1 Product

Definition 1.1 (Product). *Let $\mathcal{C}$ a category and $A, B \in \mathcal{C}$, $(A \times B, \pi_0, \pi_1)$ forms a product of A and B where $A \times B \in \mathcal{C}$, $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$, if for any $X \in \mathcal{C}$ with $\pi'_0 : X \to A$ and $\pi'_1 : X \to B$, there is a unique arrow $u : X \to A \times B$ such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\pi'_1"] \arrow[ddr, bend right, "\pi'_0"'] & & \\
& A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & B \\
& A &
\end{tikzcd}$$

Furthermore, a product of A and B is a limit of diagram:

$$\begin{tikzcd} (A & & B) \end{tikzcd}$$

One may trying to show that $0 \times X \simeq 0$ by:

$$\begin{tikzcd}
0 \times X \arrow[dr, red, "\pi_0"] \arrow[ddrrr, red, bend left, "\pi_1"] \arrow[dddrr, bend right, "\pi_0"'] & & & \\
& 0 \arrow[drr, red, bend left, "!"] \arrow[dr, dashed, "!"] \arrow[ddr, bend right, "1_0"'] & & \\
& & 0 \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & X \\
& & 0 &
\end{tikzcd}$$

However, the red triangle needs not to commutes, that is, the arrow π_0 from $(0 \times X, \pi_0, \pi_1)$ to $(0, 1_0, !)$ may not exist.

Definition 1.2 (Product of Arrow). *Suppose $(A \times B, \pi_0, \pi_1)$ and $(C \times D, \pi_2, \pi_3)$ are two product, and $f : A \to C$, $g : B \to D$. The product of arrow $f \times g$ is a* *unique arrow from $A \times B$ to $C \times D$ such that the following diagram commutes:*

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, dashed, "f \times g"] & B \arrow[d, "g"] \\
C & C \times D \arrow[l, "\pi_2"'] \arrow[r, "\pi_3"] & D
\arrow[from=1-1, to=2-1, "f"']
\end{tikzcd}$$

2 Exponential

Definition 2.1. *Let $\mathcal{C}$ a category. For any $B, C \in \mathcal{C}$, (C^B, ev) forms an exponential where $C^B \in \mathcal{C}$ and $ev : C^B \times B \to C$, if for any object $A \in \mathcal{C}$ and $f : A \times B \to C$, there is a unique $u : A \to C^B$ such that $f = ev \circ (u \times 1_B)$. In other words, the follow diagram commutes.*

$$\begin{tikzcd}
A \times B \arrow[dd, dashed, "u \times 1_B"] \arrow[dr, "f"] & \\
& C \\
C^B \times B \arrow[ur, "ev"] &
\end{tikzcd}$$

3 Pullback

Theorem 3.1. *Suppose we have two joined commuting squares like:*

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\
X \arrow[r, "h"] & Y \arrow[r, "j"] & Z
\end{tikzcd}$$

Then:

1. *The outer rectangle is a pullback square if two inner squares are pullback squares.*

2. *The inner-left square is a pullback square if the ouer rectangle and the inner-right square are pullback squares.*

Proof.

1. For any (A, a, b) such that $j \circ h \circ a = n \circ b$, then there is a unique $u : A \to M$ such that $h \circ a = m \circ u$ and $b = g \circ u$.

$$\begin{tikzcd}
A \arrow[dr, dashed, "v"] \arrow[drr, dashed, "u"] \arrow[drrr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & & \\
& L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\
& X \arrow[r, "h"] & Y \arrow[r, "j"] & Z
\end{tikzcd}$$

$\square$