

# 1 Pullback

**Theorem 1.1.** *Suppose we have two joined commuting squares like:*

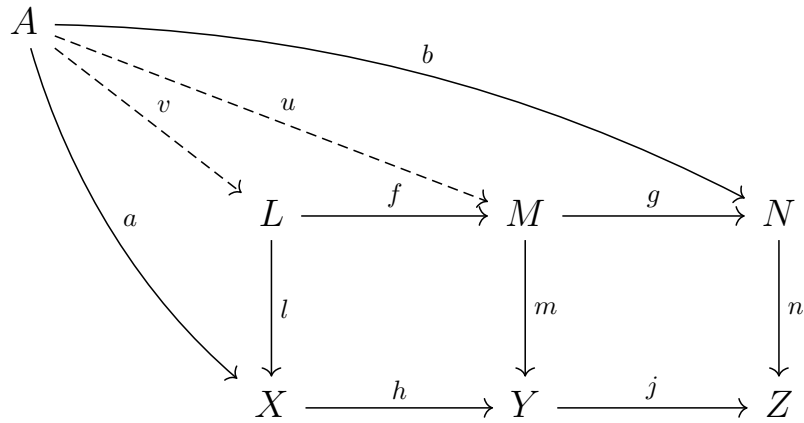
$$\begin{array}{ccccc}
 L & \xrightarrow{f} & M & \xrightarrow{g} & N \\
 \downarrow l & & \downarrow m & & \downarrow n \\
 X & \xrightarrow{h} & Y & \xrightarrow{j} & Z
 \end{array}$$

*Then:*

1. *The outer rectangle is a pullback square if two inner squares are pullback squares.*
2. *The inner-left square is a pullback square if the outer rectangle and the inner-right square are pullback squares.*

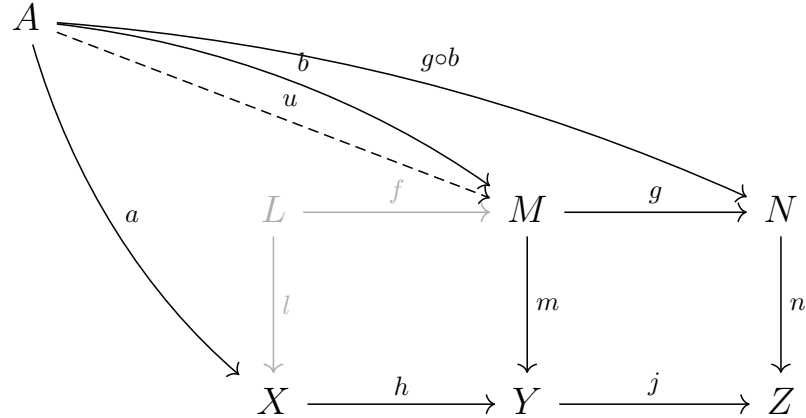
*Proof.*

1. For any  $(A, a, b)$  such that  $j \circ h \circ a = n \circ b$ , then there is a unique  $u : A \rightarrow M$  such that  $h \circ a = m \circ u$  and  $b = g \circ u$ . Then there is a unique  $v : A \rightarrow L$  such that  $l \circ a = v$  and  $f \circ v = u$ , which makes  $(A, a, b)$  against to the outer rectangle commutes.

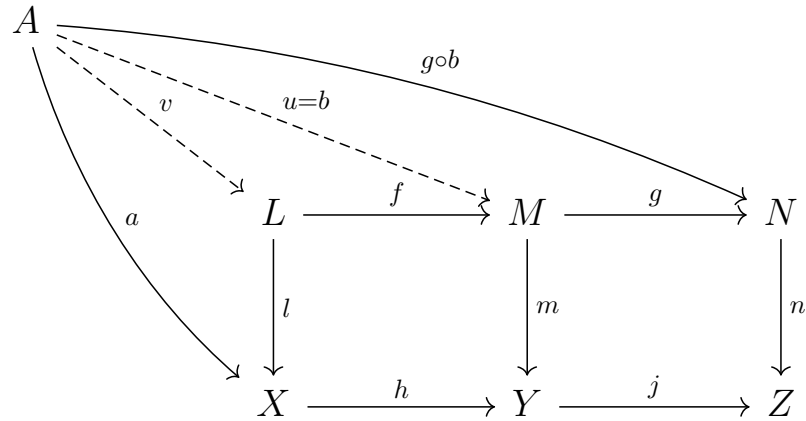


2. For any  $(A, a, b)$  such that  $h \circ a = m \circ b$ , consider the inner-right pullback,

then we have a unique  $u : A \rightarrow M$  such that the diagram commutes:



However, if we replace  $u$  with  $b$ , we have  $g \circ b = g \circ b$  and  $h \circ a = m \circ b$ , that means  $b$  can do  $u$ 's job, but we know  $u$  is unique, so  $b = u$ . Now consider the outer pullback, we have a unique  $v : A \rightarrow L$  such that the diagram commutes:



That is,  $l \circ v = a$  and  $g \circ f \circ v = g \circ b$ , we claim that  $v$  is the unique factorization from  $(A, a, u = b)$  to  $(L, l, f)$ . It is obvious that  $l \circ v = a$ , we need to show  $f \circ v = u = b$ . We may use the trick we just used, we can see that  $g \circ f \circ v = g \circ u$  and  $m \circ f \circ v = h \circ l \circ v = h \circ a$ . So  $f \circ v$  can do  $b$ 's job, so  $f \circ v = b$ .

For any arrow  $w : A \rightarrow L$  such that  $l \circ a = w$  and  $f \circ w = b$ , then we have also  $g \circ f \circ w = g \circ b$ , which implies  $w$  is the unique arrow from  $A \rightarrow L$  such that the outer diagram commutes, so  $w = v$ .

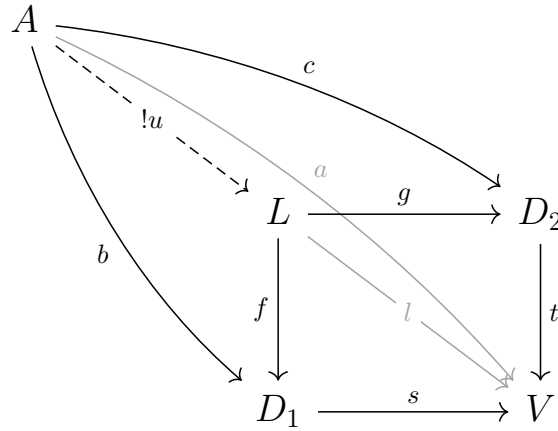
□

**Theorem 1.2.** *A pullback square for the corner  $D_1 \rightarrow V \leftarrow D_2$  is a product of  $D_1 \rightarrow V$  and  $V \leftarrow D_2$  in the slice category  $\mathcal{C}/V$ .*

*Proof.* Suppose  $(L, f, g)$  is the pullback of such corner, then we first need to show that there is an arrow  $l : L \rightarrow V$  such that  $s \circ f = l$  (therefore a morphism from  $(L, l)$  to  $(D_1, s)$ ) and  $t \circ g = l$  (a morphism from  $(L, l)$  to  $(D_2, t)$ ).

Since  $(L, f, g)$  makes the pullback square commutes, we know  $s \circ f = t \circ g$ , therefore we let  $l = s \circ f$  (or equivalently  $t \circ g$ ).

We need to show that  $((L, l), f, g)$  forms a product of  $(D_1, s)$  and  $(D_2, t)$ , consider any  $((A, a), b, c)$  where  $a : A \rightarrow V$  such that  $s \circ b = a$  and  $t \circ c = a$ . Just like  $l$  for  $L$ ,  $a$  is redundant, so we may omit it. Now, the diagram looks like:



Since  $(L, f, g)$  is a pullback, we know there is a unique  $u : A \rightarrow L$  such that two triangle commutes. However, we must first show that  $u$  is an arrow from  $(A, a)$  to  $(L, l)$ , that is,  $l \circ u = a$ . It is easy to see that  $l \circ u = s \circ f \circ u = s \circ b = a$ .  $\square$

**Theorem 1.3.** *if a category has all binary products and all equalizers for every pair of parallel arrows, then it has a pullback for any corners.*

*Proof.* Suppose  $X \rightarrow Z \leftarrow Y$  a corner, then consider the product  $X \times Y$ :

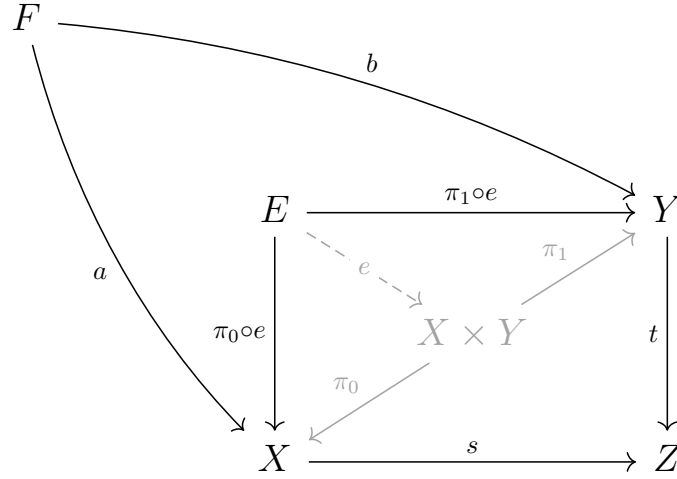
$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_1} & Y \\ \pi_0 \downarrow & & \downarrow t \\ X & \xrightarrow{s} & Z \end{array}$$

Now, consider the equalizer for the parallel arrows  $t \circ \pi_1$  and  $s \circ \pi_0$ :

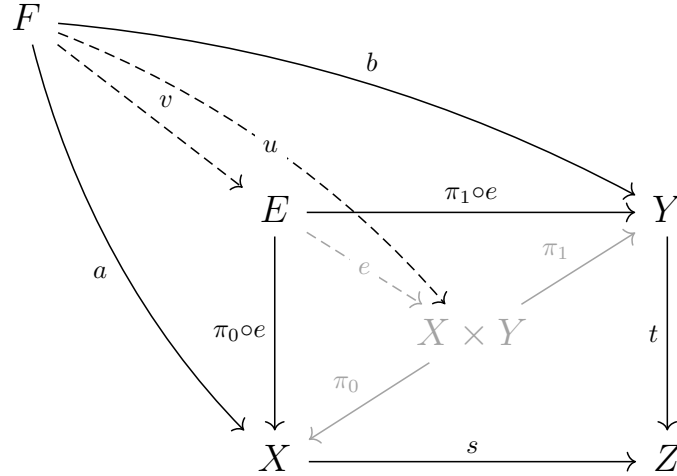
$$E \xrightarrow{e} X \times Y \rightrightarrows Z$$

$\begin{array}{c} s \circ \pi_0 \\ \hline t \circ \pi_1 \end{array}$

We claim  $(E, \pi_0 \circ e, \pi_1 \circ e)$  is a pullback of such corner. For any  $(F, f, g)$  such that the outer diagram commutes:



it is easy to see that there is a unique arrow  $u : F \rightarrow X \times Y$  such that  $\pi_0 \circ u = a$  and  $\pi_1 \circ u = b$  since  $X \times Y$  is a product. Then there is another unique arrow  $v : F \rightarrow E$  such that  $e \circ v = u$  since  $E$  is a equalizer.



Obviously, (commute)  $\pi_0 \circ e \circ v = \pi_0 \circ u = a$  and  $\pi_1 \circ e \circ v = \pi_1 \circ u = b$ .  
(unique) If an arrow  $w : F \rightarrow E$  can do the job, then  $e \circ w : F \rightarrow X \times Y$  is another factorization from  $F$  to the product  $X \times Y$ , so  $e \circ w = u$ , but that means  $w$  is also a factorization from  $F$  to the equalizer  $E$ , which means  $v = w$ .

So  $(E, \pi_0 \circ e, \pi_1 \circ e)$  is a pullback of such corner.  $\square$