

Exercise 3.1. Suppose V is a finite vector space, Show that the only two ideal of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$. A subspace \mathcal{E} of $\mathcal{L}(V)$ is called an ideal, if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for any $T \in \mathcal{L}(V)$ and $E \in \mathcal{E}$.

Proof. We will use the concept Matrix. Suppose $\lambda_0 v_0 + \cdots + \lambda_n v_n$ the basis of V . We want to construct T_i that $T(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_i v_i$ for all $0 \leq i < n$, which is a matrix with all zero but 1 at i, i .

For any matrix, we can always select a non-zero value at a, b and place it at i, b , this can be done by left multiply a matrix with 1 at i, a (this produce a vector at line i with values from line a), then right multiply a matrix with 1 at i, b (this produce a vector at column b with values from line i).

Also, we can always select a non-zero value at a, b and place it at a, i , this can be done by right multiply a matrix with 1 at b, i , then left multiply a matrix with 1 at a, i .

By combining these two operations, we can select a non-zero value at a, b and place it at i, i . Now, consider any non-zero $E \in \mathcal{E}$, we can construct a matrix with non-zero value at i, i for every $0 \leq i < \dim V$. These matrix are in \mathcal{E} since \mathcal{E} is an ideal, then we can multiply an appropriate scalar to them so that they are matrices with 1 at i, i . By adds up these matrices, we get I , we know $I \in \mathcal{E}$ since \mathcal{E} is a vector space, and now all $T \in \mathcal{L}(V)$ is also in \mathcal{E} since \mathcal{E} is an ideal, then $\mathcal{E} = \mathcal{L}(V)$.

The only exception is $\mathcal{E} = \{0\}$, in this case we can't pick any non-zero element. □