FONDIND

1. Preliminary

Definition 1.1. A Reedy category A consists of the following data:

- A small category A.
- A strict factorisation system (A_-, A_+) . We consider A_- and A_+ as wide subcategories of A.
- A function $d: \mathsf{Obj}\mathbb{A} \longrightarrow \omega$ that extends to identity-reflecting functors $d: \mathbb{A}^{\mathsf{op}}_{-} \longrightarrow \omega$ and $d: \mathbb{A}_{+} \longrightarrow \omega$, where the set of natural numbers ω is seen as a linearly ordered set.

Notation 1.2. Let \mathbb{A} be a Reedy category and $X \in \mathbf{Psh}[\mathbb{A}]$ be a presheaf. We view the category \mathbb{A} as a full subcategory of $\mathbf{Psh}[\mathbb{A}]$.

- We mean by \longrightarrow a morphism in \mathbb{A}_{-} , while by \longrightarrow , we mean a morphism in \mathbb{A}_{+} .
- A *cell* is a morphism $a \longrightarrow X$ in $\mathbf{Psh}[A]$ whose domain a is in A.
- A cell $x: a \longrightarrow X$ is degenerate if there is a factorisation $a \xrightarrow{\sigma} b \xrightarrow{y} X$ of x such that σ is non-identity morphism in \mathbb{A}_{-} .
- For each cell $x: a \longrightarrow X$, a face of x is a cell $y: b \longrightarrow X$ that factors as $b \xrightarrow{\iota} a \xrightarrow{x} X$ with ι in \mathbb{A}_+ , while a degeneracy of x is a cell $y: b \longrightarrow X$ that factors as $b \xrightarrow{\sigma} a \xrightarrow{x} X$ with σ in \mathbb{A}_- .

Lemma 1.3. Let \mathbb{A} be a Reedy category. For each presheaf $X \in \mathbf{Psh}[\mathbb{A}]$ and any cell $x \colon a \longrightarrow X$, there exits a factorisation $a \xrightarrow{\sigma} b \xrightarrow{y} X$ of x such that y is non-degenerate.

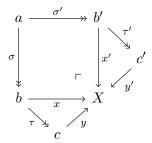
Proof. Consider the set of factorisations of x by morphisms in A_- . This is not empty because (id_a, x) gives one such factorisation. Therefore, there exists a factorisation (σ, y) whose degree of the domain of y is minimum among this set. It suffices to show y is non-degenerate. To this end, suppose y factors as $z \circ \tau$ where τ is a morphism in A_- . Since $(\tau \circ \sigma, z)$ gives a factorisation of x, the degree of the domain of z is not less than that of y. Therefore, τ must be an identity because $d: A_-^{op} \longrightarrow \omega$ reflects identity.

Proposition 1.4. For a Reedy category A, the following are equivalent.

- i) For each presheaf $X \in \mathbf{Psh}[\mathbb{A}]$, any cell $x \colon a \longrightarrow X$ factors uniquely as $a \xrightarrow{\sigma} b \xrightarrow{y} X$ where σ is in \mathbb{A}_- and y is non-degenerate.
- ii) Any span in \mathbb{A}_{-} admits a pushout in \mathbb{A} that is preserved by the yoneda $\mathbb{A} \hookrightarrow \mathbf{Psh}[\mathbb{A}]$.
- iii) Any span in A_{-} admits an absolute pushout in A_{-} .

Proof. The equivalence of ii and iii follows from a folklore result: a small colimit is absolute if and only if it is preserved by the cocompletion.

[i \Rightarrow ii] Suppose i and that we are given a span $b \stackrel{\sigma}{\longleftarrow} a \stackrel{\sigma'}{\longrightarrow} b'$ in \mathbb{A}_- , and take a pushout in $\mathbf{Psh}[\mathbb{A}]$ as in the diagram below. Since the yoneda embedding is fully faithful, it suffices to show the pushout is representable. Let us factorise x and x' by i as follows so that the cells y and y' are non-degenerate.



By the uniqueness of the factorisation of $x \circ \sigma$, we have $(\tau \circ \sigma, y) = (\tau' \circ \sigma', y')$. By the universality of the pushout applying to the square $\tau \circ \sigma = \tau' \circ \sigma'$, we obtain a section u of y = y'. Therefore,

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we have a factorisation $(u \circ y, y)$ of y. Since y is non-degenerate, through factorising $u \circ y$ by the factorisation system, we conclude $u \circ y$ is an endo-morphism in \mathbb{A}_+ , which shows $u \circ y = \mathrm{id}_c$ and that X is representable.

[ii \Rightarrow i] By Lemma 1.3, it suffices to show the uniqueness of such factorisation (σ, y) for any cell x. Suppose ii and that we are given a cell x and its two factorisations (σ, y) and (σ', y') where y and y' are non-degenerate. By taking the pushout of the span formed by σ and σ' , we obtain a square $\tau \circ \sigma = \tau' \circ \sigma'$ in \mathbb{A} and factorisations (τ, z) and (τ', z') of y and y' respectively. Since the left class of an orthogonal factorisation system must be closed under pushout, τ and τ' are in \mathbb{A}_- . Moreover, since y and y' are non-degenerate, τ and τ' are identities, and this shows $(\sigma, y) = (\sigma', y')$.

Definition 1.5. A Reedy category is *elegant* if the equivalent conditions in Proposition 1.4 hold. ◆

Proposition 1.6. For an elegant Reedy category \mathbb{A} , the left class \mathbb{A}_{-} coincides with the wide subcategory of split epimorphisms.

Proof.

2. FONDIND

Let A be a Reedy category. The following definition is after Yuki Maehara.

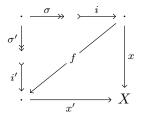
Definition 2.1. A presheaf X is fondind if any face of a non-degenerate cell is non-degenerate: i.e., X is fondind if for each factorisation $b \stackrel{\iota}{\longrightarrow} a \stackrel{x}{\longrightarrow} X$ of a cell $y : b \longrightarrow X$ with ι in \mathbb{A}_+ , the cell y is non-degenerate whenever x is as well.

Proposition 2.2. The class of fondind presheaves is closed under limits and subobjects.

Proof. It suffices to show the class is closed under products and subobjects. straightforward \Box

Note that for each presheaf X, the category of elements \mathbb{A}/X has as its objects cells of X. We write $(\mathbb{A}/X)_{nd}$ for the full subcategory of \mathbb{A}/X consisting of all non-degenerate cells.

Lemma 2.3. Let X be a presheaf. Consider the following commutative diagram in Psh[A].



Suppose that x is non-degenerate. Then we have $\sigma = \sigma'$ and $x \circ i = x' \circ i'$.

Proof. Considering the factorisation of the morphism f in \mathbb{A} , we can conclude f is a morphism in \mathbb{A}_+ since x is non-degenerate. Now since both (σ', i') and $(\sigma, f \circ i)$ give the $(\mathbb{A}_-, \mathbb{A}_+)$ -factorisation of the same morphism, they coincide. This implies $\sigma = \sigma'$ and $x \circ i = x' \circ f \circ i' = x' \circ i'$.

Proposition 2.4. A presheaf $X \in \mathbf{Psh}[\mathbb{A}]$ is fondind if $(\mathbb{A}/X)_{nd}$ is a final full subcategory of \mathbb{A}/X .

Proof. Suppose $(\mathbb{A}/X)_{nd}$ is a final full subcategory of \mathbb{A}/X . To show X is fondind, take a factorisation $a \xrightarrow{i} b \xrightarrow{x} X$ of a cell $a \xrightarrow{y} X$ by a morphism i in \mathbb{A}_+ and a non-degenerate cell x. We will show y is non-degenerate. To this end, consider another factorisation $a \xrightarrow{\sigma} c \xrightarrow{z} X$ of y obtained by Lemma 1.3. Since z is non-degenerate, it suffices to show $\sigma = id$. Since $(\mathbb{A}/X)_{nd}$ is final in \mathbb{A}/X , the comma category $y \downarrow (\mathbb{A}/X)_{nd}$ is connected. In particular, (σ, z) and (i, x) are connected by a zigzag in this comma category. Now, iterated use of Lemma 2.3 to this zigzag shows $\sigma = id$.

Proposition 2.5. The converse of Proposition 2.4 is true when \mathbb{A} is elegant; i.e., if \mathbb{A} is elegant, a presheaf X is fondind if and only if $(\mathbb{A}/X)_{nd}$ is final in \mathbb{A}/X .

Proof. Suppose we are given a fondind presheaf X and a cell $a \xrightarrow{x} X$. It suffices to show the comma category $x \downarrow (A/X)_{nd}$ is connected. Since it is inhabited by Lemma 2.3, it suffices to show that any two factorisations of x by non-degenerate cells are connected in this category. To this end, take two

factorisations $a \xrightarrow{f} \xrightarrow{y} X$ and $a \xrightarrow{f'} \xrightarrow{y'} X$ of x where y and y' are non-degenerate. Moreover, let us

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decompose f and f' by the factorisation system as $f = i \circ \sigma$ and $f' = i' \circ \sigma'$. Since X is fondind, $y \circ i$ and $y' \circ i'$ are non-degenerate, and hence $(\sigma, y \circ i)$ and $(\sigma', y' \circ i')$ become objects of the comma category. Moreover, since $\mathbb A$ is elegant, those two factorisations coincide, and now we have a span $(f, y) \stackrel{i}{\longleftarrow} (\sigma, y \circ i) = (\sigma', y' \circ i') \stackrel{i'}{\longrightarrow} (f', y')$ in $x \downarrow (\mathbb A/X)_{\rm nd}$.