NOTE ON LEMMA5.5.16 OF “HIGHER CATEGORIES AND HOMOTOPICAL ALGEBRA”

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[Cis19] The main objective is Proposition 14.

Notation 1. We employ the following notations.

- Given any category $\mathbf{C}$ and two objects $A, B \in \mathbf{C}$, we write $\mathbf{C}(A,B)$ for the homset.
- $\mathbb{\Delta}$ is the category of simplices. We write Δ^n for the n-simplex, which is also seen as a category.
- We write $\mathbf{C}^{\Delta^1}$ for the arrow category of $\mathbf{C}$. We write $\square$ for $\Delta^1 \times \Delta^1$.
- We write $\mathbf{Set}$ for the category of sets.
- $\mathbf{sSet}$ is the category of simplicial sets; i.e., $\mathbf{sSet} = [\mathbb{\Delta}^{\mathsf{op}}, \mathbf{Set}]$. Every set is seen as a simplicial set by the diagonal functor $\mathbf{Set} \longrightarrow \mathbf{sSet}$.
- $\mathbf{bisSet}$ is the category of bisimplicial sets; i.e., $\mathbf{bisSet} = [\mathbb{\Delta}^{\mathsf{op}} \times \mathbb{\Delta}^{\mathsf{op}}, \mathbf{Set}]$
- For each small category $\mathbb{A}$, we write $よ\colon \mathbb{A} \longrightarrow [\mathbb{A}^{\mathsf{op}}, \mathbf{Set}]$ for the yoneda embedding.
- We write $- \boxtimes - \colon \mathbf{sSet} \times \mathbf{sSet} \longrightarrow \mathbf{bisSet}$ for the nerve functor of $よ \times よ\colon \mathbb{\Delta}^{\mathsf{op}} \times \mathbb{\Delta}^{\mathsf{op}} \longrightarrow \mathbf{sSet} \times \mathbf{sSet}$.
- There are functors $- \multimap - \colon \mathbf{bisSet} \times \mathbf{sSet}^{\mathsf{op}} \longrightarrow \mathbf{sSet}$ and $- \multimapinv - \colon \mathbf{sSet}^{\mathsf{op}} \times \mathbf{bisSet} \longrightarrow \mathbf{sSet}$ such that there exists the following family of bijections natural in $A, B \in \mathbf{sSet}^{\mathsf{op}}$ and $X \in \mathbf{bisSet}$.

$$\mathbf{bisSet}(A \boxtimes B, X) \cong \mathbf{sSet}(A, X \multimapinv B) \cong \mathbf{sSet}(B, A \multimap X) \tag{1}$$

- We write $- \boxtimes^{\mathrm{L}} - \colon \mathbf{sSet}^{\Delta^1} \times \mathbf{sSet}^{\Delta^1} \longrightarrow \mathbf{bisSet}^{\Delta^1}$ for the *pushout product* or *Leibniz product* with respect to $\boxtimes$; given two morphisms $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ in $\mathbf{sSet}$, the morphism $f \boxtimes^{\mathrm{L}} g$ is the canonical one of the form $A \boxtimes D \cup_{A \boxtimes C} B \boxtimes C \longrightarrow B \boxtimes D$ whose domain is the fibred coproduct of $f \boxtimes \mathtt{id}_C$ and $\mathtt{id}_A \boxtimes g$.
- We write $- \overset{\mathrm{L}}{\multimap} - \colon \mathbf{sSet}^{\Delta^1,\mathsf{op}} \times \mathbf{bisSet}^{\Delta^1} \longrightarrow \mathbf{sSet}^{\Delta^1}$ and $- \overset{\mathrm{L}}{\multimapinv} - \colon \mathbf{bisSet}^{\Delta^1} \times \mathbf{sSet}^{\Delta^1,\mathsf{op}} \longrightarrow \mathbf{sSet}^{\Delta^1}$ for the *pullback product* for $\multimap$ and $\multimapinv$ respectively. Therefore, given a morphism $f\colon A \longrightarrow B$ in $\mathbf{sSet}$ and a morphism $h\colon X \longrightarrow Y$ in $\mathbf{bisSet}$, $f \overset{\mathrm{L}}{\multimap} h$ is the canonical morphism towards the fibred product of $f \multimap \mathtt{id}_Y$ and $\mathtt{id}_A \multimap h$.
- Let i and k be morphisms. We write $i \pitchfork k$ if i has left lifting property to k, or equivalently, k has right lifting property to i.
- Given a class of morphisms I, we write $\ell(\mathsf{I})$ for the class of morphisms that have left lifting property to those in I. Dually, we write $r(\mathsf{I})$ for the class of morphisms that have right lifting property to those in I.
- Given two classes of morphisms I and K in $\mathbf{sSet}$, we obtain a class of morphisms $\mathsf{I} \boxtimes^{\mathrm{L}} \mathsf{K} = \{ i \boxtimes^{\mathrm{L}} k \mid i \in \mathsf{I},\ k \in \mathsf{K} \}$.
- We write $\mathsf{I}_{\mathsf{mono}}$ for the canonical cellular model for $\mathbf{sSet}$; i.e., the set of boundary inclusions.
- We write $\mathsf{I}_{\mathsf{lft}}$ for the set of generating left anodyne extensions; i.e. the set of horn inclusions lifting initial objects.
- We write $\mathsf{I}_{\mathsf{lftbi}}$ for $\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}} \cup \mathsf{I}_{\mathsf{mono}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{lft}}$. The set $\mathsf{I}_{\mathsf{lftbi}}$ is the generator for left bi-anodyne extensions.
- For each simplicial set A, we write $(\mathbf{sSet}/A)_{\mathsf{co}}$ for the slice category equipped with the homotopical structure defined for the covariant model structure,
- For each bisimplicial set X, we write $(\mathbf{bisSet}/X)_{\mathsf{co}}$ for the slice category equipped with the homotopical structure for the bicovariant model structure.
- We write ω for the least infinite ordinal seen as a category by its order.
- We regard the horn Λ^2_2 as a subcategory of Δ^2.
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Observation 2. Δ^1 can be seen as a (boolean) lattice, and there are the meet $\wedge\colon \square \longrightarrow \Delta^1$ and the join $\vee\colon \square \longrightarrow \Delta^1$. Given a category $\mathbf{C}$ with pushout and pullback, we obtain $L_\wedge\colon \mathbf{C}^\square \longrightarrow \mathbf{C}^{\Delta^1}$ as the right Kan extension of the meet $\wedge$, while we have $L_\vee\colon \mathbf{C}^\square \longrightarrow \mathbf{C}^{\Delta^1}$ as the left Kan extension of the join $\vee$.

On the other hand, suppose we are given a functor $\otimes\colon \mathbf{A} \times \mathbf{B} \longrightarrow \mathbf{C}$. The currying of the composite

$$\Delta^1 \times \Delta^1 \times \mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \xrightarrow{\text{evaluation}} \mathbf{A} \times \mathbf{B} \xrightarrow{\otimes} \mathbf{C}$$

gives a functor $\bar{\otimes}\colon \mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \longrightarrow \mathbf{C}^\square$, where the first one is the canonical evaluation functor that sends (i, j, X, Y) to $(X(i), Y(j))$.

Now we have two functors $L_\wedge \circ \bar{\otimes}$ and $L_\vee \circ \bar{\otimes}$ of the form $\mathbf{A}^{\Delta^1} \times \mathbf{B}^{\Delta^1} \longrightarrow \mathbf{C}^{\Delta^1}$. The first one is the *pullback product* of $\otimes$, while the secound one is the *pushout product* or the *Leibniz product* of $\otimes$. ■

Definition 3. Suppose that we are given

- a complete category $\mathbf{C}$,
- a weak factorisation system $(\mathsf{Cof}, \mathsf{TFib})$ whose elements are called *cofibrations* and *trivial fibrations* respectively,
- a small category $\mathbb{I}$ that is a discrete category or either of ω^{op} or Λ^2_2, and
- two functors $F, G\colon \mathbb{I} \longrightarrow \mathbf{C}$.

A natural transformation $\alpha\colon F \Longrightarrow G$ is a ***Reedy trivial fibration*** if it satisfies the followings.

- For each object $i \in \mathbb{I}$, $\alpha_i\colon F(i) \longrightarrow G(i)$ is a trivial fibration.
- Let $u\colon i \longrightarrow j$ in $\mathbb{I}$ be either of $n+1 \longrightarrow n$ in ω^{op} or $1 \longrightarrow 2$ in Λ^2_2. Then the naturality square

$$\begin{tikzcd} F(i) \arrow[r, "F(u)"] \arrow[d, "\alpha_i"'] & F(j) \arrow[d, "\alpha_j"] \\ G(i) \arrow[r, "G(u)"'] & G(j) \end{tikzcd}$$

is sent by $L_\wedge$ (see Observation 2) to a trivial fibration. ■

Proposition 4. *The limit of a Reedy trivial fibration is a trivial fibration.*

Proof. ongoing, citation □

Lemma 5. *For each morphisms i, j in* **sSet** *and a morphism h in* **bisSet**, *the following are equivalent.*

- $i \boxtimes^L j \pitchfork h$
- $i \pitchfork h \multimapinv^L j$
- $j \pitchfork i \multimap^L h$

Proof. Follows from (1) and the universality of the pushout and the pullbacks defining $i \boxtimes^{\mathrm{L}} j$, $h \multimapinv^{\mathrm{L}} j$, and $i \multimap^{\mathrm{L}} h$. □

Lemma 6. *Suppose we are given two small classes I and J of morphisms in* **sSet**, *and a morphism f in* **bisSet**. *The following are equivalent.*

i) $h \in r(\mathsf{I} \boxtimes^L \mathsf{J})$.
ii) *For each* $j \in \mathsf{J}$, $h \multimapinv^L j \in r(\mathsf{I})$.
iii) *For each* $i \in \mathsf{I}$, $i \multimap^L h \in r(\mathsf{J})$.
iv) *For each* $j \in \ell r(\mathsf{J})$, $h \multimapinv^L j \in r(\mathsf{I})$.
v) *For each* $i \in \ell r(\mathsf{I})$, $i \multimap^L h \in r(\mathsf{J})$.

Proof. By Lemma 5, the first three arguments are equivalent to the following.

$$j \pitchfork i \multimap^{\mathrm{L}} h \quad \text{for any } i \in \mathsf{I} \text{ and } j \in \mathsf{J}.$$

By the small object argument, this is also equivalent to

$$j \pitchfork i \multimap^{\mathrm{L}} h \quad \text{for any } i \in \mathsf{I} \text{ and } j \in \ell r(\mathsf{J}),$$

which is equivalent to *iv)*. *v)* is also equivalent to them in the same way. □

Corollary 7 (Lemma 5.5.6 of [Cis19]). *A morphism q in* **bisSet** *is a trivial fibration if and only if $q \multimapinv i$ is a trivial fibration for each monomorphism i in* **sSet**.

Proof. $\mathsf{I}_{\mathsf{mono}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}$ is a cellular model for **bisSet** citation, and this is a direct consequence of Lemma 6, where both I and J are $\mathsf{I}_{\mathsf{mono}}$. ☐

Lemma 8. $\mathsf{dom}\colon (\mathbf{sSet}/A)_{\mathsf{co}} \longrightarrow (\mathbf{sSet})_{\mathsf{co}}$ *preserves and lifts trivial fibrations and naive fibrations.*

Proof. Observe that dom preserves and lifts both monomorphisms and anodyne extensions.

Let $f\colon S \longrightarrow T$ be a morphism in $\mathbf{sSet}/A$. We show that f is a trivial/naive fibration if and only if $\mathsf{dom}(f)$ is a trivial/naive fibration. The if part is easily checked considering that dom preserves monomorphisms/anodyne extensions.

Let i be a monomorphism/anodyne extension in $(\mathbf{sSet})_{\mathsf{co}}$ and $(u, v)\colon i \longrightarrow \mathsf{dom}(f)$ be a morphism in $\mathbf{sSet}^{\Delta^1}$. It suffices to show there is a diagonal filler for this square. The morphisms towards A from f is followed by (u, v), and this enables us to see (u, v) as a square in $\mathbf{sSet}/A$. Moreover, its domain is a monomorphism/anodyne extension. Therefore, desired diagonal filler exists if f is a trivial/naive fibration in $\mathbf{sSet}/A$. ☐

Lemma 9. *A left fibration between trivial fibrations is a trivial fibration.*

Proof. Suppose we are given a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, two heads, "h"', "\sim"] & & B \arrow[dl, two heads, "g", "\sim"'] \\ & C & \end{tikzcd}$$

in **sSet** such that f is a left fibration (i.e., a naive fibration in $(\mathbf{sSet})_{\mathsf{co}}$) and g and h are trivial fibrations. By Lemma 8, this lifts to another triangle

$$\begin{tikzcd} (A, h) \arrow[rr, "f"] \arrow[dr, two heads, "h"', "\sim"] & & (B, g) \arrow[dl, two heads, "g", "\sim"'] \\ & (C, \mathtt{id}) & \end{tikzcd}$$

in $(\mathbf{sSet}/C)_{\mathsf{co}}$ satisfying the same condition. Since (A, h) and (B, g) are fibrant, f is a fibration. Therefore, the 2 out of 3 property for weak equivalences ensures that f is a trivial fibration. ☐

Lemma 10. *Suppose we are given a morphism* $q\colon X \longrightarrow Y$ *in* **bisSet** *that belongs to* $r(\mathsf{I}_{\mathsf{lft}} \boxtimes^{L} \mathsf{I}_{\mathsf{mono}})$ *and a monomorphism* $i\colon K \rightarrowtail L$ *in* **sSet**. $q \multimapinv^{L} i$ *is a trivial fibration if* $q \multimapinv K$ *and* $q \multimapinv L$ *are trivial fibrations.*

Proof. By Lemma 6, $q \multimapinv^{\mathrm{L}} i$ is a left fibration filling the following diagram.

$$\begin{tikzcd} X \multimapinv L & X \multimapinv K \arrow[l, "X \multimapinv i"'] & \\ Y \multimapinv L \arrow[u, "q \multimapinv L"] & \cdot \arrow[l] \arrow[u, "x"'] \arrow[ul, phantom, "\lrcorner", very near start] & \\ & & Y \multimapinv K \arrow[ul, "q \multimapinv^{\mathrm{L}} i"] \arrow[uul, bend right, "q \multimapinv K"'] \arrow[ull, bend left] \end{tikzcd}$$

The pullback square assures x is also a trivial fibration, and this follows from Lemma 9. ☐

Definition 11. We say a morphism i in **bisSet** is a ***levelwise left anodyne extension*** if $i \multimapinv \Delta^n$ is a left anodyne extension for each $n \geq 0$. ■

Lemma 12. *Morphisms in* $\ell r(\mathsf{I}_{\mathsf{lft}} \boxtimes^{L} \mathsf{I}_{\mathsf{mono}})$ *are levelwise left anodyne extensions.*

Proof. Firstly, observe that the class of levelwise left anodyne extensions is described as

$$\bigcup_{n \geq 0} ((-) \multimapinv \Delta^n)^{-1}(\ell r(\mathsf{I}_{\mathsf{lft}})),$$

where $((-) \multimapinv \Delta^n)^{-1}$ means the inverse image of the function on morphisms induced from the functor $(-) \multimapinv \Delta^n\colon \mathbf{bisSet} \longrightarrow \mathbf{sSet}$. Since $((-) \multimapinv \Delta^n)^{-1}$ preserves colimits and saturated classes are closed under joins, this class is saturated. Therefore, it suffices to show that morphisms in $\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}$ are levelwise left anodyne extensions.

Let $i\colon A \longrightarrow B \in \mathsf{I}_{\mathsf{lft}}$ and $j\colon C \longrightarrow D \in \mathsf{I}_{\mathsf{mono}}$. By applying $(-) \multimapinv \Delta^n$ to the diagram defining $i \boxtimes^{\mathrm{L}} j$, we obtain the following diagram in $\mathbf{sSet}$.

$$\begin{tikzcd}
A \times C_n \arrow[r, "\mathtt{id}\times j_n"] \arrow[d, "i\times\mathtt{id}"'] & A \times D_n \arrow[d, "x"] \arrow[ddr, bend left, "i\times\mathtt{id}"] & \\
B \times C_n \arrow[r] \arrow[drr, bend right] & \cdot \arrow[ul, phantom, "\lrcorner", very near start] \arrow[dr, "(i\boxtimes^{\mathrm{L}} j)\multimapinv\Delta^n"] & \\
 & & B \times D_n
\end{tikzcd}$$

The two morphisms named $i \times \mathtt{id}$ are both left anodyne extensions since left anodyne extensions are closed under finite product citation. Since $(-) \multimapinv \Delta^n$ preserves pushouts, the marked square is a pushout square, and hence x is also a left anodyne extension. Moreover, since the outer square comprises monomorphisms, all morphisms in this diagram are monomorphisms citation. Therefore, the resulting morphism $(i \boxtimes^{\mathrm{L}} j) \multimapinv \Delta^n$ is a initial monomorphism, which is a left anodyne extension. □

Lemma 13. *A morphism i in* **bisSet** *is a monomorphism if and only if $i \multimapinv \Delta^n$ is a monomorphism for each $n \geq 0$.*

Proof. By the yoneda lemma and the fact that $(-) \multimapinv \Delta^m$ is a right adjoint. In detail, the following are equivalent.

- i is a monomorphism.
- $\mathbf{bisSet}(\Delta^n \boxtimes \Delta^m, i)$ is a monomorphism in **Set** for each $n, m \geq 0$.
- $\mathbf{sSet}(\Delta^n, i \multimapinv \Delta^m)$ is a monomorphism in **Set** for each $n, m \geq 0$.
- $i \multimapinv \Delta^m$ is a monomorphism in **sSet** for each $m \geq 0$. □

Proposition 14. *Levelwise left anodyne extensions are left bi-anodyne extensions.*

Proof. Let $i\colon X \longrightarrow Y$ be a levelwise left anodyne extension. Lemma 13 shows i is a monomorphism. By the small object argument, we have a weak factorisation system $(\ell r(\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}), r(\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}))$, and let us write $q \circ j = i$ for the factorisation of i with respect to this system. Since j is in particular a left bi-anodyne extension and i is a monomorphism, it suffices to show q is a trivial fibration. Moreover, Corollary 7 and Lemma 10 ensures that it suffices to show $q \multimapinv K$ is a trivial fibration for each simplicial set K.

Consider the case when $K = \Delta^n$ for some $n \geq 0$. Lemma 6 shows $q \multimapinv \Delta^n$ is a left fibration and Lemma 12 shows $j \multimapinv \Delta^n$ is a left anodyne extension. Since $i \multimapinv \Delta^n$ is assumed to be a left anodyne extension, $q \multimapinv \Delta^n$ is a initial left fibration, which is a trivial fibration.

By Lemma 1.3.10 of [Cis19], it suffices to show that the class $\{K \in \mathbf{sSet} \mid q \multimapinv K \text{ is a trivial fibration }\}$ is saturated by monomorphisms. Since trivial fibrations are closed under products, the class is closed under coproducts. To show that it is closed under the pushouts and transfinite compositions indicated in □

References

[Cis19] D.-C. Cisinski. *Higher Categories and Homotopical Algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019. doi:10.1017/9781108588737.

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