

NOTE ON POLYNOMIALS

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Notation 0.1. We employ the following notations.

- Sets are regarded as discrete categories, and categories are regarded as locally discrete 2-categories.
- **Set** is the large category of small sets.
- We write \mathfrak{Cat} for the huge 2-category of large categories.
- For each set S and a functor $X: S \rightarrow \mathbf{Set}$, we write X_s for the image of $s \in S$ under X . Moreover, we write $(X_s)_{s:S}$ for X .
- For each set S and functors $X, Y: S \rightarrow \mathbf{Set}$, a natural transformation $f: X \Rightarrow Y$ is denoted as a family of functions $(f_s)_{s:S}$.
- We write 1 for the terminal set, and we write $*$ for the unique element in 1 .

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1. POLYNOMIALS

Definition 1.1. We define a category **Poly** by the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-, \mathbf{Set}]^{\mathrm{op}}} \mathfrak{Cat}$$

A *polynomial* is an object in **Poly**. We write

$$(-)_{\circ}: \mathbf{Poly} \rightarrow \mathbf{Set}$$

for the fibration corresponding to the 2-functor. For each polynomial A , we write $A = \sum_{a:A_{\circ}} y^{A_a}$.

In particular, we mean by y the polynomial satisfying $(y)_{\circ} = 1$ and $(y)_* = 1$.

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Definition 1.2. We define a category **Fam** by the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-, \mathbf{Set}]} \mathfrak{Cat}$$

Note that the fibration $\mathbf{Poly} \rightarrow \mathbf{Set}$ is the fibrewise opsite of $\mathbf{Fam} \rightarrow \mathbf{Set}$.

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Definition 1.3. Consider the composite

$$\begin{array}{ccc} [S, \mathbf{Set}]^{\mathrm{op}} \times \mathbf{Set} & \xrightarrow{\langle \mathrm{id}_{[S, \mathbf{Set}]}, \Delta_{\mathrm{proj}} \rangle} & [S, \mathbf{Set}]^{\mathrm{op}} \times [S, \mathbf{Set}] \longrightarrow [S, \mathbf{Set}] \\ \downarrow \Psi & & \downarrow \Psi \\ ((A_s)_{s:S}, T) & \longmapsto & ([A_s, T])_{s:S} \end{array}$$

where $\Delta: \mathbf{Set} \rightarrow [S, \mathbf{Set}]$ in the former is the diagonal and the latter is the internal hom for $[S, \mathbf{Set}]$. This is natural in $S \in \mathbf{Set}^{\mathrm{op}}$ and induces a functor $\mathbf{Poly} \times \mathbf{Set} \rightarrow \mathbf{Fam}$. We write $\mathrm{rep}: \mathbf{Poly} \rightarrow [\mathbf{Set}, \mathbf{Set}]$ for the currying of the composite

$$\begin{array}{ccccc} \mathbf{Poly} \times \mathbf{Set} & \longrightarrow & \mathbf{Fam} & \xrightarrow{\Sigma} & \mathbf{Set} \\ \downarrow \Psi & & & & \downarrow \Psi \\ (A, S) & \longmapsto & & & \sum_{a:A_{\circ}} [A_p, S] \end{array}$$

where Σ is the coproduct (i.e., the left adjoint of the fibred terminal $\mathbf{Set} \rightarrow \mathbf{Fam}$).

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Proposition 1.4. rep is fully faithful.

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Proof. Suppose we are given $P, Q \in \mathbf{Poly}$, a morphism $\alpha: \mathbf{rep}(P) \rightarrow \mathbf{rep}(Q)$ in $[\mathbf{Set}, \mathbf{Set}]$, and $S \in \mathbf{Set}$. We have the following commutative square:

$$\begin{array}{ccc} \sum_{p \in P_\circ} [P_p, S] & \xrightarrow{\alpha_S} & \sum_{q \in Q_\circ} [Q_q, S] \\ \mathbf{rep}(P)(!_S) \downarrow & & \downarrow \mathbf{rep}(Q)(!_S) \\ P_\circ & \xrightarrow{\alpha_1} & Q_\circ \end{array}$$

where the left and the right vertical arrows are projections. Therefore, we have a family of morphisms $[P_p, S] \rightarrow [Q_{\alpha_1(p)}, S]$ natural in $p \in P_\circ$ and $S \in \mathbf{Set}$. By applying the yoneda lemma, we obtain $Q_{\alpha_1(p)} \rightarrow P_p$ for each $p \in P_\circ$, which induces a morphism $\bar{\alpha}: P \rightarrow Q$ in \mathbf{Poly} . One can readily check that $\bar{\alpha}$ is a unique morphism satisfying $\mathbf{rep}(\bar{\alpha}) = \alpha$. \square

Lemma 1.5. (The essential image of) \mathbf{rep} is closed under the monoidal structure $(\circ, \mathbf{id}_{\mathbf{Set}})$ on $[\mathbf{Set}, \mathbf{Set}]$. \blacklozenge

Proof. $\mathbf{rep}(y) \cong \mathbf{id}_{\mathbf{Set}}$ is trivial. It suffices to show \mathbf{rep} is closed under \circ . Let A, B be polynomials. Let us define another polynomial $A \triangleleft B$ as follows.

$$(A \triangleleft B)_\circ = \sum_{a:A_\circ} [A_a, B_\circ]$$

$$(A \triangleleft B)_{a,t} = \sum_{u:A_a} B_{t(u)}$$

for any $a : A_\circ$ and $t : A_a \rightarrow B_\circ$. Then, observe the following bijections.

$$\begin{aligned} \mathbf{rep}(A \triangleleft B)(S) &\cong \sum_{a:A_\circ} \sum_{t:[A_a, B_\circ]} \left[\sum_{u:A_a} B_{t(u)}, S \right] \\ &\cong \sum_{a:A_\circ} \sum_{t:[A_a, B_\circ]} \prod_{u:A_a} [B_{t(u)}, S] \\ &\cong \sum_{a:A_\circ} \prod_{u:A_a} \sum_{b:B_\circ} [B_b, S] \\ &\cong \sum_{a:A_\circ} \left[A_a, \sum_{b:B_\circ} [B_b, S] \right] \\ &= \mathbf{rep}(A)(\mathbf{rep}(B)(S)) \end{aligned}$$

The third bijection is the distributive law. Since these are natural in $S \in \mathbf{Set}$ this shows that $\mathbf{rep}(A \triangleleft B) \cong \mathbf{rep}(A) \circ \mathbf{rep}(B)$. \square

Proposition 1.6. Let \mathbf{V} be a category and \mathbf{W} be a monoidal category. Suppose there is a fully faithful functor $F: \mathbf{V} \hookrightarrow \mathbf{W}$ that is closed under the monoidal structure on \mathbf{W} . Then there exists a monoidal structure on \mathbf{V} that makes F strong monoidal, which is unique up to isomorphisms. \blacklozenge

Proof. **folklore** \square

Proposition 1.7. $(\mathbf{Poly}, \triangleleft, y)$ is a monoidal category such that $\mathbf{rep}: \mathbf{Poly} \rightarrow ([\mathbf{Set}, \mathbf{Set}], \circ, \mathbf{id}_{\mathbf{Set}})$ is a pseudo monoidal functor. \blacklozenge

Proof. Follows from [Lemma 1.5](#) and [Propositions 1.4](#) and [1.6](#). \square

Remark 1.8. The functor $- \triangleleft -: \mathbf{Poly} \times \mathbf{Poly} \rightarrow \mathbf{Poly}$ acts on morphisms as follows. Let $F: A \rightarrow X$, $G: B \rightarrow Y$ be morphisms in \mathbf{Poly} .

$$\begin{aligned} (F \triangleleft G)_\circ: \sum_{a:A_\circ} [A_a, B_\circ] &\rightarrow \sum_{x:X_\circ} [X_x, Y_\circ] : (a, t) \mapsto (F_\circ(a), F_a \circ t \circ G_\circ) \\ (F \triangleleft G)_{a,t}: \sum_{v:X_{F_\circ(a)}} Y_{G_\circ(t(F_a(v)))} &\rightarrow \sum_{u:A_a} B_{t(u)} : (v, r) \mapsto (F_a(v), G_{t(F_a(v))}(r)) \end{aligned}$$

\blacksquare

Lemma 1.9. $\mathbf{Poly} \times \mathbf{Poly} \xrightarrow{\triangleleft} \mathbf{Poly}$ restricts to a functor $\mathbf{Poly} \times \mathbf{Set} \xrightarrow{\triangleleft} \mathbf{Set}$ and coincides with the uncurrying of \mathbf{rep} . \blacklozenge

Proof. \square

2. DOUBLE CATEGORIES

By *double categories*, we mean psuedo ones.

Definition 2.1. A double category \mathbb{D} is a *company* if its tight arrows have companions. \mathbb{D} is an *opcompany* if \mathbb{D}^{hop} is a company. We say \mathbb{D} is an *equipment* if \mathbb{D} and \mathbb{D}^{hop} are companies. \blacksquare

Definition 2.2 ([Par24]). (It is defined for arbitrary bvdc whose underlying sdc is a opcompany.) Let \mathbb{D} be an opcompany. Define another double category $\mathbf{Ret}(\mathbb{D})$ as follows.

- The vertical category is the same as \mathbb{D} .
- A loose arrow is a loose arrow in \mathbb{D} .
- A cell

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ f \downarrow & \alpha & \downarrow g \\ X & \xrightarrow{q} & Y \end{array} \text{ in } \mathbf{Ret}(\mathbb{D})$$

is a cell in \mathbb{D} of the following form.

$$\begin{array}{ccccc} X & \xrightarrow{q} & Y & \xrightarrow{g^*} & B \\ \parallel & & \alpha & & \parallel \\ X & \xrightarrow{f^*} & A & \xrightarrow{p} & B \end{array}$$

- *horizontal composite*
- *vertical composite*

Proposition 2.3. The virtual double category of monoids in $\mathbf{Ret}(\mathbb{D})^{\text{vop}}$ is representable if \mathbb{D} has local reflexive coequalizers. \blacklozenge

Definition 2.4. Let \mathbb{D} be a double category with local reflexive coequalizers.

- We write $\mathbf{Mod}(\mathbb{D})$ for the double category corresponding to the representable virtual double category of monoids in $\mathbf{Ret}(\mathbb{D})$.
- We write $\mathbf{Mod}_{\text{co}}(\mathbb{D})$ for the vertical opposite of the double category corresponding to the representable virtual double category of monoids in $\mathbf{Ret}(\mathbb{D})^{\text{vop}}$ obtained in Proposition 2.3. \blacksquare

Lemma 2.5. Let \mathbb{D} be a double category with local reflexive coequalizers. There is an isomorphism $\mathfrak{H}(\mathbf{Mod}_{\text{co}}(\mathbb{D})) \cong \mathfrak{H}(\mathbf{Mod}(\mathbb{D}))$. *Where?* \blacklozenge

Definition 2.6. \blacksquare

3. POLYNOMIAL IN A GENERALISED CONTEXT

Definition 3.1. Let \mathbf{C} be a category.

- A morphism $f: X \rightarrow Y$ in \mathbf{C} is *carrable* if $\Sigma_f: \mathbf{C}_{/X} \rightarrow \mathbf{C}_{/Y}$ has a right adjoint f^* .
- A morphism $f: X \rightarrow Y$ in \mathbf{C} is *exponentiable* if f is carrable and $f^*: \mathbf{C}_{/Y} \rightarrow \mathbf{C}_{/X}$ has a right adjoint Π_f . \blacksquare

Definition 3.2. A *pre-representable map category (prMC)* is a pair $(\mathbf{R}, \mathbf{rep})$ such that \mathbf{R} is a category and \mathbf{rep} is a class of morphisms in \mathbf{R} satisfying the followings.

- \mathbf{rep} contains all identities and is closed under composition.
- All morphisms in \mathbf{rep} are exponentiable. \blacksquare

Definition 3.3. Let $\mathbf{R} = (\mathbf{R}, \mathbf{rep})$ be a prMC. For any object $X, Y \in \mathbf{R}$, define a category $\mathfrak{Poly}(\mathbf{R})(X, Y)$ as follows.

- An object is a diagram in \mathbf{R} of the following form

$$X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y$$

such that f is carrable and g is exponentiable.



REFERENCES

- [Par24] R. Paré. “Retrocells”. In: *Theory Appl. Categ.* 40 (2024), Paper No. 5, 130–179 (cit. on p. 3).

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