1 Constructions

Definition 1.1 (Induced)**.** *Let A a subset of a topological space $\mathcal{Y}$. Then*

- *all subsets $V \subseteq A$ such that $V = A \cap W$ for some open W in $\mathcal{Y}$*

forms a topology on A. This topology is called induced topology on A.

Proof. Obviously, $\varnothing = A \cap \varnothing$ and $A = A \cap A$.

For any open set $V_0 = A \cap W_0$ and $V_1 = A \cap W_1$, then it is easy to see that $V_0 \cup V_1 = (A \cap W_0) \cup (A \cap W_1) = A \cup (W_0 \cap W_1)$, therefore for a collection $\{V_\alpha\}$ of open sets in A, $\bigcup_\alpha V_\alpha = A \cap (\bigcup_\alpha W_\alpha)$, and we know $\bigcup_\alpha W_\alpha$ is still a open set in $\mathcal{Y}$, so is $\bigcup_\alpha V_\alpha$.

For any open set $V_0 = A \cap W_0$ and $V_1 = A \cap W_1$, we have $V_0 \cap V_1 = (A \cap W_0) \cap (A \cap W_1) = (A \cap W_0) \cap W_1 = A \cap (W_0 \cap W_1)$, so $V_0 \cap V_1$ is also a open set in A. $\square$

Definition 1.2 (Embedding)**.** *A map $f : \mathcal{X} \to \mathcal{Y}$ is called embedding if f defines a homeomorphism from $\mathcal{X}$ to the subspace $f(\mathcal{X})$ in $\mathcal{Y}$.*

Definition 1.3 (Product)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ are topology spaces, the product of $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} \times \mathcal{Y}$ is a topology space, which open set is a union of the product of open sets in $\mathcal{X}$ and $\mathcal{Y}$, that is, $\bigcup_i V_i \times W_i$ where V_i is an open set of $\mathcal{X}$ and W_i is an open set of $\mathcal{Y}$ (It won't work if the open set is just a product of open sets from both space).*

Proof. We need to show that $\mathcal{X} \times \mathcal{Y}$ is a topology space.

1. $\mathcal{X} \times \mathcal{Y}$ is an open set since both $\mathcal{X}$ and $\mathcal{Y}$ are open set, similarly, $\varnothing$ is an open set in $\mathcal{X} \times \mathcal{Y}$.

2. Trivial.

3. $$\bigcup_{a,b} (V_a \cap V_b) \times (W_a \cap W_b)$$

$\square$

Exercise 1.1. *Let $\mathscr{B}$ be a collection of open sets in a topological space $\mathcal{X}$. Show that $\mathscr{B}$ is base in $\mathcal{X}$ iff for any point $x \in \mathcal{X}$ and any neighborhood N, there is $B \in \mathscr{B}$ such that $x \in B \subseteq N$.*

Proof.

- ($\Rightarrow$) We know N can be expressed as a union of some open sets in $\mathscr{B}$, that is, $N = B_0 \cup B_1 \cup \dots$. Then there is B_i such that $x \in B_i$, and we know $B_i \subseteq N$, so $x \in B_i \subseteq N$.

- ($\Leftarrow$) For any open set A, consider any point $x \in A$, we know A is a neighborhood of x, so there is $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq A$. Then $\bigcup_{x \in A} B_x$ is a subset of A, but note that $x \in B_x$, so $\bigcup_{x \in A} B_x$ contains all points of A, so $\bigcup_{x \in A} B_x = A$.

$\square$

Theorem 1.1. *Let $\mathscr{B}$ be a set of subsets in some set $\mathcal{X}$. Show that $\mathscr{B}$ is a base of some topology on $\mathcal{X}$ iff it satisfies the following conditions:*

1. *$\mathscr{B}$ converse $\mathcal{X}$, that is, for any $x \in \mathcal{X}$, there is $B \in \mathscr{B}$ such that $x \in B$.*

2. *For any $B_1, B_2 \in \mathscr{B}$, $x \in B_1 \cap B_2$, then there exists a set $B \in \mathscr{B}$ such that $x \in B \subseteq B_1 \cap B_2$.*

Proof. ($\Rightarrow$) Trivial by the previous exercise.

($\Leftarrow$) Let $\mathscr{O}$ the set of all unions of sets of $\mathscr{B}$. We claim $\mathscr{O}$ is a topology on $\mathcal{X}$. First, $\emptyset \in \mathscr{O}$ since it is the result of union of no set. And $\mathcal{X} \in \mathscr{O}$ since $\mathscr{B}$ covers $\mathcal{X}$. Obviously, the union of any sets of $\mathscr{O}$ is in $\mathscr{O}$.

Let $O_0, O_1 \in \mathscr{O}$, then $O_0 = \bigcup_i B_i$ and $O_1 = \bigcup_j B_j$. For any $x \in O_0 \cap O_1$, we have $x \in B_x \subseteq O_0 \cap O_1$ where $B_x \in \mathscr{O}$ by (2). Then $O_0 \cap O_1 = \bigcup_x = B_x$, therefore $O_0 \cap O_1 \in \mathscr{O}$ cause it is a union of some set B_x. $\square$

Definition 1.4 (Prebase)**.** *Suppose $\mathscr{P}$ is a collection of subsets in $\mathcal{X}$ that covers the whole space. Show that the set of all finite intersections of sets in $\mathscr{P}$ is a base for some topology on $\mathcal{X}$.*

Proof. We denote all finite intersections of sets in $\mathscr{P}$ as P, then $\mathscr{P} \subseteq P$ since $\forall S \in P, S \cap S = S$. So P covers $\mathcal{X}$. For any $B_0, B_1 \in P$ and $x \in B_0 \cap B_1$, it is obvious that $x \in B_0 \cap B_1 \subseteq B_0 \cap B_1 \in P$. So $\mathscr{P}$ is a base for some topology on $\mathcal{X}$ by the previous theorem. $\square$

Theorem 1.2. *Let $\mathscr{P}$ be a prebase for the topology on $\mathcal{Y}$. Show that a map $f : \mathcal{X} \to \mathcal{Y}$ is continuous iff $f^{-1}(P)$ is open for any $P \in \mathscr{P}$.*

Proof. ($\Rightarrow$) Obviously, every set in $\mathscr{P}$ is open.

($\Leftarrow$) For any open set in $\mathcal{Y}$, it is a union of sets in $\mathscr{P}$, that is, $\bigcup_i P_i$. Then $f^{-1}(\bigcup_i P_i) = \bigcup_i f^{-1}(P_i)$ which is a union of open sets, so it is also open. $\square$

Theorem 1.3. *For any map $f : \mathcal{X} \to \mathcal{Y}$, the map $F : \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ given by $F(x) = (x, f(x))$. Show that $f(x)$ is continuous iff F is an embedding.*

Proof. ($\Rightarrow$) We first show that $F : \mathcal{X} \to F(\mathcal{X})$ is continuous. For any open set G in the induced topology on $F(\mathcal{X})$, it has form $F(\mathcal{X}) \cap (\bigcup_\alpha V_\alpha \times W_\alpha)$, where $(\bigcup_\alpha V_\alpha \times W_\alpha)$ is the open set in $\mathcal{X} \times \mathcal{Y}$. Clearly, $F^{-1}(G) = G.0 = \{\, x \mid \forall (x, y) \in G \,\} = X \cap (\bigcup_\alpha V_\alpha)$, so it is an open set in $\mathcal{X}$. It is easy to see $F^{-1} : F(\mathcal{X}) \to \mathcal{X}$ given by $F^{-1}(x, y) = x$ is the inverse of F, we need to show that it is continuous, or in other words, show that F sends open to open. For any open set $V \subseteq \mathcal{X}$, $F(V) = F(\mathcal{X}) \cap (V \times \mathcal{Y})$, which is open in $F(\mathcal{X})$. So F is an embedding.

($\Leftarrow$) For any open set G in $\mathcal{Y}$, we have $H = F(\mathcal{X}) \cap (\mathcal{X} \times G)$ is open in the induced topology on $F(\mathcal{X})$, then $F^{-1}(H)$ is open in $\mathcal{X}$. We claim $f^{-1}(G) = F^{-1}(H)$.

- ($\supseteq$) For any $x \in F^{-1}(H)$, we know $f(x) \in f(\mathcal{X}) \cap G$ since $F(x) = (x, f(x)) \in H$, so $x \in f^{-1}(G)$ since $f(x) \in G$.
- ($\subseteq$) For any $x \in f^{-1}(G)$, we know $f(x) \in G$, so $(x, f(x)) \in H$ since $x \in \mathcal{X} \cap \mathcal{X}$ and $f(x) \in f(\mathcal{X}) \cap G$, so $x \in F^{-1}(H)$.

Therefore f is continuous. $\square$