1 Compactness

Exercise 1.1. *Let $\{V_\alpha\}$ be an open cover of a topological space $\mathcal{X}$. Show that $W \subseteq \mathcal{X}$ is open iff $W \cap V_\alpha$ is open for any $V_\alpha \in \{V_\alpha\}$.*

Proof. $(\Rightarrow)$ is trivial.

$(\Leftarrow)$ For any point $x \in W$, we have $x \in V_x$ since $\{V_\alpha\}$ is open cover. Consider $\bigcup_{x\in W}(W \cap V_x)$, it is a union of open sets, and it is a subset of W, and it contains all points of W, so W is open. □

Theorem 1.1. *Show that a space $\mathcal{X}$ is compact iff for any collection of closed sets $\{Q_\alpha\}$ in $\mathcal{X}$ such that*

$$\bigcap_\alpha Q_\alpha = \varnothing$$

Then there is a finite subset of $\{Q_\alpha\}$ such that

$$Q_0 \cap Q_1 \cap \cdots \cap Q_n = \varnothing$$

Proof.

- $(\Rightarrow)$ For any collection of closed set $\{Q_\alpha\}$ in $\mathcal{X}$ such that $\bigcap_\alpha Q_\alpha = \varnothing$, we can see $\{\mathcal{X} \setminus Q_\alpha\}$ is a collection of open set and $\bigcup_\alpha(\mathcal{X} \setminus Q_\alpha) = \mathcal{X} \setminus (\bigcap_\alpha Q_\alpha) = \mathcal{X} \setminus \varnothing = \mathcal{X}$, therefore the collection of complements $\{\mathcal{X} \setminus Q_\alpha\}$ is an open cover of $\mathcal{X}$. So there is a finite subset of $\{\mathcal{X} \setminus Q_\alpha\}$ that also open covers $\mathcal{X}$. Then it is also a finite subset of $\{Q_\alpha\}$ since $\mathcal{X} = \bigcup_i(\mathcal{X} \setminus Q_i) = \mathcal{X} \setminus (\bigcap_i Q_i)$ and $Q_i \subseteq \mathcal{X}$ implies $\bigcap_i Q_i = \varnothing$.

- $(\Leftarrow)$ For any open cover $\{V_\alpha\}$ of $\mathcal{X}$, consider the collection of complements $\{\mathcal{X} \setminus V_\alpha\}$, we have $\bigcap_\alpha(\mathcal{X} \setminus V_\alpha) = \mathcal{X} \setminus (\bigcup_\alpha V_\alpha) = \varnothing$. So there is a finite subset $\{\mathcal{X} \setminus V_i\}$ such that $\varnothing = \bigcap_i(\mathcal{X} \setminus V_i) = \mathcal{X} \setminus (\bigcup_i V_i)$, therefore $\bigcup_i V_i = \mathcal{X}$, the space $\mathcal{X}$ is compact.

□

Theorem 1.2. *Let $Q_0 \supseteq Q_1 \supseteq \ldots$ be a nested sequence of closed nonempty sets in a compact space $\mathcal{K}$. Show that there is a point $q \in \mathcal{K}$ such that $\forall i, q \in Q_i$.*

Proof. If there is no such point, we know $\bigcap_i Q_i = \varnothing$, which means there is a finite subsequence such that $\bigcap_j Q_j = \varnothing$. Since the sequence Q_i is a nested sequence of nonempty sets, so $\bigcap_j Q_j$ must equal to some "smallest" Q_j, but that means this Q_j is empty set, which contradicts to the assumption. □

Theorem 1.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous mapping between topological spaces and $\mathcal{K}$ is a compact subset in $\mathcal{X}$. Show that $\mathcal{Q} = f(\mathcal{K})$ is also compact in $\mathcal{Y}$. That is, continuous mapping preserve compactness.*

Proof. For any open cover $\{V_\alpha\}$ of $f(\mathcal{K})$, then the inverse images of $\{V_\alpha\}$ cover are also open since f is continous, and also an open cover of $\mathcal{K}$ since they cover $f(\mathcal{K})$. So there is a finite subset of open cover such that covers $\mathcal{K}$, then map those cover by f, we get a finite subset of open cover of $\{V_\alpha\}$ □

Theorem 1.4. *Any closed set in a compact space is also compact.*

Proof. This proof comes from textbook.

Suppose $\mathcal{X}$ a compact space and $\mathcal{Q}$ a closed set in it. Consider any open cover $\{V_\alpha\}$ of $\mathcal{Q}$ and the complement $\mathcal{C} = \mathcal{X} \setminus \mathcal{Q}$. Obviously, $\mathcal{C}$ is open, and $\{\mathcal{C}\} \cup \{V_\alpha\}$ is an open cover of $\mathcal{X}$, therefore there is a finite open cover on $\mathcal{X}$, that open cover may or may not contains $\mathcal{W}$, but we can always add $\mathcal{W}$ to it, and it is still a finite subcover. Since the finite subcover $\{\mathcal{W}, V_{\alpha_0}, \ldots, V_{\alpha_n}\}$ covers $\mathcal{X}$, then it also covers $\mathcal{Q}$, and we can see that $\mathcal{W}$ contributes nothing for $\mathcal{Q}$, so it is safe to remove it, and $\{V_{\alpha_i}\}$ is still a finite open cover on $\mathcal{Q}$.

Furthermore, the proposition can be *iff*, since the whole space is a closed set. If any closed set in that space is compact, then the whole space is also compact. □

Definition 1.1. *Let $\{V_\alpha\}$ and $\{W_\beta\}$ be two covers of a topological space $\mathcal{X}$. We say $\{V_\alpha\}$ is inscried in $\{W_\beta\}$ if for any α, there is β such that $V_\alpha \subseteq W_\beta$*

Theorem 1.5. *A space $\mathcal{X}$ is compact iff for any cover $\{V_\alpha\}$ of $\mathcal{X}$, there is a finite cover such that inscried in $\{V_\alpha\}$.*

Proof. ($\Rightarrow$) For any cover on $\mathcal{X}$, there is a finite subcover on $\mathcal{X}$, and the finite subcover is an inscried in itself.

($\Leftarrow$) For any finite subcover $\{W_\beta\}$ on $\mathcal{X}$ that is inscried in $\{V_\alpha\}$, since for any W_β there is a V_α such that $W_\beta \subseteq V_\alpha$, we may collect these V_α and they form a cover on $\mathcal{X}$. □

Theorem 1.6. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact topological spaces, show that the product $\mathcal{X} \times \mathcal{Y}$ is also compact.*

Proof. □

Theorem 1.7. *Suppose that a product space $\mathcal{X} \times \mathcal{Y}$ is nonempty and compact. Show that $\mathcal{X}$ and $\mathcal{Y}$ are compact.*

Proof. For any open cover $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ on $\mathcal{X}$, consider the open cover $\{V_\alpha\times\mathcal{Y}\}_{\alpha\in\mathcal{I}}$ on $\mathcal{X}\times\mathcal{Y}$, there is a finite subcover $\{V_\alpha\times\mathcal{Y}\}_{\alpha\in\mathcal{J}}$. For any $x\in\mathcal{X}$, take $y\in\mathcal{Y}$ (it is possible cause $\mathcal{X}\times\mathcal{Y}$ is nonempty), we have $(x,y)\in V_\alpha\times\mathcal{Y}$ for some α, so $\{V_\alpha\}_{\alpha\in\mathcal{J}}$ is a open cover on $\mathcal{X}$. □

Definition 1.2. *A topological space $\mathcal{X}$ is called sequentially compact if any point sequence in $\mathcal{X}$ has a converging subsequence.*

Theorem 1.8. *A metric space $\mathcal{M}$ is comapct implies it is sequentially compact.*

Proof. Recall that if a sequence x_n converage to some point p, then for any ϵ, $B(p,\epsilon)$ contains infinite points in x_n.

If any subsequence in x_n is not converging, then for any point $p\in\mathcal{M}$, there is ϵ such that $B(p,\epsilon)$ contains finite points in x_n. Consider the collection of $B(p,\epsilon)$ for every $p\in\mathcal{M}$ is a cover on $\mathcal{M}$, then we have a finite subcover on $\mathcal{M}$, but then the subcover contains finite points in $\mathcal{M}$ while x_n is infinite. □

Theorem 1.9. *Show that the product of two sequentially compact spaces is sequentially compact.*

Proof. TODO □

Definition 1.3. *A sequence x_n of points in a metric space is called Cauchy if for any $\epsilon>0$ there is n such that $|x_i-x_j|<\epsilon$ for all $i,j>n$.*

Theorem 1.10. *Any converging sequence in a metric space is Cauchy.*

Proof. For any converging sequence, the distance between points becomes smaller and smaller, so for any ϵ, there is n such that $|x_i-x_j|<\epsilon$ for all $i,j>n$. □

Definition 1.4. *A metric space $\mathcal{M}$ is called complete if any Cauchy sequence in $\mathcal{M}$ converage to a point in $\mathcal{M}$.*

Theorem 1.11. *Show that any compact metric space $\mathcal{M}$ is complete.*

Proof. For any Cauchy sequence x_n, suppose it is not converage, that is, for any $p\in\mathcal{M}$, there is ϵ such that $B(p,\epsilon)$ contains finite points of x_n. Consider the cover $\{B(p,\epsilon)\}$ for all $p\in\mathcal{M}$ and corresponding ϵ on $\mathcal{M}$, we know there is a finite subcover on $\mathcal{M}$ since $\mathcal{M}$ is compact. Then these finite subcover contains all points in x_n cause it covers $\mathcal{M}$, and it contains finite points in x_n cause each $B(p,\epsilon)$ contains finite points in x_n, therefore the sequence x_n is finite, which is unacceptible. □

Definition 1.5. *Let $\mathcal{M}$ be a metric space. A subset $A \subseteq \mathcal{M}$ is called ϵ-net of $\mathcal{M}$ if for any $p \in \mathcal{M}$, there is $a \in A$ such that $|p - a|_{\mathcal{M}} < \epsilon$ (or equivalently, $p \in B(a, \epsilon)$).*

Theorem 1.12. *Let $\mathcal{M}$ be a sequentially compact metric space, then for any $\epsilon > 0$, there is a finite ϵ-net of $\mathcal{M}$.*

Proof. This proof comes from textbook.

We may trying to construct an ϵ-net of $\mathcal{M}$. We pick a point in $\mathcal{M}$ randomly, say $x_0 \in \mathcal{M}$, then we pick another point $x_1 \in \mathcal{M}$ such that $x_1 \notin B(x_0, \epsilon)$, and then we pick $x_2 \in \mathcal{M}$ such that $x_2 \notin B(x_0, \epsilon)$ and $x_2 \notin B(x_1, \epsilon)$, for any i, we pick $x_i \in \mathcal{M}$ such that $x_i \notin B(x_j, \epsilon)$ for any $j < i$. If at some point, we can't pick any x_i that satisfies the requirement, then $\{x_0, x_1, \ldots, x_{i-1}\}$ is an ϵ-net of $\mathcal{M}$. If this procedure cannot stop, then we get a sequence x_i where their distance are always greater than ϵ. Since $\mathcal{M}$ is sequentially compact, so there is a converging subsequence, however, the distance of points in the subsequence can not below ϵ, so it can't be converging. $\square$