Exercise 14.7. *Let n an integer and p a divides n and $p \neq n$. Prove that $\langle p \rangle$ is a maximal ideal in Z_n if and only if p is prime.*

Proof.

- ($\Rightarrow$) Suppose $\langle p \rangle$ is a maximal ideal in Z_n, if p is not prime, then q divides p and $q \neq 1$ and $q \neq p$. We have $\langle q \rangle$ is a ideal that properly contains p but is not Z_n since $1 \notin \langle q \rangle$.
- ($\Leftarrow$) Suppose p is a prime, and let R a ideal of Z_n. If R properly contains $\langle p \rangle$, let $q \in R$ but $q \notin \langle p \rangle$, then $\gcd(p, q) = 1$ since p is prime, then $1 \in Z_n \to R = Z_n$.

□

Exercise 14.9. *Suppose that R is a commutative ring and $a \in R$. If $\{0\}$ is a maximal ideal of R, then $aR = \{ \ ar \mid r \in R \ \} = \{0\}$ or $a \in aR$.*

Proof. We need to show that aR is an ideal, it is trivial that aR is a subring. For any $ar \in aR$ and $b \in R$, $(ar)b = a(rb) \in aR$, therefore aR is an ideal. If aR property contains $\{0\}$, then $aR = R$ since $\{0\}$ is maximal. □

Exercise 14.14. *If A and B are ideals of a ring R, show that the sum of A and B, $A + B = \{ \ a + b \mid a \in A, b \in B \ \}$ is also an ideal.*

Proof. By ideal-test:

0. $A + B$ is non-empty, since A and B are non-empty.
1. For any $x \ y \in A+B$, $x - y = a_0 + b_0 - a_1 - b_1 = (a_0 - a_1) + (b_0 - b_1) \in A + B$.
2. For any $x \in A + B$, $r \in R$, $xr = (a + b)r = ar + br \in A + B$

□

Exercise 14.16. *If A and B are ideals of a ring R, show that the product of A and B, $AB = \{ \ a_0 b_0 + a_1 b_1 + \cdots + a_n b_n \mid a_i \in A, b_i \in B, n \text{ is positive integer} \ \}$, is an ideal. (Note that $a_i = a_j$ where $i \neq j$ is possible, same for b_i)*

Proof. Trivial, similar to Exercise 14.14. □

Exercise 14.18. *Let A and B be ideals of a ring, show that $AB \subseteq A \cap B$.*

Proof. It is trivial, every element in AB is also in A, since A ideal, similarly, is also in B, since B ideal. □

Exercise 14.22. *If R is a finite commutative ring with unity, prove that every prime ideal of R is also a maximal ideal.*

Proof. For any prime ideal I of R, we know R/I is an integral ideal, since R is finite, so is R/I, then we know R/I is a field. Therefore I is a maximal ideal. □

Exercise 14.39. *Prove that the only ideals of a field F are $\{0\}$ and F.*

Proof. Suppose I is an ideal of F that contains non-zero elements, otherwise, $I = \{0\}$. Let $a \in I$ where a is non-zero, then $aa^{-1} = 1 \in I$ since I is an ideal, then $I = F$ since $1 \in I$. □

Exercise 14.40. *Let R a commutative ring with unity, if the only ideals of R are $\{0\}$ and R, show that R is a field.*

Proof. Since the only ideals of R are $\{0\}$ and R, we know $\{0\}$ is the maximal ideal of R, then $R/\{0\}$ is a field, so is R.

But unfortunately, we can't use ring isomorphism for now. □

Exercise 14.41. *Prove that every idempontent ($a^2 = a$) in a commutative ring with unity other than 0 and 1 is a zero divisor.*

Proof. For any idempontent a, $a + (1 - a) = 1 \to a^2 + (1 - a)a = a \to a + (1-a)a = a \to (1-a)a = 0$. Therefore, a is a zero divisor with $(1-a)$. □

Exercise 14.42. *Show that $\boldsymbol{R}[x]/\langle x^2 + 1\rangle$ is a field.*

Proof. We need to show $\langle x^2 + 1\rangle$ is maximal in $\mathbf{R}[x]$. We denote $\langle x^2 + 1\rangle$ by I, observe that in $\mathbf{R}[x]/I$, x^2 is treated as -1 since $x^2 + 1 + I = 0 + I$, therefore any element in $\mathbf{R}[x]/I$ has form $ax+b+I$. Let J an ideal that properly contains I and $ax + b$ a non-zero element in J, then

$$\begin{aligned}
&0 + J \\
=&(ax + b)(ax - b) + J \\
=&(a^2x^2 - b^2) + J \\
=&-a^2 - b^2 + J \quad \text{(since } I \subset J\text{)} \\
=&(-a^2 - b^2)\left(\frac{1}{-a^2 - b^2}\right) + J \\
=&1 + J
\end{aligned}$$

Therefore $1 \in J$ and $J = \mathbf{R}[x]$, which proves that $I = \langle x^2 + 1\rangle$ is maximal. □

Exercise 14.45. *Let R be the ring of continuous functions from $\boldsymbol{R}$ to $\boldsymbol{R}$. Show that $I = \{ f \in R \mid f(0) = 0 \}$ is maximal ideal of R.*

Proof. It is trivial that I is an ideal. Let J an ideal that properly contains I, then there is $f \in J$ where $f(0) \neq 0$. Let $g(x) = f(0) - f(x)$, then $g(0) = f(0) - f(0) = 0$ and $g \in J$. Let $h(x) = f(x) + g(x) = f(x) + f(0) - f(x) = f(0)$, then $h(x)$ is a constant function. It is easy to find $h^{-1}(x) = \dfrac{1}{f(0)}$ and show that $h^{-1}(x)h(x) = 1 \in J$. The continuity of g is trivial. □

Exercise 14.50. *Let R be a ring and I an ideal of R. Prove that R/I is commutative iff $rs - sr \in I$ for all $r\ s \in R$.*

Proof.

- ($\Rightarrow$) For any $r\ s \in R$, $rs + I = (r + I)(s + I) = (s + I)(r + I) = sr + I$, therefore $rs - sr \in I$.
- ($\Leftarrow$) For any $r\ s \in R$, $(r + I)(s + I) = rs + I = sr + I = (s + I)(r + I)$, since $rs - sr \in I$ implies $rs + I = sr + I$.

□

Exercise 14.57. *An integral domain D is called a principal ideal domain, if every ideal of D has form $\langle a \rangle = \{ ar \mid r \in D \}$ for some $a \in D$. Show that Z is a principal ideal domain.*

Proof. Admit. □

Exercise 14.60. *Let R a principal ideal domain, show that every non-trivial prime ideal is maximal.*

Proof. Let $\langle p \rangle$ a non-trivial prime ideal, note that $p \neq 0$ since $\langle p \rangle$ is non-trivial. Then for any ideal $\langle r \rangle$ that properly contains $\langle p \rangle$, we can show $r \notin \langle p \rangle$, if so, then $\langle r \rangle$ is the smallest ideal that contains r, but $\langle p \rangle$ is smaller and $r \in \langle p \rangle$.

Since $p \in \langle r \rangle$, we know there is k such that $rk = p$, note that $k \neq 0$, since $p \neq 0$. Now by $\langle p \rangle$ is prime and $r \notin \langle p \rangle$, we know $k \in \langle p \rangle$ and there is $q \in R$ such that $pq = k$, similarly, $q \neq 0$. Then $rkq = pq = k$, by cancellation we know $rq = 1$ and $1 \in \langle r \rangle$, therefore $\langle r \rangle = R$ and $\langle p \rangle$ is maximal. □

Exercise 14.61. *Let R a commutative ring and $A \subseteq R$. Show that the annihilator of A, $\operatorname{Ann}(A) = \{ r \in R \mid ra = 0\ \forall a \in A \}$ is an ideal.*

Proof.

0. $\text{Ann}(A)$ is non-empty, since $0a = 0$.
1. For any $s\ t \in \text{Ann}(A)$ and $a \in A$, $(s-t)a = sa - ta = 0 - 0 = 0$.
2. For any $s \in \text{Ann}(A)$, $t \in R$ and $a \in A$, $sta = tsa = t0 = 0$.

$\square$

Exercise 14.81. *Let R a commutative ring with unity and for any $a \in R$, $a^2 = a$. Let I be a prime ideal of R, show that $|R/I| = 2$.*

Proof. We know R/I is an integral ideal since I is prime, then for any $a \in R$ but $a \notin I$, $a + I$ is non-zero element of R/I, then by $a^2 + I = a + I$, we know $a + I = 1 + I$, therefore $|R/I| = |\{0 + I, 1 + I\}| = 2$. $\square$