Exercise 5.1. *Let $T \in \mathcal{L}(V)$. Show that 9 is an eigenvalue of $T^2 \iff 3$ or -3 is an eigenvalue of T.*

Proof.

- ($\Rightarrow$) We have $T^2 - 9I$ is not injective since 9 is an eigenvalue of T^2, then $(T-3I)(T+3I) = T^2 - 9I$ is not injective means one of $T - 3I$ and $T + 3I$ is not injective, thus 3 or -3 is an eigenvalue of T.

- ($\Leftarrow$) Similarly, we have $(T-3I)(T+3I)v = (T^2 - 9I)v = 0$ (if 3 is an eigenvalue of T) or $(T+3I)(T-3I)v = (T^2-9I)v = 0$ (if -3 is an eigenvalue of T).

□

Exercise 5.2. *Let V a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$ has no eigenvalue. Show that any subspace of V that is invariant under T is either $\{0\}$ or infinite dimension.*

Proof. Let $U \subseteq V$ a subspace that is invariant under T, and non-zero $u \in U$. We can repeatly apply T to u, say $u, Tu, T^2u, \cdots$. Suppose $k > 0$ is minimum such that $u, Tu, \cdots, T^k u$ is linear dependent, we have $p \in \mathcal{P}(\mathbb{C})$ with $\deg p = k$ such that $p(T) = 0$. Clearly p is not constants, thus it has a zero since p is a polynomial of complex coefficient. Thus such zero is an eigenvalue of T. □

Exercise 5.3. *Let $n > 1$ an integer, and $T \in \mathcal{L}(F^n)$ is defined by:*

$$T(x_0, \cdots, x_{n-1}) = (x_0 + \cdots + x_{n-1}, \cdots, x_0 + \cdots + x_{n-1})$$

- *Find all eigenvalue and eigenvector of T.*

- *Find the minimal polynomial of T.*

Proof.

- Observe that $\operatorname{range} T = \operatorname{span}((1, \cdots, 1))$, thus $T(1, \cdots, 1) = n(1, \cdots, 1)$.

- Observe that $T(x_0, \cdots, x_{n-1}) = (x_0 + \cdots + x_{n-1})(1, \cdots, 1)$ and $T^2(x_0, \cdots, x_{n-1}) = n(x_0 + \cdots + x_{n-1})(1, \cdots, 1)$, thus $p(T) = nT - T^2 = 0$.

□

Exercise 4 is kinda hard, sorry.

Exercise 5.6. *Let $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T.*

Proof. Observe that $T^2(w, z) = T(-z, w) = (-w, -z) = (-1)(w, z)$, thus the minimal polynomial of T is $p(T) = I + T^2$. □

Exercise 5.7.

- *Given an example that the minimal polynomial of ST is not equal to TS's.*
- *Suppose V is finite and $S, T \in \mathcal{L}(V)$. Show that the minimal polynomial of ST is equal to TS's if one of S and T is invertible.*

  Hint: Show that S is invertible and $p \in \mathcal{P}(F)$ implies $p(TS) = S^{-1}p(ST)S$.

Proof.

- The idea is to find S, T such that $ST \neq 0$ but $TS = 0$. We can find $S(x, y) = (x, 0)$ and $T(x, y) = (y, 0)$ holds:

  $$(ST)(x, y) = S(y, 0) = (y, 0)$$
  $$(TS)(x, y) = T(x, 0) = (0, 0)$$

  Thus the minimal polynomial of ST is not 0 but TS one does.

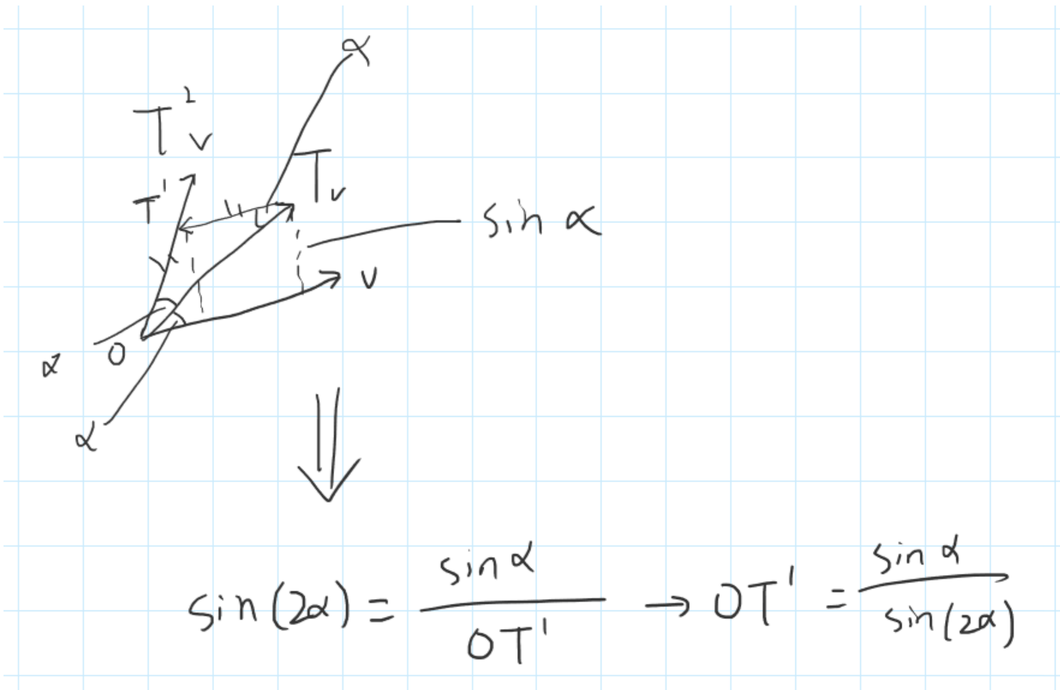
- Suppose S is invertible and $p \in \mathcal{L}(F)$ is the minimal polynomial of TS, then $p(TS) = S^{-1}p(ST)S$ since i-th term of $S^{-1}p(ST)S$ has form $S^{-1}c_i(ST)^iS = c_i(S^{-1}S)(TS)^{i-1}(TS) = c_i(TS)^i$. Thus $S^{-1}p(ST)S = 0$ and then $p(ST) = 0$. We will show that p is the minimal polynomial of ST, suppose $q \in \mathcal{L}(F)$ such that $q(ST) = 0$, then $0 = S^{-1}q(ST)S = q(TS)$, therefore $\deg q = \deg p$. Hence p is the minimal polynomial of ST.

□

Exercise 5.8. *Let $T \in \mathcal{L}(R^2)$ is the opeartor that "rotates 1 degree counter-clockwise", find the minimal polynomial of T.*

*Note that it is **NOT** $x^{180} + 1$ even $T^{180} = -I$.*

Proof. Note that there is some λ such that $Tv - \lambda v = \alpha T^2 v$ (We can show that $\lambda = \alpha$), however the calculation is too complicate.



λ should be $\frac{\sin(1^\circ)}{\sin(2^\circ)}$, thus $p(T) = -\lambda I + T - \lambda T^2$.

We suppose all v below has length 1, thus $v = (\cos\theta, \sin\theta)$, this doesn't lose the generalizability since $p(T)(\alpha v) = \alpha(p(T)v)$.

For the first component of $p(T)v = -\lambda v + Tv - \lambda T^2 v$, we have:

$$
\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - \cos(\theta + 2^\circ)) \\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - (\cos\theta\cos(2^\circ) - \sin\theta\sin(2^\circ))) \\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - \cos\theta\cos(2^\circ)) + \sin\theta\sin(1^\circ) \\
=&\cos\theta\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) + \sin\theta\sin(1^\circ)
\end{aligned}
$$

where $\sin\theta\sin(1^\circ)$ cancels a part of $(Tv)_1 = \cos(\theta + 1^\circ) = \cos\theta\cos(1^\circ) -$

$\sin\theta\sin(1^\circ)$. Thus we will show that $\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1-\cos(2^\circ)) = -\cos(1^\circ)$.

$$\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1-\cos(2^\circ))\\
=&\frac{\sin(1^\circ)}{2\sin(1^\circ)\cos(1^\circ)}(-(\cos^2(1^\circ)+\sin^2(1^\circ))-\cos^2(1^\circ)+\sin^2(1^\circ))\\
=&\frac{1}{2\cos(1^\circ)}(-\cos^2(1^\circ)-\cos^2(1^\circ))\\
=&\frac{1}{2\cos(1^\circ)}(-2\cos^2(1^\circ))\\
=&-\cos(1^\circ)
\end{aligned}$$

For the second component of $p(T)v$, we have:

$$\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\sin\theta-\sin(\theta+2^\circ))\\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\sin\theta-\sin\theta\cos(2^\circ)-\cos\theta\sin(2^\circ))\\
=&\sin\theta\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1-\cos(2^\circ))-\cos\theta\sin(1^\circ)
\end{aligned}$$

similarly, we have $p(T)v_2 = \sin(\theta+1^\circ) = \sin\theta\cos(1^\circ)+\cos\theta\sin(1^\circ)$ we will show that $\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1-\cos(2^\circ)) = -\cos(1^\circ)$, which is proven above. □

Exercise 5.9. *Let $T \in \mathcal{L}(V)$ such that for some basis of V, $\mathcal{M}(T)$ consists of rational numbers. Try to explain why the coefficients of the minimal polynomial of T is rational numbers.*

Proof. I don't know, because $\mathbb{Q}$ is also a field? □

Exercise 5.11. *Let V a vector space and $\dim V = 2$ and $T \in \mathcal{L}(V)$ such that $\mathcal{M}(T)$ for some basis of V is $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Show that:*

- $T^2 - (a+d)T + (ad-bc)I = 0$

- *the minimal polynomial of T is:*

$$\begin{cases} z-a & \textit{if } b=c=0 \textit{ and } a=d \\ z^2-(a+d)z+(ad-bc) & \textit{otherwise} \end{cases}$$

Proof.

- $$\begin{aligned} &\mathcal{M}(T^2-(a+d)T+(ad-bc)I) \\ &= \begin{bmatrix} a & c \\ b & d \end{bmatrix}^2 - (a+d)\begin{bmatrix} a & c \\ b & d \end{bmatrix} + (ad-bc)I \\ &= \begin{bmatrix} a^2+bc & ac+bd \\ ab+bd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ac+cd \\ ab+bd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \end{aligned}$$

- If $b=c=0$ and $a=d$, then T is a scalar multiple of identity operator, thus $T=aI$ and $p(T)=-aI+T=0$. Otherwise, $T^2-(a+d)T+(ad-bc)I=0$.

□

Exercise 5.13. *Let V finite, $T\in\mathcal{L}(V)$, $p\in\mathcal{P}(F)$. Show that there is a unique $r\in\mathcal{P}(F)$ such that $p(T)=r(T)$ where $\deg p$ is less than the degree of the minimal polynomial of T.*

Proof. Let q the minimal polynomial of T.

If $\deg p<\deg q$, then $r=p$. The uniqueness is guaranteed by $\deg p<\deg q$ (try $(p-s)(T)$ where $p(T)=s(T)$ and $\deg s<\deg q$).

If $\deg p>=\deg q$, then $p=sq+r$ where $s,r\in\mathcal{P}(F)$ with $\deg r<\deg q$.Then $p(T)=s(T)q(T)+r(T)=r(T)$ since $q(T)=0$. The uniqueness is guaranteed by the property of division. □

Exercise 5.14. *Let V finite, $T\in\mathcal{L}(V)$ with minimal polynomial $p(z)=4+5z-6z^2-7z^3+2z^4+z^5$. Find the minimal polynomial of T^{-1}.*

Proof. Suppose p is the minimal polynomial of T, then we can repeatly apply T^{-1} to $p(T)$, say $T^{-(\deg p)}(p(T))$, then it should be 0, and the coefficients are reversed, that is, $p_{\deg p}I+p_{\deg p-1}T^{-1}+\cdots+p_0(T^{-1})^{\deg p}$.

So the answer is $1+2z^1-7z^2-6z^3+5z^4+4z^5$. □

Exercise 5.16. *Let $a_0, \cdots, a_{n-1} \in F$ and T an operator over F^n. Its matrix (about the standard basis) is:*

$$\begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & \ddots & & & -a_2 \\ & & \ddots & & & \vdots \\ & & & 1 & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix}$$

. *Show that the minimal polynomial of T is:*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

Proof. We first need some property of this matrix, we will see it moves all number to the left when we repeatly self-multily T. We can see the k-th column of T^p is equal to $k+1$-th column of T^{p-1}, thus it is also equal to i-th column if T^j where $1 \leq i, j \leq n$ and $i + j = k + p$. In fact, j can be 0 and we have $T^0 = I$ and the property still holds.

Then, the i-th column of T^n is equal to n-th column (the last one) of T^i, and it is produced by $T^{i-1} \begin{bmatrix} -a_0 \\ -a_1 \\ \vdots \\ -a_{n-1} \end{bmatrix}$, which is equal to

$$T_i^n = -a_0 T_1^{i-1} - a_1 T_2^{i-1} - \cdots - a_{n-1} T_n^{i-1}$$

which is equal to $T^n v_i$ where v_i is i-th standard basis of F^n, that is, $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$. We may rewrite the equation into

$$T^n v = -a_0 T^0 v - a_1 T^1 v - \cdots - a_{n-1} T^{n-1} v$$

where $T^0 v = T_i^0 = T_1^{i-1}$, $T^1 v = T_i^1 = T_2^{i-1}$ and so on.

Thus all v vector in standard basis has $p(T)v = 0$, thus $p(T) = 0$.

For minimal, we can see T is invertible, thus $p(T)v_1 = 0$ (recall that T moves number to the left, thus the first column of T^i is the i-th columns of T). means there is a (non-zero) linear combination of columns of T that is equal to 0. Thus $\deg p \geq n$ since T the columns are linear independent. $\square$

Exercise 5.17. *Let V finite and $T \in \mathcal{L}(V)$ and p is the minimal polynomial of T. Let $\lambda \in F$, show that the minimal polynomial of $T - \lambda I$ is $q(z) = p(z+\lambda)$.*

Proof. $q(T - \lambda I) = p((T - \lambda I) + \lambda I) = p(T) = 0$. Suppose r is the minimal polynomial of $T - \lambda I$, then $s(z) = r(z - \lambda)$ and $s(T) = r(T - \lambda I) = 0$, thus $\deg r = \deg p = \deg q$. $\square$

Exercise 5.18. *Let V finite and $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Let $\lambda \in F$ that $\lambda \neq 0$, show that the minimal polynomial of λT is $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.*

Proof. $q(\lambda T) = \lambda^{\deg p} p(\frac{1}{\lambda}(\lambda T)) = \lambda^{\deg p} p(T) = 0$. $\lambda^{\deg p}$ only makes q a monic polynomial.

Suppose r is the minimal polynomial of λT, then $s(z) = \frac{1}{\lambda^{\deg p}} r(\lambda z)$ and $s(T) = \frac{1}{\lambda^{\deg p}} r(\lambda T) = 0$, thus $\deg s = \deg r = \deg p = \deg q$. $\square$

Exercise 5.19. *Let V finite and $T \in \mathcal{L}(V)$. Let $\mathcal{E} \subseteq \mathcal{L}(V)$ a subspace, defined by*

$$\mathcal{E} = \{ \ q(T) \mid q \in \mathcal{P}(F) \ \}$$

.

Show that $\dim \mathcal{E}$ *is equal to the degree of the minimal polynomial of T.*

Proof. We can see $I, T, T^2, \cdots, T^{\deg p - 1}$ is linear independent (since p is the minimal polynomial of T) where p is the minimal polynomial of T. For any $q \in \mathcal{P}(F)$ where $\deg q \geq \deg p$, then $q = sp + r$ where $s, r \in \mathcal{P}(F)$ and $\deg r < \deg p$, therefore $q(T) = s(T)p(T) + r(T) = r(T) \in \operatorname{span}(I, T, T^2, \cdots, T^{\deg p - 1})$. $\square$

Exercise 5.20. *let $T \in \mathcal{L}(F^4)$, which eigenvalues are $3, 5, 8$. Show that $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$.*

Proof. Suppose p is the minimal polynomial of T, then $p(z) = c(z-3)(z-5)(z-8)q(z)$ since $3, 5, 8$ are the eigenvalue of T, thus the zeros of p. Note that $\deg q \leq 1$ since $\deg p \leq \dim F^4 = 4$. Since there is no other eigenvalue (thus zero) than $3, 5, 8$, q is either 1 or one of $z - 3$, $z - 5$, $z - 8$, thus $(z - 3)^2(z - 5)^2(z - 8)^2$ is polynomial multiple of p, therefore $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$. $\square$

Exercise 5.21. *Let V finite and $T \in \mathcal{L}(V)$. Show that the degree of the minimal polynomial of T caps at $1 + \dim \operatorname{range} T$.*

Proof. IDk □

Exercise 5.22. *Let V finite and $T \in \mathcal{L}(V)$. Show that T is invertible $\iff$ $I \in \operatorname{span}(T, T^2, \cdots, T^{\dim V})$.*

Proof.

- ($\Rightarrow$) Suppose T is invertible, then the minimal polynomial p of T satisfies $p(0) \neq 0$ (since $p(0) = 0$ implies 0 is a eigenvalues of T). We know $\deg p \leq \dim V$, thus there is a linear combination of $T, T^2, \cdots, T^{\dim V}$ that is equal to a scalar multiple of I, therefore $I \in \operatorname{span}(T, T^2, \cdots, T^{\dim V})$.

- ($\Leftarrow$) Suppose $I = \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_{\dim V} T^{\dim V}$, then $I = T(\lambda_1 I + \lambda_2 T + \cdots + \lambda_{\dim V} T^{\dim V - 1}) = (\lambda_1 I + \lambda_2 T + \cdots + \lambda_{\dim V} T^{\dim V - 1})T$, thus T is invertible.

□

Exercise. *Let V a vector space and $T \in \mathcal{L}(V)$, $v, Tv, \cdots, T^k v$ a list of linear independent vectors but $v, Tv, \cdots, T^{k+1} v$ isn't. Show that $T^{k+i} v \in \operatorname{span}(v, Tv, \cdots, T^k v)$ for all $0 < i$.*

Proof. Induction on i.

- Base($i = 1$): By assumption.

- Ind($i = i+1$): $T^{k+i+1} v = T(T^{k+i} v)$, since $T^{k+i} v \in \operatorname{span}(v, Tv, \cdots, T^k v)$, thus it can be write as a linear combination of $v, Tv, \cdots, T^k v$, say $T(\lambda_0 v + \lambda_1 Tv + \cdots + \lambda_k T^k v)$, then $\lambda_0 Tv + \lambda_1 T^2 v + \cdots, + \lambda_k T^{k+1} v \in \operatorname{span}(v, Tv, \cdots, T^k v)$ since $T^{k+1} v \in \operatorname{span}(v, Tv, \cdots, T^k v)$.

□

Exercise 5.23. *Let V finite and $T \in \mathcal{L}(V)$. Let $n = \dim V$, show that for any $v \in V$, $\operatorname{span}(v, Tv, \cdots, T^{n-1} v)$ is invariant under T.*

Proof. Note that the list $v, Tv, \cdots, T^{n-1} v$ has length $n = \dim V$, thus for the list $v, Tv, \cdots, T^n v$ is linear dependent, thus $T^n v$ must be a linear combination of $v, Tv, \cdots, ^{n-1} v$.

- If $v, Tv, \cdots, T^{n-1}v$ is linear dependent, then $T^n v \in \operatorname{span}(v, Tv, \cdots, T^{n-1}v)$ (by our lemma exercise).

- Otherwise, the list $v, Tv, \cdots, T^n v$ is linear dependent while $v, Tv, \cdots, T^{n-1}v$ isn't, therefore $T^n v$ is a linear combination of $v, Tv, \cdots, T^{n-1}v$.

$\square$

Theorem 5.29. *$q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial of T.*

Proof.
- ($\Rightarrow$) Let p the minimal polynomial of T, consider $q = sp+r$ where $\deg r < \deg p$, we may suppose $r \neq 0$. Then $0 = q(T) = s(T)p(T) + r(T) = r(T)$, which contradict to the assumption that p is the minimal polynomial of T.

- ($\Leftarrow$) Trivial.

$\square$

Exercise 5.25. *Let V finite, $T \in \mathcal{L}(V)$, subspace $U \subseteq V$ is invariant under T.*

- *Show that the minimal polynomial of T is polynomial multiple of the minimal polynomial of T/U.*

- *Show that*

$$(\textit{the minimal polynomial of } T\big|_U) \times (\textit{the minimal polynomial of } T/U)$$

is a polynomial multiple of the minimal polynomial of T.

Proof.
- Let p the minimal polynomial of T, then $p(T/U)(v+U) = p(T)v + U = 0 + U$ for any $v + U \in V/U$, thus $p(T/U) = 0$, therefore p is a polynomial multiple of the minimal polynomial of T/U.

- Let p the minimal polynomial of $T\big|_U$ and q the minimal polynomial of T/U. Then $(pq)(T)v = (p(T)q(T))v = p(T)(q(T)v)$ where $q(T)v \in U$, thus $p(T)(q(T)v) = 0$.

$\square$

Exercise 5.26. *Let V finite, $T \in \mathcal{L}(V)$, U is invariant under T. Show that the set of eigenvalues of T is equal to the union of eigenvalues of $T\big|_U$ and T/U.*

Proof. This theorem separate the eigenvalues into two parts: eigenvectors in U and eigenvectors not in U (may have intersection).

- ($\subseteq$) For any $Tv = \lambda v$ where non-zero $v \in V$. If $v \in U$, then $T\big|_U(v) = Tv = \lambda v$. If $v \notin U$, then $(T/U)(v+U) = Tv + U = \lambda v + U = \lambda(v+U)$.
- ($\supseteq$) For any $T\big|_U(v) = \lambda v$, we have $T\big|_U(v) = Tv = \lambda v$. The case of T/U is proven in Exercise 5.38 of E5A.

$\square$

We will use this conclusion several times, so we prove it first.

Exercise. *Let p, q two non-constant* ***monic*** *polynomial and $p = sq$, $q = tp$ where s, t two non-zero polynomial. Show that $p = q$.*

Proof. We have $p = stp$, thus $st = 1$ and $\deg s = \deg t = 0$. Furthermore, we have $p = sq$ where p and q are monic, thus s must be 1, similar to t, hence $s = t = 1$ and $p = q$. $\square$

Exercise 5.27. *Let $F = R$ and V finite and $T \in \mathcal{L}(V)$. Show that the minimal polynomial of T_C is equal to the T one.*

Proof. Let p the minimal polynomial of T and q the minimal polynomial of T_C. We have:

$$
\begin{aligned}
&p(T_C)(v+iu)\\
=&p(T)v + ip(T)u\\
=&0v + i0u\\
=&0
\end{aligned}
$$

and

$$
\begin{aligned}
&q(T)(v)\\
=&q(T_C)(v+i0)\\
=&0
\end{aligned}
$$

thus $p = sq$ and $q = tp$ where s, t are non-zero polynomials, therefore $p = q$. $\square$

Exercise 5.28. *Let V finite and $T \in \mathcal{L}(V)$. Show that the minimal polynomial of $T' \in \mathcal{L}(V')$ is equal to the T one.*

Proof. Let p the minimal polynomial of T and p' the minimal polynomial of T'.

For any $\varphi \in V'$ and $v \in V$, we have $p(T')(\varphi)(v) = \varphi(p(T)v) = \varphi 0 = 0$ (since φ is linear), thus $p(T')(\varphi) = 0$, therefore $p(T') = 0$.

For any $v \in V$, $p'(T)v = \varphi_1(p'(T)v)v_1 + \cdots = p'(T')(\varphi)(v) + \cdots = 0$, where $v_0, \cdots, v_{\dim V-1}$ is a basis of V and $\varphi_0, \cdots, \varphi_{\dim V-1}$ is a dual basis. Thus $p'(T) = 0$.

Hence, p and p' are polynomial multiple to each other, therefore $p = p'$. $\square$

Exercise 5.29. *Let V finite, $T \in \mathcal{L}(V)$. Show that $\mathcal{M}(T)$ is upper-triangular for some basis of $V \iff \mathcal{M}(T')$ is upper-triangular for some basis of V'.*

Proof. This follows that T and T' have the same minimal polymonial. See Exercise 5.28 in E5B. $\square$