**Definition 14.1** (Ideal). *A subring $A$ of a ring $R$ is called a (two-sided) ideal of $R$ if for every $r \in R$, and every $a \in A$, both $ra$ and $ar$ are in $A$.*

**Theorem 14.1** (Ideal Test). *A non-empty subset $A$ of a ring $R$ is an ideal of $R$, if:*

- $a - b \in A$ *for all* $a\ b \in A$.
- $ar \in A$ *and* $ra \in A$ *for all* $a \in A$ *and* $r \in R$.

*Proof.* Since $A \subseteq R$, we know $ab \in A$ for all $a\ b \in A$ from the second property. Therefore, $A$ is a subring of $R$, and then it is an ideal of $R$ by the second property. □

**Example** (Trivial Ideal). *For any ring $R$, $\{0\}$ is an ideal of $R$, which is called the trivial ideal.*

**Theorem 14.2** (Factor Ring). *Let $R$ a ring and $A$ a subring of $R$. The set of coset $\{\ r + A \mid r \in R\ \}$ is a ring under:*

- *addition:* $(s + A) + (t + A) = (s + t) + A$
- *multiplication:* $(s + A)(t + A) = (st) + A$

*iff $A$ is an ideal of $R$.*

*Proof.* Newline please!!

- $(\Rightarrow)$ For any $r \in R$ and $a \in A$,

$$\begin{aligned} & 0 + A \\ =& (r + A)(0 + A) \\ =& (r + A)(a + A) \quad \text{(since } a \in A\text{)} \\ =& ra + A \end{aligned}$$

  Then $0 + A = ra + A$, and then $ra \in A$ (Recall that $a + A$ means a coset of $A$). Similarly, $ar \in A$.

- $(\Leftarrow)$ For any $s\ t \in R$,
  - Addition: $(s + A) + (t + A) = s + t + A + A = (s + t) + A$ by $(R, +)$ is Abelian group.

– Multiplication:

$$\begin{aligned}&(s+A)(t+A)\\=&(s+A)t+(s+A)A\\=&st+At+sA+AA\\=&st+A+A+A' \quad \text{(since } A \text{ is an ideal)} \quad \text{where } A' \subseteq A\\=&st+A \quad \text{(since } A \text{ is a group under addition)}\end{aligned}$$

– Assosiative and Distributive: Trivial by $R$ is a ring.

□

**Theorem 14.3.** *Let $R$ a commutative ring with unity and $A$ an ideal of $R$. Prove that $R/A$ is an integral domain iff $A$ is a prime ideal.*

*Proof.*

- ($\Rightarrow$) For any $a$ $b \in R$ and $ab \in A$, we have $ab + A = 0 + A$ and $(a+A)(b+A) = ab + A$, since $R/A$ is integral domain, we know either $a + A$ or $b + A$ is zero, in the other word, $a \in A$ or $b \in A$.
- ($\Leftarrow$) For any $a$ $b \in A$, if $(a+A)(b+A) = 0 + A$, then $ab + A = 0 + A$ which means $ab \in A$. We know $a = 0$ or $b = 0$ by $A$ is a prime ideal, therefore $a+A = 0+A$ or $b+A = 0+A$, and $R/A$ is an integral domain.

□

**Theorem 14.4.** *Let $R$ a commutative ring with unity and $A$ an ideal of $R$. Prove that $R/A$ is a field iff $A$ is a maximum ideal.*

*Proof.*

- ($\Rightarrow$) Let $B$ an ideal of $R$ and $A \subseteq B \subseteq R$. Let $b \in B$ but $b \notin A$, if we can't found such element, then $B = A$. Note that $b \neq 0$, so that $(b+A)^{-1}$ exists. For any $r \in R$, we have:

$$\begin{aligned}&r+A\\=&(r+A)(b+A)(b+A)^{-1}\\=&(rb+A)(b'+A) \quad (b' \text{ is not necessary in B})\\=&(rbb'+A)\end{aligned}$$

where $rbb' \in B$, since $B$ is an ideal. Therefore $(-rbb') + r \in B$ since $A \subseteq B$ and $r \in B$ since addition is closed. Now, $B \subseteq R$ and $R \subseteq B$, then $B = R$.

- ($\Leftarrow$) The following proof come from textbook.

  Let $b \in R$ but $b \notin A$, consider the set $B = \{ \ br + a \mid r \in R, a \in A \ \}$. It is eazy to show that $B$ is an ideal of $R$. Since $B$ properly contains $A$, $B$ must be $R$, so $1 \in B$. Then $1 = br + a$ and $1 + A = (br + a) + A = br + A = (b + A)(r + A)$, so $r + A$ is the inverse of $b + A$, now every non-zero element in $R/A$ has an inverse. We must show that $R/A$ is integral domain. For any $a + A$ and $b + A$, if $ab \in A$, and $a \notin A$, then $0 + A = (a + A)^{-1}(ab + A) = (a + A)^{-1}(a + A)(b + A) = b + A$, we know $b \in A$, and $R/A$ is an integral domain.

$\square$

Note that the magic construction $B = \{ \ br + a \mid r \in R, a \in A \ \}$ is the minimal ideal that contains $b$ and $A$.

**Corollary 14.1.** *Let $R$ a commutative ring with unity and $A$ a maximal ideal of $R$, then $A$ is also a prime ideal.*