

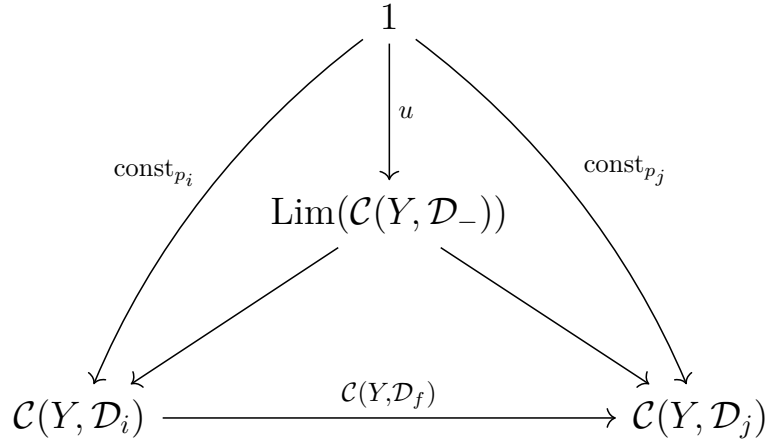
1 Adjoint

Theorem 1.1. *Show that the hom-functor preserves limit, that is, for any $Y \in \mathcal{C}$ and diagram \mathcal{D} , we have:*

$$\text{Lim}(\mathcal{C}(Y, \mathcal{D}_-)) \cong \mathcal{C}(Y, \text{Lim } \mathcal{D})$$

Proof. The idea comes from *The Dao of FP* and nlab.

We may consider the cone with singleton set as vertex:



where const_{p_i} is the function that takes a morphism $p_i \in \mathcal{C}(Y, \mathcal{D}_i)$.

We know there is a one-to-one corresponding between u and the pair $\langle \text{const}_{p_i}, \text{const}_{p_j} \rangle$:

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, \mathcal{C}(Y, \mathcal{D}_-)) \cong \mathbf{Set}(1, \text{Lim}(\mathcal{C}(Y, \mathcal{D}_-)))$$

or we can simplify the equation by defining $F_j = \mathcal{C}(Y, \mathcal{D}_j) : \mathcal{J} \rightarrow \mathbf{Set}$.

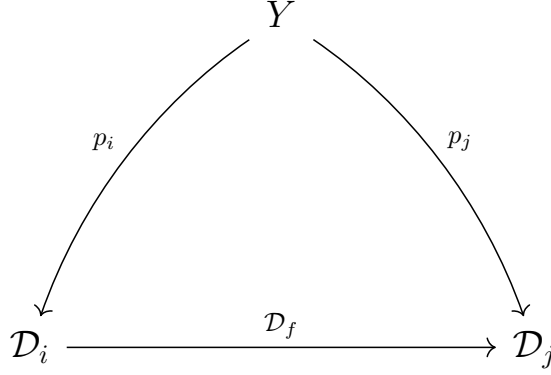
$$[\mathcal{J}, \mathbf{Set}](\Delta_1, F) \cong \mathbf{Set}(1, \text{Lim } F)$$

We may recall that the hom-set maps out from 1 is isomorphic to the target, that is:

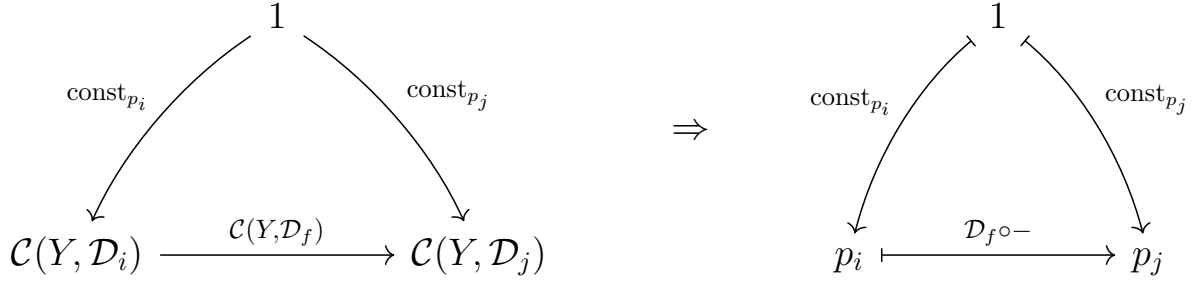
$$\mathbf{Set}(1, \text{Lim } F) \cong \text{Lim } F$$

Similarly, the cone with 1 as vertex is a pair of selection of $\mathcal{C}(Y, \mathcal{D}_i)$, it

forms a cone with Y as vertex:



the diagram is indeed commute since



Also we can make a pair of selection of $\mathcal{C}(Y, \mathcal{D}_-)$ from a cone with Y as vertex.

Then the cone of $\mathcal{C}(Y, \mathcal{D}_-)$ with vertex 1, is isomorphic to the cone of \mathcal{D}_- with vertex Y :

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, F) \cong [\mathcal{J}, \mathcal{C}](\Delta_Y, D)$$

While the later one is naturally isomorphic to the limit of \mathcal{J}_\bullet :

$$[\mathcal{J}, \mathcal{C}](\Delta_Y, D) \cong \mathcal{C}(Y, \text{Lim } D)$$

Finally, we have:

$$\begin{aligned}
 & \text{Lim } \mathcal{C}(Y, \mathcal{D}_-) \\
 & \cong \\
 & \mathbf{Set}(1, \text{Lim}(\mathcal{C}(Y, \mathcal{D}_-))) \\
 & \cong \\
 & [\mathcal{J}, \mathbf{Set}](\Delta_1, \mathcal{C}(Y, \mathcal{D}_-)) \\
 & \cong \\
 & [\mathcal{J}, \mathcal{C}](\Delta_Y, D) \\
 & \cong \\
 & \mathcal{C}(Y, \text{Lim } D)
 \end{aligned}$$

□

Dually, we also have:

$$\text{Lim}(\mathcal{C}(\mathcal{D}_-, Y)) \cong \mathcal{C}(\text{Colim } \mathcal{D}, Y)$$

1.1 Unit and Counit

Suppose $L \dashv R$, that is, for any $a \in \mathcal{D}$ and $b \in \mathcal{C}$, we have:

$$\mathcal{C}(La, b) \cong \mathcal{D}(a, Rb)$$

If you keep Yoneda trick in mind, you may try to replace b by La or a by Rb , which produce two morphisms:

$$\begin{aligned} \eta_a &: a \rightarrow R(La) \\ \epsilon_b &: L(Rb) \rightarrow b \end{aligned}$$

or we can regard them as natural transformations:

$$\begin{aligned} \eta &: \text{Id} \rightarrow R \circ L \\ \epsilon &: L \circ R \rightarrow \text{Id} \end{aligned}$$

1.2 Back to Adjoint

What condition those unit and counit should hold such that we can construct an adjoint back? Let's see how we can construct an adjoint from unit and counit, suppose $f : La \rightarrow b$ a morphism, we need to provide a morphism $a \rightarrow Rb$. One obvious way is:

$$a \xrightarrow{\eta_a} RLa \xrightarrow{Rf} Rb$$

Similarly, for given $g : a \rightarrow Rb$, we have:

$$La \xrightarrow{Lg} LRb \xrightarrow{\epsilon_b} b$$

Since the adjoint is an isomorphism between hom-set, so if we send the morphism we mapped from f back to $\mathcal{C}(La, b)$, it should still be f :

$$\begin{array}{ccc}
 La & \xrightarrow{f} & b \\
 \downarrow L(\eta_a) & & \uparrow \epsilon_b \\
 L(RLa) & \xrightarrow{LRf} & L(Rb)
 \end{array}$$

or in symbol:

$$\epsilon_b \circ LRf \circ L\eta_a = f$$

Since ϵ is a natural transformation, we have this diagram commutes:

$$\begin{array}{ccc}
 LR(La) & \xrightarrow{LR(f)} & LR(b) \\
 \downarrow \epsilon_{La} & & \downarrow \epsilon_b \\
 La & \xrightarrow{f} & b
 \end{array}$$

Then we can rewrite the equation:

$$f \circ \epsilon_{La} \circ L\eta_a = f$$

or

$$f \circ (\epsilon \cdot L)_a \circ (L \cdot \eta)_a = f$$