Exercise 5.1. *Prove or disprove: $T \in \mathcal{L}(V)$ and $\mathcal{M}(T^2)$ is upper-triangular for some basis of V, then $\mathcal{M}(T)$ is upper-triangular for some basis of V (not necessary the same as the $\mathcal{M}(T^2)$ one).*

Proof. WoBuHui. □

Exercise 5.2. *Let A, B are upper-triangular matrices with same size, the diagonal of A is $\alpha_0, \cdots, \alpha_{n-1}$ and the diagonal of B is $\beta_0, \cdots, \beta_{n-1}$. Show that*

- *$A + B$ is upper-triangular and the diagonal is $\alpha_0 + \beta_0, \cdots, \alpha_{n-1} + \beta_{n-1}$.*
- *AB is upper-triangular and the diagonal is $\alpha_0\beta_0, \cdots, \alpha_{n-1}\beta_{n-1}$.*

Proof.

- Trivial.
- Take the standard basis of F^n, we have $Bv_i \in \operatorname{span}(v_0, \cdots, v_i)$ and then $A(Bv_i) \in \operatorname{span}(v_0, \cdots, v_i)$ since both A and B are upper-triangular, thus AB is upper-triangular. For the diagonal, we know $AB_{i,i} = A_{i,-}B_{-,i}$, however, components before i-th of $A_{i,-}$ are 0 and componetns since $i-th$ of $B_{-,i}$ are 0, therefore $AB_{i,i} = A_{i,i}B_{i,i} = \alpha_i\beta_i$.

□

Exercise 5.3. *Let $T \in \mathcal{L}(V)$ invertible, and $\mathcal{M}(T)$ with respect to the basis $v_0, \cdots, v_{n-1}$ of V is upper-triangular, while the diagonal is $\lambda_0, \cdots, \lambda_{n-1}$. Show that $\mathcal{M}(T^{-1})$ with respect to that basis is also upper-triangular, and the diagonal is $\frac{1}{\lambda_0}, \cdots, \frac{1}{\lambda_{n-1}}$.*

Proof. For any $i = 1, \cdots, n$, $\operatorname{span}(v_0, \cdots, v_{i-1})$ is invariant under T, thus it is invariant under T^{-1} since T^{-1} is the inverse of T.

For the diagonal, $TT^{-1} = I$, which diagonal is $1, \cdots, 1$, which is equal to $\lambda_0\beta_0, \cdots, \lambda_{n-1}\beta_{n-1}$ where β_i is the diagonal of T^{-1}. Thus $\beta_i = \frac{1}{\lambda_i}$. □

Exercise 5.4. *Give an example that T an invertible operator, where the diagonal of $\mathcal{M}(T)$ is all 0.*

Proof.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

□

Exercise 5.5. *Give an example that T an singular operator, where the diagonal of $\mathcal{M}(T)$ is all non-zero.*

Proof.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

□

Exercise 5.6. *Let $F = C$ and V finite, and $T \in \mathcal{L}(V)$. Show that $k = 1, \cdots, \dim V$, then there is a k-dimension subspace of V that is invariant under T.*

Proof. If $F = C$, then $\mathcal{M}(T)$ is upper-triangular for some basis of V. Thus $\operatorname{span}(v_0, \cdots, v_{k-1})$ is invariant under T where v_i is such basis. □

Exercise 5.7. *Let V finite and $T \in \mathcal{L}(V)$ and $v \in V$. Show that:*

- *There is a unique monic polymonial p_v with minimal degree such that $p_v(T)v = 0$*
- *Show that the minimal polymonial of T is polymonial multiple of p_v.*

Proof.

- $p(T)v = 0$, therefore we only need to show the uniqueness. Let s, t a monic polymonial with minimal degree such that $s(T)v = t(T)v = 0$, then $(s - t)(T)v = 0$, therefore $s = t$, otherwise there is a polymonial $s - t$ with lower degree such that $(s - t)(T)v = 0$.
- We divide p by p_v, then $p = sp_v + r$ where $s, r \in \mathcal{P}(F)$ and $\deg r < \deg p_v$. Therefore $r = 0$, otherwise r is a lower polymonial such that $r(T)v = 0$, which contradict the property of p_v. Thus $p = sp_v$.

□

Exercise 5.8. *Let V finite and $T \in \mathcal{L}(V)$, and non-zero $v \in V$ such that $T^2 + 2Tv + 2v = 0$. Show that*

- *If $F = R$, then $\mathcal{M}(T)$ is **NOT** upper-triangular for all basis of V.*
- *If $F = C$, then the diagonal of upper-triangular $\mathcal{M}(T)$ contains $-1 + i$ and $-1 - i$.*

Proof.

- Note that $p_v(z) = z^2+2z+2$ is a minimal polymonial of Tv, it is minimal since p_v has no zero, therefore cannot have lower degree.

  Then the minimal polymonial p of T is a polymonial multiple of p_v, thus p is **NOT** in form of $(z - \lambda_0)\cdots(z - \lambda_{n-1})$ since p_v has no zero, thus there is no upper-triangular matrix for T for any basis of V.

- $-1 + i$ and $-1 - i$ are two zeros of p_v, thus are zeros of p, therefore are in the diagonal.

$\square$

Exercise 5.9. *Let B square matrix with complex elements. Show that there is a square matrix A with complex elements such that $A^{-1}BA$ is a upper-triangular matrix.*

Proof. We can find an operator T such that its matrix is B with respect to the standard basis. Then we can find a basis such that $\mathcal{M}(T)$ with respect to such basis is upper-triangular since B is complex. Then
$A = \mathcal{M}(I, \text{standard basis}, \text{upper-trianguler basis})$, and $A^{-1}BA$ is upper-triangular, this is the change-of-basis formula. $\square$

Exercise 5.10. *Let $T \in \mathcal{L}(V)$ and $v_0, \cdots, v_{n-1}$ a basis of V, show that the following statements are equivalent:*

- *the matrix of T with respect to $v_0, \cdots, v_{n-1}$ is lower-triangular.*
- *For any $k = 1, \cdots, n$, $\text{span}(v_{k-1}, \cdots, v_{n-1})$ is invariant under T.*
- *For any $k = 1, \cdots, n$, $Tv_{k-1} \in \text{span}(v_{k-1}, \cdots, v_{n-1})$.*

Proof. The proof is similar to the upper-triangular one.

- (1) $\Rightarrow$ (2) For any $i \leq j$, $Tv_{j-1} \in \text{span}(v_{j-1}, \cdots, v_{n-1}) \subseteq \text{span}(v_{i-1}, \cdots, v_{n-1})$, thus $\text{span}(v_{k-1}, \cdots, v_{n-1})$ is invariant under T.
- (2) $\Rightarrow$ (3) Tirival.
- (3) $\Rightarrow$ (1) Basically the definition.

$\square$

Exercise 5.11. *Let $F = C$ and V finite. Show that $T \in \mathcal{L}(V)$, then $\mathcal{M}(T)$ is lower-triangular with respect to some basis of V.*

Proof. Consider the dual map T', we know there is a basis of V' such that $\mathcal{M}(T')$ is upper-triangular, then $\mathcal{M}(T') = \mathcal{M}(T)^T$ which means $\mathcal{M}(T)^T$ is a upper-triangular, thus $(\mathcal{M}(T)^T)^T = \mathcal{M}(T)$ is lower-triangular. □

Exercise 5.12. *Let V finite and the matrix of $T \in \mathcal{L}(V)$ is upper-triangular with respect to some basis of V, and $U \subseteq V$ is invariant under T. Show that*

- *The matrix of $T\big|_U$ is upper-triangular with respect to some basis of U.*
- *The matrix of T/U is upper-triangular with respect to some basis of V/U.*

Proof.

- Since $\mathcal{M}(T)$ is upper-triangular, then the minimal polymonial of T is in form of $p(z) = (z - \lambda_0) \cdots (z - \lambda_{n-1})$. Then $p(T\big|_U) = 0$, thus p is polymonial multiple of the minimal polymonial q of $T\big|_U$. therefore q is also in form of $(z - \lambda_0) \cdots (z - \lambda_{k-1})$. Thus there is a basis of U such that the matrix of $T\big|_U$ is upper-triangular.
- Let q the minimal polymonial of T/U, and p the minimal polymonial of T, then p is polymonial multiple of q (see Exercise 5.25 in E5B). Then follow the same step as last proof.

□

Exercise 5.13. *Let V finite, $T \in \mathcal{L}(V)$, $U \subseteq V$ invariant under T, $\mathcal{M}(T\big|_U)$ is upper-triangular for some basis of U, $\mathcal{M}(T/U)$ is upper-triangular for some basis of V/U. Show that $\mathcal{M}(T)$ is upper-triangular for some basis of V.*

Proof. We will use the conclusion of Exercise 5.25 in E5B:

$$st = (\text{the minimal polymonial of } T\big|_U) \times (\text{the minimal polymonial of } T/U)$$

is a polymonial multiple of the minimal polymonial p of T. Thus st is in form of $(z-\lambda_0) \cdots (z-\lambda_{n-1})$ since both $\mathcal{M}(T\big|_U)$ and $\mathcal{M}(T/U)$ are upper-triangular for some basis, therefore p is also in form of $(z-\lambda_0) \cdots (z-\lambda_{k-1})$, hence $\mathcal{M}(T)$ is upper-triangular for some basis of V. □