \mathbf{lmono}$. By applying $(-) \circ \Delta^n$ to the diagram defining $i \boxtimes^L j$, we obtain the following diagram in \mathbf{sSet} .

$$\begin{array}{ccc}
 A \times C_n & \xrightarrow{\text{id} \times j_n} & A \times D_n \\
 i \times \text{id} \downarrow & \lrcorner & \downarrow x \\
 B \times C_n & \xrightarrow{\quad} & \cdot \\
 & & (i \boxtimes^L j) \circ \Delta^n \\
 & \searrow & \downarrow \\
 & & B \times D_n
 \end{array}$$

(A curved arrow from $B \times C_n$ to $B \times D_n$ is also present, representing the morphism $(i \boxtimes^L j) \circ \Delta^n$.)

The two morphisms named $i \times \text{id}$ are both left anodyne extensions since left anodyne extensions are closed under finite product ([Cis19, Proposition 3.4.3]). Since $(-) \circ \Delta^n$ preserves pushouts, the marked square is a pushout square, and hence x is also a left anodyne extension. Moreover, since the outer square comprises monomorphisms, all morphisms in this diagram are monomorphisms [citation](#). Therefore, the resulting morphism $(i \boxtimes^L j) \circ \Delta^n$ is a initial monomorphism, which is a left anodyne extension. \square

Lemma 3.3. *A morphism i in \mathbf{bisSet} is a monomorphism if and only if $i \circ \Delta^n$ is a monomorphism for each $n \geq 0$.*

Proof. By the yoneda lemma and the fact that $(-) \circ \Delta^m$ is a right adjoint. In detail, the following are equivalent.

- i is a monomorphism.
- $\mathbf{bisSet}(\Delta^n \boxtimes \Delta^m, i)$ is a monomorphism in \mathbf{Set} for each $n, m \geq 0$.
- $\mathbf{sSet}(\Delta^n, i \circ \Delta^m)$ is a monomorphism in \mathbf{Set} for each $n, m \geq 0$.
- $i \circ \Delta^m$ is a monomorphism in \mathbf{sSet} for each $m \geq 0$. \square

Proposition 3.4. *Levelwise left anodyne extensions are left bi-anodyne extensions.*

Proof. Let $i: X \rightarrow Y$ be a levelwise left anodyne extension. [Lemma 3.3](#) shows i is a monomorphism. By the small object argument, we have a weak factorisation system $(\ell r(\mathbf{lft} \boxtimes^L \mathbf{lmono}), r(\mathbf{lft} \boxtimes^L \mathbf{lmono}))$, and let us write

$$\begin{array}{ccc}
 X & \xrightarrow{i} & Y \\
 & \searrow j & \nearrow q \\
 & Z &
 \end{array}$$

for the factorisation of i with respect to this system. Since j is in particular a left bi-anodyne extension and i is a monomorphism, it suffices to show q is a trivial fibration. Moreover, [Corollary 1.3](#) and [Proposition 1.6](#) ensures that it suffices to show $q \circ \Delta^n$ is a trivial fibration for each simplicial set K .

Consider the case when $K = \Delta^n$ for some $n \geq 0$. [Lemma 1.2](#) shows $q \circ \Delta^n$ is a left fibration and [Lemma 3.2](#) shows $j \circ \Delta^n$ is a left anodyne extension. Since $i \circ \Delta^n$ is assumed to be a left anodyne extension, $q \circ \Delta^n$ is a initial left fibration, which is a trivial fibration.

By [Corollary 1.3.10](#) of [Cis19] (see [Fact 2.3](#)), it suffices to show that the class

$$S = \{K \in \mathbf{sSet} \mid q \circ \Delta^n \text{ is a trivial fibration} \}$$

is saturated by monomorphisms. Let $F: \mathbb{I}^{\text{op}} \rightarrow \mathbf{sSet}$ be a diagram satisfying the conditions described in [Definition 2.2](#). Since $Z \circ -$ sends colimits to limits for any $Z \in \mathbf{bisSet}$, by virtue of [Proposition 2.5](#), it suffices to show that the natural transformation

$$(2) \quad q \circ F(-): Z \circ F(-) \Rightarrow Y \circ F(-): \mathbb{I} \rightarrow \mathbf{sSet}$$

is a Reedy trivial fibration. Now considering [Observation 2.1](#), one can check that [Proposition 1.6](#) shows that the naturality square for the transformation $q \circ -$ at a monomorphism between objects in S is sent by L_\wedge to a trivial fibration. This shows (2) is a Reedy trivial fibration, which completes the proof. \square

REFERENCES

- [Cis19] D.-C. Cisinski. *Higher Categories and Homotopical Algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019. doi:[10.1017/9781108588737](https://doi.org/10.1017/9781108588737).
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