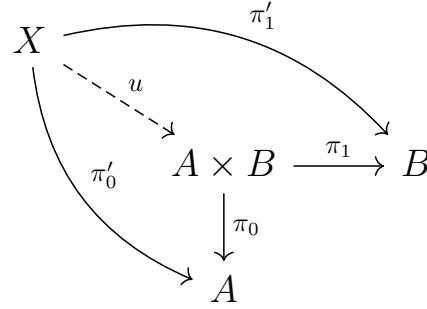


1 Product

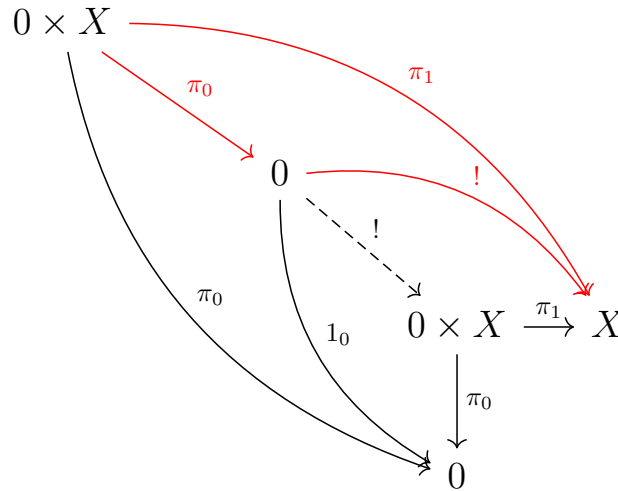
Definition 1.1 (Product). *Let \mathcal{C} a category and $A, B \in \mathcal{C}$, $(A \times B, \pi_0, \pi_1)$ forms a product of A and B where $A \times B \in \mathcal{C}$, $\pi_0 : A \times B \rightarrow A$ and $\pi_1 : A \times B \rightarrow B$, if for any $X \in \mathcal{C}$ with $\pi'_0 : X \rightarrow A$ and $\pi'_1 : X \rightarrow B$, there is a unique arrow $u : X \rightarrow A \times B$ such that the following diagram commutes:*



Furthermore, a product of A and B is a limit of diagram:

$$(A \quad B)$$

One may trying to show that $0 \times X \simeq 0$ by:



However, the red triangle needs not to commutes, that is, the arrow π_0 from $(0 \times X, \pi_0, \pi_1)$ to $(0, 1_0, !)$ may not exist.

Definition 1.2 (Product of Arrow). *Suppose $(A \times B, \pi_0, \pi_1)$ and $(C \times D, \pi_2, \pi_3)$ are two product, and $f : A \rightarrow C$, $g : B \rightarrow D$. The product of arrow $f \times g$ is a*

unique arrow from $A \times B$ to $C \times D$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 A & \xleftarrow{\pi_0} & A \times B & \xrightarrow{\pi_1} & B \\
 \downarrow f & & \downarrow f \times g & & \downarrow g \\
 C & \xleftarrow{\pi_2} & C \times D & \xrightarrow{\pi_3} & D
 \end{array}$$

2 Exponential

Definition 2.1. Let \mathcal{C} a category. For any $B, C \in \mathcal{C}$, (C^B, ev) forms an exponential where $C^B \in \mathcal{C}$ and $ev : C^B \times B \rightarrow C$, if for any object $A \in \mathcal{C}$ and $f : A \times B \rightarrow C$, there is a unique $u : A \rightarrow C^B$ such that $f = ev \circ (u \times 1_B)$. In other words, the follow diagram commutes.

$$\begin{array}{ccc}
 A \times B & & \\
 \downarrow u \times 1_B & \searrow f & \\
 C^B \times B & \xrightarrow{ev} & C
 \end{array}$$

3 Pullback

Theorem 3.1. Suppose we have two joined commuting squares like:

$$\begin{array}{ccccc}
 L & \xrightarrow{f} & M & \xrightarrow{g} & N \\
 \downarrow l & & \downarrow m & & \downarrow n \\
 X & \xrightarrow{h} & Y & \xrightarrow{j} & Z
 \end{array}$$

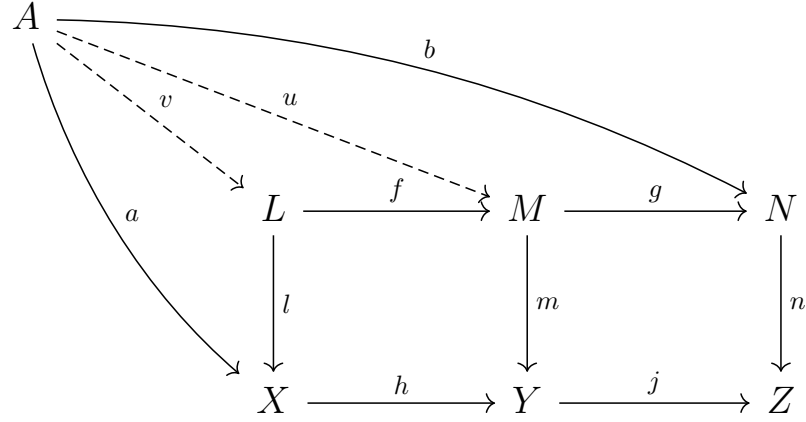
Then:

1. The outer rectangle is a pullback square if two inner squares are pullback squares.

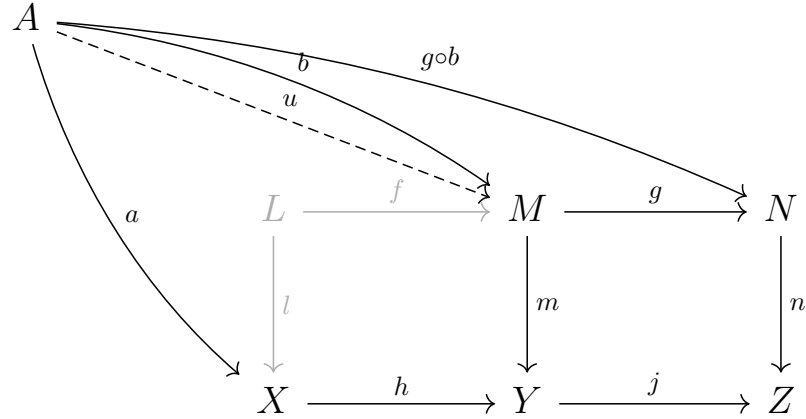
2. The inner-left square is a pullback square if the outer rectangle and the inner-right square are pullback squares.

Proof.

1. For any (A, a, b) such that $j \circ h \circ a = n \circ b$, then there is a unique $u : A \rightarrow M$ such that $h \circ a = m \circ u$ and $b = g \circ u$. Then there is a unique $v : A \rightarrow L$ such that $l \circ a = v$ and $f \circ v = u$, which makes (A, a, b) against to the outer rectangle commutes.

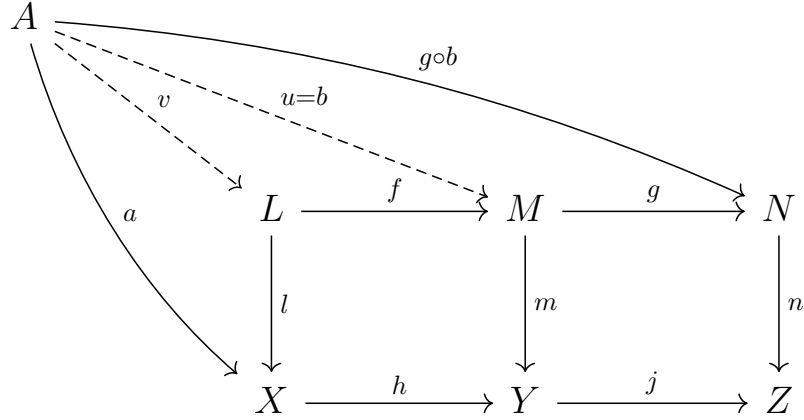


2. For any (A, a, b) such that $h \circ a = m \circ b$, consider the inner-right pullback, then we have a unique $u : A \rightarrow M$ such that the diagram commutes:



However, if we replace u with b , we have $g \circ b = g \circ b$ and $h \circ a = m \circ b$, that means b can do u 's job, but we know u is unique, so $b = u$. Now consider the outer pullback, we have a unique $v : A \rightarrow L$ such that the

diagram commutes:



That is, $l \circ v = a$ and $g \circ f \circ v = g \circ b$, we claim that v is the unique factorization from $(A, a, u = b)$ to (L, l, f) . It is obvious that $l \circ v = a$, we need to show $f \circ v = u = b$. We may use the trick we just used, we can see that $g \circ f \circ v = g \circ u$ and $m \circ f \circ v = h \circ l \circ v = h \circ a$. So $f \circ v$ can do b 's job, so $f \circ v = b$.

For any arrow $w : A \rightarrow L$ such that $l \circ a = w$ and $f \circ w = b$, then we have also $g \circ f \circ w = g \circ b$, which implies w is the unique arrow from $A \rightarrow L$ such that the outer diagram commutes, so $w = v$.

□

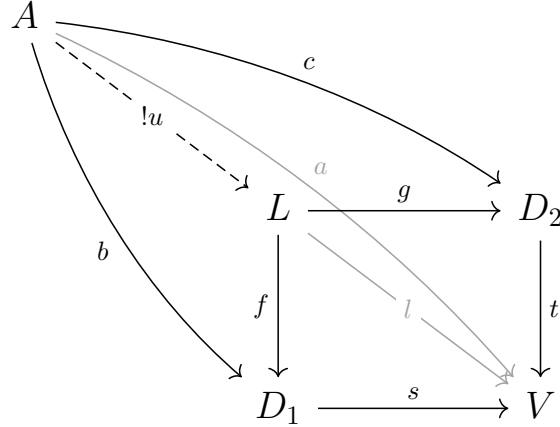
Theorem 3.2. *A pullback square for the corner $D_1 \rightarrow V \leftarrow D_2$ is a product of $D_1 \rightarrow V$ and $V \leftarrow D_2$ in the slice category \mathcal{C}/V .*

Proof. Suppose (L, f, g) is the pullback of such corner, then we first need to show that there is an arrow $l : L \rightarrow V$ such that $s \circ f = l$ (therefore a morphism from (L, l) to (D_1, s)) and $t \circ g = l$ (a morphism from (L, l) to (D_2, t)).

Since (L, f, g) makes the pullback square commutes, we know $s \circ f = t \circ g$, therefore we let $l = s \circ f$ (or equivalently $t \circ g$).

We need to show that $((L, l), f, g)$ forms a product of (D_1, s) and (D_2, t) , consider any $((A, a), b, c)$ where $a : A \rightarrow V$ such that $s \circ b = a$ and $t \circ c = a$. Just like l for L , a is redundant, so we may omit it. Now, the diagram looks

like:



Since (L, f, g) is a pullback, we know there is a unique $u : A \rightarrow L$ such that two triangle commutes. However, we must first show that u is an arrow from (A, a) to (L, l) , that is, $l \circ u = a$. It is easy to see that $l \circ u = s \circ f \circ u = s \circ b = a$. \square

Theorem 3.3. *if a category has all binary products and all equalizers for every pair of parallel arrows, then it has a pullback for any corners.*

Proof. Suppose $X \rightarrow Z \leftarrow Y$ a corner, then consider the product $X \times Y$:

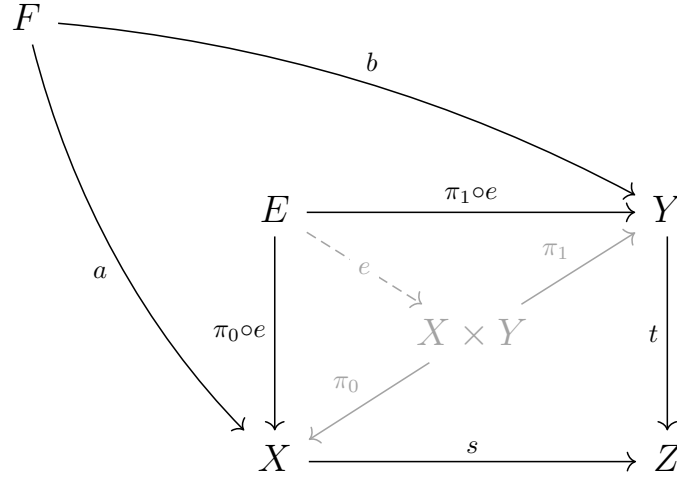
$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_1} & Y \\ \pi_0 \downarrow & & \downarrow t \\ X & \xrightarrow{s} & Z \end{array}$$

Now, consider the equalizer for the parallel arrows $t \circ \pi_1$ and $s \circ \pi_0$:

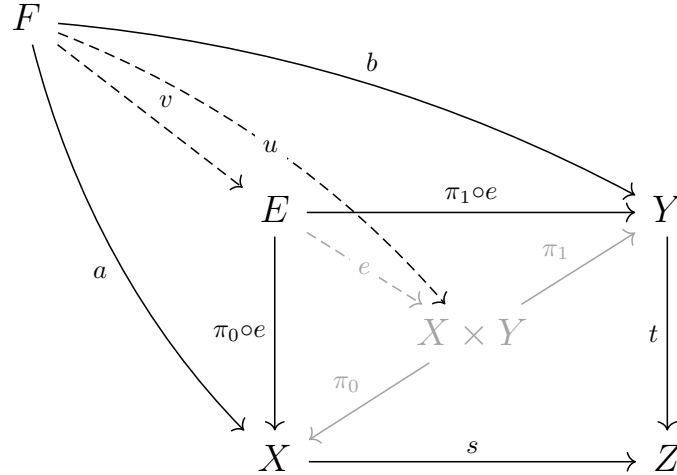
$$E \xrightarrow{e} X \times Y \rightrightarrows Z$$

$\begin{array}{c} s \circ \pi_0 \\ \hline t \circ \pi_1 \end{array}$

We claim $(E, \pi_0 \circ e, \pi_1 \circ e)$ is a pullback of such corner. For any (F, f, g) such that the outer diagram commutes:



it is easy to see that there is a unique arrow $u : F \rightarrow X \times Y$ such that $\pi_0 \circ u = a$ and $\pi_1 \circ u = b$ since $X \times Y$ is a product. Then there is another unique arrow $v : F \rightarrow E$ such that $e \circ v = u$ since E is a equalizer.



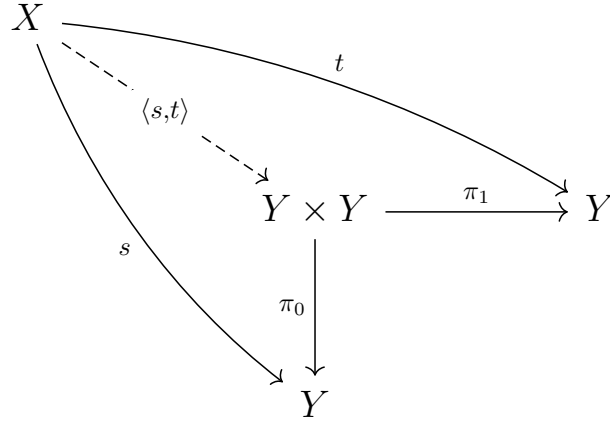
Obviously, (commute) $\pi_0 \circ e \circ v = \pi_0 \circ u = a$ and $\pi_1 \circ e \circ v = \pi_1 \circ u = b$.
(unique) If an arrow $w : F \rightarrow E$ can do the job, then $e \circ w : F \rightarrow X \times Y$ is another factorization from F to the product $X \times Y$, so $e \circ w = u$, but that means w is also a factorization from F to the equalizer E , which means $v = w$.

So $(E, \pi_0 \circ e, \pi_1 \circ e)$ is a pullback of such corner. \square

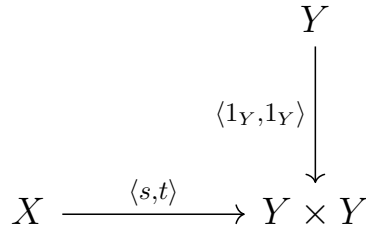
Theorem 3.4. *If a category has a terminal object and has a pullback for every corner, then it has all binary product.*

Theorem 3.5. *If a category has a terminal object and has a pullback for every corner, then it has a equalizer for every parallel arrwos.*

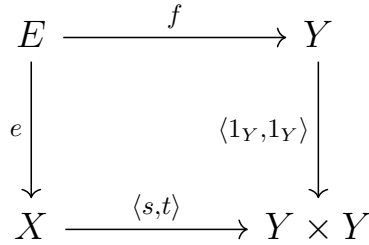
Proof. Suppose $s, t : X \rightarrow Y$ are parallel arrows, then the following diagram commutes:



Note that we have $Y \times Y$ since this category has all binary products. Then consider this corner:



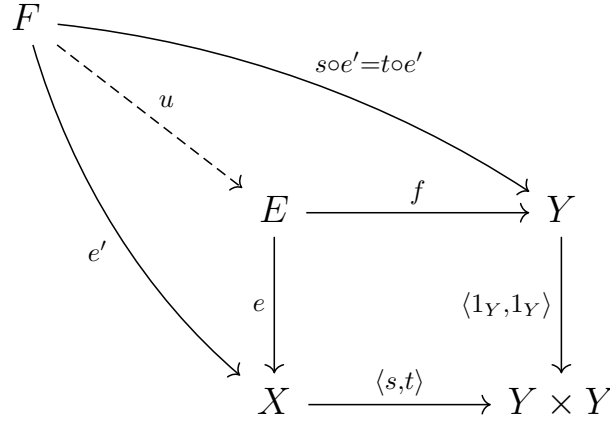
We have an object E , $e : E \rightarrow X$ and $f : E \rightarrow Y$ such that the square commutes:



(Proof comes from textbook until here)

We can see that $\pi_0 \circ \langle s, t \rangle \circ e = s \circ e$ while $\pi_0 \circ \langle 1_Y, 1_Y \rangle \circ f = 1_Y \circ f = f$, therefore $s \circ e = f$, similarly $t \circ e = f$, so $s \circ e = t \circ e$. We claim E is the equalizer for the parallel arrows $s, t : X \rightarrow Y$. For any (F, e') such that $s \circ e' = t \circ e'$, then we have a unique arrow $u : F \rightarrow E$ such that this diagram

commutes:



where $e \circ u = e'$. Suppose $v : F \rightarrow E$ where $e \circ v = e'$, then $f \circ v = s \circ e \circ v = s \circ e'$. \square

$$| -|x - z|_X + |y - z|_X |_{\approx \cap \approx} \geq \epsilon$$

4 Functors

Definition 4.1 (Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called full, if for any $a, b \in \mathcal{C}$, the mapping on morphism $F : \mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$ is surjective.

Definition 4.2 (Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called faithful, if for any $a, b \in \mathcal{C}$, the mapping on morphism $F : \mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$ is injective.

Definition 4.3 (Essentially Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called Essentially full, if for any $a \in \mathcal{C}$, the mapping on object $F : \mathcal{C} \rightarrow \mathcal{D}$ is surjective.

Theorem 4.1. Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor, and $f : a \rightarrow b$ a morphism in \mathcal{C} . Then f is an isomorphism iff Ff is an isomorphism.

Proof. (\Rightarrow) We claim $F(f^{-1}) : Fb \rightarrow Fa$ is an inverse, we can see that $F(f^{-1} \circ f) = F(id_a) = id_{Fa}$ and $F(f \circ f^{-1}) = F(id_b) = id_{Fb}$.

(\Leftarrow) Suppose Fg is the inverse of Ff , and we can retrieve g from Fg cause F is full faithful. Then $F(g \circ f) = Fg \circ Ff = id_{Fa} = F(id_a)$ therefore $g \circ f = id_a$ since F is full faithful, similar to $F(f \circ g)$, so f is indeed an isomorphism. \square

Corollary 4.1. Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ is full and faithful, show that F is injective on object.

Proof. Trivial by previous theorem. \square

5 Yoneda

This chapter combines arguments from some books:

- The Dao of FP
- The Joy of Abstraction

Definition 5.1. $H_x = \mathcal{C}(-, x)$ and $H^x = \mathcal{C}(x, -)$.

Take H^x as an example, it sends \mathcal{C} to **Set**, the interesting part is the mapping on morphism. For any morphism $f : a \rightarrow b$ of \mathcal{C} , H^f must be a mapping $\mathcal{C}(x, a) \rightarrow \mathcal{C}(x, b)$, we can see that $g \mapsto f \circ g$ would be a choice.

We have to show that it satisfies the functoriality:

- $H^{id_a}(g) = id_a \circ g = g$
- $H^{f \circ g}(h) = (f \circ g) \circ h = f \circ (g \circ h) = H^f(g \circ h) = H^f(H^g(h)) = (H^f \circ H^g)(h).$

Similar to H_x , the only difference is that H_x is a contrafunctor.

Suppose $f : a \rightarrow b$ an isomorphism, we can see that H^x gives an isomorphism between two hom-sets: $\mathcal{C}(x, a)$ and $\mathcal{C}(x, b)$.

Furthermore, we can no more fix x , that is, make H_\bullet (or H^\bullet) a functor from \mathcal{C} to $[\mathcal{C}^{op}, \mathbf{Set}]$, a functor to a functor!

The problem we need to solve is that what should H_\bullet do on a morphism $f : a \rightarrow b$. Since H_\bullet produce a functor, H_f must produce a natural transformation between H_a and H_b . Suppose $x, y \in \mathcal{C}$ and $g : x \rightarrow y$, note that H_a and H_b are contrafunctor, so we need to reverse the arrows!

$$\begin{array}{ccc}
 H_a(y) & \xrightarrow{H_a(g)} & H_a(x) \\
 \downarrow (H_f)_y & & \downarrow (H_f)_x \\
 H_b(y) & \xrightarrow{(H_b)(g)} & H_b(x)
 \end{array}$$

and we can unfold the definitions

$$\begin{array}{ccc}
\mathcal{C}(y, a) & \xrightarrow{\mathcal{C}(g, a)} & \mathcal{C}(x, a) \\
\downarrow (H_f)_y & & \downarrow (H_f)_x \\
\mathcal{C}(y, b) & \xrightarrow{\mathcal{C}(g, b)} & \mathcal{C}(x, b)
\end{array}$$

and suppose $s \in \mathcal{C}(y, a)$, we know the top-right corner would be $s \circ g$ since $\mathcal{C}(g, a) = - \circ g$ (same to $\mathcal{C}(g, b)$). In order to construct an arrow in $\mathcal{C}(y, b)$, we can pre-compose the arrow $f : a \rightarrow b$. Then the bottom-left corner would be $f \circ s$, and the bottom-right corner would be $(f \circ s) \circ g$ (by left-bottom path) and $f \circ (s \circ g)$ (by top-right path), which is exactly the same!

$$\begin{array}{ccc}
s & \xrightarrow{- \circ g} & s \circ g \\
\downarrow f \circ - & & \downarrow f \circ - \\
f \circ s & \xrightarrow{- \circ g} & (f \circ s) \circ g = f \circ (s \circ g)
\end{array}$$

Note that the condition here $(f \circ -) \circ (- \circ g) = (- \circ g) \circ (f \circ -)$ is the naturality condition which is mentioned in *The Dao of FP*. A bijection between two hom-sets $\alpha_x = \mathcal{C}(x, a) \rightarrow \mathcal{C}(x, b)$ that satisfies the naturality condition $\alpha_y \circ (- \circ g) = (- \circ g) \circ \alpha_x$ can retrieve the isomorphism between a and b . This will be unsurprised if we notice that such bijection with naturality condition forms a natural transformation, then we can retrieve the morphism (not isomorphism yet) from it. The morphism becomes iso- when we know H_\bullet is full and faithful (see below and chapter *functor*),

Definition 5.2. H_\bullet is called *Yoneda embedding*.

Theorem 5.1. Shows H_\bullet is an embedding by showing it is full and faithful.

Proof. (Full) For any $a, b \in \mathcal{C}$, suppose $\alpha : [\mathcal{C}^{op}, \mathbf{Set}](H_a, H_b)$ a morphism (natural transformation). Use the Yoneda trick, we have $\alpha_a : \mathcal{C}(a, a) \rightarrow \mathcal{C}(a, b)$ and then $\alpha_a(id_a) : \mathcal{C}(a, b)$. As we see the definition of H_\bullet on morphism,

we should expect that α has form $f \circ -$ for some $f : a \rightarrow b$. But how coincident, we have a morphism $\alpha_a(id_a) : \mathcal{C}(a, b)$. So we claim $H_{\alpha_a(id_a)} = \alpha$. (In the other hand, if α has form $f \circ -$, then $\alpha_a(id_a) = f \circ id_a = f$). For any $x \in \mathcal{C}$, we need to show $(H_{\alpha_a(id_a)})_x = \alpha_x : \mathcal{C}(x, a) \rightarrow \mathcal{C}(x, b)$. So we suppose $g \in \mathcal{C}(x, a)$, then the following diagram commutes since α is natural:

$$\begin{array}{ccc} \mathcal{C}(a, a) & \xrightarrow{- \circ g} & \mathcal{C}(x, a) \\ \alpha_a \downarrow & & \downarrow \alpha_x \\ \mathcal{C}(a, b) & \xrightarrow{- \circ g} & \mathcal{C}(x, b) \end{array}$$

Then

$$\begin{array}{ccc} id_a & \xrightarrow{- \circ g} & id_a \circ g = g \\ \alpha_a \downarrow & & \downarrow \alpha_x \\ \alpha_a(id_a) & \xrightarrow{- \circ g} & \alpha_a(id_a) \circ g = \alpha_x(g) \end{array}$$

is a proof of $H_{\alpha_a(id_a)}(g) = \alpha_a(id_a) \circ g = \alpha_x(g)$.

(Faithful) Suppose $f \circ - = H_f = H_g = g \circ -$, then $f = f \circ id_a = H_f(id_a) = H_g(id_a) = g \circ id_a = g$. \square

As we see in the proof of H_\bullet is a full functor, the natural transformation α at some x (therefore any $x \in \mathcal{C}$) is completely determined by the value $\alpha_a(id_a)$, cause for any g , we have $\alpha_a(id_a) \circ g = \alpha_x(g)$.

We may rename H_b with F , then

$$\begin{array}{ccc}
\mathcal{C}(a, a) & \xrightarrow{\mathcal{C}(g, a)} & \mathcal{C}(x, a) \\
\downarrow \alpha_a & & \downarrow \alpha_x \\
& \begin{array}{ccc}
id_a & \xrightarrow{- \circ g} & id_a \circ g = g \\
\downarrow \alpha_a & & \downarrow \alpha_x \\
\alpha_a(id_a) & \xrightarrow{Fg} & (Fg)(\alpha_a(id_a)) = \alpha_x(g)
\end{array} & \\
Fa & \xrightarrow{Fg} & Fx
\end{array}$$

It seems that H_b can be replaced with any functor in $[\mathcal{C}^{op}, \mathbf{Set}]$, furthermore, the natural transformation is still determined by $\alpha_a(id_a)$ (and $\alpha_a(id_a)$ is determined by α , trivial though, but it implies that there is a precisely corresponding).

Theorem 5.2 (Yoneda Lemma). *Show that the natural transformation between H_a and any functor $F \in [\mathcal{C}^{op}, \mathbf{Set}]$ correspond precisely to the elements of Fa . In other words,*

$$[\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \cong Fa$$

Proof. The arrow $f : [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \rightarrow Fa$ is obvious by the Yoneda trick.

$$\alpha \mapsto \alpha_a(id_a) : [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \rightarrow Fa$$

For any $x \in Fa$, as we see before, we may expect there is a natural transformation α such that $\alpha_a(id_a) = x$, then the action on other objects is completely determined by $\alpha_a(id_a)$ as we see before.

$$x, g \mapsto (Fg)(x) : Fa \rightarrow [\mathcal{C}^{op}, \mathbf{Set}](H_a, F)$$

Note that $(F(id_a))(x) = id_{Fa}(x) = x$.

We need to show that they are inverse to each other. For any natural transformation α :

$$[\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \xrightarrow{f} Fa \xrightarrow{f^{-1}} [\mathcal{C}^{op}, \mathbf{Set}](H_a, F)$$

$$\alpha \mapsto \xrightarrow{-_a(id_a)} \alpha_a(id_a) \mapsto \xrightarrow{(F-)(-)} (F-)(\alpha_a(id_a))$$

And we can see $(Fg)(\alpha_a(id_a)) = \alpha_x(g)$ for any $x \in \mathcal{C}$ and $g \in \mathcal{C}(x, a)$ by the same way as the proof which shows H_\bullet is a full functor.

In another direction, we get:

$$Fa \xrightarrow{f^{-1}} [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \xrightarrow{f} Fa$$

$$x \mapsto \xrightarrow{(F-)(-)} (F-)(x) \mapsto \xrightarrow{-_a(id_a)} F(id_a)(x)$$

It is obvious that $F(id_a)x = id_{Fa}x = x$. □

6 Adjoint

Theorem 6.1. *Show that the hom-functor preserves limit, that is:*

$$Limit(\mathcal{C}(Y, \mathcal{J}_\bullet)) \cong \mathcal{C}(Y, Limit)$$