Exercise 3.1. *Suppose V is a finite vector space, Show that the only two ideal of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$. A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called an ideal, if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for any $T \in \mathcal{L}(V)$ and $E \in \mathcal{E}$.*

Proof. We will use the concept Matrix. Suppose $\lambda_0 v_0 + \cdots + \lambda_n v_v$ the basis of V. We want to construct T_i that $T(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_i v_i$ for all $0 \leq i < n$, which is a matrix with all zero but 1 at i, i.

For any matrix, we can always select a non-zero value at a, b and place it at i, b, this can be done by left multiply a matrix with 1 at i, a (this produce a vector at line i with values from line a), then right multiply a matrix with 1 at i, b (this produce a vector at column b with values from line i).

Also, we can always select a non-zero value at a, b and place it at a, i, this can be done by right multiply a matrix with 1 at b, i, then left multiply a matrix with 1 at a, i.

By combining these two operations, we ca select a non-zero value at a, b and place it at i, i. Now, consider any non-zero $E \in \mathcal{E}$, we can construct a matrix with non-zero value at i, i for every $0 \leq i < \dim V$. These matrix are in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then we can multiply an apropriate scalar to them so that they are matrices with 1 at i, i. By adds up these matrices, we get I, we know $I \in \mathcal{E}$ since $\mathcal{E}$ is a vector space, and now all $T \in \mathcal{L}(V)$ is also in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then $\mathcal{E} = \mathcal{L}(V)$.

The only exception is $\mathcal{E} = \{0\}$, in this case we can't pick any non-zero element.

Another solution, hope this one is more simple.

Suppose $\mathcal{E}$ an ideal of $\mathcal{L}(V)$ and non-zero, non-surjective $E \in \mathcal{E}$. Let $v_0, \cdots, v_{k-1}$ a basis of $\operatorname{null} E$ and $v_k, \cdots, v_{k+n}$ such that Tv_{k+i} is a basis of $\operatorname{range} E$, then we have $n \neq 0$ and $k \neq 0$.

Define A a linear transformation which maps v_i to v_{k+i} for $0 \leq i < \min\{k, n\}$ and maps others to 0, then $\dim \operatorname{range} EA = \min\{k, n\}$.

Expand the basis $w_i = Ev_{k+i}$ of $\operatorname{range} E$ to a basis of V, say $w_0, \cdots, w_{m-1}$, define B maps Ev_{k+i} to $w_{\min\{k,n\}+i}$, we always have enough $w_{\min\{k,n\}+i}$ since $m - 1 = \dim V = \dim \operatorname{null} E + \dim \operatorname{range} E$ while $\min k, n \leq \dim \operatorname{null} E$, then $\dim \operatorname{range} BE = \operatorname{range} E$ since we just re-map the range E.

Now consider $S = EA + BE$, we have $Sv_i = EAv_i = Ev_{k+i} = w_i \in \operatorname{range} E$ for all $0 \leq i < \min\{k, n\}$ and $Sv_{\min\{k,n\}+i} = BEv_{\min\{k,n\}+i} = w_{\min\{k,n\}+i} \in \operatorname{range} BE$ for all $0 \leq i < \dim \operatorname{range} E$. We can see $\operatorname{range} EA \cap \operatorname{range} BE = \{0\}$ and $\dim \operatorname{range}(EA + BE) = \operatorname{range} E + \min\{k, n\}$, where $k = \operatorname{null} E$ and $n = \operatorname{range} E$, the range of $EA + BE$ gets larger and $EA + BE \in \mathcal{E}$ since $EA, BE \in \mathcal{E}$, if $k > n$ (this is the only case that $EA + BE$ is not surjective), then we continue this process with $E = EA + BE$, the procedure will finally terminate since $\mathcal{L}(V)$ is finite (cause V is finite).

Now we show that any $\mathcal{E}$ with non-zero, non-surjective $E \in \mathcal{E}$ implies a surjective (thus injective and invertible) $T \in \mathcal{E}$.

For any ideal with an invertible element $E \in \mathcal{E}$, we have $E^{-1}E = I \in \mathcal{E}$, which causes $\mathcal{E} = \mathcal{L}(V)$ since $IT = T$ for all $T \in \mathcal{L}(V)$.

Therefore, only $\{0\}$ and $\mathcal{L}(V)$ are ideals of $\mathcal{L}(V)$. $\square$