

Exercise 5.1. Let $T \in \mathcal{L}(V)$. Show that 9 is an eigenvalue of $T^2 \iff 3$ or -3 is an eigenvalue of T .

Proof.

- (\Rightarrow) We have $T^2 - 9I$ is not injective since 9 is an eigenvalue of T^2 , then $(T - 3I)(T + 3I) = T^2 - 9I$ is not injective means one of $T - 3I$ and $T + 3I$ is not injective, thus 3 or -3 is an eigenvalue of T .
- (\Leftarrow) Similarly, we have $(T - 3I)(T + 3I)v = (T^2 - 9I)v = 0$ (if 3 is an eigenvalue of T) or $(T + 3I)(T - 3I)v = (T^2 - 9I)v = 0$ (if -3 is an eigenvalue of T).

□

Exercise 5.2. Let V a vector space over \mathbb{C} and $T \in \mathcal{L}(V)$ has no eigenvalue. Show that any subspace of V that is invariant under T is either $\{0\}$ or infinite dimension.

Proof. Let $U \subseteq V$ a subspace that is invariant under T , and non-zero $u \in U$. We can repeatedly apply T to u , say u, Tu, T^2u, \dots . Suppose $k > 0$ is minimum such that $u, Tu, \dots, T^k u$ is linear dependent, we have $p \in \mathcal{P}(\mathbb{C})$ with $\deg p = k$ such that $p(T) = 0$. Clearly p is not constants, thus it has a zero since p is a polynomial of complex coefficient. Thus such zero is an eigenvalue of T . □

Exercise 5.3. Let $n > 1$ an integer, and $T \in \mathcal{L}(F^n)$ is defined by:

$$T(x_0, \dots, x_{n-1}) = (x_0 + \dots + x_{n-1}, \dots, x_0 + \dots + x_{n-1})$$

- Find all eigenvalue and eigenvector of T .
- Find the minimal polynomial of T .

Proof.

- Observe that $\text{range } T = \text{span}((1, \dots, 1))$, thus $T(1, \dots, 1) = n(1, \dots, 1)$.
- Observe that $T(x_0, \dots, x_{n-1}) = (x_0 + \dots + x_{n-1})(1, \dots, 1)$ and $T^2(x_0, \dots, x_{n-1}) = n(x_0 + \dots + x_{n-1})(1, \dots, 1)$, thus $p(T) = nT - T^2 = 0$.

□

Exercise 4 is kinda hard, sorry.

Exercise 5.6. Let $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T .

Proof. Observe that $T^2(w, z) = T(-z, w) = (-w, -z) = (-1)(w, z)$, thus the minimal polynomial of T is $p(T) = I + T^2$. \square

Exercise 5.7. • Given an example that the minimal polynomial of ST is not equal to TS 's.

- Suppose V is finite and $S, T \in \mathcal{L}(V)$. Show that the minimal polynomial of ST is equal to TS 's if one of S and T is invertible.

Hint: Show that S is invertible and $p \in \mathcal{P}(F)$ implies $p(TS) = S^{-1}p(ST)S$.

Proof.

- The idea is to find S, T such that $ST \neq 0$ but $TS = 0$. We can find $S(x, y) = (x, 0)$ and $T(x, y) = (y, 0)$ holds:

$$\begin{aligned}(ST)(x, y) &= S(y, 0) = (y, 0) \\ (TS)(x, y) &= T(x, 0) = (0, 0)\end{aligned}$$

Thus the minimal polynomial of ST is not 0 but TS one does.

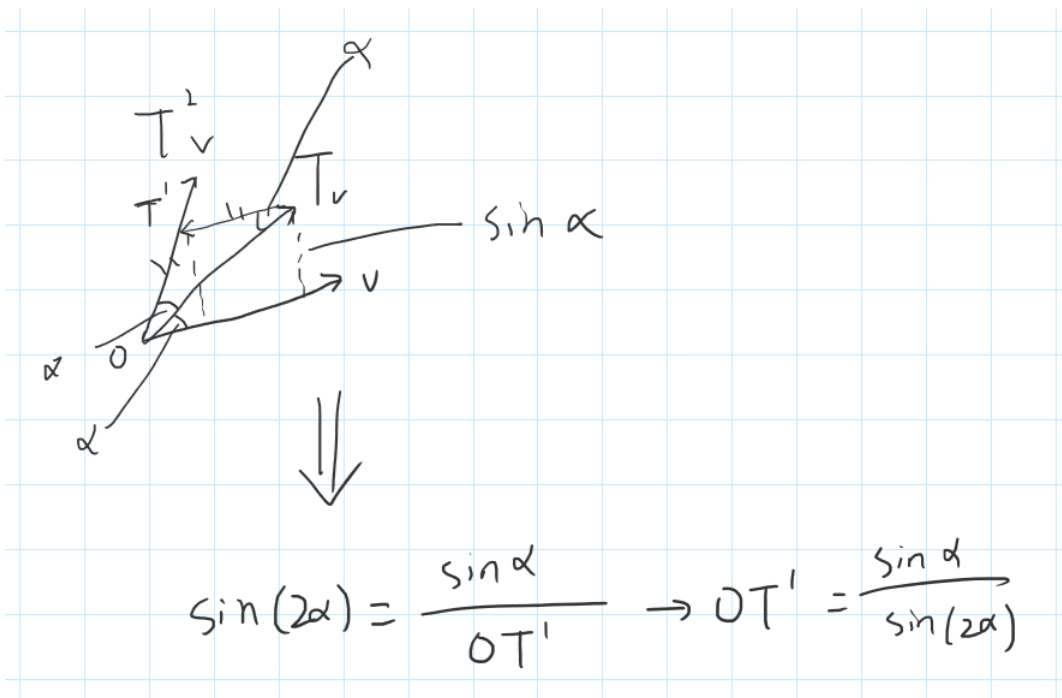
- Suppose S is invertible and $p \in \mathcal{L}(F)$ is the minimal polynomial of TS , then $p(TS) = S^{-1}p(ST)S$ since i -th term of $S^{-1}p(ST)S$ has form $S^{-1}c_i(ST)^iS = c_i(S^{-1}S)(TS)^{i-1}(TS) = c_i(TS)^i$. Thus $S^{-1}p(ST)S = 0$ and then $p(ST) = 0$. We will show that p is the minimal polynomial of ST , suppose $q \in \mathcal{L}(F)$ such that $q(ST) = 0$, then $0 = S^{-1}q(ST)S = q(TS)$, therefore $\deg q = \deg p$. Hence p is the minimal polynomial of ST .

\square

Exercise 5.8. Let $T \in \mathcal{L}(R^2)$ is the opearator that "rotates 1 degree counter-clockwise", find the minimal polynomial of T .

Note that it is **NOT** $x^{180} + 1$ even $T^{180} = -I$.

Proof. Note that there is some λ such that $Tv - \lambda v = \alpha T^2v$ (We can show that $\lambda = \alpha$), however the calculation is too complicate.



λ should be $\frac{\sin(1^\circ)}{\sin(2^\circ)}$, thus $p(T) = -\lambda I + T - \lambda T^2$.

We suppose all v below has length 1, thus $v = (\cos \theta, \sin \theta)$, this doesn't lose the generalizability since $p(T)(\alpha v) = \alpha(p(T)v)$.

For the first component of $p(T)v = -\lambda v + Tv - \lambda T^2v$, we have:

$$\begin{aligned} & \frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos \theta - \cos(\theta + 2^\circ)) \\ &= \frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos \theta - (\cos \theta \cos(2^\circ) - \sin \theta \sin(2^\circ))) \\ &= \frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos \theta - \cos \theta \cos(2^\circ)) + \sin \theta \sin(1^\circ) \\ &= \cos \theta \frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) + \sin \theta \sin(1^\circ) \end{aligned}$$

where $\sin \theta \sin(1^\circ)$ cancels a part of $(Tv)_1 = \cos(\theta + 1^\circ) = \cos \theta \cos(1^\circ) -$

$\sin \theta \sin(1^\circ)$. Thus we will show that $\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) = -\cos(1^\circ)$.

$$\begin{aligned}
& \frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) \\
&= \frac{\sin(1^\circ)}{2 \sin(1^\circ) \cos(1^\circ)}(-(\cos^2(1^\circ) + \sin^2(1^\circ)) - \cos^2(1^\circ) + \sin^2(1^\circ)) \\
&= \frac{1}{2 \cos(1^\circ)}(-\cos^2(1^\circ) - \cos^2(1^\circ)) \\
&= \frac{1}{2 \cos(1^\circ)}(-2 \cos^2(1^\circ)) \\
&= -\cos(1^\circ)
\end{aligned}$$

For the second component of $p(T)v$, we have:

$$\begin{aligned}
& \frac{\sin(1^\circ)}{\sin(2^\circ)}(-\sin \theta - \sin(\theta + 2^\circ)) \\
&= \frac{\sin(1^\circ)}{\sin(2^\circ)}(-\sin \theta - \sin \theta \cos(2^\circ) - \cos \theta \sin(2^\circ)) \\
&= \sin \theta \frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) - \cos \theta \sin(1^\circ)
\end{aligned}$$

similarly, we have $p(T)v_2 = \sin(\theta + 1^\circ) = \sin \theta \cos(1^\circ) + \cos \theta \sin(1^\circ)$ we will show that $\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) = -\cos(1^\circ)$, which is proven above. \square

Exercise 5.9. Let $T \in \mathcal{L}(V)$ such that for some basis of V , $\mathcal{M}(T)$ consists of rational numbers. Try to explain why the coefficients of the minimal polynomial of T is rational numbers.

Proof. I don't know, because \mathbb{Q} is also a field? \square

Exercise 5.11. Let V a vector space and $\dim V = 2$ and $T \in \mathcal{L}(V)$ such that $\mathcal{M}(T)$ for some basis of V is $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Show that:

- $T^2 - (a + d)T + (ad - bc)I = 0$

- the minimal polynomial of T is:

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d \\ z^2 - (a + d)z + (ad - bc) & \text{otherwise} \end{cases}$$

Proof.

•

$$\begin{aligned} & \mathcal{M}(T^2 - (a + d)T + (ad - bc)I) \\ &= \begin{bmatrix} a & c \\ b & d \end{bmatrix}^2 - (a + d) \begin{bmatrix} a & c \\ b & d \end{bmatrix} + (ad - bc)I \\ &= \begin{bmatrix} a^2 + bc & ac + bd \\ ab + bd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ac + cd \\ ab + bd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \end{aligned}$$

- If $b = c = 0$ and $a = d$, then T is a scalar multiple of identity operator, thus $T = aI$ and $p(T) = -aI + T = 0$. Otherwise, $T^2 - (a + d)T + (ad - bc)I = 0$.

□

Exercise 5.13. Let V finite, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(F)$. Show that there is a unique $r \in \mathcal{P}(F)$ such that $p(T) = r(T)$ where $\deg p$ is less than the degree of the minimal polynomial of T .

Proof. Let q the minimal polynomial of T .

If $\deg p < \deg q$, then $r = p$. The uniqueness is guaranteed by $\deg p < \deg q$ (try $(p - s)(T)$ where $p(T) = s(T)$ and $\deg s < \deg q$).

If $\deg p \geq \deg q$, then $p = sq + r$ where $s, r \in \mathcal{P}(F)$ with $\deg r < \deg q$. Then $p(T) = s(T)q(T) + r(T) = r(T)$ since $q(T) = 0$. The uniqueness is guaranteed by the property of division. □

Exercise 5.14. Let V finite, $T \in \mathcal{L}(V)$ with minimal polynomial $p(z) = 4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$. Find the minimal polynomial of T^{-1} .

Proof. Suppose p is the minimal polynomial of T , then we can repeatedly apply T^{-1} to $p(T)$, say $T^{-(\deg p)}(p(T))$, then it should be 0, and the coefficients are reversed, that is, $p_{\deg p}I + p_{\deg p-1}T^{-1} + \cdots + p_0(T^{-1})^{\deg p}$.

So the answer is $1 + 2z^1 - 7z^2 - 6z^3 + 5z^4 + 4z^5$. □

Exercise 5.16. Let $a_0, \dots, a_{n-1} \in F$ and T an operator over F^n . Its matrix (about the standard basis) is:

$$\begin{bmatrix} 0 & & & & -a_0 \\ 1 & 0 & & & -a_1 \\ & 1 & \ddots & & -a_2 \\ & & \ddots & & \vdots \\ & & & 1 & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix}$$

. Show that the minimal polynomial of T is:

$$p(z) = a_0 + a_1z + \dots + a_{n-1}z^{n-1} + z^n$$

Proof. We first need some property of this matrix, we will see it moves all number to the left when we repeatedly self-multily T . We can see the k -th column of T^p is equal to $k+1$ -th column of T^{p-1} , thus it is also equal to i -th column if T^j where $1 \leq i, j \leq n$ and $i+j = k+p$. In fact, j can be 0 and we have $T^0 = I$ and the property still holds.

Then, the i -th column of T^n is equal to n -th column (the last one) of T^i ,

and it is produced by $T^{i-1} \begin{bmatrix} -a_0 \\ -a_1 \\ \vdots \\ -a_{n-1} \end{bmatrix}$, which is equal to

$$T_i^n = -a_0T_1^{i-1} - a_1T_2^{i-1} - \dots - a_{n-1}T_n^{i-1}$$

which is equal to $T^n v_i$ where v_i is i -th standard basis of F^n , that is, $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$. We

may rewrite the equation into

$$T^n v = -a_0T^0 v - a_1T^1 v - \dots - a_{n-1}T^{n-1} v$$

where $T^0 v = T_i^0 = T_1^{i-1}$, $T^1 v = T_i^1 = T_2^{i-1}$ and so on.

Thus all v vector in standard basis has $p(T)v = 0$, thus $p(T) = 0$.

For minimal, we can see T is invertible, thus $p(T)v_1 = 0$ (recall that T moves number to the left, thus the first column of T^i is the i -th columns of T). means there is a (non-zero) linear combination of columns of T that is equal to 0. Thus $\deg p \geq n$ since T the columns are linear independent. \square

Exercise 5.17. Let V finite and $T \in \mathcal{L}(V)$ and p is the minimal polynomial of T . Let $\lambda \in F$, show that the minimal polynomial of $T - \lambda I$ is $q(z) = p(z + \lambda)$.

Proof. $q(T - \lambda I) = p((T - \lambda I) + \lambda I) = p(T) = 0$. Suppose r is the minimal polynomial of $T - \lambda I$, then $s(z) = r(z - \lambda)$ and $s(T) = r(T - \lambda I) = 0$, thus $\deg r = \deg p = \deg q$. \square

Exercise 5.18. Let V finite and $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T . Let $\lambda \in F$ that $\lambda \neq 0$, show that the minimal polynomial of λT is $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.

Proof. $q(\lambda T) = \lambda^{\deg p} p(\frac{1}{\lambda}(\lambda T)) = \lambda^{\deg p} p(T) = 0$. $\lambda^{\deg p}$ only makes q a monic polynomial.

Suppose r is the minimal polynomial of λT , then $s(z) = \frac{1}{\lambda^{\deg p}} r(\lambda z)$ and $s(T) = \frac{1}{\lambda^{\deg p}} r(\lambda T) = 0$, thus $\deg s = \deg r = \deg p = \deg q$. \square

Exercise 5.19. Let V finite and $T \in \mathcal{L}(V)$. Let $\mathcal{E} \subseteq \mathcal{L}(V)$ a subspace, defined by

$$\mathcal{E} = \{ q(T) \mid q \in \mathcal{P}(F) \}$$

.

Show that $\dim \mathcal{E}$ is equal to the degree of the minimal polynomial of T .

Proof. We can see $I, T, T^2, \dots, T^{\deg p - 1}$ is linear independent (since p is the minimal polynomial of T) where p is the minimal polynomial of T . For any $q \in \mathcal{P}(F)$ where $\deg q \geq \deg p$, then $q = sp + r$ where $s, r \in \mathcal{P}(F)$ and $\deg r < \deg p$, therefore $q(T) = s(T)p(T) + r(T) = r(T) \in \text{span}(I, T, T^2, \dots, T^{\deg p - 1})$. \square

Exercise 5.20. let $T \in \mathcal{L}(F^4)$, which eigenvalues are 3, 5, 8. Show that $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$.

Proof. Suppose p is the minimal polynomial of T , then $p(z) = c(z - 3)(z - 5)(z - 8)q(z)$ since 3, 5, 8 are the eigenvalue of T , thus the zeros of p . Note that $\deg q \leq 1$ since $\deg p \leq \dim F^4 = 4$. Since there is no other eigenvalue (thus zero) than 3, 5, 8, q is either 1 or one of $z - 3, z - 5, z - 8$, thus $(z - 3)^2(z - 5)^2(z - 8)^2$ is polynomial multiple of p , therefore $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$. \square

Exercise 5.21. Let V finite and $T \in \mathcal{L}(V)$. Show that the degree of the minimal polynomial of T caps at $1 + \dim \text{range } T$.

Proof. IDK □

Exercise 5.22. Let V finite and $T \in \mathcal{L}(V)$. Show that T is invertible $\iff I \in \text{span}(T, T^2, \dots, T^{\dim V})$.

Proof.

- (\Rightarrow) Suppose T is invertible, then the minimal polynomial p of T satisfies $p(0) \neq 0$ (since $p(0) = 0$ implies 0 is a eigenvalues of T). We know $\deg p \leq \dim V$, thus there is a linear combination of $T, T^2, \dots, T^{\dim V}$ that is equal to a scalar multiple of I , therefore $I \in \text{span}(T, T^2, \dots, T^{\dim V})$.
- (\Leftarrow) Suppose $I = \lambda_1 T + \lambda_2 T^2 + \dots + \lambda_{\dim V} T^{\dim V}$, then $I = T(\lambda_1 I + \lambda_2 T + \dots + \lambda_{\dim V} T^{\dim V - 1}) = (\lambda_1 I + \lambda_2 T + \dots + \lambda_{\dim V} T^{\dim V - 1})T$, thus T is invertible.

□

Exercise. Let V a vector space and $T \in \mathcal{L}(V)$, $v, Tv, \dots, T^k v$ a list of linear independent vectors but $v, Tv, \dots, T^{k+1} v$ isn't. Show that $T^{k+i} v \in \text{span}(v, Tv, \dots, T^k v)$ for all $0 < i$.

Proof. Induction on i .

- Base($i = 1$): By assumption.
- Ind($i = i+1$): $T^{k+i+1} v = T(T^{k+i} v)$, since $T^{k+i} v \in \text{span}(v, Tv, \dots, T^k v)$, thus it can be write as a linear combination of $v, Tv, \dots, T^k v$, say $T(\lambda_0 v + \lambda_1 Tv + \dots + \lambda_k T^k v)$, then $\lambda_0 Tv + \lambda_1 T^2 v + \dots + \lambda_k T^{k+1} v \in \text{span}(v, Tv, \dots, T^k v)$ since $T^{k+1} v \in \text{span}(v, Tv, \dots, T^k v)$.

□

Exercise 5.23. Let V finite and $T \in \mathcal{L}(V)$. Let $n = \dim V$, show that for any $v \in V$, $\text{span}(v, Tv, \dots, T^{n-1} v)$ is invariant under T .

Proof. Note that the list $v, Tv, \dots, T^{n-1} v$ has length $n = \dim V$, thus for the list $v, Tv, \dots, T^n v$ is linear dependent, thus $T^n v$ must be a linear combination of $v, Tv, \dots, T^{n-1} v$.

- If $v, Tv, \dots, T^{n-1}v$ is linear dependent, then $T^n v \in \text{span}(v, Tv, \dots, T^{n-1}v)$ (by our lemma exercise).
- Otherwise, the list $v, Tv, \dots, T^n v$ is linear dependent while $v, Tv, \dots, T^{n-1}v$ isn't, therefore $T^n v$ is a linear combination of $v, Tv, \dots, T^{n-1}v$.

□

Theorem 5.29. $q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial of T .

Proof. • (\Rightarrow) Let p the minimal polynomial of T , consider $q = sp + r$ where $\deg r < \deg p$, we may suppose $r \neq 0$. Then $0 = q(T) = s(T)p(T) + r(T) = r(T)$, which contradict to the assumption that p is the minimal polynomial of T .

- (\Leftarrow) Trivial.

□

Exercise 5.25. Let V finite, $T \in \mathcal{L}(V)$, subspace $U \subseteq V$ is invariant under T .

- Show that the minimal polynomial of T is polynomial multiple of the minimal polynomial of T/U .
- Show that

$$(\text{the minimal polynomial of } T|_U) \times (\text{the minimal polynomial of } T/U)$$

is a polynomial multiple of the minimal polynomial of T .

Proof. • Let p the minimal polynomial of T , then $p(T/U)(v + U) = p(T)v + U = 0 + U$ for any $v + U \in V/U$, thus $p(T/U) = 0$, therefore p is a polynomial multiple of the minimal polynomial of T/U .

- Let p the minimal polynomial of $T|_U$ and q the minimal polynomial of T/U . Then $(pq)(T)v = (p(T)q(T))v = p(T)(q(T)v)$ where $q(T)v \in U$, thus $p(T)(q(T)v) = 0$.

□

Exercise 5.26. Let V finite, $T \in \mathcal{L}(V)$, U is invariant under T . Show that the set of eigenvalues of T is equal to the union of eigenvalues of $T|_U$ and T/U .

Proof. This theorem separate the eigenvalues into two parts: eigenvectors in U and eigenvectors not in U (may have intersection).

- (\subseteq) For any $Tv = \lambda v$ where non-zero $v \in V$. If $v \in U$, then $T|_U(v) = Tv = \lambda v$. If $v \notin U$, then $(T/U)(v+U) = Tv+U = \lambda v+U = \lambda(v+U)$.
- (\supseteq) For any $T|_U(v) = \lambda v$, we have $T|_U(v) = Tv = \lambda v$. The case of T/U is proven in Exercise 5.38 of E5A.

□

We will use this conclusion several times, so we prove it first.

Exercise. Let p, q two non-constant **monic** polynomial and $p = sq$, $q = tp$ where s, t two non-zero polynomial. Show that $p = q$.

Proof. We have $p = stp$, thus $st = 1$ and $\deg s = \deg t = 0$. Furthermore, we have $p = sq$ where p and q are monic, thus s must be 1, similar to t , hence $s = t = 1$ and $p = q$. □

Exercise 5.27. Let $F = R$ and V finite and $T \in \mathcal{L}(V)$. Show that the minimal polynomial of T_C is equal to the T one.

Proof. Let p the minimal polynomial of T and q the minimal polynomial of T_C . We have:

$$\begin{aligned} & p(T_C)(v + iu) \\ &= p(T)v + ip(T)u \\ &= 0v + i0u \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} & q(T)(v) \\ &= q(T_C)(v + i0) \\ &= 0 \end{aligned}$$

thus $p = sq$ and $q = tp$ where s, t are non-zero polynomials, therefore $p = q$. □

Exercise 5.28. Let V finite and $T \in \mathcal{L}(V)$. Show that the minimal polynomial of $T' \in \mathcal{L}(V')$ is equal to the T one.

Proof. Let p the minimal polynomial of T and p' the minimal polynomial of T' .

For any $\varphi \in V'$ and $v \in V$, we have $p(T')(\varphi)(v) = \varphi(p(T)v) = \varphi 0 = 0$ (since φ is linear), thus $p(T')(\varphi) = 0$, therefore $p(T') = 0$.

For any $v \in V$, $p'(T)v = \varphi_1(p'(T)v)v_1 + \cdots = p'(T')(\varphi)(v) + \cdots = 0$, where $v_0, \dots, v_{\dim V-1}$ is a basis of V and $\varphi_0, \dots, \varphi_{\dim V-1}$ is a dual basis. Thus $p'(T) = 0$.

Hence, p and p' are polynomial multiple to each other, therefore $p = p'$. \square

Exercise 5.29. Let V finite, $T \in \mathcal{L}(V)$. Show that $\mathcal{M}(T)$ is upper-triangular for some basis of $V \iff \mathcal{M}(T')$ is upper-triangular for some basis of V' .

Proof. This follows that T and T' have the same minimal polynomial. See Exercise 5.28 in E5B. \square