This chapter involves group action, but the knowleage is from untrusted sources [0] [1].

Definition -1.1 (Group Action). *For any group G and any set X. A **group action** of G on X is a mapping $- \cdot - : G \times X \to X$ such that:*

- *Preserve Identity: $\forall x \in X, e \cdot x = x$*
- *Associativity: $\forall g\ h \in G, x \in X, (gh) \cdot x = g \cdot (h \cdot x)$*

We say G is left action on X, and the set X is called G-set.

Definition -1.2 (Orbit). *For any group G, and a set X where X is a G-set. For any $x \in X$, the orbit of x:*

$$\mathrm{orb}(x) = \{g \cdot x \mid g \in G\}$$

In other words, the orbit of x is all the possible 'positions' that x can be sent to.

Exercise 7.67 (Orbit is Partition). *Show that the intersection of distinct orbits is empty set.*

Proof. Suppose G a group and X a G-set, $x\ y \in X$. If there is a $a \in X$ such that $a \in \mathrm{orb}(x) \cap \mathrm{orb}(y)$, then exists $g \in G$ such that $g \cdot x = a$ and exists $g' \in G$ such that $g' \cdot y = a$. Then $((g')^{-1}g) \cdot x = y$, therefore $\mathrm{orb}(y) \subseteq \mathrm{orb}(x)$. Similarly $\mathrm{orb}(x) \subseteq \mathrm{orb}(y)$. □

Definition -1.3 (Stablizer). *For any group G, and X a G-set. For any $x \in X$, the stablizer of x:*

$$\mathrm{stab}(x) = \{g \in G \mid g \cdot x = x\}$$

Lemma -1.1 (Stablizer is Subgroup). *For any group G, and X a G-set. For any $x \in X$, $\mathrm{stab}(x)$ is a subgroup of G.*

Proof. By two-steps:

0. $\mathrm{stab}(x)$ is not empty since $e \cdot x = x$
1. For any $g\ h \in \mathrm{stab}(x)$, $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$.

[0] https://zhuanlan.zhihu.com/p/165163924

[1] https://www.bananaspace.org/wiki/%E7%BE%A4%E4%BD%9C%E7%94%A8

2. For any $g \in \operatorname{stab}(x)$,

$$\begin{aligned} g \cdot x &= x \\ g^{-1} \cdot (g \cdot x) &= g^{-1} \cdot x \\ (g^{-1} g) \cdot x &= g^{-1} \cdot x \\ x &= g^{-1} \cdot x \end{aligned}$$

Thus $\operatorname{stab}(x)$ is a subgroup of G. $\square$

Lemma -1.2. *For any group G, and X a G-set. For any $x \in X$, $|G| = |\operatorname{orb}(x)||\operatorname{stab}(x)|$ (or equivalently, $|\operatorname{orb}(x)| = [G : \operatorname{stab}(x)]$).*

Proof. Consider the function $\phi(g \operatorname{stab}(x)) = g \cdot x$ from the cosets of $\operatorname{stab}(x)$ to $\operatorname{orb}(x)$, we will show that it is bijective, but first, we need to show that it **is** a function. For any $g \operatorname{stab}(x)$ and $h \operatorname{stab}(x)$, if $g \operatorname{stab}(x) = h \operatorname{stab}(x)$, then $g \in h \operatorname{stab}(x)$, which means $g = hs$ where $s \in \operatorname{stab}(x)$. Then $\phi(g \operatorname{stab}(x)) = g \cdot x = (hs) \cdot x = h \cdot (s \cdot x) = h \cdot x = \phi(h \operatorname{stab}(x))$.

- One-to-one: For any $g\ h \in G$, $\phi(g \operatorname{stab}(x)) = \phi(h \operatorname{stab}(x))$, then $g \cdot x = h \cdot x$, therefore $g^{-1} \cdot (h \cdot x) = x$, which implies $(g^{-1} h) \cdot x = x$ and $g^{-1} h \in \operatorname{stab}(x)$. Then $g \operatorname{stab}(x) = h \operatorname{stab}(x)$.
- Onto: For any $a \in \operatorname{orb}(x)$, then there is $g \in G$ such that $g \cdot x = a$. Then $\phi(g \operatorname{stab}(x)) = g \cdot x = a$.

Thus ϕ is bijective and then $|\operatorname{orb}(x)| = [G : \operatorname{stab}(x)]$ $\square$