NOTE ON POLYNOMIALS

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Notation 0.1. We employ the following notations.

- Sets are regarded as discrete categories, and categories are regarded as locally discrete 2-categories.
- **Set** is the large category of small sets.
- We write $\mathfrak{CAT}$ for the huge 2-category of large categories.
- For each set S and a functor $X\colon S \longrightarrow \mathbf{Set}$, we write X_s for the image of $s \in S$ under X. Moreover, we write $(X_s)_{s:S}$ for X.
- For each set S and functors $X, Y\colon S \longrightarrow \mathbf{Set}$, a natural transformation $f\colon X \Longrightarrow Y$ is denoted as a family of functions $(f_s)_{s:S}$.
- We write 1 for the terminal set, and we write $*$ for the unique element in 1.

■

1. POLYNOMIALS

Definition 1.1. We define a category **Poly** by the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-,\mathbf{Set}]^{\mathrm{op}}} \mathfrak{CAT}$$

A ***polynomial*** is an object in **Poly**. We write

$$(-)_{\circ}\colon \mathbf{Poly} \longrightarrow \mathbf{Set}$$

for the fibration corresponding to the 2-functor. For each polynomial A, we write $A = \sum_{a:A_\circ} y^{A_a}$.

In particular, we mean by y the polynomial satisfying $(y)_0 = 1$ and $(y)_* = 1$. ■

Definition 1.2. We define a category **Fam** by the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-,\mathbf{Set}]} \mathfrak{CAT}$$

Note that the fibration $\mathbf{Poly} \longrightarrow \mathbf{Set}$ is the fibrewise opposit of $\mathbf{Fam} \longrightarrow \mathbf{Set}$. ■

Definition 1.3. Consider the composite

$$\begin{tikzcd}
{[S,\mathbf{Set}]^{\mathrm{op}} \times \mathbf{Set}} \arrow[r, "{\langle \mathtt{id}_{[S,\mathbf{Set}]}, \Delta\circ\mathtt{proj}\rangle}"] & {[S,\mathbf{Set}]^{\mathrm{op}} \times [S,\mathbf{Set}]} \arrow[r] & {[S,\mathbf{Set}]} \\
{((A_s)_{s:S}, T)} \arrow[rr, mapsto] \arrow[u, phantom, "\in" rotate=90] & & {([A_s, T])_{s:S}} \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

where $\Delta\colon \mathbf{Set} \longrightarrow [S, \mathbf{Set}]$ in the former is the diagonal and the latter is the internal hom for $[S, Set]$. This is natural in $S \in \mathbf{Set}^{\mathrm{op}}$ and induces a functor $\mathbf{Poly} \times \mathbf{Set} \longrightarrow \mathbf{Fam}$. We write $\mathtt{rep}\colon \mathbf{Poly} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$ for the currying of the composite

$$\begin{tikzcd}
\mathbf{Poly} \times \mathbf{Set} \arrow[r] & \mathbf{Fam} \arrow[r, "\Sigma"] & \mathbf{Set} \\
(A, S) \arrow[rr, mapsto] \arrow[u, phantom, "\in" rotate=90] & & \sum_{a:A_\circ} [A_p, S] \arrow[u, phantom, "\in" rotate=90]
\end{tikzcd}$$

where $\sum$ is the coproduct (i.e., the left adjoint of the fibred terminal $\mathbf{Set} \longrightarrow \mathbf{Fam}$). ■

Proposition 1.4. $\mathtt{rep}$ is fully faithful. ♦

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Proof. Suppose we are given $P, Q \in \mathbf{Poly}$, a morphism $\alpha\colon \mathtt{rep}(P) \longrightarrow \mathtt{rep}(Q)$ in $[\mathbf{Set}, \mathbf{Set}]$, and $S \in \mathbf{Set}$. We have the following commutative square:

$$\begin{tikzcd}
\sum_{p\in P_\circ}[P_p, S] \arrow[r, "\alpha_S"] \arrow[d, "\mathtt{rep}(P)(!_S)"'] & \sum_{q\in Q_\circ}[Q_q, S] \arrow[d, "\mathtt{rep}(Q)(!_S)"] \\
P_\circ \arrow[r, "\alpha_1"'] & Q_\circ
\end{tikzcd}$$

where the left and the right vertical arrows are projections. Therefore, we have a family of morphisms $[P_p, S] \longrightarrow \left[Q_{\alpha_1(p)}, S\right]$ natural in $p \in P_\circ$ and $S \in \mathbf{Set}$. By applying the yoneda lemma, we obtain $Q_{\alpha_1(p)} \longrightarrow P_p$ for each $p \in P_\circ$, which induces a morphism $\bar{\alpha}\colon P \longrightarrow Q$ in $\mathbf{Poly}$. One can readily check that $\bar{\alpha}$ is a unique morphism satisfying $\mathtt{rep}(\bar{\alpha}) = \alpha$. □

Lemma 1.5. (The essential image of) $\mathtt{rep}$ is closed under the monoidal structure $(\circ, \mathtt{id}_{\mathbf{Set}})$ on $[\mathbf{Set}, \mathbf{Set}]$. ♦

Proof. $\mathtt{rep}(y) \cong \mathtt{id}_{\mathbf{Set}}$ is trivial. It suffices to show $\mathtt{rep}$ is closed under $\circ$. Let A, B be polynomials. Let us define another polynomial $A \triangleleft B$ as follows.

$$(A \triangleleft B)_\circ = \sum_{a:A_\circ} [A_a, B_\circ]$$

$$(A \triangleleft B)_{a,t} = \sum_{u:A_a} B_{t(u)}$$

for any $a : A_\circ$ and $t\colon A_a \longrightarrow B_\circ$. Then, observe the following bijections.

$$\begin{aligned}
\mathtt{rep}(A \triangleleft B)(S) &\cong \sum_{a:A_\circ} \sum_{t:[A_a,B_\circ]} \left[\sum_{u:A_a} B_{t(u)}, S\right] \\
&\cong \sum_{a:A_\circ} \sum_{t:[A_a,B_\circ]} \prod_{u:A_a} [B_{t(u)}, S] \\
&\cong \sum_{a:A_\circ} \prod_{u:A_a} \sum_{b:B_\circ} [B_b, S] \\
&\cong \sum_{a:A_\circ} \left[A_a, \sum_{b:B_\circ} [B_b, S]\right] \\
&= \mathtt{rep}(A)(\mathtt{rep}(B)(S))
\end{aligned}$$

The third bijection is the distributive law. Since these are natural in $S \in \mathbf{Set}$ this shows that $\mathtt{rep}(A \triangleleft B) \cong \mathtt{rep}(A) \circ \mathtt{rep}(B)$. □

Proposition 1.6. Let $\mathbf{V}$ be a category and $\mathbf{W}$ be a monoidal category. Suppose there is a fully faithful functor $F\colon \mathbf{V} \hookrightarrow \mathbf{W}$ that is closed under the monoidal structure on $\mathbf{W}$. Then there exists a monoidal structure on $\mathbf{V}$ that makes F strong monoidal, which is unique up to isomorphisms. ♦

Proof. folklore □

Remark 1.7. The functor $- \triangleleft -\colon \mathbf{Poly} \times \mathbf{Poly} \longrightarrow \mathbf{Poly}$ acts on morphisms as follows. Let $F\colon A \longrightarrow X$, $G\colon B \longrightarrow Y$ be morphisms in $\mathbf{Poly}$.

$$(F \triangleleft G)_\circ\colon \sum_{a:A_\circ} [A_a, B_\circ] \longrightarrow \sum_{x:X_\circ} [X_x, Y_\circ] : (a, t) \mapsto (F_\circ(a), F_a \fatsemi t \fatsemi G_\circ)$$

$$(F \triangleleft G)_{a,t}\colon \sum_{v:X_{F_\circ(a)}} Y_{G_\circ(t(F_a(v)))} \longrightarrow \sum_{u:A_a} B_{t(u)} : (v, r) \mapsto (F_a(v), G_{t(F_a(v))}(r))$$

■

Lemma 1.8. $\mathbf{Poly} \times \mathbf{Poly} \xrightarrow{\triangleleft} \mathbf{Poly}$ restricts to a functor $\mathbf{Poly} \times \mathbf{Set} \xrightarrow{\triangleleft} \mathbf{Set}$ and coincide with $\mathtt{rep}(-)(-)$. ♦

Proof. □

Proposition 1.9. $(\mathbf{Poly}, \triangleleft, y)$ is a monoidal category such that $\mathtt{rep}\colon \mathbf{Poly} \longrightarrow ([\mathbf{Set}, \mathbf{Set}], \circ, \mathtt{id}_{\mathbf{Set}})$ is a pseudo monoidal functor. ♦

Proof. Follows from Lemma 1.5 and Propositions 1.4 and 1.6. □

References

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