1 Connected Spaces

Definition 1.1. *A subset of a topological space is called clopen if it is open and closed.*

Definition 1.2. *A topological space $\mathcal{X}$ is called connected if it has exactly two clopen sets: $\varnothing$ and $\mathcal{X}$.*

Note that the empty space $\varnothing$ is not connected.

A subset of topological space is called connected or disconnected if so is the corresponding subspace.

Definition 1.3. *A subset S of a topological space is called disconnected if it is empty or there are two open sets V and W such that:*

- $(V \cap S) \cap (W \cap S) = V \cap W \cap S = \varnothing$
- $V \cap S \neq \varnothing$ *and* $W \cap S \neq \varnothing$
- $V \cap W \cap S = S$ *(or equivalently,* $S \subseteq V \cap W$*)*

Otherwise, we say S is connected.

Theorem 1.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces. Show that f preserves connectness.*

Theorem 1.2. *Suppose $\mathcal{X}$ is a connected space, show that the quotient space $\mathcal{X}/\sim$ is connected for any equivalence relation $\sim$ on $\mathcal{X}$.*

Proof. Consider the quotient map $f : \mathcal{X} \to \mathcal{X}/\sim$ which is onto, then $f(\mathcal{X})$ is connected since $\mathcal{X}$ is connected. □

Theorem 1.3. *Suppose $\{A_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of connected subsets of a topological space. Suppose that $\bigcap_\alpha A_\alpha \neq \varnothing$, show that $A = \bigcup_\alpha A_\alpha$ is connected.*

Proof. Suppose A is disconnected, then there are two splitting V and W. We take $p \in \bigcap_\alpha A_\alpha \neq \varnothing$, since $A \subseteq V \cup W$, then $p \in V \cup W$, we may suppose $p \in V$. Then for any α, we have $V \cap A_\alpha \neq \varnothing$ cuase $p \in V \cap A_\alpha$, therefore $W \cap A_\alpha = \varnothing$, otherwise A_α is no longer connected. □

Theorem 1.4. *Let A be a connected set in a topological space. Suppose $A \subseteq B \subseteq \bar{A}$, show that B is connected.*

Proof. Suppose B is disconnected and V, W is a splitting of B. We may suppose $A \subseteq V$, otherwise V, W also splits A. Then $W \subseteq \partial A$ while W is open, which means there is a smaller close set $\bar{A} \setminus W$ that contains A, which is unacceptible. □

Definition 1.4. *Suppose $\mathcal{X}$ a topological space and $x \in \mathcal{X}$, the intersection of all clopen neighborhoods of x is called connected component of x. Note that the space $\mathcal{X}$ is connected iff $\mathcal{X}$ is a connected component of some point in $\mathcal{X}$.*

Theorem 1.5. *Show that any connected component is closed. Show that connected component is not necessary open.*

Proof. The intersection of closed sets is closed. However, the infinite intersection of open sets is not necessary open. □

Lemma 1.1. *Any connected component is connected.*

Proof. Suppose X is a connected component of point x and V, W are splitting of X. We may suppose $x \in V$, then V is a clopen neighborhood of x and $X \subseteq V$, so $W = \varnothing$, which contradicts to the assumption that W is splitting. □

Lemma 1.2. *Suppose V is a connected component. Show that for any $y \in V$, V is the connected component of y.*

Proof. Suppose $y \in W$ is the connected component of y, then $V \subseteq W$ cause every clopen neighborhood of x is also a clopen neighborhood of y. Suppose there is a clopen neighborhood of y that makes W a proper subset of V, then this clopen neighborhood forms a splitting on V while V is connected. □

Theorem 1.6. *Show that two connected components either coincide or disjoint.*

Proof. If two connected components is not disjoint, then any point in the intersection of them will have the same connected component. □