## 1 Functors

**Definition 1.1** (Full). A functor  $F : \mathcal{C} \to \mathcal{D}$  is called full, if for any  $a, b \in \mathcal{C}$ , the mapping on morphism  $F : \mathcal{C}(a,b) \to \mathcal{D}(Fa,Fb)$  is surjective.

**Definition 1.2** (Faithful). A functor  $F: \mathcal{C} \to \mathcal{D}$  is called faithful, if for any  $a, b \in \mathcal{C}$ , the mapping on morphism  $F: \mathcal{C}(a, b) \to \mathcal{D}(Fa, Fb)$  is injective.

**Definition 1.3** (Essentially Full). A functor  $F: \mathcal{C} \to \mathcal{D}$  is called Essentially full, if for any  $a \in \mathcal{C}$ , the mapping on object  $F: \mathcal{C} \to \mathcal{D}$  is surjective.

**Theorem 1.1.** Suppose  $F: \mathcal{C} \to \mathcal{D}$  a functor, and  $f: a \to b$  a morphism in  $\mathcal{C}$ . Then f is an isomorphism iff Ff is an isomorphism.

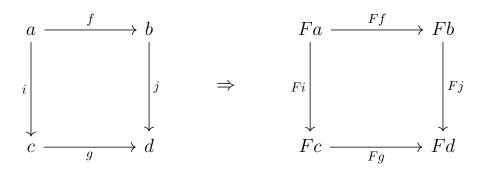
*Proof.* ( $\Rightarrow$ ) We claim  $F(f^{-1}): Fb \to Fa$  is an inverse, we can see that  $F(f^{-1} \circ f) = F(id_a) = id_{Fa}$  and  $F(f \circ f^{-1}) = F(id_b) = id_{Fb}$ .

 $(\Leftarrow)$  Suppose Fg is the inverse of Ff, and we can retrieve g from Fg cause F is full faithful. Then  $F(g \circ f) = Fg \circ Ff = id_{Fa} = F(id_a)$  therefore  $g \circ f = id_a$  since F is full faithful, similar to  $F(f \circ g)$ , so f is indeed an isomorphism.

**Corollary 1.1.** Suppose  $F: \mathcal{C} \to \mathcal{D}$  is full and faithful, show that F is injective on object.

*Proof.* Trivial by previous theorem.

Note that a commuting diagram applied to a functor is still commutes, due to the functoriality:



**Definition 1.4** (Natural Transform). Suppose  $F, G : \mathcal{C} \to \mathcal{D}$  are functors, then  $\alpha : F \Rightarrow G$  is called a natural transform from F to G, if:

• For any  $x \in \mathcal{C}$ ,  $\alpha_x : Fx \to Gx$  a morphism in  $\mathcal{D}$ .

• Furthermore, for any morphism  $f: x \to y$  in C, the following square commutes:

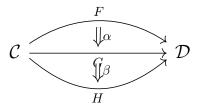
$$Fx \xrightarrow{Ff} Fy$$

$$\downarrow \qquad \qquad \downarrow$$

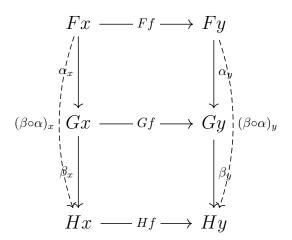
$$\downarrow \qquad \qquad \downarrow$$

$$Gx \xrightarrow{Gf} Gy$$

The one of composition of two natural transforms is vertical composition:



which is indeed a natural transform cause:



the outer diagram commutes.

There is another way to compose two natural transforms, the horizontal composition:

$$\mathcal{C} \xrightarrow{F'} \mathcal{D} \xrightarrow{G'} \mathcal{E}$$

We would expect that there is a natural transform  $\beta \cdot \alpha : G \circ F \Rightarrow G' \circ F'$ , but how? Firstly, we have the following diagram commutes cause  $\alpha$  is a

natural transform:

$$Fx \xrightarrow{Ff} Fy$$

$$\downarrow \qquad \qquad \downarrow$$

$$\downarrow \qquad \qquad \downarrow$$

$$F'x \xrightarrow{F'f} F'y$$

Then we apply it to the functor G.

$$G(Fx) \xrightarrow{G(Ff)} G(Fy)$$

$$\downarrow \qquad \qquad \downarrow$$

$$G(\alpha_x) \qquad \qquad \downarrow$$

$$G(G^y) \qquad \downarrow$$

$$G(F'x) \xrightarrow{G(F'f)} G(F'y)$$

It is similar to what we want, beside the bottom arrow, it is time to use  $\beta$ .

$$G(Fx) \xrightarrow{G(Ff)} G(Fy)$$

$$\downarrow \qquad \qquad \downarrow$$

$$G(\alpha_x) \qquad \qquad \downarrow$$

$$G(F'x) \xrightarrow{G(F'f)} G(F'y)$$

$$\downarrow \qquad \qquad \downarrow$$

$$\downarrow \qquad \qquad \downarrow$$

$$\downarrow \qquad \qquad \downarrow$$

$$G'(F'x) \xrightarrow{G'(F'f)} G'(F'y)$$

And the  $\beta_{F'-} \circ G(\alpha_-)$  is the definition of  $\beta \cdot \alpha$ .

Also, if one of the natural transform is the identity transform, say  $id_G \cdot \alpha$ , then it can be denoted by  $G \cdot \alpha$ . Notice that  $G \cdot \alpha$  has type  $G \circ F \Rightarrow G \circ F'$ , which "modifies" only one side.

You can see that the horizontal composition is much different from vertical composition, the former one is much like a product of morphism (if you treat  $\circ$  as some kind of product):

$$\alpha: F \Rightarrow F'$$
$$\beta: G \Rightarrow G'$$
$$\beta \cdot \alpha: F \circ G \Rightarrow F' \circ G'$$

While the later one is much like a composition of morphism:

$$\alpha: F \Rightarrow G$$
$$\beta: G \Rightarrow H$$
$$\beta \circ \alpha: F \Rightarrow H$$

It looks like we can write horizontal composition in vertical composition of two horizontal compositions:



In symbol, it is  $G(\alpha_{-})$  (the former one) and  $\beta_{F'-}$  (the later one), and finally  $\beta_{F'-} \circ G(\alpha_{-})$ , which is exactly the horizontal composition  $\beta \cdot \alpha$ . Similarly, we might suppose there is another definition of horizontal composition:  $G'(\alpha_{-}) \circ \beta_{F-}$  which is the vertical composition of:



The corresponding diagram would be: apply the naturality diagram of  $\alpha$  to G', then put  $\beta$  above it.