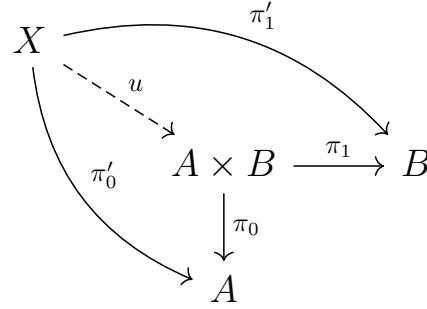


1 Product

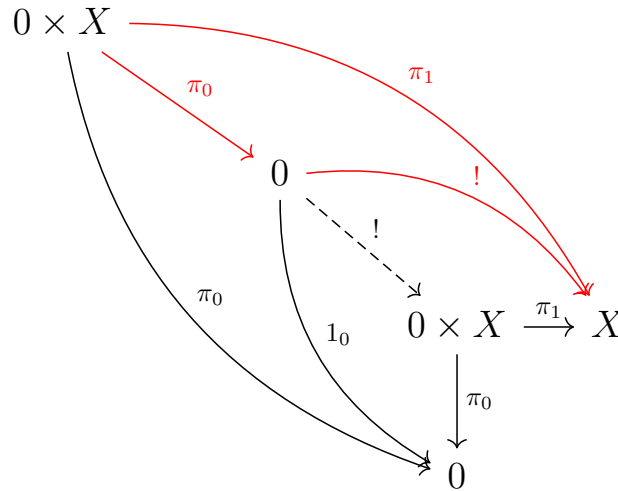
Definition 1.1 (Product). *Let \mathcal{C} a category and $A, B \in \mathcal{C}$, $(A \times B, \pi_0, \pi_1)$ forms a product of A and B where $A \times B \in \mathcal{C}$, $\pi_0 : A \times B \rightarrow A$ and $\pi_1 : A \times B \rightarrow B$, if for any $X \in \mathcal{C}$ with $\pi'_0 : X \rightarrow A$ and $\pi'_1 : X \rightarrow B$, there is a unique arrow $u : X \rightarrow A \times B$ such that the following diagram commutes:*



Furthermore, a product of A and B is a limit of diagram:

$$(A \quad B)$$

One may trying to show that $0 \times X \simeq 0$ by:



However, the red triangle needs not to commutes, that is, the arrow π_0 from $(0 \times X, \pi_0, \pi_1)$ to $(0, 1_0, !)$ may not exist.

Definition 1.2 (Product of Arrow). *Suppose $(A \times B, \pi_0, \pi_1)$ and $(C \times D, \pi_2, \pi_3)$ are two product, and $f : A \rightarrow C$, $g : B \rightarrow D$. The product of arrow $f \times g$ is a*

unique arrow from $A \times B$ to $C \times D$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 A & \xleftarrow{\pi_0} & A \times B & \xrightarrow{\pi_1} & B \\
 \downarrow f & & \downarrow f \times g & & \downarrow g \\
 C & \xleftarrow{\pi_2} & C \times D & \xrightarrow{\pi_3} & D
 \end{array}$$

2 Exponential

Definition 2.1. Let \mathcal{C} a category. For any $B, C \in \mathcal{C}$, (C^B, ev) forms an exponential where $C^B \in \mathcal{C}$ and $ev : C^B \times B \rightarrow C$, if for any object $A \in \mathcal{C}$ and $f : A \times B \rightarrow C$, there is a unique $u : A \rightarrow C^B$ such that $f = ev \circ (u \times 1_B)$. In other words, the follow diagram commutes.

$$\begin{array}{ccc}
 A \times B & & \\
 \downarrow u \times 1_B & \searrow f & \\
 C^B \times B & \xrightarrow{ev} & C
 \end{array}$$

3 Pullback

Theorem 3.1. Suppose we have two joined commuting squares like:

$$\begin{array}{ccccc}
 L & \xrightarrow{f} & M & \xrightarrow{g} & N \\
 \downarrow l & & \downarrow m & & \downarrow n \\
 X & \xrightarrow{h} & Y & \xrightarrow{j} & Z
 \end{array}$$

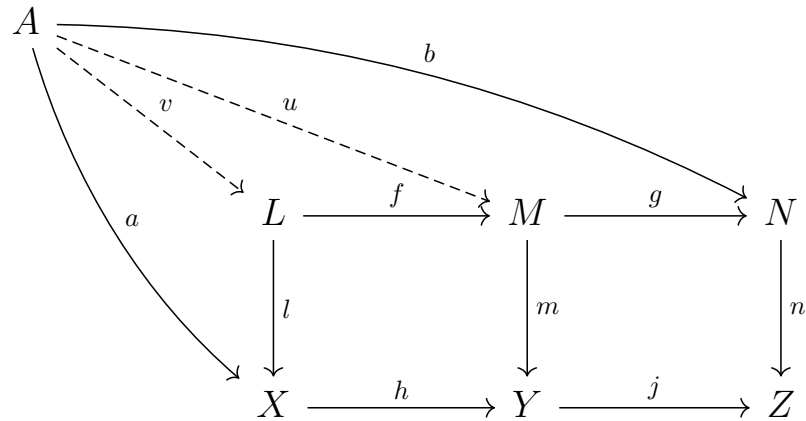
Then:

1. The outer rectangle is a pullback square if two inner squares are pullback squares.

2. The inner-left square is a pullback square if the outer rectangle and the inner-right square are pullback squares.

Proof.

1. For any (A, a, b) such that $j \circ h \circ a = n \circ b$, then there is a unique $u : A \rightarrow M$ such that $h \circ a = m \circ u$ and $b = g \circ u$.



□