1 Product

Definition 1.1 (Product). *Let $\mathcal{C}$ a category and $A, B \in \mathcal{C}$, $(A \times B, \pi_0, \pi_1)$ forms a product of A and B where $A \times B \in \mathcal{C}$, $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$, if for any $X \in \mathcal{C}$ with $\pi_0' : X \to A$ and $\pi_1' : X \to B$, there is a unique arrow $u : X \to A \times B$ such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\pi_1'"] \arrow[ddr, bend right, "\pi_0'"'] & & \\
& A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & B \\
& A &
\end{tikzcd}$$

Furthermore, a product of A and B is a limit of diagram:

$$(A \qquad B)$$

One may trying to show that $0 \times X \simeq 0$ by:

$$\begin{tikzcd}
0 \times X \arrow[dr, red, "\pi_0"] \arrow[ddrrr, red, bend left, "\pi_1"] \arrow[dddrr, bend right, "\pi_0"] & & & \\
& 0 \arrow[dr, dashed, "!"] \arrow[drr, red, bend left, "!"] \arrow[ddr, bend right, "1_0"] & & \\
& & 0 \times X \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & X \\
& & 0 &
\end{tikzcd}$$

However, the red triangle needs not to commutes, that is, the arrow π_0 from $(0 \times X, \pi_0, \pi_1)$ to $(0, 1_0, !)$ may not exist.

Definition 1.2 (Product of Arrow). *Suppose $(A \times B, \pi_0, \pi_1)$ and $(C \times D, \pi_2, \pi_3)$ are two product, and $f : A \to C$, $g : B \to D$. The product of arrow $f \times g$ is a* *unique arrow from $A \times B$ to $C \times D$ such that the following diagram commutes:*

$$\begin{tikzcd}
A & A \times B \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, dashed, "f\times g"] & B \arrow[d, "g"] \\
C & C \times D \arrow[l, "\pi_2"'] \arrow[r, "\pi_3"] & D
\arrow[from=1-1, to=2-1, "f"']
\end{tikzcd}$$

We may consider $\times$ a functor from $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ if $\mathcal{C}$ has any product. Then the $\times$ acts on any morphism $\langle f, g\rangle : (a, b) \to (c, d)$ is the product of arrows we showed above. The functoriality can be proved by the uniqueness of the factor morphism.