

# NOTE ON POLYNOMIALS

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**Notation 0.1.** We employ the following notations.

- Sets are regarded as discrete categories, and categories are regarded as locally discrete 2-categories.
- **Set** is the large category of small sets.
- We write  $\mathfrak{Cat}$  for the huge 2-category of large categories.
- For each set  $S$  and a functor  $X: S \rightarrow \mathbf{Set}$ , we write  $X_s$  for the image of  $s \in S$  under  $X$ . Moreover, we write  $(X_s)_{s:S}$  for  $X$ .
- For each set  $S$  and functors  $X, Y: S \rightarrow \mathbf{Set}$ , a natural transformation  $f: X \Rightarrow Y$  is denoted as a family of functions  $(f_s)_{s:S}$ .
- We write  $1$  for the terminal set, and we write  $*$  for the unique element in  $1$ .

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## 1. POLYNOMIALS

**Definition 1.1.** We define a category **Poly** by the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-, \mathbf{Set}]^{\mathrm{op}}} \mathfrak{Cat}$$

A *polynomial* is an object in **Poly**. We write

$$(-)_{\circ}: \mathbf{Poly} \rightarrow \mathbf{Set}$$

for the fibration corresponding to the 2-functor. For each polynomial  $A$ , we write  $A = \sum_{a:A_{\circ}} y^{A_a}$ .

In particular, we mean by  $y$  the polynomial satisfying  $(y)_{\circ} = 1$  and  $(y)_* = 1$ .

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**Definition 1.2.** We define a category **Fam** by the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-, \mathbf{Set}]} \mathfrak{Cat}$$

Note that the fibration  $\mathbf{Poly} \rightarrow \mathbf{Set}$  is the fibrewise opsite of  $\mathbf{Fam} \rightarrow \mathbf{Set}$ .

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**Definition 1.3.** Consider the composite

$$\begin{array}{ccc} [S, \mathbf{Set}]^{\mathrm{op}} \times \mathbf{Set} & \xrightarrow{\langle \mathrm{id}_{[S, \mathbf{Set}]}, \Delta_{\mathrm{proj}} \rangle} & [S, \mathbf{Set}]^{\mathrm{op}} \times [S, \mathbf{Set}] \longrightarrow [S, \mathbf{Set}] \\ \downarrow \Psi & & \downarrow \Psi \\ ((A_s)_{s:S}, T) & \longmapsto & ([A_s, T])_{s:S} \end{array}$$

where  $\Delta: \mathbf{Set} \rightarrow [S, \mathbf{Set}]$  in the former is the diagonal and the latter is the internal hom for  $[S, \mathbf{Set}]$ . This is natural in  $S \in \mathbf{Set}^{\mathrm{op}}$  and induces a functor  $\mathbf{Poly} \times \mathbf{Set} \rightarrow \mathbf{Fam}$ . We write  $\mathrm{rep}: \mathbf{Poly} \rightarrow [\mathbf{Set}, \mathbf{Set}]$  for the currying of the composite

$$\begin{array}{ccccc} \mathbf{Poly} \times \mathbf{Set} & \longrightarrow & \mathbf{Fam} & \xrightarrow{\Sigma} & \mathbf{Set} \\ \downarrow \Psi & & & & \downarrow \Psi \\ (A, S) & \longmapsto & & & \sum_{a:A_{\circ}} [A_p, S] \end{array}$$

where  $\Sigma$  is the coproduct (i.e., the left adjoint of the fibred terminal  $\mathbf{Set} \rightarrow \mathbf{Fam}$ ).

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**Proposition 1.4.**  $\mathrm{rep}$  is fully faithful.

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*Proof.* Suppose we are given  $P, Q \in \mathbf{Poly}$ , a morphism  $\alpha: \mathbf{rep}(P) \rightarrow \mathbf{rep}(Q)$  in  $[\mathbf{Set}, \mathbf{Set}]$ , and  $S \in \mathbf{Set}$ . We have the following commutative square:

$$\begin{array}{ccc} \sum_{p \in P_{\circ}} [P_p, S] & \xrightarrow{\alpha_S} & \sum_{q \in Q_{\circ}} [Q_q, S] \\ \mathbf{rep}(P)(!_S) \downarrow & & \downarrow \mathbf{rep}(Q)(!_S) \\ P_{\circ} & \xrightarrow{\alpha_1} & Q_{\circ} \end{array}$$

where the left and the right vertical arrows are projections. Therefore, we have a family of morphisms  $[P_p, S] \rightarrow [Q_{\alpha_1(p)}, S]$  natural in  $p \in P_{\circ}$  and  $S \in \mathbf{Set}$ . By applying the yoneda lemma, we obtain  $Q_{\alpha_1(p)} \rightarrow P_p$  for each  $p \in P_{\circ}$ , which induces a morphism  $\bar{\alpha}: P \rightarrow Q$  in  $\mathbf{Poly}$ . One can readily check that  $\bar{\alpha}$  is a unique morphism satisfying  $\mathbf{rep}(\bar{\alpha}) = \alpha$ .  $\square$

**Lemma 1.5.** (The essential image of)  $\mathbf{rep}$  is closed under the monoidal structure  $(\circ, \mathbf{id}_{\mathbf{Set}})$  on  $[\mathbf{Set}, \mathbf{Set}]$ .  $\blacklozenge$

*Proof.*  $\mathbf{rep}(y) \cong \mathbf{id}_{\mathbf{Set}}$  is trivial. It suffices to show  $\mathbf{rep}$  is closed under  $\circ$ . Let  $A, B$  be polynomials. Let us define another polynomial  $A \triangleleft B$  as follows.

$$(A \triangleleft B)_{\circ} = \sum_{a: A_{\circ}} [A_a, B_{\circ}]$$

$$(A \triangleleft B)_{a,t} = \sum_{u: A_a} B_{t(u)}$$

for any  $a: A_{\circ}$  and  $t: A_a \rightarrow B_{\circ}$ . Then, observe the following bijections.

$$\begin{aligned} \mathbf{rep}(A \triangleleft B)(S) &\cong \sum_{a: A_{\circ}} \sum_{t: [A_a, B_{\circ}]} \left[ \sum_{u: A_a} B_{t(u)}, S \right] \\ &\cong \sum_{a: A_{\circ}} \sum_{t: [A_a, B_{\circ}]} \prod_{u: A_a} [B_{t(u)}, S] \\ &\cong \sum_{a: A_{\circ}} \prod_{u: A_a} \sum_{b: B_{\circ}} [B_b, S] \\ &\cong \sum_{a: A_{\circ}} \left[ A_a, \sum_{b: B_{\circ}} [B_b, S] \right] \\ &= \mathbf{rep}(A)(\mathbf{rep}(B)(S)) \end{aligned}$$

The third bijection is the distributive law. Since these are natural in  $S \in \mathbf{Set}$  this shows that  $\mathbf{rep}(A \triangleleft B) \cong \mathbf{rep}(A) \circ \mathbf{rep}(B)$ .  $\square$

**Proposition 1.6.** Let  $\mathbf{V}$  be a category and  $\mathbf{W}$  be a monoidal category. Suppose there is a fully faithful functor  $F: \mathbf{V} \hookrightarrow \mathbf{W}$  that is closed under the monoidal structure on  $\mathbf{W}$ . Then there exists a monoidal structure on  $\mathbf{V}$  that makes  $F$  strong monoidal, which is unique up to isomorphisms.  $\blacklozenge$

*Proof.* **folklore**  $\square$

**Proposition 1.7.**  $(\mathbf{Poly}, \triangleleft, y)$  is a monoidal category such that  $\mathbf{rep}: \mathbf{Poly} \rightarrow ([\mathbf{Set}, \mathbf{Set}], \circ, \mathbf{id}_{\mathbf{Set}})$  is a pseudo monoidal functor.  $\blacklozenge$

*Proof.* Follows from [Lemma 1.5](#) and [Propositions 1.4](#) and [1.6](#).  $\square$

**Remark 1.8.** The functor  $- \triangleleft -: \mathbf{Poly} \times \mathbf{Poly} \rightarrow \mathbf{Poly}$  acts on morphisms as follows. Let  $F: A \rightarrow X$ ,  $G: B \rightarrow Y$  be morphisms in  $\mathbf{Poly}$ .

$$\begin{aligned} (F \triangleleft G)_{\circ}: \sum_{a: A_{\circ}} [A_a, B_{\circ}] &\rightarrow \sum_{x: X_{\circ}} [X_x, Y_{\circ}] : (a, t) \mapsto (F_{\circ}(a), F_a \circ t \circ G_{\circ}) \\ (F \triangleleft G)_{a,t}: \sum_{v: X_{F_{\circ}(a)}} Y_{G_{\circ}(t(F_a(v)))} &\rightarrow \sum_{u: A_a} B_{t(u)} : (v, r) \mapsto (F_a(v), G_{t(F_a(v))}(r)) \end{aligned}$$

$\blacksquare$

**Lemma 1.9.**  $\mathbf{Poly} \times \mathbf{Poly} \xrightarrow{\triangleleft} \mathbf{Poly}$  restricts to a functor  $\mathbf{Poly} \times \mathbf{Set} \xrightarrow{\triangleleft} \mathbf{Set}$  and coincides with the uncurrying of  $\mathbf{rep}$ .  $\blacklozenge$

*Proof.*  $\square$

## 2. DOUBLE CATEGORIES

By *double categories*, we mean psuedo ones.

**Definition 2.1.** A double category  $\mathbb{D}$  is a *company* if its tight arrows have companions.  $\mathbb{D}$  is an *opcompany* if  $\mathbb{D}^{\text{hop}}$  is a company. We say  $\mathbb{D}$  is an *equipment* if  $\mathbb{D}$  and  $\mathbb{D}^{\text{hop}}$  are companies.  $\blacksquare$

**Definition 2.2** ([Par24]). (It is defined for arbitrary bvdc whose underlying sdc is a opcompany.) Let  $\mathbb{D}$  be an opcompany. Define another double category  $\mathbf{Ret}(\mathbb{D})$  as follows.

- The vertical category is the same as  $\mathbb{D}$ .
- A loose arrow is a loose arrow in  $\mathbb{D}$ .
- A cell

$$\begin{array}{ccc} A & \xrightarrow{p} & B \\ f \downarrow & \alpha & \downarrow g \\ X & \xrightarrow{q} & Y \end{array} \text{ in } \mathbf{Ret}(\mathbb{D})$$

is a cell in  $\mathbb{D}$  of the following form.

$$\begin{array}{ccccc} X & \xrightarrow{q} & Y & \xrightarrow{g^*} & B \\ \parallel & & \alpha & & \parallel \\ X & \xrightarrow{f^*} & A & \xrightarrow{p} & B \end{array}$$

- horizontal composite
- vertical composite

**Proposition 2.3.** The virtual double category of monoids in  $\mathbf{Ret}(\mathbb{D})^{\text{vop}}$  is representable if  $\mathbb{D}$  has local reflexive coequalizers.  $\blacklozenge$

**Definition 2.4.** Let  $\mathbb{D}$  be a double category with local reflexive coequalizers.

- We write  $\mathbf{Mod}(\mathbb{D})$  for the double category corresponding to the representable virtual double category of monoids in  $\mathbf{Ret}(\mathbb{D})$ .
- We write  $\mathbf{Mod}_{\text{co}}(\mathbb{D})$  for the vertical opposite of the double category corresponding to the representable virtual double category of monoids in  $\mathbf{Ret}(\mathbb{D})^{\text{vop}}$  obtained in Proposition 2.3.  $\blacksquare$

**Lemma 2.5.** Let  $\mathbb{D}$  be a double category with local reflexive coequalizers. There is an isomorphism  $\mathfrak{H}(\mathbf{Mod}_{\text{co}}(\mathbb{D})) \cong \mathfrak{H}(\mathbf{Mod}(\mathbb{D}))$ . *Where?*  $\blacklozenge$

**Definition 2.6.**  $\blacksquare$

## 3. POLYNOMIAL IN A GENERALISED CONTEXT

**Definition 3.1.** Let  $\mathbf{C}$  be a category.

- A morphism  $f: X \rightarrow Y$  in  $\mathbf{C}$  is *carrable* if  $\Sigma_f: \mathbf{C}_{/X} \rightarrow \mathbf{C}_{/Y}$  has a right adjoint  $f^*$ .
- A morphism  $f: X \rightarrow Y$  in  $\mathbf{C}$  is *exponentiable* if  $f$  is carrable and  $f^*: \mathbf{C}_{/Y} \rightarrow \mathbf{C}_{/X}$  has a right adjoint  $\Pi_f$ .  $\blacksquare$

**Definition 3.2.** A *pre-representable map category (prMC)* is a pair  $(\mathbf{R}, \mathbf{rep})$  such that  $\mathbf{R}$  is a category and  $\mathbf{rep}$  is a class of morphisms in  $\mathbf{R}$  satisfying the followings.

- $\mathbf{rep}$  contains all identities and is closed under composition.
- All morphisms in  $\mathbf{rep}$  are exponentiable.  $\blacksquare$

**Definition 3.3.** Let  $\mathbf{R} = (\mathbf{R}, \mathbf{rep})$  be a prMC. For any object  $X, Y \in \mathbf{R}$ , define a category  $\mathfrak{Poly}(\mathbf{R})(X, Y)$  as follows.

- An object is a diagram in  $\mathbf{R}$  of the following form

$$X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y$$

such that  $f$  is carrable and  $g$  is exponentiable.



## REFERENCES

- [Par24] R. Paré. Retrocells. *Theory Appl. Categ.*, 40:Paper No. 5, 130–179, 2024. [3](#)  
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