**Definition 3.1** (Notation: $v+U$)**.** *Let $v \in V$ and $U \subseteq V$, then $v+U = \{ v+u \mid u \in U \}$.*

Such sets also called *coset* in group theory.

**Definition 3.97** (Translate)**.** *Let $v \in V$ and $U \subseteq V$, we say $v+U$ is a translate of $U$.*

**Definition 3.98** (Quotient Space)**.** *Let $U \subseteq V$ a subspace, then the quotient space $V/U$ is a set with translates of $U$, that is:*

$$V/U = \{ v+U \mid v \in V \}$$

**Theorem 3.101.** *Let $U \subseteq V$ a subspace and $v, w \in V$, then the following statements are equivalent.*

1. $v - w \in U$
2. $v + U = w + U$
3. $(v+U) \cap (w+U) \neq \varnothing$

*Proof.*

- If $v-w \in U$, for any $v+u \in v+U$, we have $v+u = v+(v-w)-(v-w)+u = v-w+w+u = w+(v-w)+u \in w+U$ since $v-w \in U$. Similarly, for any $w+u \in w+U$, we have $w+u = w+(v-w)-(v-w)+u = v-v+w+u = v-(v-w)+u = v+(-(v-w)+u) \in v+U$.
- If $v+U = w+U$, then $v = w+u$ since $v \in v+U$, therefore $v-w = u \in U$.
- if $v+U = w+U$, then $(v+U) \cap (w+U) = v+U = w+U \neq \varnothing$
- If $(v+U) \cap (w+U) \neq \varnothing$, then for any $v+u_0 = w+u_1 \in (v+U) \cap (w+U)$, we have $(v-w)+(u0-u_1) = 0$ and then $v-w = u_1 - u_0 \in U$, so $v+U = w+U$.

□

**Definition 3.102.** *Let $U \subseteq V$, then addition and scalar multiplication on $V/U$ is defined by:*

$$\begin{aligned}(v+U)+(w+U) &= (v+w)+U \\ \lambda(v+U) &= (\lambda v)+U\end{aligned}$$

**Theorem 3.103.** *Let $U \subseteq V$ a subspace, then $V/U$ is a vector space with addition and scalar multiplication we defined in previous definition.*

*Proof.* We must first show that the addition and the sclar multiplication we introduce are functions.

For any $a, b, c, d \in V$, we will show $(a+b)+U = (c+d)+U$ if $a+U = c+U$ and $b+U = d+U$. We can show $(a+b)-(c+d) \in U$ by $a-c \in U$ and $b-d \in U$.

For any $v, w \in V$ and $\lambda \in F$, we will show $(\lambda v)+U = (\lambda w)+U$ if $v+U = w+U$. We know $v-w \in U$, then $\lambda(v-w) = \lambda v - \lambda w \in U$, therefore $(\lambda v)+U = (\lambda w)+U$.

We have identity of addition $0+U$ and inverse of addition $(-v)+U$ for all $v \in V$. $\square$

**Definition 3.104.** *Let $U \subseteq V$ a subspace, the quotient map $\pi : V \to V/U$ is a linear mapping defined by:*

$$\pi(v) = v + U$$

*Proof.* We will show $\pi$ is a linear mapping, $\pi(v+w) = (v+w)+U = v+U+w+U = \pi(v)+\pi(w)$ and $\lambda\pi(v) = \lambda(v+U) = (\lambda v)+U = \pi(\lambda v)$. $\square$

**Theorem 3.105.** *Let $V$ finite and $U \subseteq V$ a subspace, show that* $\dim(V/U) = \dim V - \dim U$.

*Proof.* We can rewrite the equation as $\dim V = \dim(V/U) + \dim U$, and it is easy to see that $\operatorname{range}\pi = \dim(V/U)$ and $\operatorname{null}\pi = \dim U$. $\square$

**Definition 3.106.** *Let $T \in \mathcal{L}(V, W)$, define $\tilde{T} : V/(\operatorname{null} T) \to W$ by $\tilde{T}(v + \operatorname{null} T) = Tv$.*

**Theorem 3.107.** *Let $T \in \mathcal{L}(V, W)$, then:*

1. $\tilde{T} \circ \pi = T$
2. $\tilde{T}$ *is injective*
3. $\operatorname{range}\tilde{T} = \operatorname{range} T$
4. $V/(\operatorname{null} T) \cong \operatorname{range} T$

*Proof.*

1. For all $v \in V$, $\tilde{T}(\pi(v)) = \tilde{T}(v + \operatorname{null} T) = Tv$
2. If $\tilde{T}(v + \operatorname{null} T) = \tilde{T}(w + \operatorname{null} T)$, then $T(v - w) = 0$, which means $v - w \in \operatorname{null} T$, therefore $v + \operatorname{null} T = w + \operatorname{null} T$.
3. For any $Tv \in \operatorname{range} T$, we have $\tilde{T}(v + \operatorname{null} T) \in \operatorname{range} \tilde{T}$. For any $\tilde{T}(v + \operatorname{null} T) = Tv \in \operatorname{range} \tilde{T}$, we have $Tv \in \operatorname{range} T$.
4. Restrict the range of $\tilde{T}$ on $\operatorname{range} T$, say $\varphi(v + \operatorname{null} T) = \tilde{T}(v + \operatorname{null} T) : V/(\operatorname{null} T) \to \operatorname{range} T$, then $\varphi$ is injective since (2) and surjective since (3), therefore $\varphi$ is an isomorphism, thus $V/(\operatorname{null} T) \simeq \operatorname{range} T$.

$\square$