NOTE ON HYPERGRAPHS

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[?]

Notation 0.1. We employ the following notations.

- For each category $\mathbf{C}$, we write $\mathbf{C}^{\cong}$ for the core groupoid of $\mathbf{C}$.
- For each groupoid $\mathbf{G}$, we write $\pi_0(\mathbf{G})$ for the set of connected components of $\mathbf{G}$.
- For each category $\mathbf{C}$ and an object $c \in \mathbf{C}$, we write $\mathbf{C}_{c/}$ and $\mathbf{C}_{/c}$ for the under category and the over category.
- For each category $\mathbf{C}$ and an object $c \in \mathbf{C}$, the domain part defines a functor $\mathbf{C}_{/c} \xrightarrow{\Sigma_c} \mathbf{C}$, and we write c^* and Π_c for the right adjoints of Σ_c and c^* respectively, if exists.
  The dual of Σ_c, c^*, and Π_c are denoted by $\reflectbox{$\Sigma$}_c \colon \mathbf{C}_{c/} \longrightarrow \mathbf{C}$, $c_* \colon \mathbf{C} \longrightarrow \mathbf{C}_{c/}$, and $\coprod_c \colon \mathbf{C} \longrightarrow \mathbf{C}_{c/}$.
- We write **Set** for the catgeory of sets.
- For each nonnegative integer n, we write $\bar{n}$ for the set $\{1, \ldots, n\}$.
- We write **FinSet** for the catgeory defined as follows.
  - $\mathrm{Obj}(\mathbf{FinSet}) = \mathbb{N}$.
  - $\mathbf{FinSet}(n, m) = \mathbf{Set}(\bar{n}, \bar{m})$.
  - $n \mapsto \bar{n}$ extends to a fully faithful functor $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$.
- For each category $\mathbf{C}$ with finite coproducts, we write $- + -$ for the binary coporduct and $\emptyset$ for the initial object.
  We write $\nabla_c \colon c + c \longrightarrow c$ for the codiagonal.

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Definition 0.2. Suppose we are given a set $\Lambda \in \mathbf{Set}$. We define a category $\mathbf{FS}(\Lambda)$ as the following Grothendieck construction. by the following pullback.

$$\begin{tikzcd}
\mathbf{FS}(\Lambda) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_{/\Lambda} \arrow[d, "\Sigma_\Lambda"] \\
\mathbf{FinSet} \arrow[r, hook] & \mathbf{Set}
\end{tikzcd}$$

An element of $\mathbf{FS}(\Lambda)$ is called a ***λ-labelled finite set***. ■

Proposition 0.3. For each $\Lambda \in \mathbf{Set}$, $\mathbf{FS}(\Lambda)$ has finite colimits. ♦

Proof. Both $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ and $\mathbf{Set}_{/\Lambda} \xrightarrow{\Sigma_\Lambda} \mathbf{Set}$ create finite colimits. □

Definition 0.4. Let $\mathbf{C}$ be a category with finite colimits. We write $\mathbf{Cosp}[\mathbf{C}]$ for the category defined as follows.

- $\mathrm{Obj}(\mathbf{Cosp}[\mathbf{C}]) = \mathrm{Obj}\,\mathbf{C}$.
- For each $c, c' \in \mathbf{C}$, the homset is defined as
$$\mathbf{Cosp}[\mathbf{C}](c, c') = \pi_0\left((\mathbf{C}_{c+c'/})^{\cong}\right).$$
- The identity on c is represented by the codiagonal $\nabla \colon c + c \longrightarrow c$.
- The composition is obtained by applying $\pi_0((-)^{\cong})$ to the following functor.
$$\mathbf{C}_{c+c'/} \times \mathbf{C}_{c'+c''/} \xrightarrow{+} \mathbf{C}_{c+c'+c'+c''/} \xrightarrow{(c+\nabla_{c'}+c'')_*} \mathbf{C}_{c+c'+c''/} \xrightarrow{\reflectbox{Σ}_{\mathtt{inc}_{1,3}}} \mathbf{C}_{c+c''/}$$

■

Definition 0.5. Let $\Lambda \in \mathbf{Set}$. We write $\mathbf{Cosp}_\Lambda$ for $\mathbf{Cosp}[\mathbf{FS}(\Lambda)]$. ■

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Proposition 0.6. Let $\mathbf{C}$ be a category with finite colimits. Then the coproduct of $\mathbf{C}$ lifts to a dagger compact monoidal structure on $\mathbf{Cosp}[\mathbf{C}]$. ♦

Proof. Every object c is self-dual by applying $\pi_0((-)^{\cong})$ to the cacnonical isomorphism $\mathbf{C}_{(c_0+c)+c_1} \cong \mathbf{C}_{c_0+(c+c_1)}$. The involution $\dagger_{c,c'} \colon \mathbf{Cosp}[\mathbf{C}](c,c') \longrightarrow \mathbf{Cosp}[\mathbf{C}](c',c)$ is obtained by applying $\pi_0((-)^{\cong})$ to the canonical isomorphism $\mathbf{C}_{c+c'/} \cong \mathbf{C}_{c'+c/}$ for any $c, c' \in \mathbf{C}$. □

Definition 0.7. ■

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