NOTE ON POLYNOMIALS

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Notation 0.1. We employ the following notations.

- Sets are regarded as discrete categories, and categories are regarded as locally discrete 2-categories.
- **Set** is the large category of small sets.
- We write $\mathfrak{CAT}$ for the huge 2-category of large categories.
- For each set S and a functor $X\colon S \longrightarrow \mathbf{Set}$, we write X_s for the image of $s \in S$ under X. Moreover, we write $(X_s)_{s:S}$ for X.
- For each set S and functors $X, Y\colon S \longrightarrow \mathbf{Set}$, a natural transformation $f\colon X \Longrightarrow Y$ is denoted as a family of functions $(f_s)_{s:S}$.
- We write 1 for the terminal set, and we write $*$ for the unique element in 1.

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1. POLYNOMIALS

Definition 1.1. We define a category **Poly** as the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-,\mathbf{Set}]^{\mathrm{op}}} \mathfrak{CAT}$$

A ***polynomial*** is an object in **Poly**. We write

$$(-)_\circ\colon \mathbf{Poly} \longrightarrow \mathbf{Set}$$

for the fibration corresponding to the 2-functor. For each polynomial A, we write $A = \sum_{a:A_\circ} y^{A_a}$.

In particular, we mean by y the polynomial satisfying $(y)_0 = 1$ and $(y)_* = 1$. ■

Definition 1.2. Define a functor $- \triangleleft -\colon \mathbf{Poly} \times \mathbf{Poly} \longrightarrow \mathbf{Poly}$ as follows.

- Let A, B be polynomials.
$$(A \triangleleft B)_\circ = \sum_{a:A_\circ} [A_a, B_\circ]$$
$$(A \triangleleft B)_{a,t} = \sum_{u:A_a} B_{t(u)}$$
for any $a : A_\circ$ and $t\colon A_a \longrightarrow B_\circ$.
- Let $F\colon A \longrightarrow X$, $G\colon B \longrightarrow Y$ be morphisms in **Poly**.
$$(F \triangleleft G)_\circ\colon \sum_{a:A_\circ} [A_a, B_\circ] \longrightarrow \sum_{x:X_\circ} [X_x, Y_\circ] : (a, t) \mapsto (F_\circ(a), F_a \fatsemi t \fatsemi G_\circ)$$
$$(F \triangleleft G)_{a,t}\colon \sum_{v:X_{F_\circ(a)}} Y_{G_\circ(t(F_a(v)))} \longrightarrow \sum_{u:A_a} B_{t(u)} : (v, r) \mapsto (F_a(v), G_{t(F_a(v))}(r))$$

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Proposition 1.3. For each polynomial A, there are natural isomorphisms $y \triangleleft A \cong A$ and $A \triangleleft y \cong A$ in **Poly**. ⧫

Proposition 1.4. Let $\mathbf{V}$ be a category and $\mathbf{W} = (\mathbf{W}, \otimes', I')$ be a monoidal category. Consider a functor $\otimes\colon \mathbf{V} \times \mathbf{V} \longrightarrow \mathbf{V}$ and an object $I \in \mathbf{V}$. Suppose that we are given a functor $F\colon \mathbf{V} \longrightarrow \mathbf{W}$ such that there are a family of isomorphisms $F(A \otimes B) \cong F(A) \otimes F(B)$ natural in $A, B \in \mathbf{V}$ and an isomorphism $F(I) \cong I$. If F induces a fully faithful functor on the core groupoids, then $(\mathbf{V}, \otimes, I)$ becomes a monoidal category and F is a pseudo monoidal functor. ⧫

Date: April 29, 2024.

Definition 1.5. Define $\mathbf{Set} \hookrightarrow \mathbf{Poly}$ as a functor induced from the 2-natural transformation $\Delta(\mathbf{1}) \Longrightarrow [-, \mathbf{Set}]^{\mathrm{op}}$ whose component at $S \in \mathbf{Set}^{\mathrm{op}}$ is the name for the terminal object in $[S, \mathbf{Set}]^{\mathrm{op}}$. In other words, the fibration $\mathbf{Poly} \xrightarrow{(-)_\circ} \mathbf{Set}$ has the fibred terminal. This functor is automatically fully faithful, and we regard sets as polynomials through this functor. ■

Lemma 1.6. $\mathbf{Poly} \times \mathbf{Poly} \xrightarrow{\triangleleft} \mathbf{Poly}$ *restricts to a functor* $\mathbf{Poly} \times \mathbf{Set} \xrightarrow{\triangleleft} \mathbf{Set}$. ♦

Proof. □

Definition 1.7. Define a functor $\mathtt{rep}\colon \mathbf{Poly} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$ as the currying of the functor $\mathbf{Poly} \times \mathbf{Set} \xrightarrow{\triangleleft} \mathbf{Set}$ obtained in Lemma 1.6. ■

Proposition 1.8. $\mathtt{rep}$ *induces a fully faithful functor on the core groupoids.* ♦

Proof. Suppose we are given $P, Q \in \mathbf{Poly}$, an isomorphism $\alpha\colon \mathtt{rep}(P) \cong \mathtt{rep}(Q)$ in $[\mathbf{Set}, \mathbf{Set}]$, and $S \in \mathbf{Set}$. We have the following commutative diagram:

$$\begin{tikzcd}
\sum_{p \in P_\circ}[P_p, S] \arrow[r, phantom, "\cong"] \arrow[d] & P \triangleleft S \arrow[r, "\alpha_S", "\sim"'] \arrow[d, "P \triangleleft !_S"'] & Q \triangleleft S \arrow[r, phantom, "\cong"] \arrow[d, "Q \triangleleft !_S"] & \sum_{q \in Q_\circ}[Q_q, S] \arrow[d] \\
P_\circ \arrow[r, phantom, "\cong"] & P \triangleleft 1 \arrow[r, "\sim", "\alpha_1"'] & Q \triangleleft 1 \arrow[r, phantom, "\cong"] & Q_\circ
\end{tikzcd}$$

where the left and the right vertical arrows are projections. Therefore, we have $[P_p, S] \cong \left[Q_{\alpha_1(p)}, S\right]$ for each $p \in P_\circ$ and $S \in \mathbf{Set}$. By applying the yoneda lemma, we obtain $Q_{\alpha_1(p)} \cong P_p$ for each $p \in P_\circ$, which induces an isomophism $\bar{\alpha}\colon P \cong Q$ in $\mathbf{Poly}$. One can readily check that $\bar{\alpha}$ is a unique isomorphism satisfying $\mathtt{rep}(\bar{\alpha}) = \alpha$. □

Lemma 1.9. *There is a family of isomorphisms* $\mathtt{rep}(P \triangleleft Q) \cong \mathtt{rep}(P) \circ \mathtt{rep}(Q)$ *natural in* $P, Q \in \mathbf{Poly}$. *Moreover, we have* $\mathtt{rep}(y) \cong \mathtt{id}_{\mathbf{Set}}$ *in* $[\mathbf{Set}, \mathbf{Set}]$. ♦

Proposition 1.10. $(\mathbf{Poly}, \triangleleft, y)$ *is a monoidal category such that* $\mathtt{rep}\colon \mathbf{Poly} \longrightarrow ([\mathbf{Set}, \mathbf{Set}], \circ, \mathtt{id}_{\mathbf{Set}})$ *is a pseudo monoidal functor.* ♦

Proof. Follows from Lemma 1.9 and Propositions 1.4 and 1.8. □

References

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