Exercise 5.1. *Give an example that $S, T \in \mathcal{L}(F^4)$, such that there is a subspace that is invariant under S but not T, and another subspace that is invariant under T but not S.*

Proof. $S(a, b, c, d) = (a, b, d, -c)$ and $T(a, c, -b, d)$, then $U = \operatorname{span}((0, 0, 1, 0), (0, 0, 0, 1))$ and $W = \operatorname{span}((0, 0, 1, 0), (0, 1, 0, 0))$, obviously U is invariant under S and W is invariant under T, and $T(0, 0, 1, 1) = (0, 1, -0, 1) \notin U$ and $S(0, 1, 1, 0) = (0, 1, 0, -1) \notin W$. □

Exercise 5.3. *Let $S, T \in \mathcal{L}(V)$ and S, T commute. Let $p \in \mathcal{P}(F)$.*

- *Show that* $\operatorname{null} p(S)$ *is invariant under T.*
- *Show that* $\operatorname{range} p(S)$ *is invariant under T.*

Proof.

- For any $v \in \operatorname{null} p(S)$, we have $p(S)(Tv) = T(p(S)v) = T0 = 0$ (cause S, T commute), thus $Tv \in \operatorname{null} p(S)$.
- For any $v \in \operatorname{range} p(S)$, we have $Tv = T(p(S)w) = p(S)(Tw) \in \operatorname{range} p(S)$.

□

Exercise 5.4. *Prove or disporve: Let A diagonal matrix and B upper-triangular matrix with same size as A, then A, B is commute.*

Proof. This can be disprove by Exercise 5.13 in E5D: A, B commute means B is diagonal matrix, as long as elements in the diagonal A are distinct. □

Exercise 5.5. *Show that a pair of operators in a finite vector space are commute $\iff$ their dual operators are commute.*

Proof.

$$\begin{aligned}
ST = TS &\iff \mathcal{M}(S)\mathcal{M}(T) = \mathcal{M}(T)\mathcal{M}(S) \\
&\iff (\mathcal{M}(T)^T\mathcal{M}(S)^T)^T = (\mathcal{M}(S)^T\mathcal{M}(T)^T)^T \\
&\iff \mathcal{M}(T)^T\mathcal{M}(S)^T = \mathcal{M}(S)^T\mathcal{M}(T)^T \\
&\iff \mathcal{M}(T')\mathcal{M}(S') = \mathcal{M}(S')\mathcal{M}(T') \\
&\iff T'S' = S'T'
\end{aligned}$$

□

Exercise 5.6. *Let V a non-zero, finite, complex vector space, and $S, T \in \mathcal{L}(V)$ are commute. Show that there is $\alpha, \lambda \in F$ such that*

$$\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V$$

Proof. The goal is find a common eigenvector of S, T, fortunately, there is common eigenvector for commute operators of non-zero, finite, complex vector. Thus α, λ are the eigenvalues of that common eigenvector. □

Exercise 5.7. *Let V complex vector space, $S \in \mathcal{L}(V)$ is diagonalizable, and $T \in \mathcal{L}(V)$ commutes with S. Show that there is a basis of V such that $\mathcal{M}(S)$ is diagonal and $\mathcal{M}(T)$ is upper-triangular.*

Proof. I guess V is finite. This proof basically a modified proof from book We induction on $\dim V$.

- Base($\dim V = 1$), trivial.
- Base($\dim V = n + 1$): We know S have at least one eigenvalue, then $V = \operatorname{span}(v) \oplus W$ for some W.

  Define $P(\alpha v + \beta w) = \beta w$ and $\hat{S}(w) = P(S(w))$ and $\hat{T}(w) = P(T(w))$ two operators in $\mathcal{L}(W)$. We will show that $\hat{S}, \hat{T}$ commute. $\hat{S}(\hat{T}w) = \hat{S}(P(Tw)) = \hat{S}(Tw - \alpha v) = P(STw - \alpha Sv) = P(TSw) - 0$, similarly, $\hat{T}(\hat{S}w) = P(STw)$, thus $\hat{S}\hat{T} = \hat{T}\hat{S}$.

  We then will show that $\hat{S}$ is diagonalizable, suppose $v \in E(\lambda_0, S)$, then $V = E(\lambda_0, S) \oplus E(\lambda_1, S) \oplus \cdots \oplus E(\lambda_{m-1}, S)$, where $E(\lambda_0, S) = \operatorname{span}(v) \oplus \operatorname{span}(w) \oplus \cdots$, where $v, w, \cdots$ are basis of $E(\lambda_0, S)$, thus we have $W = (\operatorname{span}(w) \oplus \cdots) \oplus E(\lambda_1, S) \oplus \cdots$. For any vector in $E(\lambda_k, S)$ is also an eigenvector of $\hat{S}$, and for any $v \in \operatorname{span}(w) \oplus \cdots$, $\hat{S}(w + \cdots) = S(w + \cdots) = \lambda w + \cdots = \lambda(w + \cdots) \in \operatorname{span}(w) \oplus \cdots$.

  Then by induction hypothesis, there is a basis of V, say $v_1, \cdots, v_{n-1}$ that $\hat{S}$ is diagonal and $\hat{T}$ is upper-triangular. Then $v, v_1, \cdots, v_{n-1}$ is a basis of V where they are eigenvectors of S cause $\mathcal{M}(\hat{S})$ is diagonal and v is an eigenvector of S. Also we have $Tv = \lambda v$ as v is also an eigenvector of T, and $Tv_k = \hat{T}(v_k) \in \operatorname{span}(v_1, \cdots, v_k) \subseteq \operatorname{span}(v, v_1, \cdots, v_k)$.

□

Exercise 5.9. *Let V finite, non-zero, complex vector space and $\mathcal{E} \subseteq \mathcal{L}(V)$, such that any pair in $\mathcal{E}$ is commute.*

- *Show that there is a vector in v such that it is eigenvector for all element in $\mathcal{E}$.*

- *Show that there is a basis of V such that any element in $\mathcal{E}$ is upper-triangular with respect to that basis.*

Note that $\mathcal{E}$ can be infinite.

Proof.

- Take any two element in $\mathcal{E}$, say S, T, we know there is a common eigenvector v of S and T. Then $\text{span}(v)$ is invariant under all element of $\mathcal{E}$, and any element that restrict to $\text{span}(v)$ have an eigenvector, therefore v is the common eigenvector for all element in $\mathcal{E}$.

- Induction on $\dim V$:

  Base($\dim V = 1$): trivial.
  Ind($\dim V = n+1$): Let v_0 the common eigenvector of all element of $\mathcal{E}$, then $V = \text{span}(v_0) \oplus W$ for some W. Define $P(\alpha v_0 + w) = w$, and $\hat{T}_i(w) = P(T_i(w))$ for all $T_i \in \mathcal{E}$. For any $T_i, T_j \in \mathcal{E}$, we have $\hat{T}_i\hat{T}_j(w) = \hat{T}_i(P(T_j)w) = \hat{T}_i(T_jw - \alpha v_0) = P(T_iT_jw - \alpha T_iv_0)$, recall that v_0 is the common eigenvector of all $T_k \in \mathcal{E}$, thus $P(T_iT_jw - \alpha T_iv_0) = P(T_iT_jw)$, similarly $\hat{T}_j\hat{T}_iw = P(T_jT_iw)$, therefore $\hat{T}_i, \hat{T}_j$ commute.

  By induction hypothesis, we know there is a basis of W such that $\hat{T}_i$ is upper-triangular, say $v_1, \cdots, v_{n-1}$. Then the basis $v_0, \cdots, v_{n-1}$ makes all T_i upper-triangular, cause $T_iv_0 \in \text{span}(v_0)$ and $T_iv_j \in \text{span}(v_1, \cdots, v_{n-1}) \subseteq \text{span}(v_0, \cdots, v_{n-1})$ for any $j = 1, \cdots, n-1$.

$\square$