## 1 Hausdorff spaces

**Definition 1.1.** A topological space  $\mathcal{X}$  is called Hausdorff, if for each pair of distinct points  $x, y \in \mathcal{X}$  there are disjoint neighborhoods  $x \in V$  and  $y \in W$ .

**Theorem 1.1.** Show that any converging sequence in a Hausdorff space has a unique limit.

Proof. Suppose a sequence  $x_n$  converage to x and y, then there are disjoint neighborhoods  $x \in V$  and  $y \in W$ . Since x is the limit of  $x_n$ , so there is i such that for any neighborhood of x,  $x_j$  is in that neighborhood for any j > i. Similar to y, but V and W are disjoint, and V contains infinite points of  $x_n$  from some i, therefore W contains finite points of  $x_n$ , which is unacceptible.

**Theorem 1.2.** Show that a topological space  $\mathcal{X}$  is Hausdorff iff the diagonal

$$\Delta = \{ (x, x) \in \mathcal{X} \times \mathcal{X} \}$$

is a closed set in the product space  $\mathcal{X} \times \mathcal{X}$ 

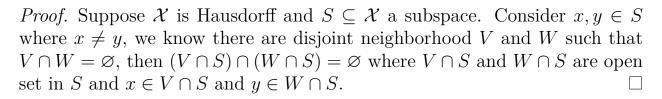
Proof. ( $\Rightarrow$ ) The set  $S = \bigcup_{x,y \in \mathcal{X}, x \neq y} V_{(x,y)} \times W_{(x,y)}$  is an open set in  $\mathcal{X} \times \mathcal{X}$  where  $V_{(x,y)}$  and  $W_{(x,y)}$  are disjoint neighborhoods of x and y, respectively. It is easy to see that  $(x,x) \notin S$  for any  $x \in \mathcal{X}$ , otherwise (x,x) must belongs to some  $V_{(x,x)} \times W_{(x,x)}$  while  $V_{(x,x)}$  and  $W_{(x,x)}$  are disjoint and  $x \in V_{(x,x)} \cap W_{(x,x)}$ . Also, there is no point that is not in S beside the point in  $\Delta$ . So  $\Delta$  is the complement of S, and S is open, so  $\Delta$  is closed.

( $\Leftarrow$ ) For any distinct  $x, y \in \mathcal{X}$ , we have  $(x, y) \in \Delta^C$ . We know  $\Delta^C$  is a union of products of open sets in  $\mathcal{X}$ , so we may suppose  $x \in V$  and  $y \in W$  where V and W are open sets in  $\mathcal{X}$ . Suppose  $z \in V \cap W$ , then  $(z, z) \in \Delta^C$  and then  $(z, z) \notin \Delta$ , which is unacceptible.

Corollary. Any one-point set in a Hausdorff space is closed.

*Proof.* For any Hausdorff space  $\mathcal{X}$  and  $p \in \mathcal{X}$ , for any  $y \in \mathcal{X}$  that  $x \neq y$ , we have a pair of disjoint neighborhood  $V_y$  of x and  $W_y$  of y. Consider the union of these  $W_y$ , obviously it is an open set that contains every point in  $\mathcal{X}$  beside x, so the one-point set  $\{x\}$  is closed.

Corollary. Any subspace of Hausdorff space is Hausdorff.



**Theorem 1.3.** Let  $\mathcal{X}$  be a Hausdorff space and  $K \subseteq \mathcal{X}$  be a compact subset. Then for any  $y \notin K$ , there are open sets  $K \subseteq V$  and  $y \in W$  such that  $V \cap W = \emptyset$ .

*Proof.* For any point  $z \in K$ , we have disjoint neighborhoods  $y \in W_z$  and  $z \in V_z$ , consider the union of  $V_z$ , obviously it is an open cover on K, so there is a finite subcover  $\{V_{z_\alpha}\}$ , then we consider the intersection of the corresponding  $W_{z_\alpha}$ , that is,  $\bigcap_{\alpha} W_{z_\alpha}$ , we can do this intersection cause the subcover is finite. Then let  $V = \bigcup_{\alpha} V_{z_\alpha}$  and  $W = \bigcap_{\alpha} W_{z_\alpha}$ , obviously  $V \cap W = \emptyset$ .

**Theorem 1.4.** Any compact subset of Hausdorff is closed.

*Proof.* Suppose  $\mathcal{X}$  is Hausdorff and  $K \subseteq \mathcal{X}$  a compact subset. For any  $y \notin K$ , we have  $K \subseteq V_y$  and  $y \in W_y$  such that  $V_y \cap W_y = \emptyset$ . Consider the union of these  $W_y$ , we can see that it is an open set and contains every point in  $\mathcal{X} \setminus K$ .

**Theorem 1.5.** Let  $\mathcal{X}$  be a Hausdorff space and  $K, L \subseteq \mathcal{X}$  be two compact subsets that  $K \cap L = \emptyset$ . Show that there are open sets  $K \subseteq V$  and  $L \subseteq W$  such that  $V \cap W = \emptyset$ 

Proof. For any  $l \in L$ , we have  $K \subseteq V_l$  and  $l \in W_l$  where  $V_l \cap W_l = \emptyset$  by Theorem 1.3. Consider the union of these  $W_l$ , it is an open cover on L, therefore there is a finite subcover  $\{W_{l_{\alpha}}\}$ . Then we can take the intersection of the corresponding  $V_l$ , that is,  $V = \bigcap_{\alpha} V_{l_{\alpha}}$ , and the union of the subcover  $W = \bigcup_{\alpha} W_{l_{\alpha}}$ . They are disjoint, otherwise there is  $x \in V_{l_{\alpha}}$  and  $x \in W_{l_{\alpha}}$  for some  $\alpha$ , which contradict to the property from Theorem 1.3.