WHEN A COMPLEX IS CONSTRUCTED FROM NON-DEGENERATE CELLS (IN A SENSE)

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1. 導入

Simplicial set の議論をするとき、ある一つの simplicial set の simplex たちを議論するのに人々はしばしばダイアグラムを使います。このダイアグラムは正確には simplicial set のダイアグラムというよりはその non-degenerate cell たちからなる semi-simplicial set のダイアグラムです。それでもそのようなダイアグラムを用いた議論が有用であるという事実は、ある意味で simplicial set は non-degenerate な cell たちだけから構成されている、ということを示唆しています。このことは、厳密には次の事実によって確かめられます: 任意の simplicial set は non-degenerate な simplex で index された boundary inclusion の colimit で表示できる (skeletal filtration).

一方で、simplicial set の圏 **sSet** は presheaf category なので、任意の simplicial set X は simplex の colimit をつかって $X \cong \mathrm{colim}(\Delta/X \to \Delta \to \mathbf{sSet})$ とかけます。ここで category of elements Δ/X は X の cell のなす圏です (Yoneda lemma)。上の観察から、「この colimit は Δ/X の non-degenerate cell のなす full subcategory に制限できるか？」という問いが自然であることがわかるでしょう。実はこれは一般には成立しません。例えば、一つの 0-simplex とその上の degenerate 1-cell を boundary にもつ一つの 2-simplex からなる simplicial set X を考えると $\mathrm{colim}((\Delta/X)_{\mathtt{nd}} \to \Delta/X \to \Delta \to \mathbf{sSet})$ は non-degenerate な 1-cell を持つことになり、もとの X と異なるものになります。

この記事では、この colimit が X を復元するための十分条件である「non-degenerate cell たちが Δ/X の final full subcategory をなす」という条件が fondind(Face Of Non-Degenerate Is Non-Degenerate) とここで呼ぶことにする条件と同値になることを、任意の elegant Reedy category に Δ を一般化した上で示します。このことは fondind な simplicial set がある強い意味で non-degenerate な cell たちだけからなっていることを意味します。さらにこのクラスは (many-sorted) quasivariety をなすことがわかります。

2. Preliminary (elegant Reedy category)

This section is mainly based on [BR13, Section 3].

Definition 2.1. A *Reedy category* $\mathbb{A}$ consists of the following data:

- A small category $\mathbb{A}$.
- A strict factorisation system $(\mathbb{A}_-, \mathbb{A}_+)$. We consider $\mathbb{A}_-$ and $\mathbb{A}_+$ as wide subcategories of $\mathbb{A}$.
- A function $d\colon \mathsf{Obj}\mathbb{A} \longrightarrow \omega$ that extends to identity-reflecting functors $d\colon \mathbb{A}_-^{\mathsf{op}} \longrightarrow \omega$ and $d\colon \mathbb{A}_+ \longrightarrow \omega$, where the set of natural numbers ω is seen as a linearly ordered set. ♦

Notation 2.2. Let $\mathbb{A}$ be a Reedy category and $X \in \mathbf{Psh}[\mathbb{A}]$ be a presheaf. We view the category $\mathbb{A}$ as a full subcategory of $\mathbf{Psh}[\mathbb{A}]$.

- We mean by $\twoheadrightarrow$ a morphism in $\mathbb{A}_-$, while by $\rightarrowtail$, we mean a morphism in $\mathbb{A}_+$.
- A *cell* is a morphism $a \longrightarrow X$ in $\mathbf{Psh}[\mathbb{A}]$ whose domain a is in $\mathbb{A}$.
- A cell $x\colon a \longrightarrow X$ is *degenerate* if there is a factorisation $a \xrightarrow{\sigma} b \xrightarrow{y} X$ of x such that σ is non-identity morphism in $\mathbb{A}_-$.
- For each cell $x\colon a \longrightarrow X$, a *face* of x is a cell $y\colon b \longrightarrow X$ that factors as $b \xrightarrow{\iota} a \xrightarrow{x} X$ with ι in $\mathbb{A}_+$, while a *degeneracy* of x is a cell $y\colon b \longrightarrow X$ that factors as $b \xrightarrow{\sigma} a \xrightarrow{x} X$ with σ in $\mathbb{A}_-$. ♦

Lemma 2.3. Let $\mathbb{A}$ be a Reedy category. For each presheaf $X \in \mathbf{Psh}[\mathbb{A}]$ and any cell $x\colon a \longrightarrow X$, there exits a factorisation $a \xrightarrow{\sigma} b \xrightarrow{y} X$ of x such that y is non-degenerate.

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Proof. Consider the set of factorisations of x by morphisms in $\mathbb{A}_-$. This is not empty because $(\mathtt{id}_a, x)$ gives one such factorisation. Therefore, there exists a factorisation (σ, y) whose degree of the domain of y is minimum among this set. It suffices to show y is non-degenerate. To this end, suppose y factors as $z \circ \tau$ where τ is a morphism in $\mathbb{A}_-$. Since $(\tau \circ \sigma, z)$ gives a factorisation of x, the degree of the domain of z is not less than that of y. Therefore, τ must be an identity because $d\colon \mathbb{A}_-^{\mathsf{op}} \longrightarrow \omega$ reflects identity. □

Proposition 2.4. For a Reedy category $\mathbb{A}$, the following are equivalent.

i) For each presheaf $X \in \mathbf{Psh}[\mathbb{A}]$, any cell $x\colon a \longrightarrow X$ factors uniquely as $a \overset{\sigma}{\twoheadrightarrow} b \overset{y}{\longrightarrow} X$ where σ is in $\mathbb{A}_-$ and y is non-degenerate.
ii) Any span in $\mathbb{A}_-$ admits a pushout in $\mathbb{A}$ that is preserved by the yoneda $\mathbb{A} \hookrightarrow \mathbf{Psh}[\mathbb{A}]$.
iii) Any span in $\mathbb{A}_-$ admits an absolute pushout in $\mathbb{A}$.

Proof. The equivalence of ii and iii follows from a folklore result: a small colimit is absolute if and only if it is preserved by the cocompletion.

[i⇒ii] Suppose i and that we are given a span $b \overset{\sigma}{\twoheadleftarrow} a \overset{\sigma'}{\twoheadrightarrow} b'$ in $\mathbb{A}_-$, and take a pushout in $\mathbf{Psh}[\mathbb{A}]$ as in the diagram below. Since the yoneda embedding is fully faithful, it suffices to show the pushout is representable. Let us factorise x and x' by i as follows so that the cells y and y' are non-degenerate.

$$\begin{tikzcd}
a \arrow[rr, two heads, "\sigma'"] \arrow[dd, two heads, "\sigma"'] & & b' \arrow[dd, "x'"] \arrow[dr, two heads, "\tau'"] & \\
& & & c' \arrow[dl, "y'"] \\
b \arrow[rr, "x"'] \arrow[dr, two heads, "\tau"'] & & X \arrow[uull, phantom, "\ulcorner", very near start] & \\
& c \arrow[ur, "y"'] & &
\end{tikzcd}$$

By the uniqueness of the factorisation of $x \circ \sigma$, we have $(\tau \circ \sigma, y) = (\tau' \circ \sigma', y')$. By the universality of the pushout applying to the square $\tau \circ \sigma = \tau' \circ \sigma'$, we obtain a section u of $y = y'$. Therefore, we have a factorisation $(u \circ y, y)$ of y. Since y is non-degenerate, through factorising $u \circ y$ by the factorisation system, we conclude $u \circ y$ is an endo-morphism in $\mathbb{A}_+$, which shows $u \circ y = \mathtt{id}_c$ and that X is representable.

[ii⇒i] By Lemma 2.3, it suffices to show the uniqueness of such factorisation (σ, y) for any cell x. Suppose ii and that we are given a cell x and its two factorisations (σ, y) and (σ', y') where y and y' are non-degenerate. By taking the pushout of the span formed by σ and σ', we obtain a square $\tau \circ \sigma = \tau' \circ \sigma'$ in $\mathbb{A}$ and factorisations (τ, z) and (τ', z') of y and y' respectively. Since the left class of an orthogonal factorisation system must be closed under pushout, τ and τ' are in $\mathbb{A}_-$. Moreover, since y and y' are non-degenerate, τ and τ' are identities, and this shows $(\sigma, y) = (\sigma', y')$. □

Definition 2.5. A Reedy category is *elegant* if the equivalent conditions in Proposition 2.4 hold. ♦

Fact 2.6 ([BR13, Proposition 4.2])**.** A Reedy category is elegant if the following condition holds:

(EZ) Any morphism in $\mathbb{A}_-$ is a split epi, and for any parallel morphisms $f\colon a \twoheadrightarrow b$ and $g\colon a \twoheadrightarrow b$ in $\mathbb{A}_-$, $f = g$ holds if any morphism $b \longrightarrow a$ is a section of f and g simultaneously. ♦

Example 2.7. The category of simplices Δ is a Reedy category by the (surjection, injection)-factorisation system and the ordinary degree function. Moreover, this is elegant because Δ satisfies the condition in Fact 2.6. ♦

3. FONDIND

Let $\mathbb{A}$ be a Reedy category. The following definition is after Yuki Maehara.

Definition 3.1. A presheaf X is *fondind* if any face of a non-degenerate cell is non-degenerate: i.e., X is fondind if for each factorisation $b \overset{\iota}{\rightarrowtail} a \overset{x}{\longrightarrow} X$ of a cell $y\colon b \longrightarrow X$ with ι in $\mathbb{A}_+$, the cell y is non-degenerate whenever x is as well. ♦

Proposition 3.2. The class of fondind presheaves is closed under limits and subobjects.

Proof. A cell in a subobject of a simplicial set X is non-degenerate if and only if it is non-degenerate in X. On the other hand, a non-degenerate cell $\langle x_i \rangle_i \colon a \longrightarrow \prod_{i \in I} X_i$ is non-degenerate if and only if x_i is non-degenerate for each i. Those observations immediately show that being fondind is closed under products and subobjects, which shows the proposition. □

Remark 3.3. The full subcategory of $\mathbf{Psh}[\mathbb{A}]$ spanned by fondind objects is a quasivariety in the sense of [AHS90, p. 16.12]. This follows from the above proposition using [AHS90, Theorem 16.8 and Theorem 16.14]. ♦

Note that for each presheaf X, the category of elements $\mathbb{A}/X$ has as its objects cells of X. We write $(\mathbb{A}/X)_{\mathtt{nd}}$ for the full subcategory of $\mathbb{A}/X$ consisting of all non-degenerate cells.

Lemma 3.4. Let X be a presheaf. Consider the following commutative diagram in $\mathbf{Psh}[\mathbb{A}]$.

$$\begin{tikzcd}
\cdot \arrow[r, two heads, "\sigma"] \arrow[d, "\sigma'"'] & {} \arrow[r, tail, "i"] & \cdot \arrow[dd, "x"] \arrow[ddll, "f"] \\
{} \arrow[d, tail, "i'"'] & & \\
\cdot \arrow[rr, "x'"'] & & X
\end{tikzcd}$$

Suppose that x is non-degenerate. Then we have $\sigma = \sigma'$ and $x \circ i = x' \circ i'$.

Proof. Considering the factorisation of the morphism f in $\mathbb{A}$, we can conclude f is a morphism in $\mathbb{A}_+$ since x is non-degenerate. Now since both (σ', i') and $(\sigma, f \circ i)$ give the $(\mathbb{A}_-, \mathbb{A}_+)$-factorisation of the same morphism, they coincide. This implies $\sigma = \sigma'$ and $x \circ i = x' \circ f \circ i' = x' \circ i'$. □

Proposition 3.5. A presheaf $X \in \mathbf{Psh}[\mathbb{A}]$ is fondind if $(\mathbb{A}/X)_{\mathtt{nd}}$ is a final full subcategory of $\mathbb{A}/X$.

Proof. Suppose $(\mathbb{A}/X)_{\mathtt{nd}}$ is a final full subcategory of $\mathbb{A}/X$. To show X is fondind, take a factorisation $a \overset{i}{\rightarrowtail} b \xrightarrow{x} X$ of a cell $a \xrightarrow{y} X$ by a morphism i in $\mathbb{A}_+$ and a non-degenerate cell x. We will show y is non-degenerate. To this end, consider another factorisation $a \overset{\sigma}{\twoheadrightarrow} c \xrightarrow{z} X$ of y obtained by Lemma 2.3. Since z is non-degenerate, it suffices to show $\sigma = \mathtt{id}$. Since $(\mathbb{A}/X)_{\mathtt{nd}}$ is final in $\mathbb{A}/X$, the comma category $y \downarrow (\mathbb{A}/X)_{\mathtt{nd}}$ is connected. In particular, (σ, z) and (i, x) are connected by a zigzag in this comma category. Now, iterated use of Lemma 3.4 to this zigzag shows $\sigma = \mathtt{id}$. □

Proposition 3.6. The converse of Proposition 3.5 is true when $\mathbb{A}$ is elegant; i.e., if $\mathbb{A}$ is elegant, a presheaf X is fondind if and only if $(\mathbb{A}/X)_{\mathtt{nd}}$ is final in $\mathbb{A}/X$.

Proof. Suppose we are given a fondind presheaf X and a cell $a \xrightarrow{x} X$. It suffices to show the comma category $x \downarrow (\mathbb{A}/X)_{\mathtt{nd}}$ is connected. Since it is inhabited by Lemma 3.4, it suffices to show that any two factorisations of x by non-degenerate cells are connected in this category. To this end, take two factorisations $a \xrightarrow{f} \xrightarrow{y} X$ and $a \xrightarrow{f'} \xrightarrow{y'} X$ of x where y and y' are non-degenerate. Moreover, let us decompose f and f' by the factorisatiion system as $f = i \circ \sigma$ and $f' = i' \circ \sigma'$. Since X is fondind, $y \circ i$ and $y' \circ i'$ are non-degenerate, and hence $(\sigma, y \circ i)$ and $(\sigma', y' \circ i')$ become objects of the comma category. Moreover, since $\mathbb{A}$ is elegant, those two factorisations coincide, and now we have a span $(f, y) \xleftarrow{i} (\sigma, y \circ i) = (\sigma', y' \circ i') \xrightarrow{i'} (f', y')$ in $x \downarrow (\mathbb{A}/X)_{\mathtt{nd}}$. □

References

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[BR13] J. E. Bergner and C. Rezk. "Reedy categories and the Θ-construction". In: *Math. Z.* 274.1-2 (2013), pp. 499–514. DOI: 10.1007/s00209-012-1082-0. URL: https://doi.org/10.1007/s00209-012-1082-0 (cit. on pp. 1, 2).