FONDIND

1. Preliminary

Definition 1.1. A *Reedy category* $\mathbb{A}$ consists of the following data:

- A small category $\mathbb{A}$.
- A strict factorisation system $(\mathbb{A}_-, \mathbb{A}_+)$. We consider $\mathbb{A}_-$ and $\mathbb{A}_+$ as wide subcategories of $\mathbb{A}$.
- A function $d\colon \mathsf{Obj}\mathbb{A} \longrightarrow \omega$ that extends to identity-reflecting functors $d\colon \mathbb{A}_-^{\mathsf{op}} \longrightarrow \omega$ and $d\colon \mathbb{A}_+ \longrightarrow \omega$, where the set of natural numbers ω is seen as a linearly ordered set. ♦

Notation 1.2. Let $\mathbb{A}$ be a Reedy category and $X \in \mathbf{Psh}[\mathbb{A}]$ be a presheaf. We view the category $\mathbb{A}$ as a full subcategory of $\mathbf{Psh}[\mathbb{A}]$.

- We mean by $\twoheadrightarrow$ a morphism in $\mathbb{A}_-$, while by $\rightarrowtail$, we mean a morphism in $\mathbb{A}_+$.
- A *cell* is a morphism $a \longrightarrow X$ in $\mathbf{Psh}[\mathbb{A}]$ whose domain a is in $\mathbb{A}$.
- A cell $x\colon a \longrightarrow X$ is *degenerate* if there is a factorisation $a \overset{\sigma}{\twoheadrightarrow} b \xrightarrow{y} X$ of x such that σ is non-identity morphism in $\mathbb{A}_-$.
- For each cell $x\colon a \longrightarrow X$, a *face* of x is a cell $y\colon b \longrightarrow X$ that factors as $b \overset{\iota}{\rightarrowtail} a \xrightarrow{x} X$ with ι in $\mathbb{A}_+$, while a *degeneracy* of x is a cell $y\colon b \longrightarrow X$ that factors as $b \overset{\sigma}{\twoheadrightarrow} a \xrightarrow{x} X$ with σ in $\mathbb{A}_-$. ♦

Lemma 1.3. Let $\mathbb{A}$ be a Reedy category. For each presheaf $X \in \mathbf{Psh}[\mathbb{A}]$ and any cell $x\colon a \longrightarrow X$, there exits a factorisation $a \overset{\sigma}{\twoheadrightarrow} b \xrightarrow{y} X$ of x such that y is non-degenerate.

Proof. Consider the set of factorisations of x by morphisms in $\mathbb{A}_-$. This is not empty because $(\mathtt{id}_a, x)$ gives one such factorisation. Therefore, there exists a factorisation (σ, y) whose degree of the domain of y is minimum among this set. It suffices to show y is non-degenerate. To this end, suppose y factors as $z \circ \tau$ where τ is a morphism in $\mathbb{A}_-$. Since $(\tau \circ \sigma, z)$ gives a factorisation of x, the degree of the domain of z is not less than that of y. Therefore, τ must be an identity because $d\colon \mathbb{A}_-^{\mathsf{op}} \longrightarrow \omega$ reflects identity. □

Proposition 1.4. For a Reedy category $\mathbb{A}$, the following are equivalent.

i) For each presheaf $X \in \mathbf{Psh}[\mathbb{A}]$, any cell $x\colon a \longrightarrow X$ factors uniquely as $a \overset{\sigma}{\twoheadrightarrow} b \xrightarrow{y} X$ where σ is in $\mathbb{A}_-$ and y is non-degenerate.
ii) Any span in $\mathbb{A}_-$ admits a pushout in $\mathbb{A}$ that is preserved by the yoneda $\mathbb{A} \hookrightarrow \mathbf{Psh}[\mathbb{A}]$.
iii) Any span in $\mathbb{A}_-$ admits an absolute pushout in $\mathbb{A}$.

Proof. The equivalence of ii and iii follows from a folklore result: a small colimit is absolute if and only if it is preserved by the cocompletion.

[i⇒ii] Suppose i and that we are given a span $b \overset{\sigma}{\twoheadleftarrow} a \overset{\sigma'}{\twoheadrightarrow} b'$ in $\mathbb{A}_-$, and take a pushout in $\mathbf{Psh}[\mathbb{A}]$ as in the diagram below. Since the yoneda embedding is fully faithful, it suffices to show the pushout is representable. Let us factorise x and x' by i as follows so that the cells y and y' are non-degenerate.

$$\begin{tikzcd}
a \arrow[rr, two heads, "\sigma'"] \arrow[dd, two heads, "\sigma"'] & & b' \arrow[dd, "x'"] \arrow[dr, two heads, "\tau'"] & \\
& & & c' \arrow[dl, "y'"] \\
b \arrow[rr, "x"] \arrow[dr, two heads, "\tau"'] & & X \arrow[ul, phantom, "\ulcorner", very near start] & \\
& c \arrow[ur, "y"'] & &
\end{tikzcd}$$

By the uniqueness of the factorisation of $x \circ \sigma$, we have $(\tau \circ \sigma, y) = (\tau' \circ \sigma', y')$. By the universality of the pushout applying to the square $\tau \circ \sigma = \tau' \circ \sigma'$, we obtain a section u of $y = y'$. Therefore,

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we have a factorisation $(u \circ y, y)$ of y. Since y is non-degenerate, through factorising $u \circ y$ by the factorisation system, we conclude $u \circ y$ is an endo-morphism in $\mathbb{A}_+$, which shows $u \circ y = \mathtt{id}_c$ and that X is representable.
[ii⇒i] By Lemma 1.3, it suffices to show the uniqueness of such factorisation (σ, y) for any cell x. Suppose ii and that we are given a cell x and its two factorisations (σ, y) and (σ', y') where y and y' are non-degenerate. By taking the pushout of the span formed by σ and σ', we obtain a square $\tau \circ \sigma = \tau' \circ \sigma'$ in $\mathbb{A}$ and factorisations (τ, z) and (τ', z') of y and y' respectively. Since the left class of an orthogonal factorisation system must be closed under pushout, τ and τ' are in $\mathbb{A}_-$. Moreover, since y and y' are non-degenerate, τ and τ' are identities, and this shows $(\sigma, y) = (\sigma', y')$. □

Definition 1.5. A Reedy category is *elegant* if the equivalent conditions in Proposition 1.4 hold. ⧫

Proposition 1.6. For an elegant Reedy category $\mathbb{A}$, the left class $\mathbb{A}_-$ coincides with the wide subcategory of split epimorphisms.

Proof. □

2. FONDIND

Let $\mathbb{A}$ be a Reedy category. The following definition is after Yuki Maehara.

Definition 2.1. A presheaf X is *fondind* if any face of a non-degenerate cell is non-degenerate: i.e., X is fondind if for each factorisation $b \overset{\iota}{\rightarrowtail} a \xrightarrow{x} X$ of a cell $y\colon b \longrightarrow X$ with ι in $\mathbb{A}_+$, the cell y is non-degenerate whenever x is as well. ⧫

Proposition 2.2. The class of fondind presheaves is closed under limits and subobjects.

Proof. It suffices to show the class is closed under products and subobjects. straightforward □

Note that for each presheaf X, the category of elements $\mathbb{A}/X$ has as its objects cells of X. We write $(\mathbb{A}/X)_{\mathtt{nd}}$ for the full subcategory of $\mathbb{A}/X$ consisting of all non-degenerate cells.

Lemma 2.3. Let X be a presheaf. Consider the following commutative diagram in $\mathbf{Psh}[\mathbb{A}]$.

$$\begin{tikzcd}
\cdot \arrow[r, two heads, "\sigma"] \arrow[d, two heads, "\sigma'"'] & {} \arrow[r, tail, "i"] & \cdot \arrow[dd, "x"] \arrow[ddll, "f"] \\
{} \arrow[d, tail, "i'"'] & & \\
\cdot \arrow[rr, "x'"'] & & X
\end{tikzcd}$$

Suppose that x is non-degenerate. Then we have $\sigma = \sigma'$ and $x \circ i = x' \circ i'$.

Proof. Considering the factorisation of the morphism f in $\mathbb{A}$, we can conclude f is a morphism in $\mathbb{A}_+$ since x is non-degenerate. Now since both (σ', i') and $(\sigma, f \circ i)$ give the $(\mathbb{A}_-, \mathbb{A}_+)$-factorisation of the same morphism, they coincide. This implies $\sigma = \sigma'$ and $x \circ i = x' \circ f \circ i' = x' \circ i'$. □

Proposition 2.4. A presheaf $X \in \mathbf{Psh}[\mathbb{A}]$ is fondind if $(\mathbb{A}/X)_{\mathtt{nd}}$ is a final full subcategory of $\mathbb{A}/X$.

Proof. Suppose $(\mathbb{A}/X)_{\mathtt{nd}}$ is a final full subcategory of $\mathbb{A}/X$. To show X is fondind, take a factorisation $a \overset{i}{\rightarrowtail} b \xrightarrow{x} X$ of a cell $a \xrightarrow{y} X$ by a morphism i in $\mathbb{A}_+$ and a non-degenerate cell x. We will show y is non-degenerate. To this end, consider another factorisation $a \overset{\sigma}{\twoheadrightarrow} c \xrightarrow{z} X$ of y obtained by Lemma 1.3. Since z is non-degenerate, it suffices to show $\sigma = \mathtt{id}$. Since $(\mathbb{A}/X)_{\mathtt{nd}}$ is final in $\mathbb{A}/X$, the comma category $y \downarrow (\mathbb{A}/X)_{\mathtt{nd}}$ is connected. In particular, (σ, z) and (i, x) are connected by a zigzag in this comma category. Now, iterated use of Lemma 2.3 to this zigzag shows $\sigma = \mathtt{id}$. □

Proposition 2.5. The converse of Proposition 2.4 is true when $\mathbb{A}$ is elegant; i.e., if $\mathbb{A}$ is elegant, a presheaf X is fondind if and only if $(\mathbb{A}/X)_{\mathtt{nd}}$ is final in $\mathbb{A}/X$.

Proof. Suppose we are given a fondind presheaf X and a cell $a \xrightarrow{x} X$. It suffices to show the comma category $x \downarrow (\mathbb{A}/X)_{\mathtt{nd}}$ is connected. Since it is inhabited by Lemma 2.3, it suffices to show that any two factorisations of x by non-degenerate cells are connected in this category. To this end, take two factorisations $a \xrightarrow{f} \xrightarrow{y} X$ and $a \xrightarrow{f'} \xrightarrow{y'} X$ of x where y and y' are non-degenerate. Moreover, let us decompose f and f' by the factorisatiion system as $f = i \circ \sigma$ and $f' = i' \circ \sigma'$. Since X is fondind, $y \circ i$ and $y' \circ i'$ are non-degenerate, and hence $(\sigma, y \circ i)$ and $(\sigma', y' \circ i')$ become objects of the comma category. Moreover, since $\mathbb{A}$ is elegant, those two factorisations coincide, and now we have a span $(f, y) \xleftarrow{i} (\sigma, y \circ i) = (\sigma', y' \circ i') \xrightarrow{i'} (f', y')$ in $x \downarrow (\mathbb{A}/X)_{\mathtt{nd}}$. $\square$