NOTE ON LEMMA5.5.16 OF "HIGHER CATEGORIES AND HOMOTOPICAL ALGEBRA"

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Notation 1. We employ the following notations.

- Given any category $\mathbf{C}$ and two objects $A, B \in \mathbf{C}$, we write $\mathbf{C}(A, B)$ for the homset.
- $\mathbb{\Delta}$ is the category of simplices. We write Δ^n for the n-simplex, which is also seen as a category.
- We write $\mathbf{C}^{\Delta^1}$ for the arrow category of $\mathbf{C}$. We write $\square$ for $\Delta^1 \times \Delta^1$.
- We write $\mathbf{Set}$ for the category of sets.
- $\mathbf{sSet}$ is the category of simplicial sets; i.e., $\mathbf{sSet} = [\mathbb{\Delta}^{\mathsf{op}}, \mathbf{Set}]$.
- $\mathbf{bisSet}$ is the category of bisimplicial sets; i.e., $\mathbf{bisSet} = [\mathbb{\Delta}^{\mathsf{op}} \times \mathbb{\Delta}^{\mathsf{op}}, \mathbf{Set}]$
- For each small category $\mathbb{A}$, we write $よ\colon \mathbb{A} \longrightarrow [\mathbb{A}^{\mathsf{op}}, \mathbf{Set}]$ for the yoneda embedding.
- We write $- \boxtimes - \colon \mathbf{sSet} \times \mathbf{sSet} \longrightarrow \mathbf{bisSet}$ for the nerve functor of $よ \times よ\colon \mathbb{\Delta}^{\mathsf{op}} \times \mathbb{\Delta}^{\mathsf{op}} \longrightarrow \mathbf{sSet} \times \mathbf{sSet}$.
- There are functors $- \multimap - \colon \mathbf{bisSet} \times \mathbf{sSet}^{\mathsf{op}} \longrightarrow \mathbf{sSet}$ and $- \multimapinv - \colon \mathbf{sSet}^{\mathsf{op}} \times \mathbf{bisSet} \longrightarrow \mathbf{sSet}$ such that there exists the following family of bijections natural in $A, B \in \mathbf{sSet}^{\mathsf{op}}$ and $X \in \mathbf{bisSet}$.
$$\mathbf{bisSet}(A \boxtimes B, X) \cong \mathbf{sSet}(A, X \multimapinv B) \cong \mathbf{sSet}(B, A \multimap X)$$
- We write $- \boxtimes^{\mathrm{L}} - \colon \mathbf{sSet}^{\Delta^1} \times \mathbf{sSet}^{\Delta^1} \longrightarrow \mathbf{bisSet}^{\Delta^1}$ for the *pushout product* or *Leibniz product* with respect to $\boxtimes$; given two morphisms $f\colon A \longrightarrow B$ and $g\colon C \longrightarrow D$ in $\mathbf{sSet}$ $f \boxtimes^{\mathrm{L}} g$ is the canonical morphism of the form $A \boxtimes D \cup_{A \boxtimes C} B \boxtimes C \longrightarrow B \boxtimes D$ whose domain is the fibred coproduct of $f \boxtimes \mathtt{id}_C$ and $\mathtt{id}_A \boxtimes g$.
- We write $- \overset{\mathrm{L}}{\multimap} - \colon \mathbf{sSet}^{\Delta^1, \mathsf{op}} \times \mathbf{bisSet}^{\Delta^1} \longrightarrow \mathbf{sSet}^{\Delta^1}$ and $- \overset{\mathrm{L}}{\multimapinv} - \colon \mathbf{bisSet}^{\Delta^1} \times \mathbf{sSet}^{\Delta^1, \mathsf{op}} \longrightarrow \mathbf{sSet}^{\Delta^1}$ for the *pullback product* for $\multimap$ and $\multimapinv$ respectively. Therefore, given a morphism $f\colon A \longrightarrow B$ in $\mathbf{sSet}$ and a morphism $h\colon X \longrightarrow Y$ in $\mathbf{bisSet}$, $f \overset{\mathrm{L}}{\multimap} h$ is the canonical morphism towards the fibred product of $f \multimap \mathtt{id}_Y$ and $\mathtt{id}_A \multimap h$.
- Let i and k be morphisms. We write $i \pitchfork k$ if i has left lifting property to k, or equivalently, k has right lifting property to i.
- Given a class of morphisms I, we write $\ell(\mathsf{I})$ for the class of morphisms that have left lifting property to those in I. Dually, we write $r(\mathsf{I})$ for the class of morphisms that have right lifting property to those in I.
- Given two classes of morphisms I and K in $\mathbf{sSet}$, we obtain a class of morphisms $\mathsf{I} \boxtimes^{\mathrm{L}} \mathsf{K} = \{ i \boxtimes^{\mathrm{L}} k \mid i \in \mathsf{I},\ k \in \mathsf{K} \}$.
- We write $\mathsf{I}_{\mathsf{mono}}$ for the canonical cellular model for $\mathbf{sSet}$; i.e., the set of boundary inclusions.
- We write $\mathsf{I}_{\mathsf{lft}}$ for the generating left anodyne extensions.
- We write Λ for the Leibniz join $\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}}$, and we write $\mathsf{I}_{\mathsf{lftbi}}$ for $\mathsf{I}_{\mathsf{lft}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{mono}} \cup \mathsf{I}_{\mathsf{mono}} \boxtimes^{\mathrm{L}} \mathsf{I}_{\mathsf{lft}}$. The set $\mathsf{I}_{\mathsf{lftbi}}$ is the generator for left bi-anodyne extensions.
- For each simplicial set A, we write $(\mathbf{sSet}/A)_{\mathsf{co}}$ for the slice category equipped with the homotopical structure defined for the covariant model structure,
- For each bisimplicial set X, we write $(\mathbf{bisSet}/X)_{\mathsf{co}}$ for the slice category equipped with the homotopical structure for the bicovariant model structure.
- We write ω for the least infinite ordinal seen as a category by its order.
- We regard the horn Λ^2_2 as a subcategory of Δ^2.
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Observation 2. Δ^1 can be seen as a (boolean) lattice, and there are the meet $\wedge\colon \square \longrightarrow \Delta^1$ and the join $\vee\colon \square \longrightarrow \Delta^1$. Given a category $\mathbf{C}$ with pushout and pullback, we obtain $L_\wedge\colon \mathbf{C}^\square \longrightarrow \mathbf{C}^{\Delta^1}$ as the right Kan extension of the meet $\wedge$, while we have $L_\vee\colon \mathbf{C}^\square \longrightarrow \mathbf{C}^{\Delta^1}$ as the left Kan extension of the join $\vee$.

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On the other hand, suppose we are given a functor $\otimes\colon \mathbf{A}\times\mathbf{B}\longrightarrow\mathbf{C}$. The currying of the composite

$$\Delta^1\times\Delta^1\times\mathbf{A}^{\Delta^1}\times\mathbf{B}^{\Delta^1}\xrightarrow{\text{evaluation}}\mathbf{A}\times\mathbf{B}\xrightarrow{\otimes}\mathbf{C}$$

gives a functor $\bar{\otimes}\colon \mathbf{A}^{\Delta^1}\times\mathbf{B}^{\Delta^1}\longrightarrow\mathbf{C}^{\square}$, where the first one is the canonical evaluation functor that sends (i,j,X,Y) to $(X(i),Y(j))$.

Now we have two functors $L_\wedge\circ\bar{\otimes}$ and $L_\vee\circ\bar{\otimes}$ of the form $\mathbf{A}^{\Delta^1}\times\mathbf{B}^{\Delta^1}\longrightarrow\mathbf{C}^{\Delta^1}$. The first one is the *pullback product* of $\otimes$, while the secound one is the *pushout product* or the *Leibniz product* of $\otimes$. ■

Definition 3. We say a morphism i in **bisSet** is a ***levelwise left anodyne extension*** if $i\multimapinv\Delta^n$ is a left anodyne extension for each $n\geq 0$. ■

Lemma 4. *For each morphisms i,j in* **sSet** *and a morphism h in* **bisSet**, *the following are equivalent.*

- $i\boxtimes^L j\pitchfork h$
- $i\pitchfork h\overset{L}{\multimapinv} j$
- $j\pitchfork i\overset{L}{\multimap} h$

Lemma 5. *Suppose we are given two small classes I and J of morphisms in* **sSet**, *and a morphism f in* **bisSet**. *The following are equivalent.*

i) $h\in r(\mathsf{I}\boxtimes^L\mathsf{J})$.
ii) *For each* $j\in\mathsf{J}$, $h\overset{L}{\multimapinv} j\in r(\mathsf{I})$.
iii) *For each* $i\in\mathsf{I}$, $i\overset{L}{\multimap} h\in r(\mathsf{J})$.
iv) *For each* $j\in\ell r(\mathsf{J})$, $h\overset{L}{\multimapinv} j\in r(\mathsf{I})$.
v) *For each* $i\in\ell r(\mathsf{I})$, $i\overset{L}{\multimap} h\in r(\mathsf{J})$.

Proof. By Lemma 4, the first three arguments are equivalent to the following.

$$j\pitchfork i\overset{L}{\multimap} h\quad\text{for any } i\in\mathsf{I}\text{ and } j\in\mathsf{J}.$$

By the small object argument, this is also equivalent to

$$j\pitchfork i\overset{L}{\multimap} h\quad\text{for any } i\in\mathsf{I}\text{ and } j\in\ell r(\mathsf{J}),$$

which is equivalent to *iv)*. *v)* is also equivalent to them in the same way. □

Corollary 6. *Morphisms in $\ell r(\mathsf{I}_{\mathsf{lft}}\boxtimes^L\mathsf{I}_{\mathsf{mono}})$ are levelwise left anodyne extensions.*

Proof. Suppose we are given a simplicial set K and a morphism i in **bisSet**. Let us write $!_K\colon\emptyset\longrightarrow K$ for the unique morphism. Then $i\overset{L}{\multimapinv}!_K$ is isomorphic to $i\multimapinv K$ and $!_K$ is in $\ell r(\mathsf{I}_{\mathsf{mono}})$ since it is a monomorphism. By applying □

Lemma 7. $\mathsf{dom}\colon(\mathbf{sSet}/A)_{\mathsf{co}}\longrightarrow(\mathbf{sSet})_{\mathsf{co}}$ *preserves and lifts trivial fibrations and naive fibrations.*

Proof. Observe that dom preserves and lifts both monomorphisms and anodyne extensions.

Let $f\colon S\longrightarrow T$ be a morphism in $\mathbf{sSet}/A$. We show that f is a trivial/naive fibration if and only if $\mathsf{dom}(f)$ is a trivial/naive fibration. The if part is easily checked considering that dom preserves monomorphisms/anodyne extensions.

Let i be a monomorphism/anodyne extension in $(\mathbf{sSet})_{\mathsf{co}}$ and $(u,v)\colon i\longrightarrow\mathsf{dom}(f)$ be a morphism in $\mathbf{sSet}^{\Delta^1}$. It suffices to show there is a diagonal filler for this square. The morphisms towards A from f is followed by (u,v), and this enables us to see (u,v) as a square in $\mathbf{sSet}/A$. Moreover, its domain is a monomorphism/anodyne extension. Therefore, desired diagonal filler exists if f is a trivial/naive fibration in $\mathbf{sSet}/A$. □

Lemma 8. *A left fibration between trivial fibrations is a trivial fibration.*

Proof. Suppose we are given a commutative triangle

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "h"', "\sim"] & & B \arrow[dl, "g", "\sim"'] \\ & C & \end{tikzcd}$$

in **sSet** such that f is a left fibration (i.e., a naive fibration in $(\mathbf{sSet})_{\mathsf{co}}$) and g and h are trivial fibrations. By the above lemma, this lifts to another triangle

$$\begin{tikzcd}
(A,h) \arrow[rr, "f"] \arrow[dr, "h"', "\sim"] & & (B,g) \arrow[dl, "g", "\sim"'] \\
& (C,\mathtt{id}) &
\end{tikzcd}$$

in $(\mathbf{sSet}/C)_{\mathsf{co}}$ satisfying the same condition. Since (A,h) and (B,g) are fibrant, f is a fibration. Therefore, the 2 out of 3 property for weak equivalences ensures that f is a trivial fibration. □

Definition 9. Suppose that we are given

- a complete category $\mathbf{C}$,
- a weak factorisation system $(\mathsf{Cof}, \mathsf{TFib})$ whose elements of are called *cofibrations* and *trivial fibrations* respectively,
- a small category $\mathbb{I}$ that is either of ω^{op} or Λ^2_2, and
- two functors $F, G\colon \mathbb{I} \longrightarrow \mathbf{C}$.

A natural transformation $\alpha\colon F \Longrightarrow G$ is a ***Reedy trivial fibration*** if it satisfies the followings.

- For each object $i \in \mathbb{I}$, $\alpha_i\colon F(i) \longrightarrow G(i)$ is a trivial fibration.
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Lemma 10. *Limit of a Reedy trivial fibration is a trivial fibration*

Proposition 11. *Levelwise left anodyne extensions are left bi-anodyne extensions.*

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