1 Pullback

Theorem 1.1. *Suppose we have two joined commuting squares like:*

$$\begin{tikzcd}
L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\
X \arrow[r, "h"] & Y \arrow[r, "j"] & Z
\end{tikzcd}$$

Then:

1. *The outer rectangle is a pullback square if two inner squares are pullback squares.*

2. *The inner-left square is a pullback square if the ouer rectangle and the inner-right square are pullback squares.*

Proof.

1. For any (A, a, b) such that $j \circ h \circ a = n \circ b$, then there is a unique $u : A \to M$ such that $h \circ a = m \circ u$ and $b = g \circ u$. Then there is a unique $v : A \to L$ such that $l \circ a = v$ and $f \circ v = u$, which makes (A, a, b) against to the outer rectangle commutes.

$$\begin{tikzcd}
A \arrow[dr, dashed, "v"] \arrow[drr, dashed, "u"] \arrow[drrr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & & \\
& L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\
& X \arrow[r, "h"] & Y \arrow[r, "j"] & Z
\end{tikzcd}$$

2. For any (A, a, b) such that $h \circ a = m \circ b$, consider the inner-right pullback,

then we have a unique $u : A \to M$ such that the diagram commutes:

$$
\begin{tikzcd}
A \arrow[ddr, bend right, "a"'] \arrow[drr, dashed, "u"] \arrow[drr, bend left, "b"] \arrow[drrr, bend left, "g \circ b"] & & & \\
& L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\
& X \arrow[r, "h"] & Y \arrow[r, "j"] & Z
\end{tikzcd}
$$

However, if we replace u with b, we have $g \circ b = g \circ b$ and $h \circ a = m \circ b$, that means b can do u's job, but we know u is unique, so $b = u$. Now consider the outer pullback, we have a unique $v : A \to L$ such that the diagram commutes:

$$
\begin{tikzcd}
A \arrow[ddr, bend right, "a"'] \arrow[dr, dashed, "v"] \arrow[drr, dashed, "u=b"] \arrow[drrr, bend left, "g \circ b"] & & & \\
& L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\
& X \arrow[r, "h"] & Y \arrow[r, "j"] & Z
\end{tikzcd}
$$

That is, $l \circ v = a$ and $g \circ f \circ v = g \circ b$, we claim that v is the unique factorization from $(A, a, u = b)$ to (L, l, f). It is obvious that $l \circ v = a$, we need to show $f \circ v = u = b$. We may use the trick we just used, we can see that $g \circ f \circ v = g \circ u$ and $m \circ f \circ v = h \circ l \circ v = h \circ a$. So $f \circ v$ can do b's job, so $f \circ v = b$.

For any arrow $w : A \to L$ such that $l \circ a = w$ and $f \circ w = b$, then we have also $g \circ f \circ w = g \circ b$, which implies w is the unique arrow from $A \to L$ such that the outer diagram commutes, so $w = v$.

$\square$