NOTE ON LEMMA5.5.16 OF "HIGHER CATEGORIES AND HOMOTOPICAL ALGEBRA"

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See [Cis19] and the question[1] in mathoverflow. The main objective is Proposition 3.4, which implies [Cis19, Lemma5.5.16].

Notation 0.1 (basics)**.** We employ the following notations.

- Given any category $\mathbf{C}$ and two objects $A, B \in \mathbf{C}$, we write $\mathbf{C}(A, B)$ for the homset.
- $\mathbb{\Delta}$ is the category of simplices. We write Δ^n for the n-simplex.
- We write $\mathbf{C}^{[1]}$ for the arrow category of $\mathbf{C}$. The category $[1]$ is the walking arrow (the 2-element chain), and we write $\square$ for $[1] \times [1]$.
- We write $\mathbf{Set}$ for the category of sets.
- $\mathbf{sSet}$ is the category of simplicial sets; i.e., $\mathbf{sSet} = [\mathbb{\Delta}^{\mathsf{op}}, \mathbf{Set}]$. Every set is seen as a simplicial set by the diagonal functor $\mathbf{Set} \longrightarrow \mathbf{sSet}$.
- $\mathbf{bisSet}$ is the category of bisimplicial sets; i.e., $\mathbf{bisSet} = [\mathbb{\Delta}^{\mathsf{op}} \times \mathbb{\Delta}^{\mathsf{op}}, \mathbf{Set}]$ ■

Definition 0.2. We write $- \boxtimes - \colon \mathbf{sSet} \times \mathbf{sSet} \longrightarrow \mathbf{bisSet}$ for the nerve functor of $\text{よ} \times \text{よ} \colon \mathbb{\Delta} \times \mathbb{\Delta} \longrightarrow \mathbf{sSet} \times \mathbf{sSet}$, where $\text{よ} \colon \mathbb{\Delta} \longrightarrow \mathbf{sSet}$ is the yoneda embedding.

There are functors $- \multimap - \colon \mathbf{sSet}^{\mathsf{op}} \times \mathbf{bisSet} \longrightarrow \mathbf{sSet}$ and $- \multimapinv - \colon \mathbf{bisSet} \times \mathbf{sSet}^{\mathsf{op}} \longrightarrow \mathbf{sSet}$ such that there exists the following family of bijections

$$\mathbf{bisSet}(A \boxtimes B, X) \cong \mathbf{sSet}(A, X \multimapinv B) \cong \mathbf{sSet}(B, A \multimap X) \tag{1}$$

that is natural in $A, B \in \mathbf{sSet}^{\mathsf{op}}$ and $X \in \mathbf{bisSet}$. ■

Definition 0.3. $[1]$ can be seen as a lattice so that there are the meet $\wedge \colon \square \longrightarrow [1]$ and the join $\vee \colon \square \longrightarrow [1]$. Given a category $\mathbf{C}$ with pushout and pullback, we obtain $L_\wedge \colon \mathbf{C}^\square \longrightarrow \mathbf{C}^{[1]}$ as the right Kan extension along the meet $\wedge$, while we have $L_\vee \colon \mathbf{C}^\square \longrightarrow \mathbf{C}^{[1]}$ as the left Kan extension along the join $\vee$. In detail, $L_\wedge(\alpha)$ is the unique morphism making the diagram on the left below, where α is the outer square and the inner square is a pullback square. Dually, $L_\vee(\alpha)$ is the canonical morphism described on the right.

$$\begin{tikzcd}
A \arrow[rrd, bend left] \arrow[rdd, bend right] \arrow[rd, dashed, "L_\wedge(\alpha)"] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[rdd, bend left] & \\
X \arrow[r] \arrow[rrd, bend right] & \cdot \arrow[ul, phantom, "\ulcorner", very near start] \arrow[rd, dashed, "L_\vee(\alpha)"] & \\
& & Y
\end{tikzcd}$$

■

Definition 0.4. Suppose we are given a functor $\otimes \colon \mathbf{A} \times \mathbf{B} \longrightarrow \mathbf{C}$. Consider the composite

$$[1] \times [1] \times \mathbf{A}^{[1]} \times \mathbf{B}^{[1]} \xrightarrow{\text{evaluation}} \mathbf{A} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{C},$$

where the former is the canonical evaluation functor that sends (i, j, X, Y) to $(X(i), Y(j))$. The currying of this composite gives a functor $\bar{\otimes} \colon \mathbf{A}^{[1]} \times \mathbf{B}^{[1]} \longrightarrow \mathbf{C}^\square$.

Now we have two functors $L_\wedge \circ \bar{\otimes}$ and $L_\vee \circ \bar{\otimes}$ of the form $\mathbf{A}^{[1]} \times \mathbf{B}^{[1]} \longrightarrow \mathbf{C}^{[1]}$. The first one is called the *pullback product* with respect to $\otimes$, while the second one is called the *pushout product* or the *Leibniz product* with respect to $\otimes$. ■

Definition 0.5. We write $- \boxtimes^{\mathrm{L}} - \colon \mathbf{sSet}^{[1]} \times \mathbf{sSet}^{[1]} \longrightarrow \mathbf{bisSet}^{[1]}$ for the pushout product with respect to $\boxtimes$; given two morphisms $f \colon A \longrightarrow B$ and $g \colon C \longrightarrow D$ in $\mathbf{sSet}$, the morphism $f \boxtimes^{\mathrm{L}} g$ is the canonical

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[1] https://mathoverflow.net/questions/467753/on-lemma-5-5-16-of-cisinskis-higher-categories-and-homotopical-algebra

one of the form $A \boxtimes D \cup_{A \boxtimes C} B \boxtimes C \longrightarrow B \boxtimes D$ whose domain is the fibred coproduct of $f \boxtimes \mathtt{id}_C$ and $\mathtt{id}_A \boxtimes g$. ■

Definition 0.6. We write $-\overset{\mathrm{L}}{\multimap}-\colon \mathbf{sSet}^{[1],\mathsf{op}} \times \mathbf{bisSet}^{[1]} \longrightarrow \mathbf{sSet}^{[1]}$ and $-\overset{\mathrm{L}}{\multimapinv}-\colon \mathbf{bisSet}^{[1]} \times \mathbf{sSet}^{[1],\mathsf{op}} \longrightarrow \mathbf{sSet}^{[1]}$ for the pullback product for $\multimap$ and $\multimapinv$ respectively. Therefore, given a morphism $f\colon A \longrightarrow B$ in **sSet** and a morphism $h\colon X \longrightarrow Y$ in **bisSet**, the morphism $f \overset{\mathrm{L}}{\multimap} h$ is the canonical one towards the fibred product of $f \multimap \mathtt{id}_Y$ and $\mathtt{id}_A \multimap h$. ■

Notation 0.7 (class of morphisms)**.** We employ the following notations.

- Let i and k be morphisms. We write $i \pitchfork k$ if i has left lifting property to k, or equivalently, k has right lifting property to i.
- Given a class of morphisms I, we write $\ell(\mathsf{I})$ for the class of morphisms that have left lifting property to those in I. Dually, we write $r(\mathsf{I})$ for the class of morphisms that have right lifting property to those in I.
- Given two classes of morphisms I and K in **sSet**, we obtain a class of morphisms $\mathsf{I} \boxtimes^{\mathrm{L}} \mathsf{K} = \{i \boxtimes^{\mathrm{L}} k \mid i \in \mathsf{I},\ k \in \mathsf{K}\}$. Similarly, we define $\mathsf{J} \overset{\mathrm{L}}{\multimapinv} \mathsf{I}$ and $\mathsf{I} \overset{\mathrm{L}}{\multimap} \mathsf{J}$ for a class J of morphisms in **bisSet**.
- We write $\mathsf{I}_{\mathsf{mono}}$ for the canonical cellular model for **sSet**; i.e., the set of boundary inclusions.
- We write $\mathsf{I}_{\mathsf{lft}}$ for the set of generating left anodyne extensions; i.e. the set of horn inclusions lifting initial objects. ■

The set $\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}} \cup \mathsf{I}_{\mathsf{mono}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{lft}}$ is the generator for left bi-anodyne extensions (see [Cis19, Definition 5.5.10]).

Notation 0.8 (Homotopical structures)**.** In this note, the following homotopical structures ([Cis19, Definition 2.4.11]) are considered.

- For each simplicial set A, we write $(\mathbf{sSet}/A)_{\mathsf{co}}$ for the slice category equipped with the homotopical structure defined for the covariant model structure ([Cis19, §4.1 and Theorem 4.4.14]).
- For each bisimplicial set X, we write $(\mathbf{bisSet}/X)_{\mathsf{co}}$ for the slice category equipped with the homotopical structure for the bicovariant model structure ([Cis19, Lemma 5.5.12 and Theorem 5.5.13]). ■

We write ω for the least infinite ordinal seen as a category by its order. The horn Λ^2_2 is seen as a category; i.e., Λ^2_2 is the walking cospan.

1. On classes of morphisms in bisSet

Lemma 1.1. *For each morphisms i, j in* **sSet** *and a morphism h in* **bisSet***, the following are equivalent.*

- $i \boxtimes^{L} j \pitchfork h$
- $i \pitchfork h \overset{L}{\multimapinv} j$
- $j \pitchfork i \overset{L}{\multimap} h$

Proof. Follows from (1) and the universality of the pushout and the pullbacks defining $i \boxtimes^{\mathrm{L}} j$, $h \overset{\mathrm{L}}{\multimapinv} j$, and $i \overset{\mathrm{L}}{\multimap} h$. □

Lemma 1.2. *Suppose we are given two small classes I and J of morphisms in* **sSet***, and a morphism h in* **bisSet***. The following are equivalent.*

i) $h \in r(\mathsf{I} \boxtimes^{L} \mathsf{J})$.
ii) *For each* $j \in \mathsf{J}$, $h \overset{L}{\multimapinv} j \in r(\mathsf{I})$.
iii) *For each* $i \in \mathsf{I}$, $i \overset{L}{\multimap} h \in r(\mathsf{J})$.
iv) *For each* $j \in \ell r(\mathsf{J})$, $h \overset{L}{\multimapinv} j \in r(\mathsf{I})$.
v) *For each* $i \in \ell r(\mathsf{I})$, $i \overset{L}{\multimap} h \in r(\mathsf{J})$.

Proof. By Lemma 1.1, the first three arguments are equivalent to the following.

$$j \pitchfork i \overset{\mathrm{L}}{\multimap} h \quad \text{for any } i \in \mathsf{I} \text{ and } j \in \mathsf{J}.$$

Since $\ell(\mathsf{I} \overset{\mathrm{L}}{\multimap} \{h\})$ is saturated, this is also equivalent to

$$j \pitchfork i \overset{\mathrm{L}}{\multimap} h \quad \text{for any } i \in \mathsf{I} \text{ and } j \in \ell r(\mathsf{J}),$$

which is equivalent to *iv)*. *v)* is also equivalent to them in the same way. □

Corollary 1.3 ([Cis19, Lemma 5.5.6])**.** *A morphism q in* **bisSet** *is a trivial fibration if and only if $q \overset{L}{\multimapinv} i$ is a trivial fibration for each monomorphism i in* **sSet***.*

Proof. $\mathsf{I}_{\mathsf{mono}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}$ is a cellular model for **bisSet** (see [Cis19, Example 1.3.4 and 2.4.5]), and this is a direct consequence of Lemma 1.2, where both I and J are $\mathsf{I}_{\mathsf{mono}}$. □

Lemma 1.4. $\mathsf{dom}\colon (\mathbf{sSet}/A)_{\mathsf{co}} \longrightarrow (\mathbf{sSet})_{\mathsf{co}}$ *preserves and lifts trivial fibrations and naive fibrations.*

Proof. Recall that dom preserves and lifts both monomorphisms and anodyne extensions.

Let $f\colon S \longrightarrow T$ be a morphism in $\mathbf{sSet}/A$. We show that f is a trivial/naive fibration if and only if $\mathsf{dom}(f)$ is a trivial/naive fibration. The if part is easily checked by considering that dom preserves monomorphisms/anodyne extensions.

Let i be a monomorphism/anodyne extension in $(\mathbf{sSet})_{\mathsf{co}}$ and $(u,v)\colon i \longrightarrow \mathsf{dom}(f)$ be a morphism in **sSet**[1]. It suffices to show there is a diagonal filler for this square. The morphisms towards A from f is followed by (u,v), and this enables us to see (u,v) as a square in $\mathbf{sSet}/A$. Moreover, its domain is a monomorphism/anodyne extension. Therefore, desired diagonal filler exists if f is a trivial/naive fibration in $\mathbf{sSet}/A$. □

Lemma 1.5. *A left fibration between trivial fibrations is a trivial fibration.*

Proof. Suppose we are given a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\sim", "h"'] & & B \arrow[dl, "\sim"', "g"] \\ & C & \end{tikzcd}$$

in **sSet** such that f is a left fibration (i.e., a naive fibration in $(\mathbf{sSet})_{\mathsf{co}}$) and g and h are trivial fibrations. By Lemma 1.4, this lifts to another triangle

$$\begin{tikzcd} (A,h) \arrow[rr, "f"] \arrow[dr, "\sim", "h"'] & & (B,g) \arrow[dl, "\sim"', "g"] \\ & (C,\mathtt{id}) & \end{tikzcd}$$

in $(\mathbf{sSet}/C)_{\mathsf{co}}$ satisfying the same condition. Since (A,h) and (B,g) are fibrant, f is a fibration. Therefore, the 2 out of 3 property for weak equivalences ensures that f is a trivial fibration. □

Proposition 1.6. *Suppose we are given a morphism $q\colon X \longrightarrow Y$ in* **bisSet** *that belongs to $r(\mathsf{I}_{\mathsf{lft}} \boxtimes^{L} \mathsf{I}_{\mathsf{mono}})$ and a monomorphism $i\colon K \rightarrowtail L$ in* **sSet***. $q \overset{L}{\multimapinv} i$ is a trivial fibration if $q \multimapinv K$ and $q \multimapinv L$ are trivial fibrations.*

Proof. By Lemma 1.2, $q \overset{\mathrm{L}}{\multimapinv} i$ is a left fibration filling the following diagram.

$$\begin{tikzcd} X \multimapinv L & X \multimapinv K \arrow[l, "X \multimapinv i"'] & \\ Y \multimapinv L \arrow[u, "q \multimapinv L"] & \cdot \arrow[l] \arrow[u, "x"'] \arrow[ul, phantom, "\lrcorner"] & \\ & & Y \multimapinv K \arrow[ul, "q \overset{\mathrm{L}}{\multimapinv} i"] \arrow[ul, bend right, "q \multimapinv K"'] \arrow[ull, bend left, ""] \end{tikzcd}$$

The pullback square assures x is also a trivial fibration, and this follows from Lemma 1.5. □

2. Insights from Reedy category theory

Definition 2.1. Suppose that we are given a cocomplete category $\mathbf{C}$ and a class S of objects in $\mathbf{C}$. We say S is ***saturated by monomorphisms*** if for each diagram $F\colon \mathbb{I}^{\mathsf{op}} \longrightarrow \mathbf{C}$ satisfying the following conditions, the colimit of F is in S;

- $\mathbb{I}$ is a discrete category, or either of ω^{op} or Λ^2_2.
- If $u\colon i \longrightarrow j$ in $\mathbb{I}$ is of the form $n+1 \longrightarrow n$ in ω^{op} or $1 \longrightarrow 2$ in Λ^2_2, then $F(u)$ is a monomorphism.
- The image of F is in S. ■

Fact 2.2 ([Cis19, Corollary 1.3.10])**.** *A class of simplicial sets contains all simplicial sets if it is saturated by monomorphisms and contains representables.*

Definition 2.3. Suppose that we are given

- a complete category $\mathbf{C}$,
- a weak factorisation system $(\mathsf{Cof}, \mathsf{TFib})$ whose elements are called *cofibrations* and *trivial fibrations* respectively,
- a small category $\mathbb{I}$ that is a discrete category or either of ω^{op} or Λ^2_2, and
- two functors $F, G \colon \mathbb{I} \longrightarrow \mathbf{C}$.

A natural transformation $\alpha \colon F \Longrightarrow G$ is a ***Reedy trivial fibration*** if it satisfies the followings.

- For each object $i \in \mathbb{I}$, $\alpha_i \colon F(i) \longrightarrow G(i)$ is a trivial fibration.
- Let $u \colon i \longrightarrow j$ in $\mathbb{I}$ be either of $n+1 \longrightarrow n$ in ω^{op} or $1 \longrightarrow 2$ in Λ^2_2. Then the naturality square

$$\begin{tikzcd} F(i) \arrow[r, "F(u)"] \arrow[d, "\alpha_i"'] & F(j) \arrow[d, "\alpha_j"] \\ G(i) \arrow[r, "G(u)"'] & G(j) \end{tikzcd}$$

  is sent by $L_\wedge$ (see Definition 0.3) to a trivial fibration. ■

Proposition 2.4. *The limit of a Reedy trivial fibration is a trivial fibration.*

Proof. Let us write $\Delta_{\mathbb{I}} \colon \mathbf{C} \longrightarrow [\mathbb{I}, \mathbf{C}]$ for the diagonal functor, and write $\lim_{\mathbb{I}}$ for its right adjoint. Suppose that we are given a cofibration $f \colon K \rightarrowtail L$ in $\mathbf{C}$ and a Reedy trivial fibration $\alpha \colon F \longrightarrow G$. We show $f \pitchfork \lim_{\mathbb{I}}(\alpha)$, which is equivalent to $\Delta_{\mathbb{I}}(f) \pitchfork \alpha$ in $[\mathbb{I}, \mathbf{C}]$. To this end, let $(u, v) \colon \Delta_{\mathbb{I}}(f) \longrightarrow \alpha$ be a morphism in $[\mathbb{I}, \mathbf{C}]^{[1]}$. It suffices to find the diagonal filler $k \colon \Delta_{\mathbb{I}}(L) \longrightarrow F$ for this square.

- When $\mathbb{I}$ is a discrete category, take diagonal fillers $k_i \colon L \longrightarrow G(i)$ for each squares $(u_i, v_i) \colon f \longrightarrow \alpha_i$. Such fillers exist because α_i is a trivial fibration for each $i \in \mathbb{I}$.
- Suppose $\mathbb{I} = \omega^{\mathsf{op}}$. We construct the diagonal filler $(k_n)_{n \geq 0}$ by induction on n.
  (n=0) k_0 is the diagonal filler for the square $(u_0, v_0) \colon f \longrightarrow \alpha_0$.
  (n>0) Suppose we have constructed k_{n-1} that fits into the following commutative diagram.

$$\begin{tikzcd} K \arrow[r, "u_n"] \arrow[d, tail, "f"'] & F(n) \arrow[r] \arrow[d, dashed] & F(n-1) \arrow[d, "\alpha_{n-1}"] \\ L \arrow[r, "v_n"'] \arrow[urr, "k_{n-1}"] & G(n) \arrow[r] & G(n-1) \end{tikzcd}$$

  Since the square on the right is sent by $L_\wedge$ to a trivial fibration, we have another square whose right side is a tribial fibration;

$$\begin{tikzcd} K \arrow[r, "u_n"] \arrow[d, tail, "f"'] & F(n) \arrow[d] \\ K \arrow[r, "\bar{k}_{n-1}"'] & \cdot \end{tikzcd}$$

  where the bottom side is the morphism induced from k_{n-1} and v_n by the universality of the pullback defining the right side. We define k_n as a filler for this square.
- Suppose $\mathbb{I} = \Lambda^2_2$. Define $(k_x)_{x=0,1,2}$ as follows.
  - k_0 is the filler for $(u_0, v_0) \colon f \longrightarrow \alpha_0$, and define k_2 as the composite

$$L \xrightarrow{k_0} F(0) \longrightarrow F(2)$$

    where the latter is the image under F of $0 \longrightarrow 2$.
  - Now we have the following commutative diagram.

$$\begin{tikzcd} K \arrow[r, "u_1"] \arrow[d, tail, "f"'] & F(1) \arrow[r] \arrow[d, dashed] & F(2) \arrow[d, "\alpha_2"] \\ L \arrow[r, "v_1"'] \arrow[urr, "k_2"] & G(1) \arrow[r] & G(2) \end{tikzcd}$$

We define k_1 in the same way as the definition of k_n above for $\mathbb{I} = \omega^{\mathsf{op}}$. □

3. The main proposition

Definition 3.1. We say a morphism i in **bisSet** is a ***levelwise left anodyne extension*** if $i \multimapinv \Delta^n$ is a left anodyne extension for each $n \geq 0$. ■

Lemma 3.2. *Morphisms in $\ell r(\mathsf{I}_{\mathsf{lft}} \boxtimes^L \mathsf{I}_{\mathsf{mono}})$ are levelwise left anodyne extensions.*

Proof. Firstly, observe that the class of levelwise left anodyne extensions is described as

$$\bigcup_{n \geq 0} ((-) \multimapinv \Delta^n)^{-1}(\ell r(\mathsf{I}_{\mathsf{lft}})),$$

where $((-) \multimapinv \Delta^n)^{-1}$ means the inverse image of the function on morphisms induced from the functor $(-) \multimapinv \Delta^n \colon \mathbf{bisSet} \longrightarrow \mathbf{sSet}$. Since $(-) \multimapinv \Delta^n$ preserves colimits and saturated classes are closed under unions, this class is saturated. Therefore, it suffices to show that morphisms in $\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}$ are levelwise left anodyne extensions.

Let $i \colon A \longrightarrow B \in \mathsf{I}_{\mathsf{lft}}$ and $j \colon C \longrightarrow D \in \mathsf{I}_{\mathsf{mono}}$. By applying $(-) \multimapinv \Delta^n$ to the diagram defining $i \boxtimes^{\mathrm{L}} j$, we obtain the following diagram in **sSet**.

$$\begin{tikzcd}
A \times C_n \arrow[r, "\mathtt{id} \times j_n"] \arrow[d, "i \times \mathtt{id}"'] & A \times D_n \arrow[d, "x"] \arrow[ddr, bend left, "i \times \mathtt{id}"] & \\
B \times C_n \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\lrcorner" very near start] \arrow[dr, "(i \boxtimes^{\mathrm{L}} j) \multimapinv \Delta^n"] & \\
& & B \times D_n
\end{tikzcd}$$

The two morphisms named $i \times \mathtt{id}$ are both left anodyne extensions since left anodyne extensions are closed under finite product ([Cis19, Proposition 3.4.3]). Since $(-) \multimapinv \Delta^n$ preserves pushouts, the marked square is a pushout square, and hence x is also a left anodyne extension. Moreover, since the outer square comprises monomorphisms, all morphisms in this diagram are monomorphisms citation. Therefore, the resulting morphism $(i \boxtimes^{\mathrm{L}} j) \multimapinv \Delta^n$ is a initial monomorphism, which is a left anodyne extension. □

Lemma 3.3. *A morphism i in* **bisSet** *is a monomorphism if and only if $i \multimapinv \Delta^n$ is a monomorphism for each $n \geq 0$.*

Proof. By the yoneda lemma and the fact that $(-) \multimapinv \Delta^m$ is a right adjoint. In detail, the following are equivalent.

- i is a monomorphism.
- $\mathbf{bisSet}(\Delta^n \boxtimes \Delta^m, i)$ is a monomorphism in **Set** for each $n, m \geq 0$.
- $\mathbf{sSet}(\Delta^n, i \multimapinv \Delta^m)$ is a monomorphism in **Set** for each $n, m \geq 0$.
- $i \multimapinv \Delta^m$ is a monomorphism in **sSet** for each $m \geq 0$. □

Proposition 3.4. *Levelwise left anodyne extensions are left bi-anodyne extensions.*

Proof. Let $i \colon X \longrightarrow Y$ be a levelwise left anodyne extension. Lemma 3.3 shows i is a monomorphism. By the small object argument, we have a weak factorisation system $(\ell r(\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}), r(\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}))$, and let us write

$$\begin{tikzcd}
X \arrow[rr, "i"] \arrow[dr, "j"'] & & Y \\
& Z \arrow[ur, "q"'] &
\end{tikzcd}$$

for the factorisation of i with respect to this system. Since j is in particular a left bi-anodyne extension and i is a monomorphism, it suffices to show q is a trivial fibration. Moreover, Corollary 1.3 and Proposition 1.6 ensures that it suffices to show $q \multimapinv K$ is a trivial fibration for each simplicial set K.

Consider the case when $K = \Delta^n$ for some $n \geq 0$. Lemma 1.2 shows $q \multimapinv \Delta^n$ is a left fibration and Lemma 3.2 shows $j \multimapinv \Delta^n$ is a left anodyne extension. Since $i \multimapinv \Delta^n$ is assumed to be a left anodyne extension, $q \multimapinv \Delta^n$ is a initial left fibration, which is a trivial fibration.

By Corollary 1.3.10 of [Cis19] (see Fact 2.2), it suffices to show that the class

$$S = \{K \in \mathbf{sSet} \mid q \multimapinv K \text{ is a trivial fibration } \}$$

is saturated by monomorphisms. Let $F \colon \mathbb{I}^{\mathsf{op}} \longrightarrow \mathbf{sSet}$ be a diagram satisfying the conditions described in Definition 2.1. Since $Z \multimapinv -$ sends colimits to limits for any $Z \in \mathbf{bisSet}$, by virtue of Proposition 2.4, it suffices to show that the natural transformation

$$q \multimapinv F(-) \colon Z \multimapinv F(-) \Longrightarrow Y \multimapinv F(-) \colon \mathbb{I} \longrightarrow \mathbf{sSet} \tag{2}$$

is a Reedy trivial fibration. Now by the definition of the pullback product $\overset{\llcorner}{\multimapinv}$, one can check that Proposition 1.6 shows that the naturality square for the transformation $q \multimapinv -$ at a monomorphism between objects in S is sent by $L_\wedge$ to a trivial fibration. This shows (2) is a Reedy trivial fibration, which completes the proof. □

References

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