

NOTE ON LEMMA 5.5.16 OF “HIGHER CATEGORIES AND HOMOTOPICAL ALGEBRA”

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[Cis19] The main objective is Proposition 14.

Notation 1. We employ the following notations.

- Given any category \mathbf{C} and two objects $A, B \in \mathbf{C}$, we write $\mathbf{C}(A, B)$ for the homset.
- Δ is the category of simplices. We write Δ^n for the n -simplex, which is also seen as a category.
- We write \mathbf{C}^{Δ^1} for the arrow category of \mathbf{C} . We write \square for $\Delta^1 \times \Delta^1$.
- We write \mathbf{Set} for the category of sets.
- \mathbf{sSet} is the category of simplicial sets; i.e., $\mathbf{sSet} = [\Delta^{\text{op}}, \mathbf{Set}]$. Every set is seen as a simplicial set by the diagonal functor $\mathbf{Set} \rightarrow \mathbf{sSet}$.
- \mathbf{bisSet} is the category of bisimplicial sets; i.e., $\mathbf{bisSet} = [\Delta^{\text{op}} \times \Delta^{\text{op}}, \mathbf{Set}]$.
- For each small category \mathbb{A} , we write $\mathfrak{y}: \mathbb{A} \rightarrow [\Delta^{\text{op}}, \mathbf{Set}]$ for the yoneda embedding.
- We write $- \boxtimes -: \mathbf{sSet} \times \mathbf{sSet} \rightarrow \mathbf{bisSet}$ for the nerve functor of $\mathfrak{y} \times \mathfrak{y}: \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \mathbf{sSet} \times \mathbf{sSet}$.
- There are functors $- \multimap -: \mathbf{bisSet} \times \mathbf{sSet}^{\text{op}} \rightarrow \mathbf{sSet}$ and $- \multimap -: \mathbf{sSet}^{\text{op}} \times \mathbf{bisSet} \rightarrow \mathbf{sSet}$ such that there exists the following family of bijections natural in $A, B \in \mathbf{sSet}^{\text{op}}$ and $X \in \mathbf{bisSet}$.

$$(1) \quad \mathbf{bisSet}(A \boxtimes B, X) \cong \mathbf{sSet}(A, X \multimap B) \cong \mathbf{sSet}(B, A \multimap X)$$

- We write $- \boxtimes^L -: \mathbf{sSet}^{\Delta^1} \times \mathbf{sSet}^{\Delta^1} \rightarrow \mathbf{bisSet}^{\Delta^1}$ for the *pushout product* or *Leibniz product* with respect to \boxtimes ; given two morphisms $f: A \rightarrow B$ and $g: C \rightarrow D$ in \mathbf{sSet} , the morphism $f \boxtimes^L g$ is the canonical one of the form $A \boxtimes D \cup_{A \boxtimes C} B \boxtimes C \rightarrow B \boxtimes D$ whose domain is the fibred coproduct of $f \boxtimes \text{id}_C$ and $\text{id}_A \boxtimes g$.
- We write $- \overset{L}{\circ} -: \mathbf{sSet}^{\Delta^1, \text{op}} \times \mathbf{bisSet}^{\Delta^1} \rightarrow \mathbf{sSet}^{\Delta^1}$ and $- \circ^L -: \mathbf{bisSet}^{\Delta^1} \times \mathbf{sSet}^{\Delta^1, \text{op}} \rightarrow \mathbf{sSet}^{\Delta^1}$ for the *pullback product* for \multimap and \multimap respectively. Therefore, given a morphism $f: A \rightarrow B$ in \mathbf{sSet} and a morphism $h: X \rightarrow Y$ in \mathbf{bisSet} , $f \overset{L}{\circ} h$ is the canonical morphism towards the fibred product of $f \multimap \text{id}_Y$ and $\text{id}_A \multimap h$.
- Let i and k be morphisms. We write $i \pitchfork k$ if i has left lifting property to k , or equivalently, k has right lifting property to i .
- Given a class of morphisms \mathbf{l} , we write $\ell(\mathbf{l})$ for the class of morphisms that have left lifting property to those in \mathbf{l} . Dually, we write $r(\mathbf{l})$ for the class of morphisms that have right lifting property to those in \mathbf{l} .
- Given two classes of morphisms \mathbf{l} and \mathbf{K} in \mathbf{sSet} , we obtain a class of morphisms $\mathbf{l} \boxtimes^L \mathbf{K} = \{i \boxtimes^L k \mid i \in \mathbf{l}, k \in \mathbf{K}\}$.
- We write \mathbf{l}_{mono} for the canonical cellular model for \mathbf{sSet} ; i.e., the set of boundary inclusions.
- We write \mathbf{l}_{ft} for the set of generating left anodyne extensions; i.e. the set of horn inclusions lifting initial objects.
- We write \mathbf{l}_{ftbi} for $\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}} \cup \mathbf{l}_{\text{mono}} \boxtimes^L \mathbf{l}_{\text{ft}}$. The set \mathbf{l}_{ftbi} is the generator for left bi-anodyne extensions.
- For each simplicial set A , we write $(\mathbf{sSet}/A)_{\text{co}}$ for the slice category equipped with the homotopical structure defined for the covariant model structure,
- For each bisimplicial set X , we write $(\mathbf{bisSet}/X)_{\text{co}}$ for the slice category equipped with the homotopical structure for the bicovariant model structure.
- We write ω for the least infinite ordinal seen as a category by its order.
- We regard the horn Λ_2^2 as a subcategory of Δ^2 .
- ■

Observation 2. Δ^1 can be seen as a (boolean) lattice, and there are the meet $\wedge: \square \rightarrow \Delta^1$ and the join $\vee: \square \rightarrow \Delta^1$. Given a category \mathbf{C} with pushout and pullback, we obtain $L_\wedge: \mathbf{C}^\square \rightarrow \mathbf{C}^{\Delta^1}$ as the right Kan extension of the meet \wedge , while we have $L_\vee: \mathbf{C}^\square \rightarrow \mathbf{C}^{\Delta^1}$ as the left Kan extension of the join \vee .

On the other hand, suppose we are given a functor $\otimes: \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{C}$. The currying of the composite

$$\Delta^1 \times \Delta^1 \times \mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \xrightarrow{\text{evaluation}} \mathbf{A} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{C}$$

gives a functor $\bar{\otimes}: \mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \rightarrow \mathbf{C}^\square$, where the first one is the canonical evaluation functor that sends (i, j, X, Y) to $(X(i), Y(j))$.

Now we have two functors $L_\wedge \circ \bar{\otimes}$ and $L_\vee \circ \bar{\otimes}$ of the form $\mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \rightarrow \mathbf{C}^{\Delta^1}$. The first one is the *pullback product* of \otimes , while the second one is the *pushout product* or the *Leibniz product* of \otimes . ■

Definition 3. Suppose that we are given

- a complete category \mathbf{C} ,
- a weak factorisation system $(\text{Cof}, \text{TFib})$ whose elements are called *cofibrations* and *trivial fibrations* respectively,
- a small category \mathbb{I} that is a discrete category or either of ω^{op} or Λ_2^2 , and
- two functors $F, G: \mathbb{I} \rightarrow \mathbf{C}$.

A natural transformation $\alpha: F \Rightarrow G$ is a **Reedy trivial fibration** if it satisfies the followings.

- For each object $i \in \mathbb{I}$, $\alpha_i: F(i) \rightarrow G(i)$ is a trivial fibration.
- Let $u: i \rightarrow j$ in \mathbb{I} be either of $n+1 \rightarrow n$ in ω^{op} or $1 \rightarrow 2$ in Λ_2^2 . Then the naturality square

$$\begin{array}{ccc} F(i) & \xrightarrow{F(u)} & F(j) \\ \alpha_i \downarrow & & \downarrow \alpha_j \\ G(i) & \xrightarrow{G(u)} & G(j) \end{array}$$

is sent by L_\wedge (see [Observation 2](#)) to a trivial fibration. ■

Proposition 4. *The limit of a Reedy trivial fibration is a trivial fibration.*

Proof. [ongoing, citation](#) □

Lemma 5. *For each morphisms i, j in \mathbf{sSet} and a morphism h in \mathbf{bisSet} , the following are equivalent.*

- $i \boxtimes^L j \pitchfork h$
- $i \pitchfork h \circ^L j$
- $j \pitchfork i \circ^L h$

Proof. Follows from [\(1\)](#) and the universality of the pushout and the pullbacks defining $i \boxtimes^L j$, $h \circ^L j$, and $i \circ^L h$. □

Lemma 6. *Suppose we are given two small classes \mathbf{I} and \mathbf{J} of morphisms in \mathbf{sSet} , and a morphism f in \mathbf{bisSet} . The following are equivalent.*

- $h \in r(\mathbf{I} \boxtimes^L \mathbf{J})$.
- For each $j \in \mathbf{J}$, $h \circ^L j \in r(\mathbf{I})$.
- For each $i \in \mathbf{I}$, $i \circ^L h \in r(\mathbf{J})$.
- For each $j \in \ell r(\mathbf{J})$, $h \circ^L j \in r(\mathbf{I})$.
- For each $i \in \ell r(\mathbf{I})$, $i \circ^L h \in r(\mathbf{J})$.

Proof. By [Lemma 5](#), the first three arguments are equivalent to the following.

$$j \pitchfork i \circ^L h \quad \text{for any } i \in \mathbf{I} \text{ and } j \in \mathbf{J}.$$

By the small object argument, this is also equivalent to

$$j \pitchfork i \circ^L h \quad \text{for any } i \in \mathbf{I} \text{ and } j \in \ell r(\mathbf{J}),$$

which is equivalent to *iv*). *v*) is also equivalent to them in the same way. □

Corollary 7 (Lemma 5.5.6 of [\[Cis19\]](#)). *A morphism q in \mathbf{bisSet} is a trivial fibration if and only if $q \circ -$ is a trivial fibration for each monomorphism i in \mathbf{sSet} .*

Proof. $\mathbf{l}_{\text{mono}} \boxtimes^L \mathbf{l}_{\text{mono}}$ is a cellular model for **bisSet** [citation](#), and this is a direct consequence of [Lemma6](#), where both \mathbf{l} and \mathbf{j} are \mathbf{l}_{mono} . \square

Lemma 8. $\text{dom}: (\mathbf{sSet}/A)_{\text{co}} \longrightarrow (\mathbf{sSet})_{\text{co}}$ preserves and lifts trivial fibrations and naive fibrations.

Proof. Observe that dom preserves and lifts both monomorphisms and anodyne extensions.

Let $f: S \longrightarrow T$ be a morphism in \mathbf{sSet}/A . We show that f is a trivial/naive fibration if and only if $\text{dom}(f)$ is a trivial/naive fibration. The if part is easily checked considering that dom preserves monomorphisms/anodyne extensions.

Let i be a monomorphism/anodyne extension in $(\mathbf{sSet})_{\text{co}}$ and $(u, v): i \longrightarrow \text{dom}(f)$ be a morphism in \mathbf{sSet}^{Δ^1} . It suffices to show there is a diagonal filler for this square. The morphisms towards A from f is followed by (u, v) , and this enables us to see (u, v) as a square in \mathbf{sSet}/A . Moreover, its domain is a monomorphism/anodyne extension. Therefore, desired diagonal filler exists if f is a trivial/naive fibration in \mathbf{sSet}/A . \square

Lemma 9. A left fibration between trivial fibrations is a trivial fibration.

Proof. Suppose we are given a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \searrow h & & \swarrow g \\ & C & \end{array}$$

in \mathbf{sSet} such that f is a left fibration (i.e., a naive fibration in $(\mathbf{sSet})_{\text{co}}$) and g and h are trivial fibrations. By [Lemma8](#), this lifts to another triangle

$$\begin{array}{ccc} (A, h) & \xrightarrow{f} & (B, g) \\ \searrow h & & \swarrow g \\ & (C, \text{id}) & \end{array}$$

in $(\mathbf{sSet}/C)_{\text{co}}$ satisfying the same condition. Since (A, h) and (B, g) are fibrant, f is a fibration. Therefore, the 2 out of 3 property for weak equivalences ensures that f is a trivial fibration. \square

Lemma 10. Suppose we are given a morphism $q: X \longrightarrow Y$ in **bisSet** that belongs to $r(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}})$ and a monomorphism $i: K \hookrightarrow L$ in \mathbf{sSet} . $q \circ^{\perp} i$ is a trivial fibration if $q \circ - K$ and $q \circ - L$ are trivial fibrations.

Proof. By [Lemma6](#), $q \circ^{\perp} i$ is a left fibration filling the following diagram.

$$\begin{array}{ccccc} X \circ - L & \xleftarrow{X \circ - i} & X \circ - K & & \\ q \circ - L \uparrow & & \uparrow x & \swarrow q \circ - K & \\ Y \circ - L & \xleftarrow{\quad} & \cdot & \swarrow q \circ^{\perp} i & \\ & & & \searrow & Y \circ - K \end{array}$$

The pullback square assures x is also a trivial fibration, and this follows from [Lemma9](#). \square

Definition 11. We say a morphism i in **bisSet** is a *levelwise left anodyne extension* if $i \circ - \Delta^n$ is a left anodyne extension for each $n \geq 0$. \blacksquare

Lemma 12. Morphisms in $lr(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}})$ are levelwise left anodyne extensions.

Proof. Firstly, observe that the class of levelwise left anodyne extensions is described as

$$\bigcup_{n \geq 0} ((-) \circ - \Delta^n)^{-1}(lr(\mathbf{l}_{\text{ft}})),$$

where $((-) \circ - \Delta^n)^{-1}$ means the inverse image of the function on morphisms induced from the functor $(-) \circ - \Delta^n: \mathbf{bisSet} \longrightarrow \mathbf{sSet}$. Since $((-) \circ - \Delta^n)^{-1}$ preserves colimits and saturated classes are closed under joins, this class is saturated. Therefore, it suffices to show that morphisms in $\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}}$ are levelwise left anodyne extensions.

Let $i: A \rightarrow B \in \mathbf{l}_{\text{ft}}$ and $j: C \rightarrow D \in \mathbf{l}_{\text{mono}}$. By applying $(-) \circ \Delta^n$ to the diagram defining $i \boxtimes^L j$, we obtain the following diagram in \mathbf{sSet} .

$$\begin{array}{ccc}
 A \times C_n & \xrightarrow{\text{id} \times j_n} & A \times D_n \\
 i \times \text{id} \downarrow & \lrcorner \downarrow x & \downarrow i \times \text{id} \\
 B \times C_n & \longrightarrow & \cdot \\
 & & (i \boxtimes^L j) \circ \Delta^n \\
 & \searrow & \downarrow \\
 & & B \times D_n
 \end{array}$$

The two morphisms named $i \times \text{id}$ are both left anodyne extensions since left anodyne extensions are closed under finite product [citation](#). Since $(-) \circ \Delta^n$ preserves pushouts, the marked square is a pushout square, and hence x is also a left anodyne extension. Moreover, since the outer square comprises monomorphisms, all morphisms in this diagram are monomorphisms [citation](#). Therefore, the resulting morphism $(i \boxtimes^L j) \circ \Delta^n$ is a initial monomorphism, which is a left anodyne extension. \square

Lemma 13. *A morphism i in \mathbf{bisSet} is a monomorphism if and only if $i \circ \Delta^n$ is a monomorphism for each $n \geq 0$.*

Proof. By the yoneda lemma and the fact that $(-) \circ \Delta^m$ is a right adjoint. In detail, the following are equivalent.

- i is a monomorphism.
- $\mathbf{bisSet}(\Delta^n \boxtimes \Delta^m, i)$ is a monomorphism in \mathbf{Set} for each $n, m \geq 0$.
- $\mathbf{sSet}(\Delta^n, i \circ \Delta^m)$ is a monomorphism in \mathbf{Set} for each $n, m \geq 0$.
- $i \circ \Delta^m$ is a monomorphism in \mathbf{sSet} for each $m \geq 0$. \square

Proposition 14. *Levelwise left anodyne extensions are left bi-anodyne extensions.*

Proof. Let $i: X \rightarrow Y$ be a levelwise left anodyne extension. [Lemma 13](#) shows i is a monomorphism. By the small object argument, we have a weak factorisation system $(\ell r(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}}), r(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}}))$, and let us write $q \circ j = i$ for the factorisation of i with respect to this system. Since j is in particular a left bi-anodyne extension and i is a monomorphism, it suffices to show q is a trivial fibration. Moreover, [Corollary 7](#) and [Lemma 10](#) ensures that it suffices to show $q \circ \Delta^n$ is a trivial fibration for each simplicial set K .

Consider the case when $K = \Delta^n$ for some $n \geq 0$. [Lemma 6](#) shows $q \circ \Delta^n$ is a left fibration and [Lemma 12](#) shows $j \circ \Delta^n$ is a left anodyne extension. Since $i \circ \Delta^n$ is assumed to be a left anodyne extension, $q \circ \Delta^n$ is a initial left fibration, which is a trivial fibration.

By Lemma 1.3.10 of [\[Cis19\]](#), it suffices to show that the class $\{K \in \mathbf{sSet} \mid q \circ \Delta^n \text{ is a trivial fibration}\}$ is saturated by monomorphisms. Since trivial fibrations are closed under products, the class is closed under coproducts. To show that it is closed under the pushouts and transfinite compositions indicated in \square

REFERENCES

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