

See [Cis19] and the question<sup>1</sup> in mathoverflow. The main objective is Proposition 3.4, which implies [Cis19, Lemma5.5.16].

**Notation 0.1** (basics). We employ the following notations.

- Given any category  $\mathbf{C}$  and two objects  $A, B \in \mathbf{C}$ , we write  $\mathbf{C}(A, B)$  for the homset.
- $\Delta$  is the category of simplices. We write  $\Delta^n$  for the  $n$ -simplex.
- We write  $\mathbf{C}^{[1]}$  for the arrow category of  $\mathbf{C}$ . The category  $[1]$  is the walking arrow (the 2-element chain), and we write  $\square$  for  $[1] \times [1]$ .
- We write  $\mathbf{Set}$  for the category of sets.
- $\mathbf{sSet}$  is the category of simplicial sets; i.e.,  $\mathbf{sSet} = [\Delta^{\text{op}}, \mathbf{Set}]$ . Every set is seen as a simplicial set by the diagonal functor  $\mathbf{Set} \rightarrow \mathbf{sSet}$ .
- $\mathbf{bisSet}$  is the category of bisimplicial sets; i.e.,  $\mathbf{bisSet} = [\Delta^{\text{op}} \times \Delta^{\text{op}}, \mathbf{Set}]$  ■

**Definition 0.2.** We write  $- \boxtimes - : \mathbf{sSet} \times \mathbf{sSet} \rightarrow \mathbf{bisSet}$  for the nerve functor of  $\mathfrak{y} \times \mathfrak{y} : \Delta \times \Delta \rightarrow \mathbf{sSet} \times \mathbf{sSet}$ , where  $\mathfrak{y} : \Delta \rightarrow \mathbf{sSet}$  is the yoneda embedding.

There are functors  $- \circ - : \mathbf{sSet}^{\text{op}} \times \mathbf{bisSet} \rightarrow \mathbf{sSet}$  and  $- \circ - : \mathbf{bisSet} \times \mathbf{sSet}^{\text{op}} \rightarrow \mathbf{sSet}$  such that there exists the following family of bijections

$$(1) \quad \mathbf{bisSet}(A \boxtimes B, X) \cong \mathbf{sSet}(A, X \circ - B) \cong \mathbf{sSet}(B, A \circ - X)$$

that is natural in  $A, B \in \mathbf{sSet}^{\text{op}}$  and  $X \in \mathbf{bisSet}$ . ■

**Definition 0.3.**  $[1]$  can be seen as a lattice so that there are the meet  $\wedge : \square \rightarrow [1]$  and the join  $\vee : \square \rightarrow [1]$ . Given a category  $\mathbf{C}$  with pushout and pullback, we obtain  $L_{\wedge} : \mathbf{C}^{\square} \rightarrow \mathbf{C}^{[1]}$  as the right Kan extension along the meet  $\wedge$ , while we have  $L_{\vee} : \mathbf{C}^{\square} \rightarrow \mathbf{C}^{[1]}$  as the left Kan extension along the join  $\vee$ . In detail,  $L_{\wedge}(\alpha)$  is the unique morphism making the diagram on the left below, where  $\alpha$  is the outer square and the inner square is a pullback square. Dually,  $L_{\vee}(\alpha)$  is the canonical morphism described on the right.

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow \scriptstyle L_{\wedge}(\alpha) & \searrow & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\quad} & Y \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\quad} & \cdot \\ \searrow & \swarrow \scriptstyle L_{\vee}(\alpha) & \downarrow \\ & & Y \end{array}$$

**Definition 0.4.** Suppose we are given a functor  $\otimes : \mathbf{A} \times \mathbf{B} \rightarrow \mathbf{C}$ . Consider the composite

$$[1] \times [1] \times \mathbf{A}^{[1]} \times \mathbf{B}^{[1]} \xrightarrow{\text{evaluation}} \mathbf{A} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{C},$$

where the former is the canonical evaluation functor that sends  $(i, j, X, Y)$  to  $(X(i), Y(j))$ . The currying of this composite gives a functor  $\bar{\otimes} : \mathbf{A}^{[1]} \times \mathbf{B}^{[1]} \rightarrow \mathbf{C}^{\square}$ .

Now we have two functors  $L_{\wedge} \circ \bar{\otimes}$  and  $L_{\vee} \circ \bar{\otimes}$  of the form  $\mathbf{A}^{[1]} \times \mathbf{B}^{[1]} \rightarrow \mathbf{C}^{[1]}$ . The first one is called the *pullback product* with respect to  $\otimes$ , while the second one is called the *pushout product* or the *Leibniz product* with respect to  $\otimes$ . ■

**Definition 0.5.** We write  $- \boxtimes^L - : \mathbf{sSet}^{[1]} \times \mathbf{sSet}^{[1]} \rightarrow \mathbf{bisSet}^{[1]}$  for the *pushout product* with respect to  $\boxtimes$ ; given two morphisms  $f : A \rightarrow B$  and  $g : C \rightarrow D$  in  $\mathbf{sSet}$ , the morphism  $f \boxtimes^L g$  is the canonical

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<sup>1</sup><https://mathoverflow.net/questions/467753/on-lemma-5-5-16-of-cisinskis-higher-categories-and-homotopical-algebra>

one of the form  $A \boxtimes D \cup_{A \boxtimes C} B \boxtimes C \rightarrow B \boxtimes D$  whose domain is the fibred coproduct of  $f \boxtimes \text{id}_C$  and  $\text{id}_A \boxtimes g$ . ■

**Definition 0.6.** We write  $-\overset{\text{L}}{\circ}- : \mathbf{sSet}^{[1], \text{op}} \times \mathbf{bisSet}^{[1]} \rightarrow \mathbf{sSet}^{[1]}$  and  $-\circ^{\text{L}}- : \mathbf{bisSet}^{[1]} \times \mathbf{sSet}^{[1], \text{op}} \rightarrow \mathbf{sSet}^{[1]}$  for the pullback product for  $\rightarrow$  and  $\circ$  respectively. Therefore, given a morphism  $f : A \rightarrow B$  in  $\mathbf{sSet}$  and a morphism  $h : X \rightarrow Y$  in  $\mathbf{bisSet}$ , the morphism  $f \overset{\text{L}}{\circ} h$  is the canonical one towards the fibred product of  $f \rightarrow \text{id}_Y$  and  $\text{id}_A \rightarrow h$ . ■

**Notation 0.7** (class of morphisms). We employ the following notations.

- Let  $i$  and  $k$  be morphisms. We write  $i \pitchfork k$  if  $i$  has left lifting property to  $k$ , or equivalently,  $k$  has right lifting property to  $i$ .
- Given a class of morphisms  $\mathbf{I}$ , we write  $\ell(\mathbf{I})$  for the class of morphisms that have left lifting property to those in  $\mathbf{I}$ . Dually, we write  $r(\mathbf{I})$  for the class of morphisms that have right lifting property to those in  $\mathbf{I}$ .
- Given two classes of morphisms  $\mathbf{I}$  and  $\mathbf{K}$  in  $\mathbf{sSet}$ , we obtain a class of morphisms  $\mathbf{I} \boxtimes^{\text{L}} \mathbf{K} = \{i \boxtimes^{\text{L}} k \mid i \in \mathbf{I}, k \in \mathbf{K}\}$ . Similarly, we define  $\mathbf{J} \circ^{\text{L}} \mathbf{I}$  and  $\mathbf{I} \overset{\text{L}}{\circ} \mathbf{J}$  for a class  $\mathbf{J}$  of morphisms in  $\mathbf{bisSet}$ .
- We write  $\mathbf{I}_{\text{mono}}$  for the canonical cellular model for  $\mathbf{sSet}$ ; i.e., the set of boundary inclusions.
- We write  $\mathbf{I}_{\text{ift}}$  for the set of generating left anodyne extensions; i.e. the set of horn inclusions lifting initial objects. ■

**Fact 0.8.** The set  $\mathbf{I}_{\text{ift}} \boxtimes^{\text{L}} \mathbf{I}_{\text{mono}} \cup \mathbf{I}_{\text{mono}} \boxtimes^{\text{L}} \mathbf{I}_{\text{ift}}$  is the generator for left bi-anodyne extensions.

**Notation 0.9** (Homotopical structures). In this note, the following homotopical structures are considered.

- For each simplicial set  $A$ , we write  $(\mathbf{sSet}/A)_{\text{co}}$  for the slice category equipped with the homotopical structure defined for the covariant model structure.
- For each bisimplicial set  $X$ , we write  $(\mathbf{bisSet}/X)_{\text{co}}$  for the slice category equipped with the homotopical structure for the bicovariant model structure. ■

We write  $\omega$  for the least infinite ordinal seen as a category by its order. The horn  $\Lambda_2^2$  is seen as a category; i.e.,  $\Lambda_2^2$  is the walking cospan.

## 1. ON CLASSES OF MORPHISMS IN $\mathbf{bisSet}$

**Lemma 1.1.** For each morphisms  $i, j$  in  $\mathbf{sSet}$  and a morphism  $h$  in  $\mathbf{bisSet}$ , the following are equivalent.

- $i \boxtimes^{\text{L}} j \pitchfork h$
- $i \pitchfork h \circ^{\text{L}} j$
- $j \pitchfork i \overset{\text{L}}{\circ} h$

*Proof.* Follows from (1) and the universality of the pushout and the pullbacks defining  $i \boxtimes^{\text{L}} j$ ,  $h \circ^{\text{L}} j$ , and  $i \overset{\text{L}}{\circ} h$ . □

**Lemma 1.2.** Suppose we are given two small classes  $\mathbf{I}$  and  $\mathbf{J}$  of morphisms in  $\mathbf{sSet}$ , and a morphism  $h$  in  $\mathbf{bisSet}$ . The following are equivalent.

- $h \in r(\mathbf{I} \boxtimes^{\text{L}} \mathbf{J})$ .
- For each  $j \in \mathbf{J}$ ,  $h \circ^{\text{L}} j \in r(\mathbf{I})$ .
- For each  $i \in \mathbf{I}$ ,  $i \overset{\text{L}}{\circ} h \in r(\mathbf{J})$ .
- For each  $j \in \ell r(\mathbf{J})$ ,  $h \circ^{\text{L}} j \in r(\mathbf{I})$ .
- For each  $i \in \ell r(\mathbf{I})$ ,  $i \overset{\text{L}}{\circ} h \in r(\mathbf{J})$ .

*Proof.* By Lemma 1.1, the first three arguments are equivalent to the following.

$$j \pitchfork i \overset{\text{L}}{\circ} h \quad \text{for any } i \in \mathbf{I} \text{ and } j \in \mathbf{J}.$$

Since  $\ell(\mathbf{I} \overset{\text{L}}{\circ} \{h\})$  is saturated, this is also equivalent to

$$j \pitchfork i \overset{\text{L}}{\circ} h \quad \text{for any } i \in \mathbf{I} \text{ and } j \in \ell r(\mathbf{J}),$$

which is equivalent to *iv*). *v*) is also equivalent to them in the same way. □

**Corollary 1.3** ([Cis19, Lemma 5.5.6]). A morphism  $q$  in  $\mathbf{bisSet}$  is a trivial fibration if and only if  $q \circ^{\text{L}} i$  is a trivial fibration for each monomorphism  $i$  in  $\mathbf{sSet}$ .

*Proof.*  $\mathbf{l}_{\text{mono}} \boxtimes^L \mathbf{l}_{\text{mono}}$  is a cellular model for **bisSet** (see [Cis19, Example 1.3.4 and 2.4.5]), and this is a direct consequence of Lemma 1.2, where both  $\mathbf{l}$  and  $\mathbf{j}$  are  $\mathbf{l}_{\text{mono}}$ .  $\square$

**Lemma 1.4.**  $\text{dom}: (\mathbf{sSet}/A)_{\text{co}} \rightarrow (\mathbf{sSet})_{\text{co}}$  preserves and lifts trivial fibrations and naive fibrations.

*Proof.* Recall that  $\text{dom}$  preserves and lifts both monomorphisms and anodyne extensions.

Let  $f: S \rightarrow T$  be a morphism in  $\mathbf{sSet}/A$ . We show that  $f$  is a trivial/naive fibration if and only if  $\text{dom}(f)$  is a trivial/naive fibration. The if part is easily checked by considering that  $\text{dom}$  preserves monomorphisms/anodyne extensions.

Let  $i$  be a monomorphism/anodyne extension in  $(\mathbf{sSet})_{\text{co}}$  and  $(u, v): i \rightarrow \text{dom}(f)$  be a morphism in  $\mathbf{sSet}^{[1]}$ . It suffices to show there is a diagonal filler for this square. The morphisms towards  $A$  from  $f$  is followed by  $(u, v)$ , and this enables us to see  $(u, v)$  as a square in  $\mathbf{sSet}/A$ . Moreover, its domain is a monomorphism/anodyne extension. Therefore, desired diagonal filler exists if  $f$  is a trivial/naive fibration in  $\mathbf{sSet}/A$ .  $\square$

**Lemma 1.5.** A left fibration between trivial fibrations is a trivial fibration.

*Proof.* Suppose we are given a commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \searrow h & & \swarrow g \\ & C & \end{array}$$

in  $\mathbf{sSet}$  such that  $f$  is a left fibration (i.e., a naive fibration in  $(\mathbf{sSet})_{\text{co}}$ ) and  $g$  and  $h$  are trivial fibrations. By Lemma 1.4, this lifts to another triangle

$$\begin{array}{ccc} (A, h) & \xrightarrow{f} & (B, g) \\ \searrow h & & \swarrow g \\ & (C, \text{id}) & \end{array}$$

in  $(\mathbf{sSet}/C)_{\text{co}}$  satisfying the same condition. Since  $(A, h)$  and  $(B, g)$  are fibrant,  $f$  is a fibration. Therefore, the 2 out of 3 property for weak equivalences ensures that  $f$  is a trivial fibration.  $\square$

**Proposition 1.6.** Suppose we are given a morphism  $q: X \rightarrow Y$  in **bisSet** that belongs to  $r(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}})$  and a monomorphism  $i: K \rightarrow L$  in  $\mathbf{sSet}$ .  $q \circ^L i$  is a trivial fibration if  $q \circ K$  and  $q \circ L$  are trivial fibrations.

*Proof.* By Lemma 1.2,  $q \circ^L i$  is a left fibration filling the following diagram.

$$\begin{array}{ccccc} X \circ L & \xleftarrow{X \circ i} & X \circ K & & \\ q \circ L \uparrow & & \uparrow x & \swarrow q \circ K & \\ Y \circ L & \xleftarrow{\quad} & \cdot & \swarrow q \circ^L i & \\ & & & \searrow & Y \circ K \end{array}$$

The pullback square assures  $x$  is also a trivial fibration, and this follows from Lemma 1.5.  $\square$

## 2. INSIGHTS FROM REEDY CATEGORY THEORY

**Definition 2.1.** Suppose that we are given a cocomplete category  $\mathbf{C}$  and a class  $S$  of objects in  $\mathbf{C}$ . We say  $S$  is **saturated by monomorphisms** if for each diagram  $F: \mathbb{I}^{\text{op}} \rightarrow \mathbf{C}$  satisfying the following conditions, the colimit of  $F$  is in  $S$ ;

- $\mathbb{I}$  is a discrete category, or either of  $\omega^{\text{op}}$  or  $\Lambda_2^2$ .
- If  $u: i \rightarrow j$  in  $\mathbb{I}$  is of the form  $n+1 \rightarrow n$  in  $\omega^{\text{op}}$  or  $1 \rightarrow 2$  in  $\Lambda_2^2$ , then  $F(u)$  is a monomorphism.
- The image of  $F$  is in  $S$ .  $\blacksquare$

**Fact 2.2** ([Cis19, Corollary 1.3.10]). A class of simplicial sets contains all simplicial sets if it is saturated by monomorphisms and contains representables.

**Definition 2.3.** Suppose that we are given

- a complete category  $\mathbf{C}$ ,
- a weak factorisation system  $(\mathbf{Cof}, \mathbf{TFib})$  whose elements are called *cofibrations* and *trivial fibrations* respectively,
- a small category  $\mathbb{I}$  that is a discrete category or either of  $\omega^{\text{op}}$  or  $\Lambda_2^2$ , and
- two functors  $F, G: \mathbb{I} \rightarrow \mathbf{C}$ .

A natural transformation  $\alpha: F \Rightarrow G$  is a **Reedy trivial fibration** if it satisfies the followings.

- For each object  $i \in \mathbb{I}$ ,  $\alpha_i: F(i) \rightarrow G(i)$  is a trivial fibration.
- Let  $u: i \rightarrow j$  in  $\mathbb{I}$  be either of  $n+1 \rightarrow n$  in  $\omega^{\text{op}}$  or  $1 \rightarrow 2$  in  $\Lambda_2^2$ . Then the naturality square

$$\begin{array}{ccc} F(i) & \xrightarrow{F(u)} & F(j) \\ \alpha_i \downarrow & & \downarrow \alpha_j \\ G(i) & \xrightarrow{G(u)} & G(j) \end{array}$$

is sent by  $L_\wedge$  (see [Definition 0.3](#)) to a trivial fibration. ■

**Proposition 2.4.** *The limit of a Reedy trivial fibration is a trivial fibration.*

*Proof.* Let us write  $\Delta_{\mathbb{I}}: \mathbf{C} \rightarrow [\mathbb{I}, \mathbf{C}]$  for the diagonal functor, and write  $\lim_{\mathbb{I}}$  for its right adjoint. Suppose that we are given a cofibration  $f: K \rightarrowtail L$  in  $\mathbf{C}$  and a Reedy trivial fibration  $\alpha: F \rightarrow G$ . We show  $f \pitchfork \lim_{\mathbb{I}}(\alpha)$ , which is equivalent to  $\Delta_{\mathbb{I}}(f) \pitchfork \alpha$  in  $[\mathbb{I}, \mathbf{C}]$ . To this end, let  $(u, v): \Delta_{\mathbb{I}}(f) \rightarrow \alpha$  be a morphism in  $[\mathbb{I}, \mathbf{C}]^{[1]}$ . It suffices to find the diagonal filler  $k: \Delta_{\mathbb{I}}(L) \rightarrow F$  for this square.

- When  $\mathbb{I}$  is a discrete category, take diagonal fillers  $k_i: L \rightarrow G(i)$  for each squares  $(u_i, v_i): f \rightarrow \alpha_i$ . Such fillers exist because  $\alpha_i$  is a trivial fibration for each  $i \in \mathbb{I}$ .
- Suppose  $\mathbb{I} = \omega^{\text{op}}$ . We construct the diagonal filler  $(k_n)_{n \geq 0}$  by induction on  $n$ .  
 (n=0)  $k_0$  is the diagonal filler for the square  $(u_0, v_0): f \rightarrow \alpha_0$ .  
 (n>0) Suppose we have constructed  $k_{n-1}$  that fits into the following commutative diagram.

$$\begin{array}{ccccc} K & \xrightarrow{u_n} & F(n) & \longrightarrow & F(n-1) \\ f \downarrow & & \downarrow & \nearrow k_{n-1} & \downarrow \alpha_{n-1} \\ L & \xrightarrow{v_n} & G(n) & \longrightarrow & G(n-1) \end{array}$$

Since the square on the right is sent by  $L_\wedge$  to a trivial fibration, we have another square whose right side is a trivial fibration;

$$\begin{array}{ccc} K & \xrightarrow{u_n} & F(n) \\ f \downarrow & & \downarrow \\ K & \xrightarrow{k_{n-1}} & \cdot \end{array}$$

where the bottom side is the morphism induced from  $k_{n-1}$  and  $v_n$  by the universality of the pullback defining the right side. We define  $k_n$  as a filler for this square.

- Suppose  $\mathbb{I} = \Lambda_2^2$ . Define  $(k_x)_{x=0,1,2}$  as follows.
  - $k_0$  is the filler for  $(u_0, v_0): f \rightarrow \alpha_0$ , and define  $k_2$  as the composite

$$L \xrightarrow{k_0} F(0) \rightarrow F(2)$$

where the latter is the image under  $F$  of  $0 \rightarrow 2$ .

- Now we have the following commutative diagram.

$$\begin{array}{ccccc} K & \xrightarrow{u_1} & F(1) & \longrightarrow & F(2) \\ f \downarrow & & \downarrow & \nearrow k_2 & \downarrow \alpha_2 \\ L & \xrightarrow{v_1} & G(1) & \longrightarrow & G(2) \end{array}$$

We define  $k_1$  in the same way as the definition of  $k_n$  above for  $\mathbb{I} = \omega^{\text{op}}$ . □

## 3. THE MAIN PROPOSITION

**Definition 3.1.** We say a morphism  $i$  in **bisSet** is a *levelwise left anodyne extension* if  $i \circ \Delta^n$  is a left anodyne extension for each  $n \geq 0$ .  $\blacksquare$

**Lemma 3.2.** *Morphisms in  $\ell r(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}})$  are levelwise left anodyne extensions.*

*Proof.* Firstly, observe that the class of levelwise left anodyne extensions is described as

$$\bigcup_{n \geq 0} ((-) \circ \Delta^n)^{-1}(\ell r(\mathbf{l}_{\text{ft}})),$$

where  $((-) \circ \Delta^n)^{-1}$  means the inverse image of the function on morphisms induced from the functor  $(-) \circ \Delta^n: \mathbf{bisSet} \rightarrow \mathbf{sSet}$ . Since  $(-) \circ \Delta^n$  preserves colimits and saturated classes are closed under unions, this class is saturated. Therefore, it suffices to show that morphisms in  $\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}}$  are levelwise left anodyne extensions.

Let  $i: A \rightarrow B \in \mathbf{l}_{\text{ft}}$  and  $j: C \rightarrow D \in \mathbf{l}_{\text{mono}}$ . By applying  $(-) \circ \Delta^n$  to the diagram defining  $i \boxtimes^L j$ , we obtain the following diagram in **sSet**.

$$\begin{array}{ccc} A \times C_n & \xrightarrow{\text{id} \times j_n} & A \times D_n \\ i \times \text{id} \downarrow & \lrcorner & \downarrow x \\ B \times C_n & \xrightarrow{\quad} & \cdot \\ & & \searrow (i \boxtimes^L j) \circ \Delta^n \\ & & B \times D_n \end{array}$$

(A curved arrow from  $B \times C_n$  to  $B \times D_n$  is also present, labeled  $i \times \text{id}$ .)

The two morphisms named  $i \times \text{id}$  are both left anodyne extensions since left anodyne extensions are closed under finite product ([Cis19, Proposition 3.4.3]). Since  $(-) \circ \Delta^n$  preserves pushouts, the marked square is a pushout square, and hence  $x$  is also a left anodyne extension. Moreover, since the outer square comprises monomorphisms, all morphisms in this diagram are monomorphisms [citation](#). Therefore, the resulting morphism  $(i \boxtimes^L j) \circ \Delta^n$  is a initial monomorphism, which is a left anodyne extension.  $\square$

**Lemma 3.3.** *A morphism  $i$  in **bisSet** is a monomorphism if and only if  $i \circ \Delta^n$  is a monomorphism for each  $n \geq 0$ .*

*Proof.* By the yoneda lemma and the fact that  $(-) \circ \Delta^m$  is a right adjoint. In detail, the following are equivalent.

- $i$  is a monomorphism.
- $\mathbf{bisSet}(\Delta^n \boxtimes \Delta^m, i)$  is a monomorphism in **Set** for each  $n, m \geq 0$ .
- $\mathbf{sSet}(\Delta^n, i \circ \Delta^m)$  is a monomorphism in **Set** for each  $n, m \geq 0$ .
- $i \circ \Delta^m$  is a monomorphism in **sSet** for each  $m \geq 0$ .  $\square$

**Proposition 3.4.** *Levelwise left anodyne extensions are left bi-anodyne extensions.*

*Proof.* Let  $i: X \rightarrow Y$  be a levelwise left anodyne extension. [Lemma 3.3](#) shows  $i$  is a monomorphism. By the small object argument, we have a weak factorisation system  $(\ell r(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}}), r(\mathbf{l}_{\text{ft}} \boxtimes^L \mathbf{l}_{\text{mono}}))$ , and let us write

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow j & \nearrow q \\ & Z & \end{array}$$

for the factorisation of  $i$  with respect to this system. Since  $j$  is in particular a left bi-anodyne extension and  $i$  is a monomorphism, it suffices to show  $q$  is a trivial fibration. Moreover, [Corollary 1.3](#) and [Proposition 1.6](#) ensures that it suffices to show  $q \circ K$  is a trivial fibration for each simplicial set  $K$ .

Consider the case when  $K = \Delta^n$  for some  $n \geq 0$ . [Lemma 1.2](#) shows  $q \circ \Delta^n$  is a left fibration and [Lemma 3.2](#) shows  $j \circ \Delta^n$  is a left anodyne extension. Since  $i \circ \Delta^n$  is assumed to be a left anodyne extension,  $q \circ \Delta^n$  is a initial left fibration, which is a trivial fibration.

By Corollary 1.3.10 of [Cis19] (see Fact 2.2), it suffices to show that the class

$$S = \{K \in \mathbf{sSet} \mid q \circ - K \text{ is a trivial fibration} \}$$

is saturated by monomorphisms. Let  $F: \mathbb{I}^{\text{op}} \rightarrow \mathbf{sSet}$  be a diagram satisfying the conditions described in Definition 2.1. Since  $Z \circ -$  sends colimits to limits for any  $Z \in \mathbf{bisSet}$ , by virtue of Proposition 2.4, it suffices to show that the natural transformation

$$(2) \quad q \circ - F(-): Z \circ - F(-) \Rightarrow Y \circ - F(-): \mathbb{I} \rightarrow \mathbf{sSet}$$

is a Reedy trivial fibration. Now by the definition of the pullback product  $\circ^{\perp}$ , one can check that Proposition 1.6 shows that the naturality square for the transformation  $q \circ -$  at a monomorphism between objects in  $S$  is sent by  $L_{\wedge}$  to a trivial fibration. This shows (2) is a Reedy trivial fibration, which completes the proof.  $\square$

## REFERENCES

- [Cis19] D.-C. Cisinski. *Higher Categories and Homotopical Algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019. doi:10.1017/9781108588737.  
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