NOTE ON POLYNOMIALS

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Notation 0.1. We employ the following notations.

- Sets are regarded as discrete categories, and categories are regarded as locally discrete 2-categories.
- **Set** is the large category of small sets.
- We write $\mathfrak{CAT}$ for the huge 2-category of large categories.
- For each set S and a functor $X\colon S \longrightarrow \mathbf{Set}$, we write X_s for the image of $s \in S$ under X. Moreover, we write $(X_s)_{s:S}$ for X.
- For each set S and functors $X, Y\colon S \longrightarrow \mathbf{Set}$, a natural transformation $f\colon X \Longrightarrow Y$ is denoted as a family of functions $(f_s)_{s:S}$.
- We write 1 for the terminal set, and we write $*$ for the unique element in 1.

■

1. POLYNOMIALS

Definition 1.1. We define a category **Poly** by the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-,\mathbf{Set}]^{\mathrm{op}}} \mathfrak{CAT}$$

A ***polynomial*** is an object in **Poly**. We write

$$(-)_\circ\colon \mathbf{Poly} \longrightarrow \mathbf{Set}$$

for the fibration corresponding to the 2-functor. For each polynomial A, we write $A = \sum_{a:A_\circ} y^{A_a}$.

In particular, we mean by y the polynomial satisfying $(y)_\circ = 1$ and $(y)_* = 1$. ■

Definition 1.2. We define a category **Fam** by the Grothendieck construction of the following 2-functor.

$$\mathbf{Set}^{\mathrm{op}} \xrightarrow{[-,\mathbf{Set}]} \mathfrak{CAT}$$

Note that the fibration $\mathbf{Poly} \longrightarrow \mathbf{Set}$ is the fibrewise opposit of $\mathbf{Fam} \longrightarrow \mathbf{Set}$. ■

Definition 1.3. Consider the composite

$$\begin{array}{ccccc} [S,\mathbf{Set}]^{\mathrm{op}} \times \mathbf{Set} & \xrightarrow{\langle \mathtt{id}_{[S,\mathbf{Set}]}, \Delta\circ\mathtt{proj}\rangle} & [S,\mathbf{Set}]^{\mathrm{op}} \times [S,\mathbf{Set}] & \longrightarrow & [S,\mathbf{Set}] \\ \uplus & & & & \uplus \\ ((A_s)_{s:S}, T) & \longmapsto & & & ([A_s, T])_{s:S} \end{array}$$

where $\Delta\colon \mathbf{Set} \longrightarrow [S,\mathbf{Set}]$ in the former is the diagonal and the latter is the internal hom for $[S, Set]$. This is natural in $S \in \mathbf{Set}^{\mathrm{op}}$ and induces a functor $\mathbf{Poly} \times \mathbf{Set} \longrightarrow \mathbf{Fam}$. We write $\mathtt{rep}\colon \mathbf{Poly} \longrightarrow [\mathbf{Set}, \mathbf{Set}]$ for the currying of the composite

$$\begin{array}{ccccc} \mathbf{Poly} \times \mathbf{Set} & \longrightarrow & \mathbf{Fam} & \xrightarrow{\Sigma} & \mathbf{Set} \\ \uplus & & & & \uplus \\ (A, S) & \longmapsto & & & \sum_{a:A_\circ} [A_p, S] \end{array}$$

where $\sum$ is the coproduct (i.e., the left adjoint of the fibred terminal $\mathbf{Set} \longrightarrow \mathbf{Fam}$). ■

Proposition 1.4. $\mathtt{rep}$ is fully faithful. ♦

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Proof. Suppose we are given $P, Q \in \mathbf{Poly}$, a morphism $\alpha\colon \mathtt{rep}(P) \longrightarrow \mathtt{rep}(Q)$ in $[\mathbf{Set}, \mathbf{Set}]$, and $S \in \mathbf{Set}$. We have the following commutative square:

$$\begin{tikzcd}
\sum_{p\in P_\circ}[P_p, S] \arrow[r, "\alpha_S"] \arrow[d, "\mathtt{rep}(P)(!_S)"'] & \sum_{q\in Q_\circ}[Q_q, S] \arrow[d, "\mathtt{rep}(Q)(!_S)"] \\
P_\circ \arrow[r, "\alpha_1"'] & Q_\circ
\end{tikzcd}$$

where the left and the right vertical arrows are projections. Therefore, we have a family of morphisms $[P_p, S] \longrightarrow \left[Q_{\alpha_1(p)}, S\right]$ natural in $p \in P_\circ$ and $S \in \mathbf{Set}$. By applying the yoneda lemma, we obtain $Q_{\alpha_1(p)} \longrightarrow P_p$ for each $p \in P_\circ$, which induces a morphism $\bar{\alpha}\colon P \longrightarrow Q$ in $\mathbf{Poly}$. One can readily check that $\bar{\alpha}$ is a unique morphism satisfying $\mathtt{rep}(\bar{\alpha}) = \alpha$. □

Lemma 1.5. (The essential image of) $\mathtt{rep}$ is closed under the monoidal structure $(\circ, \mathtt{id}_{\mathbf{Set}})$ on $[\mathbf{Set}, \mathbf{Set}]$. ♦

Proof. $\mathtt{rep}(y) \cong \mathtt{id}_{\mathbf{Set}}$ is trivial. It suffices to show $\mathtt{rep}$ is closed under $\circ$. Let A, B be polynomials. Let us define another polynomial $A \triangleleft B$ as follows.

$$(A \triangleleft B)_\circ = \sum_{a:A_\circ} [A_a, B_\circ]$$

$$(A \triangleleft B)_{a,t} = \sum_{u:A_a} B_{t(u)}$$

for any $a : A_\circ$ and $t\colon A_a \longrightarrow B_\circ$. Then, observe the following bijections.

$$\begin{aligned}
\mathtt{rep}(A \triangleleft B)(S) &\cong \sum_{a:A_\circ} \sum_{t:[A_a,B_\circ]} \left[\sum_{u:A_a} B_{t(u)}, S\right] \\
&\cong \sum_{a:A_\circ} \sum_{t:[A_a,B_\circ]} \prod_{u:A_a} [B_{t(u)}, S] \\
&\cong \sum_{a:A_\circ} \prod_{u:A_a} \sum_{b:B_\circ} [B_b, S] \\
&\cong \sum_{a:A_\circ} \left[A_a, \sum_{b:B_\circ} [B_b, S]\right] \\
&= \mathtt{rep}(A)(\mathtt{rep}(B)(S))
\end{aligned}$$

The third bijection is the distributive law. Since these are natural in $S \in \mathbf{Set}$ this shows that $\mathtt{rep}(A \triangleleft B) \cong \mathtt{rep}(A) \circ \mathtt{rep}(B)$. □

Proposition 1.6. Let $\mathbf{V}$ be a category and $\mathbf{W}$ be a monoidal category. Suppose there is a fully faithful functor $F\colon \mathbf{V} \hookrightarrow \mathbf{W}$ that is closed under the monoidal structure on $\mathbf{W}$. Then there exists a monoidal structure on $\mathbf{V}$ that makes F strong monoidal, which is unique up to isomorphisms. ♦

Proof. folklore □

Proposition 1.7. $(\mathbf{Poly}, \triangleleft, y)$ is a monoidal category such that $\mathtt{rep}\colon \mathbf{Poly} \longrightarrow ([\mathbf{Set}, \mathbf{Set}], \circ, \mathtt{id}_{\mathbf{Set}})$ is a pseudo monoidal functor. ♦

Proof. Follows from Lemma 1.5 and Propositions 1.4 and 1.6. □

Remark 1.8. The functor $- \triangleleft -\colon \mathbf{Poly} \times \mathbf{Poly} \longrightarrow \mathbf{Poly}$ acts on morphisms as follows. Let $F\colon A \longrightarrow X$, $G\colon B \longrightarrow Y$ be morphisms in $\mathbf{Poly}$.

$$(F \triangleleft G)_\circ\colon \sum_{a:A_\circ} [A_a, B_\circ] \longrightarrow \sum_{x:X_\circ} [X_x, Y_\circ] : (a, t) \mapsto (F_\circ(a), F_a \mathbin{⨾} t \mathbin{⨾} G_\circ)$$

$$(F \triangleleft G)_{a,t}\colon \sum_{v:X_{F_\circ(a)}} Y_{G_\circ(t(F_a(v)))} \longrightarrow \sum_{u:A_a} B_{t(u)} : (v, r) \mapsto (F_a(v), G_{t(F_a(v))}(r))$$

■

Lemma 1.9. $\mathbf{Poly} \times \mathbf{Poly} \xrightarrow{\triangleleft} \mathbf{Poly}$ restricts to a functor $\mathbf{Poly} \times \mathbf{Set} \xrightarrow{\triangleleft} \mathbf{Set}$ and coincides with the uncurrying of $\mathtt{rep}$. ♦

Proof. □

2. Double categories

By ***double categories***, we mean psuedo ones.

Definition 2.1. A double category $\mathbb{D}$ is a ***company*** if its tight arrows have companions. $\mathbb{D}$ is an ***opcompany*** if $\mathbb{D}^{\mathsf{hop}}$ is a company. We say $\mathbb{D}$ is an ***equipment*** if $\mathbb{D}$ and $\mathbb{D}^{\mathsf{hop}}$ are companies. ■

Definition 2.2 ([Par24]). (It is defined for arbitrary bvdc whose underlying sdc is a opcompany.) Let $\mathbb{D}$ be an opcompany. Define another double category $\mathbb{R}\mathrm{et}(\mathbb{D})$ as follows.

- The vertical category is the same as $\mathbb{D}$.
- A loose arrow is a loose arrow in $\mathbb{D}$.
- A cell

$$\begin{tikzcd} A \arrow[r, "p", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "g"] \\ X \arrow[r, "q"', "\shortmid"{marking}] & Y \end{tikzcd} \quad \text{in } \mathbb{R}\mathrm{et}(\mathbb{D})$$

  is a cell in $\mathbb{D}$ of the following form.

$$\begin{tikzcd} X \arrow[r, "q", "\shortmid"{marking}] \arrow[d, equal] & Y \arrow[r, "g^*", "\shortmid"{marking}] \arrow[d, phantom, "\alpha"] & B \arrow[d, equal] \\ X \arrow[r, "f^*"', "\shortmid"{marking}] & A \arrow[r, "p"', "\shortmid"{marking}] & B \end{tikzcd}$$

- horizontal composite
- vertical composite ■

Proposition 2.3. The virtual double category of monoids in $\mathbb{R}\mathrm{et}(\mathbb{D})^{\mathsf{vop}}$ is representable if $\mathbb{D}$ has local reflexive coequalizers. ♦

Definition 2.4. Let $\mathbb{D}$ be a double category with local reflexive coequalizers.

- We write $\mathbb{M}\mathrm{od}(\mathbb{D})$ for the double category corresponding to the representable virtual double category of monoids in $\mathbb{R}\mathrm{et}(\mathbb{D})$.
- We write $\mathbb{M}\mathrm{od}_{\mathsf{co}}(\mathbb{D})$ for the vertical opposite of the double category corresponding to the representable virtual double category of monoids in $\mathbb{R}\mathrm{et}(\mathbb{D})^{\mathsf{vop}}$ obtained in Proposition 2.3. ■

Lemma 2.5. Let $\mathbb{D}$ be a double category with local reflexive coequalizers. There is an isomorphism $\mathfrak{H}(\mathbb{M}\mathrm{od}_{\mathsf{co}}(\mathbb{D})) \cong \mathfrak{H}(\mathbb{M}\mathrm{od}(\mathbb{D}))$. Where? ♦

Definition 2.6. ■

3. Polynomial in a generalised context

Definition 3.1. Let $\mathbf{C}$ be a category.

- A morphism $f\colon X \longrightarrow Y$ in $\mathbf{C}$ is ***carrable*** if $\Sigma_f\colon \mathbf{C}_{/X} \longrightarrow \mathbf{C}_{/Y}$ has a right adjoint f^*.
- A morphism $f\colon X \longrightarrow Y$ in $\mathbf{C}$ is ***exponentiable*** if f is carrable and $f^*\colon \mathbf{C}_{/Y} \longrightarrow \mathbf{C}_{/X}$ has a right adjoint Π_f. ■

Definition 3.2. A ***pre-representable map category (pRMC)*** is a pair $(\mathbf{R}, \mathtt{rep})$ such that $\mathbf{R}$ is a category and $\mathtt{rep}$ is a class of morphisms in $\mathbf{R}$ satisfying the followings.

- $\mathtt{rep}$ contains all identities and is closed under composition.
- All morphisms in $\mathtt{rep}$ are exponentiable. ■

Definition 3.3. Let $\mathbf{R} = (\mathbf{R}, \mathtt{rep})$ be a pRMC. For any object $X, Y \in \mathbf{R}$, define a category $\mathfrak{P}oly(\mathbf{R})(X, Y)$ as follows.

- An object is a diagram in $\mathbf{R}$ of the following form

$$X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y$$

  such that f is carrable and g is exponentiable.

■

References

[Par24] R. Paré. “Retrocells”. In: ***Theory Appl. Categ.*** 40 (2024), Paper No. 5, 130–179 (cit. on p. 3).

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