

Definition 3.110 (Dual Space). Let V a vector space, then we denote V' the dual space of V , where

$$V' = \mathcal{L}(V, F)$$

Theorem 3.111. Let V a finite vector space, then $\dim V' = \dim V$

Proof. $\dim V' = \dim \mathcal{L}(V, F) = (\dim V)(\dim F) = \dim V$ □

Definition 3.112 (Dual Basis). Let v_0, \dots, v_{m-1} a basis of V , then the dual basis of v_0, \dots, v_{m-1} is $\varphi_0, \dots, \varphi_{m-1}$ such that:

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

holes for any $0 \leq i, j < m$.

We can see that the basis the dual basis extracts the coefficients of any vector in V .

Theorem 3.113. Let v_0, \dots, v_{m-1} a basis of V , and dual basis $\varphi_0, \dots, \varphi_{m-1}$ of which. Then for any $v \in V$,

$$v = \varphi_0(v)v_0 + \dots + \varphi_{m-1}(v)v_{m-1}$$

Proof. For any i , $\varphi_i(v) = \varphi_i(\lambda_0 v_0 + \dots + \lambda_{m-1} v_{m-1}) = \varphi_i(\lambda_i v_i) = \lambda_i \varphi_i(v_i) = \lambda_i \times 1$. □

Theorem 3.116. Let V a finite space, then the dual basis of basis of V is a basis of V' .

Proof. Let v_0, \dots, v_{m-1} a basis of V , then its dual basis has the same length, therefore we only need to show its dual basis is linear independent.

Suppose $\lambda_0 \varphi_0 + \dots + \lambda_{m-1} \varphi_{m-1} = 0$, then for any $0 \leq i < m$, $(\lambda_0 \varphi_0 + \dots + \lambda_{m-1} \varphi_{m-1})(v_i) = \lambda_i = 0$, therefore the dual basis is linear independent. □

Definition 3.118 (Dual Map). Let $T \in \mathcal{L}(V, W)$. A dual map of T is a linear map $T' \in \mathcal{L}(W', V')$, such that for any $\varphi \in W'$:

$$T'(\varphi) = \varphi \circ T$$

Theorem 3.128. Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that:

1. $\text{null } T' = (\text{range } T)^0$
2. $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$

Proof.

- $\text{null } T'$ is a space $\{ \varphi \in \mathcal{L}(W, F) \mid \varphi \circ T = 0 \}$, which means $\text{range } T \subseteq \text{null } \varphi$. $(\text{range } T)^0$ is a space $\{ \varphi \in \mathcal{L}(W, F) \mid \varphi(\text{range } T) = \{0\} \}$, which means $\text{range } T \subseteq \text{null } \varphi$. Therefore $\text{null } T' = (\text{range } T)^0$
- $\dim(\text{range } T)^0 = \dim W - \dim \text{range } T = \dim W - (\dim V - \dim \text{null } T)$

□

Theorem 3.129. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that*

$$T \text{ is surjective} \iff T' \text{ is injective}$$

Proof.

- Suppose T is surjective, then for any $T'(\varphi) = T'(\psi)$, we have $\varphi \circ T = \psi \circ T$. Since T is surjective, then T is an epimorphism (we proved this in E3B), therefore $\varphi = \psi$.
- Suppose T' is injective, then for any $\varphi, \psi \in \mathcal{L}(W, F)$ such that $\varphi \circ T = \psi \circ T$, we have $\varphi = \psi$ since T' is injective. Therefore T is epimorphism, thus surjective.

□

The last theorem is obviously true in category theory, but we haven't show that T' is a morphism in Vect' where $\text{Vect}' \simeq \text{Vect}^{\text{op}}$.

Theorem 3.130. *Let V, W are finite and $T \in \mathcal{L}(V, W)$, show that:*

1. $\dim \text{range } T' = \dim \text{range } T$
2. $\text{range } T' = (\text{null } T)^0$

Proof.

- $\dim \text{range } T' = \dim W' - \dim \text{null } T' = \dim W' - (\dim \text{null } T + \dim W - \dim V) = \dim V - \dim \text{null } T = \dim \text{range } T$

- For any $\varphi \circ T \in \text{range } T'$, $(\varphi \circ T)(\text{null } T) = \varphi(\{0\}) = \{0\}$, therefore $\text{range } T' \subseteq (\text{null } T)^0$. Since $\dim(\text{null } T)^0 = \dim V - \dim \text{null } T = \dim \text{range } T = \dim \text{range } T'$, therefore $\text{range } T' = (\text{null } T)^0$ since both of them are finite and $\text{range } T' \subseteq (\text{null } T)^0$.

□

Theorem 3.131. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show*

$$T \text{ is injective} \iff T' \text{ is surjective}$$

Proof.

- $\dim \text{range } T' = \dim \text{range } T = \dim V = \dim V'$ since T is injective, therefore T' is surjective.
- $\dim \text{range } T = \dim \text{range } T' = \dim V' = \dim V$ therefore T is injective.

□