1 Fiber and Fibration

I am trying to understand fiber, fibration and pullback with my stupid brain.

1.1 Fiber

I will use "intuitive" rather than "definition" cause I really don't understand fiber.

Intuitive 1.1 (Fiber)**.** *Suppose we are in a space (i.e.* **Set***), and a mapping* $f : A \to B$*, then for some point* $b \in B$*, the inverse image of* b*, which is exactly* $f^{-1}(b)$*, is called a fiber.*

We can treat a product as a pullback with apex 1, the terminal object:

$$
\begin{tikzcd}
C \arrow[dr, dashed, "!u"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & B \arrow[d, "!"] \\
& A \arrow[r, "!"'] & 1
\end{tikzcd}
$$

We can treat A as the fiber against to the only point in 1, same for B. Now, what if we replace 1 with something else?

$$
\begin{tikzcd}
C \arrow[dr, dashed, "!u"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& A \times_I B \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & B \arrow[d, "t"] \\
& A \arrow[r, "s"'] & I
\end{tikzcd}
$$

For every point $i \in I$, we have fiber $A_i \subseteq A$ and $B_i \subseteq B$, which can form a product $A_i \times B_i$. We may sum all these products, and finally get $A \times_I B$, this is why the pullback is sometimes called *fiber product*.

We can also pick certain fiber from this pullback:

$$
\begin{tikzcd}
\varphi \arrow[r, "i"] \arrow[d, "!"'] & A \arrow[d, "f"] \\
1 \arrow[r, "x"'] & B
\end{tikzcd}
$$

The morphism $x : 1 \to B$ is a global element, which "pick" an element of B, then i must maps φ to the fiber of f over point x, which should be a injection.

The collection of fiber (the source of the morphism/the domain of the function) is called *fiber bundle*.

1.2 Fibration

Some intuitive comes from this article.

Intuitive 1.2. *A fibration works like an indexed family (i.e. a function $I \to A$), but do it in fiber way (i.e. a function $A \to I$).*

1.3 Base-change Functor

These section is related to *The Dao of FP*

We can also treat the morphism on right-hand side as a fibration, and the bottom-left corner a base (the target of a fibration):

$$
\begin{tikzcd}
? \arrow[r, dashed] \arrow[d, dashed] & E \arrow[d, "p"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

Then we can treat $E \xrightarrow{p} B$ as an object in the slice category $\mathcal{C}/B$, similarly, the left-hand side morphism an object in $\mathcal{C}/A$. Then we can define a base-change functor $f^* : \mathcal{C}/B \to \mathcal{C}/A$ such that:

$$
\begin{tikzcd}
f^*E \arrow[r, dashed, "g"] \arrow[d, dashed, "f^*p"'] & E \arrow[d, "p"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

a pullback.

We denote f^*E as the source of f^*p, it doesn't mean that f^* accept a object in $\mathcal{C}$.

We need to define the action of base-change functor on the morphism of $\mathcal{C}/B$:

$$
\begin{tikzcd}
f^*E' \arrow[rrr, dashed, bend left, "g'"] \arrow[dr, "f*p'"'] & f^*E \arrow[l, "?"'] \arrow[r, dashed, "g"] \arrow[d, "f^*p"'] & E \arrow[r, "e"] \arrow[d, "p"] & E' \arrow[dl, "p'"] \\
& A \arrow[r, "f"'] & B &
\end{tikzcd}
$$

Two commute triangles are the morphisms in $\mathcal{C}/A$ and $\mathcal{C}/B$.

Since f^*E' is a pullback of $A \xrightarrow{B} \xleftarrow{E'}$, it tips us that we can find the commute square below to get the morphism we want:

$$
\begin{tikzcd}
f^*E \arrow[r] \arrow[d] & E' \arrow[d, "p'"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
$$

If we look the last diagram carefully, we can find this square commutes:

$$\begin{tikzcd}
f^*E \arrow[r, dashed, "g"] \arrow[d, "f^*p"'] & E \arrow[r, "e"] \arrow[d, "p"] & E' \arrow[dl, "p'"] \\
A \arrow[r, "f"'] & B &
\end{tikzcd}$$

therefore

$$\begin{tikzcd}
f^*E \arrow[dr, dashed, "f^*e"] \arrow[drr, bend left, "e \circ g"] \arrow[ddr, bend right, "f^*p"] & & \\
& f * E' \arrow[r, "g'"] \arrow[d, "f^*p'"'] & E' \arrow[d, "p'"] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

The functoriality follows the fact that f^*e is unique that makes the diagram commutes.