

1 Pullback

Theorem 1.1. *Suppose we have two joined commuting squares like:*

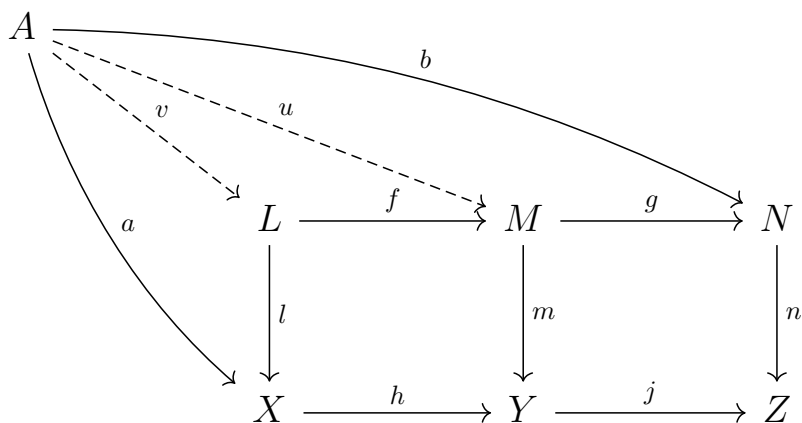
$$\begin{array}{ccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N \\ \downarrow l & & \downarrow m & & \downarrow n \\ X & \xrightarrow{h} & Y & \xrightarrow{j} & Z \end{array}$$

Then:

1. *The outer rectangle is a pullback square if two inner squares are pullback squares.*
2. *The inner-left square is a pullback square if the over rectangle and the inner-right square are pullback squares.*

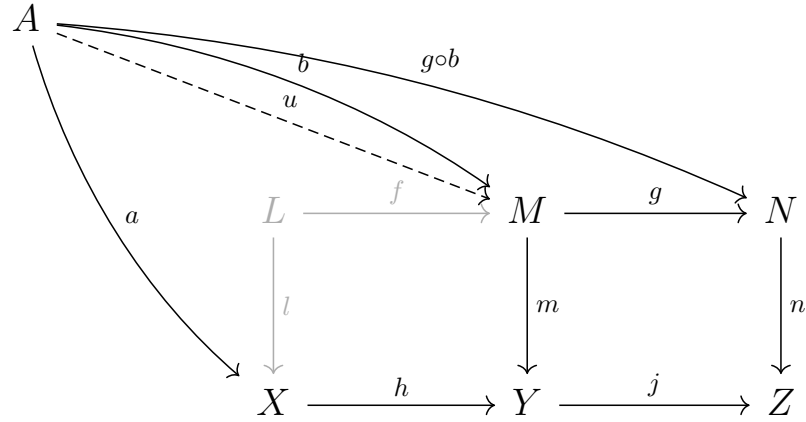
Proof.

- For any (A, a, b) such that $j \circ h \circ a = n \circ b$, then there is a unique $u : A \rightarrow M$ such that $h \circ a = m \circ u$ and $b = g \circ u$. Then there is a unique $v : A \rightarrow L$ such that $l \circ a = v$ and $f \circ v = u$, which makes (A, a, b) against to the outer rectangle commutes.

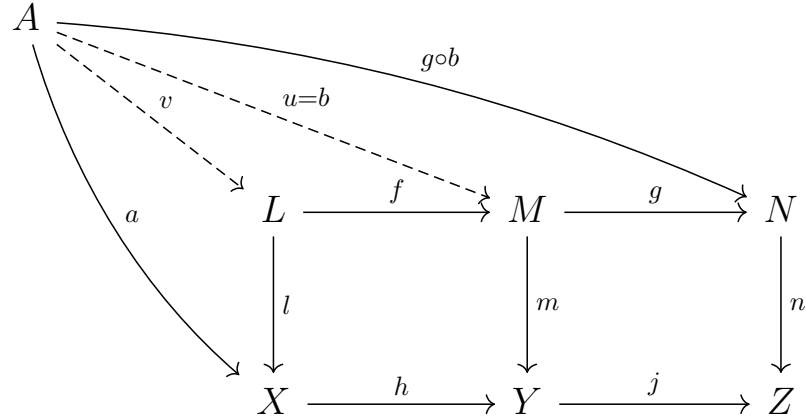


2. For any (A, a, b) such that $hoa = mob$, consider the inner-right pullback,

then we have a unique $u : A \rightarrow M$ such that the diagram commutes:



However, if we replace u with b , we have $g \circ b = g \circ b$ and $h \circ a = m \circ b$, that means b can do u 's job, but we know u is unique, so $b = u$. Now consider the outer pullback, we have a unique $v : A \rightarrow L$ such that the diagram commutes:



That is, $l \circ v = a$ and $g \circ f \circ v = g \circ b$, we claim that v is the unique factorization from $(A, a, u = b)$ to (L, l, f) . It is obvious that $l \circ v = a$, we need to show $f \circ v = u = b$. We may use the trick we just used, we can see that $g \circ f \circ v = g \circ u$ and $m \circ f \circ v = h \circ l \circ v = h \circ a$. So $f \circ v$ can do b 's job, so $f \circ v = b$.

For any arrow $w : A \rightarrow L$ such that $l \circ a = w$ and $f \circ w = b$, then we have also $g \circ f \circ w = g \circ b$, which implies w is the unique arrow from $A \rightarrow L$ such that the outer diagram commutes, so $w = v$.

□