1 Functors

Definition 1.1 (Full). *A functor $F : \mathcal{C} \to \mathcal{D}$ is called full, if for any $a, b \in \mathcal{C}$, the mapping on morphism $F : \mathcal{C}(a, b) \to \mathcal{D}(Fa, Fb)$ is surjective.*

Definition 1.2 (Faithful). *A functor $F : \mathcal{C} \to \mathcal{D}$ is called faithful, if for any $a, b \in \mathcal{C}$, the mapping on morphism $F : \mathcal{C}(a, b) \to \mathcal{D}(Fa, Fb)$ is injective.*

Definition 1.3 (Essentially Full). *A functor $F : \mathcal{C} \to \mathcal{D}$ is called Essentially full, if for any $a \in \mathcal{C}$, the mapping on object $F : \mathcal{C} \to \mathcal{D}$ is surjective.*

Theorem 1.1. *Suppose $F : \mathcal{C} \to \mathcal{D}$ a functor, and $f : a \to b$ a morphism in $\mathcal{C}$. Then f is an isomorphism iff Ff is an isomorphism.*

Proof. ($\Rightarrow$) We claim $F(f^{-1}) : Fb \to Fa$ is an inverse, we can see that $F(f^{-1} \circ f) = F(id_a) = id_{Fa}$ and $F(f \circ f^{-1}) = F(id_b) = id_{Fb}$.

($\Leftarrow$) Suppose Fg is the inverse of Ff, and we can retrieve g from Fg cause F is full faithful. Then $F(g \circ f) = Fg \circ Ff = id_{Fa} = F(id_a)$ therefore $g \circ f = id_a$ since F is full faithful, similar to $F(f \circ g)$, so f is indeed an isomorphism. □

Corollary 1.1. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, show that F is injective on object.*

Proof. Trivial by previous theorem. □

Note that a commuting diagram applied to a functor is still commutes, due to the functoriality:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & b \arrow[d, "j"] \\ c \arrow[r, "g"'] & d \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, "Fi"'] & Fb \arrow[d, "Fj"] \\ Fc \arrow[r, "Fg"'] & Fd \end{tikzcd}$$

Definition 1.4 (Natural Transform). *Suppose $F, G : \mathcal{C} \to \mathcal{D}$ are functors, then $\alpha : F \Rightarrow G$ is called a natural transform from F to G, if:*

- *For any $x \in \mathcal{C}$, $\alpha_x : Fx \to Gx$ a morphism in $\mathcal{D}$.*

- *Furthermore, for any morphism $f : x \to y$ in $\mathcal{C}$, the following square commutes:*

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x" description] & Fy \arrow[d, "\alpha_y" description] \\
Gx \arrow[r, "Gf"] & Gy
\end{tikzcd}$$

The one of composition of two natural transforms is vertical composition:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=40, "F"{name=F}] \arrow[r, "G" description, ""{name=G}] \arrow[r, bend right=40, "H"'{name=H}] & \mathcal{D}
\arrow[from=F, to=G, phantom, "\Downarrow\alpha"]
\arrow[from=G, to=H, phantom, "\Downarrow\beta"]
\end{tikzcd}$$

which is indeed a natural transform cause:

$$\begin{tikzcd}
Fx \arrow[r, "Ff" description] \arrow[d, "\alpha_x" description] \arrow[dd, bend right=30, dashed, "(\beta\circ\alpha)_x"'] & Fy \arrow[d, "\alpha_y" description] \arrow[dd, bend left=30, dashed, "(\beta\circ\alpha)_y"] \\
Gx \arrow[r, "Gf" description] \arrow[d, "\beta_x" description] & Gy \arrow[d, "\beta_y" description] \\
Hx \arrow[r, "Hf" description] & Hy
\end{tikzcd}$$

the outer diagram commutes.

There is another way to compose two natural transforms, the horizontal composition:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"{name=F}] \arrow[r, bend right, "F'"'{name=F2}] & \mathcal{D} \arrow[r, bend left, "G"{name=G}] \arrow[r, bend right, "G'"'{name=G2}] & \mathcal{E}
\arrow[from=F, to=F2, phantom, "\Downarrow\alpha"]
\arrow[from=G, to=G2, phantom, "\Downarrow\beta"]
\end{tikzcd}$$

We would expect that there is a natural transform $\beta \cdot \alpha : G \circ F \Rightarrow G' \circ F'$, but how? Firstly, we have the following diagram commutes cause α is a natural transform:

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x"] & Fy \arrow[d, "\alpha_y"] \\
F'x \arrow[r, "F'f"] & F'y
\end{tikzcd}$$

Then we apply it to the functor G.

$$\begin{tikzcd}
G(Fx) \arrow[r, "G(Ff)"] \arrow[d, "G(\alpha_x)"] & G(Fy) \arrow[d, "G(\alpha_y)"] \\
G(F'x) \arrow[r, "G(F'f)"] & G(F'y)
\end{tikzcd}$$

It is similar to what we want, beside the bottom arrow, it is time to use β.

$$\begin{tikzcd}
G(Fx) \arrow[r, "G(Ff)"] \arrow[d, "G(\alpha_x)"] & G(Fy) \arrow[d, "G(\alpha_y)"] \\
G(F'x) \arrow[r, "G(F'f)"] \arrow[d, "\beta_{F'x}"] & G(F'y) \arrow[d, "\beta_{F'y}"] \\
G'(F'x) \arrow[r, "G'(F'f)"] & G'(F'y)
\end{tikzcd}$$

And the $\beta_{F'\_} \circ G(\alpha_\_)$ is the definition of $\beta \cdot \alpha$.

Also, if one of the natural transform is the identity transform, say $id_G \cdot \alpha$, then it can be denoted by $G \cdot \alpha$. Notice that $G \cdot \alpha$ has type $G \circ F \Rightarrow G \circ F'$, which "modifies" only one side.

You can see that the horizontal composition is much different from vertical composition, the former one is much like a product of morphism (if you treat $\circ$ as some kind of product):

$$\alpha : F \Rightarrow F'$$
$$\beta : G \Rightarrow G'$$
$$\beta \cdot \alpha : F \circ G \Rightarrow F' \circ G'$$

While the later one is much like a composition of morphism:

$$
\begin{aligned}
\alpha &: F \Rightarrow G \\
\beta &: G \Rightarrow H \\
\beta \circ \alpha &: F \Rightarrow H
\end{aligned}
$$

It looks like we can write horizontal composition in vertical composition of two horizontal compositions:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "G"] & \mathcal{E}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend right, "F'"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}
$$

In symbol, it is $G(\alpha_-)$ (the former one) and $\beta_{F'-}$ (the later one), and finally $\beta_{F'-} \circ G(\alpha_-)$, which is exactly the horizontal composition $\beta \cdot \alpha$. Similarly, we might suppose there is another definition of horizontal composition: $G'(\alpha_-) \circ \beta_{F-}$ which is the vertical composition of:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend right, "G'"'] & \mathcal{E}
\end{tikzcd}
$$

The corresponding diagram would be: apply the naturality diagram of α to G', then put β above it.