1 Path-connected spaces

Definition 1.1. *Let $\mathcal{X}$ be a topological space. A continuous map $f : [0,1] \to \mathcal{X}$ is called path. If $f(0) = x$ and $f(1) = y$, we say f is a path from x to y.*

Definition 1.2. *A space $\mathcal{X}$ is called path-connected if it is nonempty and any two points in $\mathcal{X}$ can be connected by a path.*

Theorem 1.1. *Any path-connected space is connected.*

Proof. Suppose $\mathcal{X}$ a path-connected space, take $x \in \mathcal{X}$. Then we consider all path from x to y for all $y \in \mathcal{X}$, we can see that all these path is connected cause $[0,1]$ is connected. Then:

$$\bigcup_{y \in \mathcal{X}} f_y([0,1])$$

is connected cause every $f_y([0,1])$ (the path from x to y) is connected, and these path contain x. □

Definition 1.3. *Given a path $f : [0,1] \to \mathcal{X}$, one can consider the time-reversed path $\bar{f}$:*

$$\bar{f}(t) = f(1-t)$$

Note that $\bar{f}$ is continuous cause f is continuous.

Definition 1.4. *let f and g be paths in the topological space $\mathcal{X}$. If $f(1) = g(0)$, we can join these two paths into one $h : [0,1] \to \mathcal{X}$:*

$$h(t) = \begin{cases} f(2t) & \text{if } t \leq \frac{1}{2} \\ g(2t-1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

Theorem 1.2. *Show that $\sim$ is equivalence relation: $x \sim y$ iff there is a path from x to y.*

Proof.

- (Reflexivity) Obviously, there is a path from x to x.
- (Symmetry) Consider the time-reversed path.
- (Transitivity) Consider the concatenation of paths.

$\square$

The equivalence class of point x for the equivalence relation $\sim$ is called path-connected component of x.

Theorem 1.3. *Show that the product of path-connected space is path-connected.*

Proof. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are path-connected spaces, for any $(a,b),(c,d) \in \mathcal{X}\times\mathcal{Y}$, we know there are paths $a \xrightarrow{f} c$ and $b \xrightarrow{g} d$. We claim $h(t) = (f(t), g(t))$ is a path from (a,b) to (c,d). We need to show that h is continuous. For any open sets $\bigcup_\alpha V_\alpha \times W_\alpha$ in $\mathcal{X}\times\mathcal{Y}$ for every α, we claim $h^{-1}(V_\alpha \times W_\alpha) = f^{-1}(V_\alpha) \cap g^{-1}(W_\alpha)$.

($\subseteq$) For any $i \in [0,1]$ such that $h(i) \in V_\alpha \times W_\alpha$, we know $f(i) \in V_\alpha$ and $g(i) \in W_\alpha$, therefore $i \in f^{-1}(V_\alpha) \cap g^{-1}(W_\alpha)$.

($\supseteq$) For any $i \in f^{-1}(V_\alpha) \cap g^{-1}(W_\alpha)$, we know $h(i) = (f(i), g(i)) \in V_\alpha \times W_\alpha$ cause $f(i) \in V_\alpha$ and $g(i) \in W_\alpha$, therefore $i \in h^{-1}(V_\alpha \times W_\alpha)$.

We can see that $h^{-1}(V_\alpha \times W_\alpha)$ is open cause f and g are continuous. Then we claim $h^{-1}(\bigcup V_\alpha \times W_\alpha) = \bigcup h^{-1}(V_\alpha \times W_\alpha)$.

($\subseteq$) For any $i \in [0,1]$ such that $h(i) \in \bigcup V_\alpha \times W_\alpha$, we know $h(i) \in V_\alpha \times W_\alpha$ for some α, therefore $i \in h^{-1}(V_\alpha \times W_\alpha)$.

($\supseteq$) For any $i \in \bigcup h^{-1}(V_\alpha \times W_\alpha)$, we know $i \in h^{-1}(V_\alpha \times W_\alpha)$ for some α, therefore $h(i) \in V_\alpha \times W_\alpha \subseteq \bigcup V_\alpha \times W_\alpha$.

Therefore the inverse image of some open sets in $\mathcal{X}\times\mathcal{Y}$ is open, cause it is a union of some open sets, then h is continuous. $\square$

Hey, we are trying to obtain an arrow $h : [0,1] \to \mathcal{X}\times\mathcal{Y}$ from $f : [0,1] \to \mathcal{X}$ and $g : [0,1] \to \mathcal{Y}$, it is similar to the unique morphism in the product diagram!

$$
\begin{tikzcd}
{[0,1]} \arrow[dr, dashed, "!u"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
 & \mathcal{X}\times\mathcal{Y} \arrow[r, "\pi_1"] \arrow[d, "\pi_0"] & \mathcal{Y} \\
 & \mathcal{X} &
\end{tikzcd}
$$

(Although we didn't show that the product of topological spaces is really a categorical product)

Exercise 1.1. *Show that the product of topological spaces is a categorical product.*

Proof. Consider the topological product equipped with natural projections. Suppose $\mathcal{Z}$ a topological space and $f : \mathcal{Z} \to \mathcal{X}$, $g : \mathcal{Z} \to \mathcal{Y}$ are continuous maps, then we claim $u : \mathcal{Z} \to \mathcal{X} \times \mathcal{Y}$ given by $u(z) = (f(z), g(z))$ is continuous. For any open sets in $\mathcal{X} \times \mathcal{Y}$, it has form $\bigcup_\alpha V_\alpha \times W_\alpha$ where V_α and W_α are open sets in $\mathcal{X}$ and $\mathcal{Y}$, respectively. For each α, we claim $u^{-1}(V_\alpha \times W_\alpha) = f^{-1}(V_\alpha) \cap g^{-1}(W_\alpha)$.

- ($\subseteq$) For any $z \in u^{-1}(V_\alpha \times W_\alpha)$, we know $u(z) = (f(z), g(z)) \in V_\alpha \times W_\alpha$, therefore $z \in f^{-1}(V_\alpha)$ and $z \in g^{-1}(W_\alpha)$.

- ($\supseteq$) For any $z \in f^{-1}(V_\alpha) \cap g^{-1}(W_\alpha)$, $u(z) = (f(z), g(z)) \in V_\alpha \times W_\alpha$ since $f(z) \in V_\alpha$ and $g(z) \in W_\alpha$, therefore $z \in u^{-1}(V_\alpha \times W_\alpha)$.

Note that $u^{-1}(V_\alpha \times W_\alpha)$ is open cause it is the intersection of two open sets.

It is easy to show that $u^{-1}(\bigcup_\alpha V_\alpha \times W_\alpha) = \bigcup_\alpha u^{-1}(V_\alpha \times W_\alpha)$, then $u^{-1}(\bigcup_\alpha V_\alpha \times W_\alpha)$ is open cause it is the union of open sets. So u is continuous.

$u \circ \pi_0 = f$ and $u \circ \pi_1 = g$ are trivial, u is unique cause $\mathcal{X} \times \mathcal{Y}$ is product in **Set**. $\square$