Exercise 3.7. *Suppose vector space V and W are finite ($2 \leq \dim V \leq \dim W$), show that $\{\, T \in \mathcal{L}(V, W) \mid T$ is not injective $\}$ is not a subspace.*

Proof. Consider the basis $v_0 + \cdots + v_{(\dim V - 1)} \in V$, and $T(v_0 + \cdots + v_{(\dim V - 1)}) = (0 + 1_0 + \cdots + 1_{(\dim V - 1)})$ and $T'(v_0 + \cdots + v_{(\dim V - 1)}) = (v_0 + 0 + \cdots + v_{(\dim V - 1)})$. Then $(T + T')(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_{(\dim V - 1)} v_{(\dim V - 1)}) = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + 2\lambda_{(\dim V - 1)} v_{(\dim V - 1)}$, which is obviously injective. $\qquad \square$

Exercise 3.11. *Suppose V is finite and $T \in \mathcal{L}(V, W)$, show that there is a subspace $U \subset V$ such that:*

$$ U \cap \operatorname{null} T = \{0\} \quad and \quad \operatorname{range} T = \{\, Tu \mid u \in U \,\} $$

Proof. This is similar to the *isomorphism theorems* about groups! This can be done by the similar way we used in proving $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. $\qquad \square$

The next two exercises remind me the categorical injective and surjective, let try them first!

Exercise. *For any $F \in \mathcal{L}(V, W)$, F is injective $\iff$ for any $S, T \in \mathcal{L}(U, V)$, $FS = FT$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(V, W)$ that $FS = FT$, then for any $u \in U$, we have $F(Su) = F(Tu)$, since F is injective, we know $Su = Tu$, so $S = T$.

- ($\Leftarrow$) For any $v, w \in V$ such that $Fv = Fw$. Consider

$$ S(\lambda) = \lambda v $$
$$ T(\lambda) = \lambda w $$

  in $\mathcal{L}(\mathbb{R}, V)$. Then for any $\lambda \in \mathbb{R}$, we have $FS\lambda = \lambda FS1 = \lambda Fv = \lambda Fw = \lambda FT1 = FT\lambda$. so $FS = FT$ then $S = T$, which means $v = S1 = T1 = w$.

$\qquad \square$

Exercise. *Suppose W is finite, then for any $F \in \mathcal{L}(V, W)$, F is surjective $\iff$ for any $S, T \in \mathcal{L}(W, U)$, $SF = TF$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(W, U)$ such that $SF = TF$. For any $w \in W$, there is $v \in V$ such that $Fv = w$ since F is surjective. Then we have $SFv = TFv$ so $Sw = S(Fv) = T(Fv) = Tw$ then $S = T$.

- ($\Leftarrow$) Consider

$$S = I \quad \text{and} \quad T(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 w_0 + \cdots + \lambda_k w_k$$

where $w_0, \cdots, w_k$ is the basis of range F and $w_0, \cdots, w_n$ is the basis of W that expand from $w_0, \cdots, w_k$.

It is easy to show that T is a linear transformation. Then for any $v \in V$, we have $TFv = Fv$ (since T acts like identity transformation on range F) and $SFv = Fv$, so $S = T$ by the property of F. Since range $S = W$, so is range T, that means $w_0, \cdots, w_k$ spans W, so $k = n$, which means range $F = W$, therefore F is surjective.

$\square$

Exercise 3.19. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is injective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $ST = I$.*

Proof.

- ($\Rightarrow$) Consider the basis $v_0, \cdots, v_n$ of V, then $Tv_0, \cdots, Tv_n$ is a basis of range T since T is injective. We denote Tv_i as w_i and $w_0, \cdots, w_m$ as the basis of W which expand from $w_0, \cdots, w_n$. Define $S(\lambda_0 w_0 + \cdots + \lambda_m w_m) = \lambda_0 w_0 + \cdots + \lambda_n w_n$, and then for any $v \in V$, $ST(\lambda_0 v_0 + \cdots + \lambda_n v_n) = S(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 v_0 + \cdots + \lambda_n v_n$, so $ST = I$.

- ($\Leftarrow$) Suppose $A, B \in \mathcal{L}(U, V)$, such that $TA = TB$, we will show that $A = B$. $STA = IA = A$ and $STB = IB = B$ and $STA = STB$ since $TA = TB$. Then we know T is a monomorphism, and then T is injective.

$\square$

Exercise 3.20. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is surjective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $TS = I$.*

Proof. $\square$

Exercise 3.21. *Suppose V is finite, $T \in \mathcal{L}(V, W)$, $U \subseteq W$ a subspace. Show that the inverse image of U: $\{ v \in V \mid Tv \in U \}$ is a subspace of V, and*

$$\dim\{ v \in V \mid Tv \in U \} = \dim \text{null} \, T + \dim(U \cap \text{range} \, T)$$

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Proof. The second part is quite easy, we can restrict the domain of T to $\{\ v \in V \mid Tv \in U\ \}$, say $T' \in \mathcal{L}(\{\ v \in V \mid Tv \in U\ \}, W)$, so that it is in form $\dim\{\ v \in V \mid Tv \in U\ \} = \dim \operatorname{null} T' + \dim \operatorname{range} T'$. Obviously $\operatorname{range} T' = U \cap \operatorname{range} T$ and $\operatorname{null} T' = \operatorname{null} T$.

We will now show that $\{\ v \in V \mid Tv \in U\ \}$ is a subspace of V.

- $T0 \in U$.

- For any $v, w \in V$ such that $Tv, Tw \in U$, we have $T(v+w) = Tv + Tw \in U$.

- For any $v \in V$ such that $Tv \in U$ and $\lambda \in F$, we have $T(\lambda v) = \lambda Tv \in U$.

Therefore it is a subspace. □

Exercise 3.22. *Suppose U and V are finite, $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$, show that*

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$$

Proof. Consider the inverse image of $\operatorname{null} S$ on T: $K = \{\ v \in V \mid Tv \in \operatorname{null} S\ \}$, which dimension: $\dim K = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T)$, where $\dim(\operatorname{null} S \cap \operatorname{range} T)$ caps at $\dim \operatorname{null} S$.

We know show that $\operatorname{null} ST = \operatorname{null} K$. For any $STv = 0$, we know $S(Tv) = 0$, so $Tv \in K$, therefore $\operatorname{null} ST \subseteq \operatorname{null} K$; For any $Tv \in \operatorname{null} S$, that means $S(Tv) = 0$, therefore $v \in \operatorname{null} ST$, therefore $\operatorname{null} ST \supseteq \operatorname{null} K$, and $\operatorname{null} ST = \operatorname{null} K$. □

Exercise 3.25. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that $\operatorname{null} S \subseteq \operatorname{null} T \iff$ there is $E \in \mathcal{L}(W)$ such that $T = ES$.*

Proof. We define $E(S(v)) = Tv$ for any $v \in V$, so that $E \in \mathcal{L}(\operatorname{range} S, W)$. We first show that E is a mapping, and also a linear transformation.

Suppose $Sv, Sw \in W$ such that $Sv = Sw$, we need to show that $E(Sv) = E(Sw)$, or normalized $Tv = Tw$. We know $v - w \in \operatorname{null} S$ since $Sv = Sw$, so $v - w \in \operatorname{null} T$ since $\operatorname{null} S \subseteq \operatorname{null} T$, therefore $T(v - w) = 0$, and then $Tv = Tw$, so E is a mapping.

Now we show that E is a linear transformation.

- For any $Sv, Sw \in \operatorname{range} S$, $E(Sv) + E(Sw) = Tv + Tw = T(v + w) = E(S(v + w)) = E(Sv + Sw)$.

- For any $Sv \in \text{range } S$ and $\lambda \in F$, $\lambda E(Sv) = \lambda Tv = T(\lambda v) = E(S(\lambda v) = E(\lambda Sv))$.

therefore E is a linear transformation.

Now we can expand the domain of E to W such that $E'v = Ev$ for any $v \in \text{range } S$ (this is proven in previous exercise). For any $v \in V$, we have $ESv = E(Sv) = Tv$, there fore $T = ES$.

For another direction, for any $v \in \text{null } S$, we have $ESv = E0 = 0 = Tv$, so $v \in \text{null } T$. $\qquad\square$

Exercise 3.26. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that $\text{range } S \subseteq \text{range } T \iff$ there is $E \in \mathcal{L}(V)$ such that $S = TE$.*

Proof. Consider the inverse image of $\text{range } S$ with basis $w_0, \cdots, w_n$, say $v_0, \cdots, v_n$, it is easy to show $v_0, \cdots, v_n$ is linear irrelavent. Then $E(v) = \lambda_0 v_0 + \cdots + \lambda_n v_n$ where $Sv = \lambda_0 w_0 + \cdots + \lambda_n w_n$. $\qquad\square$

Exercise 3.27. *Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$, show that $V = \text{null } P \oplus \text{range } P$.*

Proof. Such element is called *idempotent* in algebra.

We will show $\text{null } P \oplus \text{range } P$ by showing $\text{null } P \cap \text{range } P = \{0\}$. For any $v \in \text{null } P \cap \text{range } P$, we know there is $w \in V$ such that $Pw = v$ since $v \in rangevP$, then $P^2(v) = P(Pv) = P0 = 0$ since $v \in \text{null } P$ and $P^2(v) = P(Pv) = Pw$, so $Pw = 0$ while $Pw = v$ therefore $v = 0$.

Then we have $\dim V = \dim \text{null } P + \dim \text{range } P$ and $\dim(\text{null } P \oplus \text{range } P) = \dim \text{null } P + \dim \text{range } P - \dim\{0\}$, so $V = \text{null } P \oplus \text{range } P$. $\qquad\square$

Exercise 3.28. *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ such that for any non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$, $\deg(Dp) = \deg p - 1$. Show that D is surjective.*

Proof. We induction on n, starts from 1, to show that $D(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_{n-1}(\mathbb{R})$.

- Base: for any $p \in \mathcal{P}(\mathbb{R})$ where $\deg p = 1$, we know $\deg Dp = 0$, so $D(\mathcal{P}_1(\mathbb{R}))$ is a non-zero subspace of $\mathcal{P}_0(\mathbb{R})$, which is $\mathcal{P}_0(\mathbb{R})$.

- Induction: We have induction hypothesis: For any $i \leq n$, we have $D(\mathcal{P}_i(\mathbb{R})) = \mathcal{P}_{i-1}(\mathbb{R})$. We want to show that $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$. For any $p \in \mathcal{P}(\mathbb{R})$ with $\deg p = n+1$, we can write p in form of $p = \lambda x^{n+1} + r$ where $\deg r \leq n$, then $Dp = D(\lambda x^{n+1} + r) = D(\lambda x^{n+1}) + Dr$ where $\deg D(\lambda x^{n+1}) = n$ and $\deg Dr \leq n-1$. So $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$ since: $\mathcal{P}_n(\mathbb{R}) \subseteq D(\mathcal{P}_{n+1}(\mathbb{R}))$ and $D(\lambda x^{n+1}) \in D(\mathcal{P}_{n+1}(\mathbb{R}))$, it is sufficient to span $\mathcal{P}_n(\mathbb{R})$.

$\square$

Exercise 3.29. *For any $p \in \mathcal{P}(\mathbb{R})$, show that there is $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.*

Proof. We can rewrite the goal as $5DDq + 3Dq = p$ where $D(p) = p'$, then $5DDq + 3Dq = D(5Dq) + D(3q) = D(5Dq + 3q) = p$. We know D is surjective by the previous exercise, the goal is now showing that $5Dq + 3q = r$ where $Dr = p$. Then we continue rewrite the goal $5Dq + 3q = (5D)q + (3I)q = (5D + 3I)q = r$, we will show that $5D + 3I$ is surjective, we use the same method in previous exercise.

We denote $5D + 3I$ by F, and induction on $n \in \mathbb{N}$ to show that $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$.

- Base: We should show that $F(\mathcal{P}_0(\mathbb{R})) = \mathcal{P}_0(\mathbb{R})$, for any $p \in \mathcal{P}_0(\mathbb{R})$, we have $Fp = 5Dp + 3p$, where $Dp = 0$ since $\deg p = 0$, so $Fp = 3p$, which means we have $1 \in F(\mathcal{P}_0(\mathbb{R}))$ since p is literally a number and $\frac{1}{3p}Fp = 1$.

- Induction: We have induction hypothesis: $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$, and we want to show $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

  For any $p \in \mathcal{P}_{n+1}(\mathbb{R})$, we have $Fp = 5Dp + 3p$ where $\deg 5Dp = n$ and $\deg 3p = n + 1$, then we can eliminate $5Dp$ and every term in p with degree less then $n + 1$ since $\mathcal{P}_n(\mathbb{R}) \subseteq \text{range } F$, then we get z^{n+1}, thus $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

Therefore there is q such that $(5D + 3I)q = r$ since $5D + 3I$ is surjective.

Another solution from internet: Define $Tq = 5q'' + 3q'$, we can see for any $q \in \mathcal{P}(\mathbb{R})$ we have $\deg Tq = \deg q - 1$, so T is surjective. Then there is q such that $Tq = 5q'' + 3q' = p$. $\square$

Exercise 3.30. *Suppose $\varphi \in \mathcal{L}(V, F)$ not zero, and $u \in V$ that $u \notin \text{null } \varphi$, show that $V = \text{null } \varphi \oplus \{\, au \mid a \in F \,\}$.*

Proof. We can see φ is surjective since $\varphi u \neq 0$, then for any $i \in F$, we have $(i(\varphi u)^{-1})\varphi u = i$.

For any $v \in V$, since φ is surjective (in a particular way), so we have $a\varphi u$ such that $a\varphi u = \varphi v$, then $\varphi(au - v) = 0$ so $au - v \in \text{null } \varphi$. That means $(-1)(au - v) + au = v$ where $(-1)(au - v) \in \text{null } \varphi$ and $au \in \{\, au \mid a \in F \,\}$, so $V = \text{null } \varphi + \{\, au \mid a \in F \,\}$.

Then $\text{null } \varphi \oplus \{\, au \mid a \in F \,\}$ since $u \notin \text{null } \varphi$. $\square$

Exercise 3.31. *Suppose V is finite (dim $V > 1$), show that if $\varphi : \mathcal{L}(V) \to F$ is a linear mapping with property $\varphi(ST) = \varphi(S)\varphi(T)$ for any $S, T \in \mathcal{L}(V)$, show that $\varphi = 0$.*

Proof. Consider null φ, since dim $V > 1$ while dim $F = 1$, so φ cannot be injective, therefore null $\varphi \neq \{0\}$.

For any non-zero $S \in \text{null } \varphi$ and $T \in \mathcal{L}(V)$, we have $\varphi(ST) = \varphi(S)\varphi(T) = 0 = \varphi(T)\varphi(S) = \varphi(TS)$ since $S \in \text{null } \varphi$, thus $ST \in \text{null } \varphi$. We show that null φ is an ideal of $\mathcal{L}(V)$, recall that the property of $\mathcal{L}(V)$, the only ideal of $\mathcal{L}(V)$ is $\{0\}$ and $LT(V)$, so null $\varphi = \mathcal{L}(V)$, which means $\varphi = 0$. □

Exercise 3.32. *Let V, W are vector spaces and $T \in \mathcal{L}(V, W)$, define $T_C : V_C \to W_C$:*
$$T_C(u + iv) = Tu + iTv$$
for any $u, v \in V$.

1. *Show that T_C is a (complex) linear mapping from V_C to W_C.*

2. *Show that T_C is injective $\iff T$ is injective.*

3. *Show that $\text{range } T_C = W_C \iff \text{range } T = W$.*

Proof.

1. For any $u, v, s, t \in V$ $\lambda \in \mathbb{C}$, we have
$$
\begin{aligned}
&T((u + iv) + (s + it)) \\
=&T(u + s + i(v + t)) \\
=&T(u + s) + iT(v + t) \\
=&Tu + Ts + iTv + iTt \\
=&T(u + iv) + T(s + it)
\end{aligned}
$$

   and

$$
\begin{aligned}
&\lambda T(u + iv) \\
=&\lambda(Tu + iTv) \\
=&\lambda Tu + \lambda iTv \\
=&T(\lambda u) + iT(\lambda v) \\
=&T(\lambda u + i(\lambda v)) \\
=&T(\lambda u + i\lambda v) \\
=&T(\lambda(u + iv))
\end{aligned}
$$

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I believe these are trivial, so the future me should be able to prove these without any effort. $\square$