Definition 0.11. *For any $T \in \mathcal{L}(V, W)$, set* $\operatorname{null} T = \{ v \mid Tv = 0 \}$ *is called the* ***null space*** *of T.*

This is also called the **kernal** of T in algebra.

Theorem 0.13. *For any $T \in \mathcal{L}(V, W)$,* $\operatorname{null} T$ *is a subspace of V.*

Proof.

- We have $0 \in \operatorname{null} T$ since $T0 = 0$, which is the property of linear transformation.
- For any $a, b \in \operatorname{null} T$, we have $0 = Ta + Tb = T(a + b)$, so $a + b \in \operatorname{null} T$.
- For any $Ta \in \operatorname{null} T$ and $\lambda \in F$, we have $\lambda Ta = T(\lambda a)$, so $\lambda a \in \operatorname{null} T$.

□

Definition 0.15. *For any $T \in \mathcal{L}(V, W)$, set* $\operatorname{range} T = T(V) = \{ Tv \mid v \in V \}$ *is called the* ***range*** *of T.*

This is also called the **image** of T in math.

Theorem 0.18. *For any $T \in \mathcal{L}(V, W)$,* $\operatorname{range} T$ *is a subsapce of W.*

Proof.

- We have $T(0) = 0 \in \operatorname{range} T$.
- For any $Ta, Tb \in \operatorname{range} T$, $Ta + Tb = T(a + b) \in \operatorname{range} T$.
- For any $Ta \in \operatorname{range} T$ and $\lambda \in F$, $\lambda Ta = T(\lambda a) \in \operatorname{range} T$.

□

Theorem 0.21. *Suppose V is finite and $T \in \mathcal{L}(V, W)$, then* $\operatorname{range} T$ *is finite, and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof. Consider the basis $v_0, \cdots, v_k$ of $\operatorname{null} T$, and the basis $v_0, \cdots, v_n$ of V that expand from $v_0, \cdots, v_k$. We will show that $T(v_{k+1}), \cdots, T(v_n)$ is the basis of $\operatorname{range} T$.

We first show that $T(v_{k+1}), \cdots, T(v_n)$ is linear independent. If it is linear independent, then

$$
\begin{aligned}
&\lambda_1 T(v_{k+1}) + \cdots + \lambda_i T(v_{k+i}) \\
=&T(\lambda_1 v_{k+1} + \cdots + \lambda_i T(v_{k+i})) \\
=&0
\end{aligned}
$$

That means a linear combation of v_{k+i} is in $\operatorname{null} T$, which is $\operatorname{span}(v_0, \cdots, v_k)$, therefore the basis $v_0, \cdots, v_n$ is linear dependent.

Then we show that $T(v_{k+1}), \cdots, T(v_n)$ spans $\operatorname{range} T$. For any $Tv \in \operatorname{range} T$, there must be $v \in V$ such that $Tv = Tv$, then v can be written in form of the linear combination of $v_0, \cdots, v_n$, and then $Tv = T(\lambda_0 v_0 + \cdots + \lambda_n v_n)$. We can drop all terms with v_i where $i \leq k$, since they are in $\operatorname{null} T$, so Tv is now represent by a linear combination of $T(v_{k+i})$ for all $0 < i \leq n - k$, therefore, it is a basis of $\operatorname{range} T$ and $\dim \operatorname{range} T$ is finite.

Finally, $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

Definition 3.16 (Notation: $v + U$)**.** *Let $v \in V$ and $U \subseteq V$, then $v + U = \{ v + u \mid u \in U \}$.*

Such sets also called *coset* in group theory.

Definition 3.97 (Translate)**.** *Let $v \in V$ and $U \subseteq V$, we say $v + U$ is a translate of U.*

Definition 3.98 (Quotient Space)**.** *Let $U \subseteq V$ a subspace, then the quotient space V/U is a set with translates of U, that is:*

$$V/U = \{ v + U \mid v \in V \}$$

Theorem 3.101. *Let $U \subseteq V$ a subspace and $v, w \in V$, then the following statements are equivalent.*

1. $v - w \in U$
2. $v + U = w + U$
3. $(v + U) \cap (w + U) \neq \varnothing$

Proof.

- If $v - w \in U$, for any $v + u \in v + U$, we have $v + u = v + (v - w) - (v - w) + u = v - w + w + u = w + (v - w) + u \in w + U$ since $v - w \in U$. Similarly, for any $w + u \in w + U$, we have $w + u = w + (v - w) - (v - w) + u = v - v + w + u = v - (v - w) + u = v + (-(v - w) + u) \in v + U$.

- If $v+U = w+U$, then $v = w+u$ since $v \in v+U$, therefore $v-w = u \in U$.
- if $v + U = w + U$, then $(v + U) \cap (w + U) = v + U = w + U \neq \varnothing$
- If $(v+U) \cap (w+U) \neq \varnothing$, then for any $v+u_0 = w+u_1 \in (v+U) \cap (w+U)$, we have $(v - w) + (u0 - u_1) = 0$ and then $v - w = u_1 - u_0 \in U$, so $v + U = w + U$.

□

Definition 3.102. *Let $U \subseteq V$, then addition and scalar multiplication on V/U is defined by:*

$$(v + U) + (w + U) = (v + w) + U$$
$$\lambda(v + U) = (\lambda v) + U$$

Theorem 3.103. *Let $U \subseteq V$ a subspace, then V/U is a vector space with addition and scalar multiplication we defined in previous definition.*

Proof. We must first show that the addition and the sclar multiplication we introduce are functions.

For any $a, b, c, d \in V$, we will show $(a+b)+U = (c+d)+U$ if $a+U = c+U$ and $b + U = d + U$. We can show $(a + b) - (c + d) \in U$ by $a - c \in U$ and $b - d \in U$.

For any $v, w \in V$ and $\lambda \in F$, we will show $(\lambda v) + U = (\lambda w) + U$ if $v+U = w+U$. We know $v-w \in U$, then $\lambda(v-w) = \lambda v - \lambda w \in U$, therefore $(\lambda v) + U = (\lambda w) + U$.

We have identity of addition $0 + U$ and inverse of addition $(-v) + U$ for all $v \in V$. □

Definition 3.104. *Let $U \subseteq V$ a subspace, the quotient map $\pi : V \to V/U$ is a linear mapping defined by:*

$$\pi(v) = v + U$$

Proof. We will show π is a linear mapping, $\pi(v + w) = (v + w) + U = v + U + w + U = \pi(v) + \pi(w)$ and $\lambda\pi(v) = \lambda(v + U) = (\lambda v) + U = \pi(\lambda v)$. □

Theorem 3.105. *Let V finite and $U \subseteq V$ a subspace, show that* $\dim(V/U) = \dim V - \dim U$.

Proof. We can rewrite the equation as $\dim V = \dim(V/U) + \dim U$, and it is easy to see that $\operatorname{range}\pi = \dim(V/U)$ and $\operatorname{null}\pi = \dim U$. □

Definition 3.106. *Let $T \in \mathcal{L}(V, W)$, define $\tilde{T} : V/(\operatorname{null} T) \to W$ by $\tilde{T}(v + \operatorname{null} T) = Tv$.*

Theorem 3.107. *Let $T \in \mathcal{L}(V, W)$, then:*

1. $\tilde{T} \circ \pi = T$
2. *$\tilde{T}$ is injective*
3. $\operatorname{range}\tilde{T} = \operatorname{range} T$
4. $V/(\operatorname{null} T) \cong \operatorname{range} T$

Proof.

1. For all $v \in V$, $\tilde{T}(\pi(v)) = \tilde{T}(v + \operatorname{null} T) = Tv$
2. If $\tilde{T}(v + \operatorname{null} T) = \tilde{T}(w + \operatorname{null} T)$, then $T(v - w) = 0$, which means $v - w \in \operatorname{null} T$, therefore $v + \operatorname{null} T = w + \operatorname{null} T$.
3. For any $Tv \in \operatorname{range} T$, we have $\tilde{T}(v + \operatorname{null} T) \in \operatorname{range}\tilde{T}$. For any $\tilde{T}(v + \operatorname{null} T) = Tv \in \operatorname{range}\tilde{T}$, we have $Tv \in \operatorname{range} T$.
4. Restrict the range of $\tilde{T}$ on $\operatorname{range} T$, say $\varphi(v + \operatorname{null} T) = \tilde{T}(v + \operatorname{null} T) : V/(\operatorname{null} T) \to \operatorname{range} T$, then φ is injective since (2) and surjective since (3), therefore φ is an isomorphism, thus $V/(\operatorname{null} T) \simeq \operatorname{range} T$.

□

Definition 3.110 (Dual Space)**.** *Let V a vector space, then we denote V' the dual space of V, where*

$$V' = \mathcal{L}(V, F)$$

Theorem 3.111. *Let V a finite vector space, then* $\dim V' = \dim V$

Proof. $\dim V' = \dim \mathcal{L}(V, F) = (\dim V)(\dim F) = \dim V$ □

Definition 3.112 (Dual Basis)**.** *Let $v_0, \cdots, v_{m-1}$ a basis of V, then the dual basis of $v_0, \cdots, v_{m-1}$ is $\varphi_0, \cdots, \varphi_{m-1}$ such that:*

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

holes for any $0 \leq i, j < m$.

We can see that the basis the dual basis extracts the coefficients of any vector in V.

Theorem 3.113. *Let $v_0, \cdots, v_{m-1}$ a basis of V, and dual basis $\varphi_0, \cdots, \varphi_{m-1}$ of which. Then for any $v \in V$,*

$$v = \varphi_0(v)v_0 + \cdots + \varphi_{m-1}(v)v_{m-1}$$

Proof. For any i, $\varphi_i(v) = \varphi_i(\lambda_0 v_0 + \cdots + \lambda_{m-1}v_{m-1}) = \varphi_i(\lambda_i v_i) = \lambda_i \varphi_i(v_i) = \lambda_i \times 1$. □

Theorem 3.116. *Let V a finite space, then the dual basis of basis of V is a basis of V'.*

Proof. Let $v_0, \cdots, v_{m-1}$ a basis of V, then its dual basis has the same length, therefore we only need to show its dual basis is linear independent.

Suppose $\lambda_0\varphi_0 + \cdots + \lambda_{m-1}\varphi_{m-1} = 0$, then for any $0 \le i < m$, $(\lambda_0\varphi_0 + \cdots + \lambda_{m-1}\varphi_{m-1})(v_i) = \lambda_i = 0$, therefore the dual basis is linear independent. □

Definition 3.118 (Dual Map)**.** *Let $T \in \mathcal{L}(V, W)$. A dual map of T is a linear map $T' \in \mathcal{L}(W', V')$, such that for any $\varphi \in W'$:*

$$T'(\varphi) = \varphi \circ T$$

Theorem 3.128. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that:*

1. $\operatorname{null} T' = (\operatorname{range} T)^0$
2. $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

Proof.

- $\operatorname{null} T'$ is a space $\{ \varphi \in \mathcal{L}(W, F) \mid \varphi \circ T = 0 \}$, which means $\operatorname{range} T \subseteq \operatorname{null} \varphi$. $(\operatorname{range} T)^0$ is a space $\{ \varphi \in \mathcal{L}(W, F) \mid \varphi(\operatorname{range} T) = \{0\} \}$, which means $\operatorname{range} T \subseteq \operatorname{null} \varphi$. Therefore $\operatorname{null} T' = (\operatorname{range} T)^0$
- $\dim(\operatorname{range} T)^0 = \dim W - \dim \operatorname{range} T = \dim W - (\dim V - \dim \operatorname{null} T)$

□

Theorem 3.129. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that*

$$T \text{ is surjective} \iff T' \text{ is injective}$$

Proof.

- Suppose T is surjective, then for any $T'(\varphi) = T'(\psi)$, we have $\varphi \circ T = \psi \circ T$. Since T is surjective, then T is an epimorphism (we proved this in E3B), therefore $\varphi = \psi$.

- Suppose T' is injective, then for any $\varphi, \psi \in \mathcal{L}(W, F)$ such that $\varphi \circ T = \psi \circ T$, we have $\varphi = \psi$ since T' is injective. Therefore T is epimorphism, thus surjective.

□

The last theorem is obviously true in category theory, but we haven't show that T' is a morphism in Vect' where $\mathrm{Vect}' \simeq \mathrm{Vect}^{\mathrm{op}}$.

Theorem 3.130. *Let V, W are finite and $T \in \mathcal{L}(V, W)$, show that:*

1. $\dim \operatorname{range} T' = \dim \operatorname{range} T$

2. $\operatorname{range} T' = (\operatorname{null} T)^0$

Proof.

- $\dim \operatorname{range} T' = \dim W' - \dim \operatorname{null} T' = \dim W' - (\dim \operatorname{null} T + \dim W - \dim V) = \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$

- For any $\varphi \circ T \in \operatorname{range} T'$, $(\varphi \circ T)(\operatorname{null} T) = \varphi(\{0\}) = \{0\}$, therefore $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$. Since $\dim(\operatorname{null} T)^0 = \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$ *dim* $\operatorname{range} T'$, therefore $\operatorname{range} T' = (\operatorname{null} T)^0$ since both of them are finite and $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$.

□

Theorem 3.131. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show*

$$T \textit{ is injective} \iff T' \textit{ is surjective}$$

Proof.

- $\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim V = \dim V'$ since T is injective, therefore T' is surjective.

- $\dim \operatorname{range} T = \dim \operatorname{range} T' = \dim V' = \dim V$ therefore T is injective.

□

This chapter is much like a note, I will record some idea about polynomial and linear algebra.

One relationship is that a polynomial is a linear combination of the standard basis of $\mathcal{P}(F)$, that is, $1, x, x^2, \cdots$. This is important when we apply p to an operator of a vector space, say $T \in \mathcal{L}(V)$ and $p(T) = c_0 I + c_1 T + c_2 T^2 + \cdots$. If we apply $p(T)$ to some $v \in V$, it becomes a linear combination of $v, Tv, T^2 v, \cdots$.

Theorem 3.16. *Let $p \in \mathcal{P}(\mathbb{R})$ is not constant, then p can be factorized into:*

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M)$$

where $c, \lambda_1, \cdots, \lambda_m, b_1, \cdots, b_M, c_1, \cdots, c_M \in \mathbb{R}$ and for any $1 \leq k \leq M$, $b_k^2 < 4c_k$.

I won't paste the proof here, but the statement can be considered as: any non-constant, real p can be factorized into:

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x - \lambda_{m+1}) \cdots (x - \lambda_{m+M})$$

where $c, \lambda_1, \cdots, \lambda_1 \in \mathbb{R}$ and $\lambda_{m+1}, \cdots, \lambda_{m+M} \in \mathbb{C}$. Those λ are zeros of p, however some are real, some are complex. This re-expression makes the statement more understandable. Note that λ_{m+k} is paired, since both λ_{m+k} and $\bar{\lambda}_{m+k}$ are zeros of p.

Theorem 3.39. *Let $T \in \mathcal{L}(V)$ and $v_0, \cdots, v_{n-1}$ a basis of V, then the following statements are equivalent to each others.*

- *$\mathcal{M}(T)$ about $v_0, \cdots, v_{n-1}$ is upper-triangular matrix*
- *For any $k = 1, \cdots, n$, $\operatorname{span}(v_0, \cdots, v_{k-1})$ is invariant under T.*
- *For any $k = 1, \cdots, n$, $Tv_{k-1} \in \operatorname{span}(v_0, \cdots, v_{k-1})$.*

Proof.

- (1) ⇒ (2) Induction on k. In breif, first k columns are in $\operatorname{span}(v_0, \cdots, v_{k-1})$, therefore $\operatorname{span}(v_0, \cdots, v_{k-1})$ is invariant under T.
- (2) ⇒ (3) Trivial, $v_{k-1} \in \operatorname{span}(v_0, \cdots, v_{k-1})$ which is invariant under T.

- (3) $\Rightarrow$ (1) Basically the definition of upper-triangular matrix, $Tv_{k-1} \in \operatorname{span}(v_0, \cdots, v_{k-1})$ means the k-th column of $\mathcal{M}(T)$ consists of first k number (the coefficients of Tv_{k-1}) and 0s.

□

Theorem 3.40. *Let $T \in \mathcal{L}(V)$ and $v_0, \cdots, v_{n-1}$ a basis of V, such that $\mathcal{M}(T)$ is upper-triangular matrix, and $\lambda_0, \cdots, \lambda_{n-1}$ are the numbers of its diagonal. Show that*

$$(T - \lambda_0 I) \cdots (T - \lambda_{n-1} I) = 0$$

Proof. All numbers since i of $(T - \lambda_i I)v$ are 0 if numbers after i of v are 0. Thus $(T - \lambda_0 I) \cdots (T - \lambda_{n-1} I)$ makes $n-1$-th number 0, then $n-2$-th number and so on. □

Theorem 3.41. *Let $T \in \mathcal{L}(V)$ and $\mathcal{M}(T)$ about some basis of V is upper-triangular matrix. Show that the eigenvalues of T are the numbers in the diagonal.*

Proof. Let $v_0, \cdots, v_{n-1}$ a basis of V and $\mathcal{M}(T)$ about this basis is upper-triangular matrix. For λ_i where $i = 0, \cdots, n-1$, we will show that $T - \lambda_i I$ is not invertible.

We will see first i columns of $T - \lambda_i I$ is linear dependent, since they have at most $i - 1$ non-zero numbers while the list they form has length i.

Another part of proof follows the book. Let $q(z) = (z - \lambda_0) \cdots (z - \lambda_{n-1})$, then $q(T) = 0$ by **??**, thus q is polynomial multiple of the minimal polynomial of T, thus any zero of the minimal polynomial of T is also a zero of q, which means it belongs to the list $\lambda_0, \cdots, \lambda_{n-1}$. □

Theorem 3.44. *Let V finite and $T \in \mathcal{L}(V)$. Show that $\mathcal{M}(T)$ is upper-triangular matrix about some basis of V $\iff$ the minimal polynomial of T is in form of $(z - \lambda_0) \cdots (z - \lambda_{n-1})$ where $\lambda_i \in F$.*

Proof. This proof comes from the book.

The ($\Rightarrow$) part follows **??**, $(z - \lambda_0) \cdots (z - \lambda_{m-1})$ ($\lambda_0, \cdots, \lambda_{m-1}$ are the numbers in the diagonal) is polynomial multiple of the minimal polynomial of T, then the minimal polynomial of T must in a similar form.

For ($\Leftarrow$), we will induction on n.

- Base($n = 0$), the minimal polynomial of T is in form $(z - \lambda_0)$, thus $T = \lambda_0 I$.

- Ind($n = n+1$), the minimal polynomial of T is in form $p(z) = (z - \lambda_0)\cdots(z-\lambda_{n+1-1})$. Consider $T - \lambda_n I$, there is non-zero $v \in V$ such that $(T - \lambda_n I)v = 0$ since λ_n is a zero of p therefore an eigenvalue of T. We define $U = \text{range}(T - \lambda_n I)$, consider $q(z) = (z - \lambda_0)\cdots(z - \lambda_{n-1})$ and then $q(T\big|_U) = 0$, recall that $U = \text{range}(T - \lambda_n I)$, therefore for any $v \in U$, there is u such that $(T - \lambda_n I)u = v$. Thus $q(T\big|_U)u = q(T)(T - \lambda_n I)v = p(T)v = 0$ where $u = (T - \lambda_n I)v$ and $u, v \in U$. Thus the matrix of $T\big|_U$ is upper-triangular.

  Then we consider $u_0, \cdots, u_{k-1}$ a basis of U, we will expand $u_0, \cdots, u_{k-1}$ to a basis of V, say $u_0, \cdots, u_{k-1}v_0, \cdots, v_{m-1}$. Then for any v_i where i, we have $Tv_i = Tv_i - \lambda_n v_i + \lambda_n v_i = (T - \lambda_n I)v_i + \lambda_n v_i$, where $(T - \lambda_n)v_i \in U$ (recall the definition of U), thus $Tv_i \in \text{span}(u_0, \cdots, u_{k-1}, v_0, \cdots, v_i)$, then by **??**, we know $\mathcal{M}(T)$ is an upper-triangular matrix about the basis $u_0, \cdots, u_{k-1}, v_0, \cdots, v_{m-1}$.

  One may confused that the "length" of u_i is greater than the size of $\mathcal{M}(T\big|_U) = \dim U$ (thus it won't be a square matrix but tall and thin), however, these two things are unrelated, a matrix only represents how to combine the basis, and doesn't care what the basis looks like.

□

Exercise 3.1. *Suppose V is a finite vector space, Show that the only two ideal of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$. A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called an ideal, if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for any $T \in \mathcal{L}(V)$ and $E \in \mathcal{E}$.*

Proof. We will use the concept Matrix. Suppose $\lambda_0 v_0 + \cdots + \lambda_n v_v$ the basis of V. We want to construct T_i that $T(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_i v_i$ for all $0 \le i < n$, which is a matrix with all zero but 1 at i, i.

For any matrix, we can always select a non-zero value at a, b and place it at i, b, this can be done by left multiply a matrix with 1 at i, a (this produce a vector at line i with values from line a), then right multiply a matrix with 1 at i, b (this produce a vector at column b with values from line i).

Also, we can always select a non-zero value at a, b and place it at a, i, this can be done by right multiply a matrix with 1 at b, i, then left multiply a matrix with 1 at a, i.

By combining these two operations, we ca select a non-zero value at a, b and place it at i, i. Now, consider any non-zero $E \in \mathcal{E}$, we can construct a matrix with non-zero value at i, i for every $0 \le i < \dim V$. These matrix are in $\mathcal{E}$

since $\mathcal{E}$ is an ideal, then we can multiply an apropriate scalar to them so that they are matrices with 1 at i, i. By adds up these matrices, we get I, we know $I \in \mathcal{E}$ since $\mathcal{E}$ is a vector space, and now all $T \in \mathcal{L}(V)$ is also in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then $\mathcal{E} = \mathcal{L}(V)$.

The only exception is $\mathcal{E} = \{0\}$, in this case we can't pick any non-zero element.

Another solution, hope this one is more simple.

Suppose $\mathcal{E}$ an ideal of $\mathcal{L}(V)$ and non-zero, non-surjective $E \in \mathcal{E}$. Let $v_0, \cdots, v_{k-1}$ a basis of $\operatorname{null} E$ and $v_k, \cdots, v_{k+n}$ such that Tv_{k+i} is a basis of $\operatorname{range} E$, then we have $n \neq 0$ and $k \neq 0$.

Define A a linear transformation which maps v_i to v_{k+i} for $0 \leq i < \min\{k, n\}$ and maps others to 0, then $\dim \operatorname{range} EA = \min\{k, n\}$.

Expand the basis $w_i = Ev_{k+i}$ of $\operatorname{range} E$ to a basis of V, say $w_0, \cdots, w_{m-1}$, define B maps Ev_{k+i} to $w_{\min\{k,n\}+i}$, we always have enough $w_{\min\{k,n\}+i}$ since $m - 1 = \dim V = \dim \operatorname{null} E + \dim \operatorname{range} E$ while $\min k, n \leq \dim \operatorname{null} E$, then $\dim \operatorname{range} BE = \operatorname{range} E$ since we just re-map the range E.

Now consider $S = EA+BE$, we have $Sv_i = EAv_i = Ev_{k+i} = w_i \in \operatorname{range} E$ for all $0 \leq i < \min\{k, n\}$ and $Sv_{\min\{k,n\}+i} = BEv_{\min\{k,n\}+i} = w_{\min\{k,n\}+i} \in \operatorname{range} BE$ for all $0 \leq i < \dim \operatorname{range} E$. We can see $\operatorname{range} EA \cap \operatorname{range} BE = \{0\}$ and $\dim \operatorname{range}(EA + BE) = \operatorname{range} E + \min\{k, n\}$, where $k = \operatorname{null} E$ and $n = \operatorname{range} E$, the range of $EA + BE$ gets larger and $EA + BE \in \mathcal{E}$ since $EA, BE \in \mathcal{E}$, if $k > n$ (this is the only case that $EA + BE$ is not surjective), then we continue this process with $E = EA + BE$, the procedure will finally terminate since $\mathcal{L}(V)$ is finite (cause V is finite).

Now we show that any $\mathcal{E}$ with non-zero, non-surjective $E \in \mathcal{E}$ implies a surjective (thus injective and invertible) $T \in \mathcal{E}$.

For any ideal with an invertible element $E \in \mathcal{E}$, we have $E^{-1}E = I \in \mathcal{E}$, which causes $\mathcal{E} = \mathcal{L}(V)$ since $IT = T$ for all $T \in \mathcal{L}(V)$.

Therefore, only $\{0\}$ and $\mathcal{L}(V)$ are ideals of $\mathcal{L}(V)$. □

Exercise 3.7. *Suppose vector space V and W are finite* $(2 \leq \dim V \leq \dim W)$, *show that* $\{ T \in \mathcal{L}(V, W) \mid T$ *is not injective* $\}$ *is not a subspace.*

Proof. Consider the basis $v_0 + \cdots + v_{(\dim V - 1)} \in V$, and $T(v_0 + \cdots + v_{(\dim V - 1)}) = (0 + 1_0 + \cdots + 1_{(\dim V - 1)})$ and $T'(v_0 + \cdots + v_{(\dim V - 1)}) = (v_0 + 0 + \cdots + v_{(\dim V - 1)})$. Then $(T + T')(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_{(\dim V - 1)} v_{(\dim V - 1)}) = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + 2\lambda_{(\dim V - 1)} v_{(\dim V - 1)}$, which is obviously injective. □

Exercise 3.11. *Suppose V is finite and $T \in \mathcal{L}(V, W)$, show that there is a* *subspace $U \subset V$ such that:*

$$U \cap \operatorname{null} T = \{0\} \quad \textit{and} \quad \operatorname{range} T = \{\ Tu \mid u \in U\ \}$$

Proof. This is similar to the *isomorphism theorems* about groups! This can be done by the similar way we used in proving $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

The next two exercises remind me the categorical injective and surjective, let try them first!

Exercise. *For any $F \in \mathcal{L}(V, W)$, F is injective $\iff$ for any $S, T \in \mathcal{L}(U, V)$, $FS = FT$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(V, W)$ that $FS = FT$, then for any $u \in U$, we have $F(Su) = F(Tu)$, since F is injective, we know $Su = Tu$, so $S = T$.
- ($\Leftarrow$) For any $v, w \in V$ such that $Fv = Fw$. Consider

$$S(\lambda) = \lambda v$$
$$T(\lambda) = \lambda w$$

  in $\mathcal{L}(\mathbb{R}, V)$. Then for any $\lambda \in \mathbb{R}$, we have $FS\lambda = \lambda FS1 = \lambda Fv = \lambda Fw = \lambda FT1 = FT\lambda$. so $FS = FT$ then $S = T$, which means $v = S1 = T1 = w$.

□

Exercise. *Suppose W is finite, then for any $F \in \mathcal{L}(V, W)$, F is surjective $\iff$ for any $S, T \in \mathcal{L}(W, U)$, $SF = TF$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(W, U)$ such that $SF = TF$. For any $w \in W$, there is $v \in V$ such that $Fv = w$ since F is surjective. Then we have $SFv = TFv$ so $Sw = S(Fv) = T(Fv) = Tw$ then $S = T$.
- ($\Leftarrow$) Consider

$$S = I \quad \text{and} \quad T(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 w_0 + \cdots + \lambda_k w_k$$

where $w_0, \cdots, w_k$ is the basis of range F and $w_0, \cdots, w_n$ is the basis of W that expand from $w_0, \cdots, w_k$.

(If we can use another way to construct T, then W is not need to be finite, for example, $W = \operatorname{range} T \oplus W_0$).

It is easy to show that T is a linear transformation. Then for any $v \in V$, we have $TFv = Fv$ (since T acts like identity transformation on range F) and $SFv = Fv$, so $S = T$ by the property of F. Since range $S = W$, so is range T, that means $w_0, \cdots, w_k$ spans W, so $k = n$, which means range $F = W$, therefore F is surjective.

□

Exercise 3.19. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is injective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $ST = I$.*

Proof.

- ($\Rightarrow$) Consider the basis $v_0, \cdots, v_n$ of V, then $Tv_0, \cdots, Tv_n$ is a basis of range T since T is injective. We denote Tv_i as w_i and $w_0, \cdots, w_m$ as the basis of W which expand from $w_0, \cdots, w_n$. Define $S(\lambda_0 w_0 + \cdots + \lambda_m w_m) = \lambda_0 w_0 + \cdots + \lambda_n w_n$, and then for any $v \in V$, $ST(\lambda_0 v_0 + \cdots + \lambda_n v_n) = S(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 v_0 + \cdots + \lambda_n v_n$, so $ST = I$.

- ($\Leftarrow$) Suppose $A, B \in \mathcal{L}(U, V)$, such that $TA = TB$, we will show that $A = B$. $STA = IA = A$ and $STB = IB = B$ and $STA = STB$ since $TA = TB$. Then we know T is a monomorphism, and then T is injective.

□

Exercise 3.20. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is surjective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $TS = I$.*

Proof. □

Exercise 3.21. *Suppose V is finite, $T \in \mathcal{L}(V, W)$, $U \subseteq W$ a subspace. Show that the inverse image of U: $\{ v \in V \mid Tv \in U \}$ is a subspace of V, and*

$$\dim\{ v \in V \mid Tv \in U \} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$$

Proof. The second part is quite easy, we can restrict the domain of T to $\{ v \in V \mid Tv \in U \}$, say $T' \in \mathcal{L}(\{ v \in V \mid Tv \in U \}, W)$, so that it is in form $\dim\{ v \in V \mid Tv \in U \} = \dim \operatorname{null} T' + \dim \operatorname{range} T'$. Obviously $\operatorname{range} T' = U \cap \operatorname{range} T$ and $\operatorname{null} T' = \operatorname{null} T$.

We will now show that $\{ v \in V \mid Tv \in U \}$ is a subspace of V.

- $T0 \in U$.
- For any $v, w \in V$ such that $Tv, Tw \in U$, we have $T(v+w) = Tv+Tw \in U$.
- For any $v \in V$ such that $Tv \in U$ and $\lambda \in F$, we have $T(\lambda v) = \lambda Tv \in U$.

Therefore it is a subspace. □

Exercise 3.22. *Suppose U and V are finite, $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$, show that*

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$$

Proof. Consider the inverse image of $\operatorname{null} S$ on T: $K = \{ v \in V \mid Tv \in \operatorname{null} S \}$, which dimension: $\dim K = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T)$, where $\dim(\operatorname{null} S \cap \operatorname{range} T)$ caps at $\dim \operatorname{null} S$.

We know show that $\operatorname{null} ST = \operatorname{null} K$. For any $STv = 0$, we know $S(Tv) = 0$, so $Tv \in K$, therefore $\operatorname{null} ST \subseteq \operatorname{null} K$; For any $Tv \in \operatorname{null} S$, that means $S(Tv) = 0$, therefore $v \in \operatorname{null} ST$, therefore $\operatorname{null} ST \supseteq \operatorname{null} K$, and $\operatorname{null} ST = \operatorname{null} K$. □

Exercise 3.25. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that* $\operatorname{null} S \subseteq \operatorname{null} T \iff$ *there is $E \in \mathcal{L}(W)$ such that $T = ES$.*

Proof. We define $E(S(v)) = Tv$ for any $v \in V$, so that $E \in \mathcal{L}(\operatorname{range} S, W)$. We first show that E is a mapping, and also a linear transformation.

Suppose $Sv, Sw \in W$ such that $Sv = Sw$, we need to show that $E(Sv) = E(Sw)$, or normalized $Tv = Tw$. We know $v - w \in \operatorname{null} S$ since $Sv = Sw$, so $v - w \in \operatorname{null} T$ since $\operatorname{null} S \subseteq \operatorname{null} T$, therefore $T(v - w) = 0$, and then $Tv = Tw$, so E is a mapping.

Now we show that E is a linear transformation.

- For any $Sv, Sw \in \operatorname{range} S$, $E(Sv) + E(Sw) = Tv + Tw = T(v + w) = E(S(v + w)) = E(Sv + Sw)$.

- For any $Sv \in \operatorname{range} S$ and $\lambda \in F$, $\lambda E(Sv) = \lambda Tv = T(\lambda v) = E(S(\lambda v) = E(\lambda Sv))$.

therefore E is a linear transformation.

Now we can expand the domain of E to W such that $E'v = Ev$ for any $v \in \operatorname{range} S$ (this is proven in previous exercise). For any $v \in V$, we have $ESv = E(Sv) = Tv$, there fore $T = ES$.

For another direction, for any $v \in \operatorname{null} S$, we have $ESv = E0 = 0 = Tv$, so $v \in \operatorname{null} T$. □

Exercise 3.26. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that* $\operatorname{range} S \subseteq \operatorname{range} T \iff$ *there is $E \in \mathcal{L}(V)$ such that $S = TE$.*

Proof. Consider the inverse image of $\operatorname{range} S$ with basis $w_0, \cdots, w_n$, say $v_0, \cdots, v_n$, it is easy to show $v_0, \cdots, v_n$ is linear independent. Then $E(v) = \lambda_0 v_0 + \cdots + \lambda_n v_n$ where $Sv = \lambda_0 w_0 + \cdots + \lambda_n w_n$. □

Exercise 3.27. *Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$, show that $V = \operatorname{null} P \oplus \operatorname{range} P$.*

Proof. Such element is called *idempotent* in algebra.

We will show $\operatorname{null} P \oplus \operatorname{range} P$ by showing $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. For any $v \in \operatorname{null} P \cap \operatorname{range} P$, we know there is $w \in V$ such that $Pw = v$ since $v \in rangevP$, then $P^2(v) = P(Pv) = P0 = 0$ since $v \in \operatorname{null} P$ and $P^2(v) = P(Pv) = Pw$, so $Pw = 0$ while $Pw = v$ therefore $v = 0$.

Then we have $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$ and $\dim(\operatorname{null} P \oplus \operatorname{range} P) = \dim \operatorname{null} P + \dim \operatorname{range} P - \dim\{0\}$, so $V = \operatorname{null} P \oplus \operatorname{range} P$. □

Exercise 3.28. *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ such that for any non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$, $\deg(Dp) = \deg p - 1$. Show that D is surjective.*

Proof. We induction on n, starts from 1, to show that $D(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_{n-1}(\mathbb{R})$.

- Base: for any $p \in \mathcal{P}(\mathbb{R})$ where $\deg p = 1$, we know $\deg Dp = 0$, so $D(\mathcal{P}_1(\mathbb{R}))$ is a non-zero subspace of $\mathcal{P}_0(\mathbb{R})$, which is $\mathcal{P}_0(\mathbb{R})$.

- Induction: We have induction hypothesis: For any $i \leq n$, we have $D(\mathcal{P}_i(\mathbb{R})) = \mathcal{P}_{i-1}(\mathbb{R})$. We want to show that $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$. For any $p \in \mathcal{P}(\mathbb{R})$ with $\deg p = n+1$, we can write p in form of $p = \lambda x^{n+1} + r$ where $\deg r \leq n$, then $Dp = D(\lambda x^{n+1} + r) = D(\lambda x^{n+1}) + Dr$ where $\deg D(\lambda x^{n+1}) = n$ and $\deg Dr \leq n - 1$. So $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$ since: $\mathcal{P}_n(\mathbb{R}) \subseteq D(\mathcal{P}_{n+1}(\mathbb{R}))$ and $D(\lambda x^{n+1}) \in D(\mathcal{P}_{n+1}(\mathbb{R}))$, it is sufficient to span $\mathcal{P}_n(\mathbb{R})$.

□

Exercise 3.29. *For any $p \in \mathcal{P}(\mathbb{R})$, show that there is $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.*

Proof. We can rewrite the goal as $5DDq + 3Dq = p$ where $D(p) = p'$, then $5DDq + 3Dq = D(5Dq) + D(3q) = D(5Dq + 3q) = p$. We know D is surjective by the previous exercise, the goal is now showing that $5Dq + 3q = r$ where $Dr = p$. Then we continue rewrite the goal $5Dq + 3q = (5D)q + (3I)q = (5D + 3I)q = r$, we will show that $5D + 3I$ is surjective, we use the same method in previous exercise.

We denote $5D+3I$ by F, and induction on $n \in \mathbb{N}$ to show that $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$.

- Base: We should show that $F(\mathcal{P}_0(\mathbb{R})) = \mathcal{P}_0(\mathbb{R})$, for any $p \in \mathcal{P}_0(\mathbb{R})$, we have $Fp = 5Dp + 3p$, where $Dp = 0$ since $\deg p = 0$, so $Fp = 3p$, which means we have $1 \in F(\mathcal{P}_0(\mathbb{R}))$ since p is literally a number and $\frac{1}{3p}Fp = 1$.

- Induction: We have induction hypothesis: $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$, and we want to show $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

  For any $p \in \mathcal{P}_{n+1}(\mathbb{R})$, we have $Fp = 5Dp + 3p$ where $\deg 5Dp = n$ and $\deg 3p = n + 1$, then we can eliminate $5Dp$ and every term in p with degree less then $n + 1$ since $\mathcal{P}_n(\mathbb{R}) \subseteq \operatorname{range} F$, then we get z^{n+1}, thus $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

Therefore there is q such that $(5D + 3I)q = r$ since $5D + 3I$ is surjective.

Another solution from internet: Define $Tq = 5q'' + 3q'$, we can see for any $q \in \mathcal{P}(\mathbb{R})$ we have $\deg Tq = \deg q - 1$, so T is surjective. Then there is q such that $Tq = 5q'' + 3q' = p$. □

Exercise 3.30. *Suppose $\varphi \in \mathcal{L}(V, F)$ not zero, and $u \in V$ that $u \notin \operatorname{null} \varphi$, show that $V = \operatorname{null} \varphi \oplus \{ au \mid a \in F \}$.*

Proof. We can see φ is surjective since $\varphi u \neq 0$, then for any $i \in F$, we have $(i(\varphi u)^{-1})\varphi u = i$.

For any $v \in V$, since φ is surjective (in a particular way), so we have $a\varphi u$ such that $a\varphi u = \varphi v$, then $\varphi(au - v) = 0$ so $au - v \in \operatorname{null} \varphi$. That means $(-1)(au - v) + au = v$ where $(-1)(au - v) \in \operatorname{null} \varphi$ and $au \in \{ au \mid a \in F \}$, so $V = \operatorname{null} \varphi + \{ au \mid a \in F \}$.

Then $\operatorname{null} \varphi \oplus \{ au \mid a \in F \}$ since $u \notin \operatorname{null} \varphi$. □

Exercise 3.31. *Suppose V is finite* ($\dim V > 1$)*, show that if $\varphi : \mathcal{L}(V) \to F$ is a linear mapping with property $\varphi(ST) = \varphi(S)\varphi(T)$ for any $S, T \in \mathcal{L}(V)$, show that $\varphi = 0$.*

Proof. Consider $\operatorname{null}\varphi$, since $\dim V > 1$ while $\dim F = 1$, so φ cannot be injective, therefore $\operatorname{null}\varphi \neq \{0\}$.

For any non-zero $S \in \operatorname{null}\varphi$ and $T \in \mathcal{L}(V)$, we have $\varphi(ST) = \varphi(S)\varphi(T) = 0 = \varphi(T)\varphi(S) = \varphi(TS)$ since $S \in \operatorname{null}\varphi$, thus $ST \in \operatorname{null}\varphi$. We show that $\operatorname{null}\varphi$ is an ideal of $\mathcal{L}(V)$, recall that the property of $\mathcal{L}(V)$, the only ideal of $\mathcal{L}(V)$ is $\{0\}$ and $LT(V)$, so $\operatorname{null}\varphi = \mathcal{L}(V)$, which means $\varphi = 0$. □

Exercise 3.32. *Let V, W are vector spaces and $T \in \mathcal{L}(V, W)$, define $T_C : V_C \to W_C$:*

$$T_C(u + iv) = Tu + iTv$$

for any $u, v \in V$.

1. *Show that T_C is a (complex) linear mapping from V_C to W_C.*
2. *Show that T_C is injective $\iff$ T is injective.*
3. *Show that* $\operatorname{range} T_C = W_C \iff \operatorname{range} T = W$.

Proof.

1. For any $u, v, s, t \in V$ $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
&T((u + iv) + (s + it)) \\
=&T(u + s + i(v + t)) \\
=&T(u + s) + iT(v + t) \\
=&Tu + Ts + iTv + iTt \\
=&T(u + iv) + T(s + it)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda T(u + iv) \\
=&\lambda(Tu + iTv) \\
=&\lambda Tu + \lambda iTv \\
=&T(\lambda u) + iT(\lambda v) \\
=&T(\lambda u + i(\lambda v)) \\
=&T(\lambda u + i\lambda v) \\
=&T(\lambda(u + iv))
\end{aligned}$$

I believe these are trivial, so the future me should be able to prove these without any effort. $\square$

Exercise 3.4. *Suppose $D(p) = p' : \mathcal{L}(\mathcal{P}_3(\mathbb{R})) \to \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$. Find a basis of $\mathcal{L}(\mathcal{P}_3(\mathbb{R}))$ and a basis of $\mathcal{L}(\mathcal{P}_2(\mathbb{R}))$, such that $\mathcal{M}(D)$ about these basis is:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Proof. Consider $x, x^2, x^3, 1$ the basis of $\mathcal{P}_3(\mathbb{R})$ and $1, x, 2x^2$. $\square$

Exercise 3.5. *Suppose V and W are finite and $T \in \mathcal{L}(V, W)$. Show that there are basis of V and W respectively, such that $\mathcal{M}(T, \text{those basis})$ is all zero except 1 at $k, k \quad (1 \le k \le \dim \operatorname{range} T)$.*

Proof. Consider the basis $w_0, \cdots, w_{k-1}$ of range T and the basis $w_0, \cdots, w_{m-1}$ of W which expands from $w_0, \cdots, w_{k-1}$. Then there must be $v_0, \cdots, v_{k-1}$ such that $Tv_i = w_i$ for all $0 \le i < k$, we know $v_0, \cdots, v_{k-1}$ is linear independent since $w_0, \cdots, w_{k-1}$ is linear independent, so we can expand it to a basis of V, say $v_0, \cdots, v_{n-1}$.

We claim that $\mathcal{M}(T, v_0, \cdots, v_{n-1}, w_0, \cdots, w_{m-1})$ is a matrix with all zero but 1 at $k, k \quad (1 \le k < \operatorname{range} T)$. For any $\lambda_0 v_0 + \cdots + \lambda_{n-1} v_{n-1} \in V$, we have $T(\lambda_0 v_0 + \cdots + \lambda_{n-1} v_{n-1}) = \lambda_0 w_0 + \cdots + \lambda_{k-1} w_{k-1}$, note that all v_i where $i \ge k$ disappear, since they maps to 0. Therefore $\mathcal{M}(T)$ is all zero but 1 at k, k (since $\lambda_i w_i$ in the last equation). $\square$

Exercise 3.6. *Show that $-^T : F^{m,n} \to F^{n,m}$ is a linear mapping.*

Proof. Trivial, sorry. $\square$

Exercise 3.7. *Show that $(AB)^T = B^T A^T$.*

Proof. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix, then for any $i \in [1, m]$ and $j \in [1, p]$, we have $(AB)^T_{i,j} = (AB)_{j,i} = \displaystyle\sum_{r=1}^{n} A_{j,r} B_{r,i} = \sum_{r=1}^{n} B^T_{i,r} A^T_{r,j} = (B^T A^T)_{i,j}$. $\square$

Exercise 3.8. *Let A a $m \times n$ matrix, show that the rank of A is 1 $\iff$ there is $c_0, \cdots, c_{m-1} \in F^m$ and $d_0, \cdots, d_{n-1} \in F^n$ such that $A_{j,k} = c_j d_k$ for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, n-1$.*

Proof. The right hand side is actually the external product of vectors, that is vw^T.

($\Rightarrow$) is easy since we can use the theorem that any $m \times n$ matrix A can be expressed by CR where C is a $m \times r$ matrix, R is a $r \times n$ matrix, r is the rank of A. In this case, $r = 1$, so C and R are just vectors.

($\Leftarrow$) is also easy since other column is a scalar multiple of the first column, therefore the rank of A is 1. □

Exercise 3.9. *Let $T \in \mathcal{L}(V)$, $u_0, \cdots, u_{n-1}$ and $v_0, \cdots, v_{n-1}$ are the bases of V, show that the following statements are equivalent:*

1. *T is injective*
2. *The columns of $\mathcal{M}(T)$ is linear independent*
3. *The columns of $\mathcal{M}(T)$ spans $F^{n,1}$*
4. *The lines of $\mathcal{M}(T)$ is linear independent*
5. *The lines of $\mathcal{M}(T)$ spans $F^{1,n}$*

Proof. (2), (3) are obviously equivalent and (4), (5) too.

Although I want to make an arrow loop, but the arrow between (1) and (4), (5) is too hard, so I will show that (1) $\iff$ (2), (3) and (2), (3) $\iff$ (4), (5).

- ($\Rightarrow$) Let $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1} = [0, \ldots, 0]$, then $T(\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}) = 0$, so λ_i are 0 since T is injective, which means $\operatorname{null} T = \{0\}$.

  ($\Leftarrow$) For any $T(\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}) = 0$, we have the linear combination of v_i is 0 where the coefficients come from $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1}$ (w_i are the columns of $\mathcal{M}(T)$), therefore the coefficients are all 0 since v_i is linear independent, thus $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1} = 0$, which means λ_i are all 0 since w_i is linear independent.

- For any matrix, its line rank is equal to its column rank, so columns independent $\iff$ lines independent.

□

Exercise 3.4. *Let V a finite vector space with $\dim V > 1$, show that $S = \{\ T \text{ is singular} \mid T \in \mathcal{L}(V)\ \}$ is **NOT** a subspace of $\mathcal{L}(V)$.*

Proof. If S is a subspace of $\mathcal{L}(V)$, then it is an ideal of $\mathcal{L}(V)$ since for any $A \in S$ and $B \in \mathcal{L}(V)$, AB and BA are singular, therefore $AB, BA \in S$. However, we know the only two ideals of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$, none of them is S. □

Exercise 3.11. *Let V finite vector space, and $S, T \in \mathcal{L}(V)$, show that*

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible}$$

Proof.

- ($\Rightarrow$) Suppose $STW = WST = I$, then $S(TW) = (TW)S = I$ since $\dim V = \dim V$, therefore $S^{-1} = TW$, also $(WS)T = T(WS) = I$ since $\dim V = \dim V$, therefore $T^{-1} = WS$.
- ($\Leftarrow$) Trivial.

□

Exercise 3.12. *Let V finite vector space, and $S, T, U \in \mathcal{L}(V)$ such that $STU = I$, Show that $T^{-1} = US$.*

Proof. Since $STU = I$ we know U is invertible (since STU is invertible), then $ST = U^{-1}$. Since U^{-1} is invertible, we know S and T are invertible therefore $T = S^{-1}U^{-1}$ and $T^{-1} = US$. □

Exercise 3.13. *Show that the conclusion of previous exercise can be false if V is not finite.*

Proof. Let $S(x_0, x_1, \dots) = (x_1, \dots)$ the backward-shift mapping and $U(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ the forward-shift mapping and $T = I$ the identity mapping.

We have $SU = I$ and $US \neq I$, T is clearly inveritble with $T^{-1} = I$, but we know $US \neq I$, so $T^{-1} = US \neq I$.

In fact, this also disprove the infinite version of **??** since SU is invertible but neither S nor U is invertible. □

Exercise 3.17. *Let V a finite vector space, $S \in \mathcal{L}(V)$, define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$, show that:*

1. $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$
2. $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$

Proof. Since $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$, we know $\dim \mathcal{L}(V) = \dim \operatorname{null} \mathcal{A} + \dim \operatorname{range} \mathcal{A}$, also, $\dim \mathcal{L}(V) = (\dim V)^2$ and $\dim V = \dim \operatorname{null} S + \dim \operatorname{range} S$. Therefore we have $\dim \operatorname{null} A + \dim \operatorname{range} A = (\dim V)(\dim \operatorname{null} S + \dim \operatorname{range} S)$, which means we only need to prove one of (1) and (2).

We will show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$. We found that $\dim \mathcal{L}(V, \operatorname{null} S) = (\dim V)(\dim \operatorname{null} S)$, so it would be nice if $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$. For any $T \in \operatorname{null} \mathcal{A}$, we have $ST = 0$, which means $\operatorname{range} T \subseteq \operatorname{null} S$, therefore $T \in \mathcal{L}(V, \operatorname{null} S)$. For any $T \in \mathcal{L}(V, \operatorname{null} S)$, we have $ST = 0$ since $\operatorname{range} T \subseteq \operatorname{null} S$, so $T \in \operatorname{null} \mathcal{A}$, therefore $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$, thus $\dim \operatorname{null} A = (\dim V)(\dim \operatorname{null} S)$. □

Exercise 3.18. *Show that V and $\mathcal{L}(F, V)$ are isomorphic.*

Proof. This can be proven by $\dim V = \dim \mathcal{L}(F, V) = 1(\dim V)$, but we can find $\varphi(v) = x \mapsto xv$ an isomorphism. For any $T \in \mathcal{L}(F, V)$, T is determined by $T(1)$. □

Exercise 3.1. *Let $T : V \to W$, the graph of T is a subset of $V \times W$ such that*

$$\textit{graph of } T = \{\ (v, Tv) \mid v \in V\ \}$$

.

Show that T is a linear mapping $\iff$ the graph of T is a subspace.

Proof.

- $(0, T0) \in$ graph of T. $(v, Tv) + (w + Tw) = (v + w, Tv + Tw) = (v + w, T(v + w))$. $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v))$
- $(v, Tv) + (w + Tw) = (v + w, T(v + w))$ since the graph of T is a subspace, therefore $Tv + Tw = T(v + w)$. Similarly, $\lambda Tv = T(\lambda v)$.

□

Exercise 3.3. *Let V_i are vector spaces, show that $\mathcal{L}(V_0 \times \cdots \times V_{m-1}, W) \simeq \mathcal{L}(V_0, W) \times \cdots \times \mathcal{L}(V_{m-1}, W)$.*

Proof. This can be proven by $A \times B$ is a categorical product, so we will show that for any A, B are vector spaces, $A \times B$ is a product.

In order to show that $A \times B$ is a product, or more specifically, $A \times B$ equipped with linear mappings

$$\pi_0(a, b) = a$$
$$\pi_1(a, b) = b$$

is a product, we have to show that for any C, $s \in \mathcal{L}(C, A)$ and $t \in \mathcal{L}(C, B)$, there is a unique $u \in \mathcal{L}(C, A \times B)$ such that $s = \pi_0 \circ u$ and $t = \pi_1 \circ u$.

Define $u(c) = (sc, tc) : C \to A \times B$, we will show that u is a linear mapping.

- For all $v, w \in C$, $u(v)+u(w) = (sv, tv)+(sw, tw) = (sv+sw, tv+tw) = (s(v+w), t(v+w)) = u(v+w)$
- For all $c \in C$ and $\lambda \in F$, $\lambda u(c) = \lambda(sc, tc) = (\lambda sc, \lambda tc) = (s(\lambda c), t(\lambda c)) = u(\lambda c)$.

Then we can see $\pi_0(u(c)) = \pi_0(sc, tc) = sc$ and $\pi_1(u(c)) = \pi_1(sc, tc) = tc$. Now we have to show that u is unique (which is trivial, I don't want to prove this, sorry). $\square$

Exercise 3.5. *Let m a positive number, define $V^m = \underbrace{V \times \cdots \times V}_{m}$, show that $V^m \simeq \mathcal{L}(F^m, V)$.*

Proof. Define $\varphi(v_0, \cdots, v_{m-1}) = i_0, \cdots, i_{m-1} \mapsto i_0 v_0 + \cdots + i_{m-1} v_{m-1}$ which accept a list of vector and a list of coefficients then produce a linear combination.

For any $T \in \mathcal{L}(F^m, V)$, T is completely determined by $T(1, \cdots, 1) = v_0 + \cdots + v_{m-1}$, therefore $\varphi(v_0, \cdots, v_{m-1}) = T$ and thus φ is surjective.

For any $(v_0, \cdots, v_{m-1}), (w_0, \cdots, w_{m-1}) \in V^m$ such that $\varphi(v_0, \cdots, v_{m-1}) = \varphi(w_0, \cdots, w_{m-1})$, then $w_0 = \varphi(v_0, \cdots, v_{m-1})(1, 0, \cdots, 0) = \varphi(v_0, \cdots, v_{m-1})(1, 0, \cdots, 0) = v_0$, same for other v_i and w_i, so $(v_0, \cdots, v_{m-1}) = (w_0, \cdots, w_{m-1})$, therefore φ is injective. $\square$

Exercise 3.6. *Let $v, x \in V$ and $U, W \subseteq V$ are subspaces such that $v + U = x + W$. Show that $U = W$.*

Proof. We know $v = x + w_0$ for some $w_0 \in W$ since $v + U = x + W$ and $v \in v + U$, then for any $u \in U$, we have $v + u = x + w$ for some $w \in W$, then $(x + w_0) + u = x + w$ therefore $u = x + w - x - w_0 = w - w_0 \in W$ thus $U \subseteq W$. Similarly $W \subseteq U$. $\square$

Exercise 3.7. *Let $U = \{ (x, y, z) \in R^3 \mid 2x + 3y + 5z = 0 \}$ and $A \subseteq R^3$. Show that A is a translate of U (that is $A = a + U$) $\iff$ there is c such that $A = \{ (x, y, z) \in R^3 \mid 2x + 3y + 5z = c \}$.*

Proof.

- ($\Rightarrow$) For any $(a_0, a_1, a_2) + (x, y, z) \in a + U$, we have $2(a_0 + x) + 3(a_1 + y) + 5(a_2 + z) = 2a_0 + 3a_1 + 5a_2$, therefore $c = 2a_0 + 3a_1 + 5a_2$.

- ($\Leftarrow$) We can see 2, 3 and 5 are coprime to each other, therefore there is $2a_0+3a_1+5a_2=1$ (I am not sure if this is true in generalized case, I just extends the theorem "$as+bt=1 \iff$ a coprime to b" to three elements case without checking), in this case we have $2(1)+3(-2)+5(1)=1$, then for any $2x+3y+5z=c$, we have $2x+3y+5z=2(ca_0)+3(ca_1)+5(ca_2)$, then $2(x-ca_0)+3(y-ca_1)+5(z-ca_2)=0$, therefore $A=((-c)(a_0,a_1,a_2))+U$.

$\square$

Exercise 3.8. *Let $T \in \mathcal{L}(V,W)$ and $c \in W$, show that $\{ v \in V \mid Tv = c \}$ is an empty set or a translate of* $\operatorname{null} T$. *Then explain why the solutions of a system of linear equations is either an empty set or a translate of some subspace of F^n.*

Proof. Let $Ta=c$ for some $a \in V$, if no such a, then $\{ v \in V \mid Tv = c \} = \varnothing$. We claim $\{ v \in V \mid Tv = c \} = a + \operatorname{null} T$. For any $v \in V$ such that $Tv=c$, then $v = a+v-a$ and $T(v-a)=Tv-Ta=c-c=0$, therefore $v-a \in \operatorname{null} T$, thus $v \in a+\operatorname{null} T$. In another direction, for any $a+v \in a+\operatorname{null} T$, we have $T(a+v)=Ta+Tv=c+0=c$. $\square$

Exercise 3.9. *Let $A \subseteq V$ a non-empty subset. Show that A is a translate of some subspace of $V \iff \lambda v+(1-\lambda)w \in A$ for any $v,w \in A$ and $\lambda \in F$.*

Proof.

- ($\Rightarrow$) Suppose $A = a+U$ for some subspace $U \subseteq V$.

- ($\Leftarrow$) Let $w \in A$, we will show that $(-w)+A$ is a subspace of V.

  For any $a-w, b-w \in (-w)+A$, we need to show that $a-w+b-w = (a+b-w)-w \in (-w)+A$ or equivalently $a+b-w \in A$. We found that the property $\lambda v+(1-\lambda)w \in A$ gives us the ability to construct something like $v-w$. Since $2v+(1-2)w = 2v-w$, we just let $w=v+a$ then $2v-(v+a)=v-a$. Therefore, we let $\lambda=2$, $v=a+b$ and $w=a+b+w$, and now $2(a+b)-(a+b+w)=a+b-w \in A$, so $a+b-w-w \in (-w)+A$.

  For any $a-w \in (-w)+A$ and $\lambda \in F$, we need to show that $\lambda(a-w) \in (-w)+A$. $\lambda(a-w) = \lambda a - \lambda w = \lambda a - (\lambda-1)w - w$. We let $\lambda = (-1)(\lambda-1) = (1-\lambda)$, $v=w$ and $w=a$ in $\lambda v+(1-\lambda w) \in A$, then

$(1-\lambda)w + (1-(1-\lambda))a = (-1)(\lambda-1)w + \lambda a = \lambda a - (\lambda-1)w \in A$, therefore $\lambda a - (\lambda-1)w - w = \lambda a - \lambda w \in (-w) + A$.

Therefore $(-w) + A$ is a subapce of V and $w + (-w) + A$ is a translate.

□

Exercise 3.10. *Let $A = a + U$ and $B = b + W$ where $a, b \in V$, $U, W \subseteq V$ are subspaces. Show that $A \cap B$ is either a translate of some subspace of V or an empty space.*

Proof. Suppose $A \cap B \neq \varnothing$, we claim that $A \cap B$ is a translate of $U \cap W$, more specificly, for any $a+u_0 = b+w_0 \in A\cap B$, we claim that $A\cap B = (a+u_0)+U\cap W$.

For any $u = w \in U \cap W$, we have $(a + u_0) + u = a + (u_0 + u) \in a + U$, similarly, we have $(b + w_0) + w = b + (w_0 + w) \in b + W$, therefore $(a + u_0) + (U \cap W)\ subseteq A \cap B$.

For any $a + u = b + w \in A \cap B$, we have $a + u - (a + u_0) = u - u_0 \in U$ and $b + w - (b + w_0) = w - w_0 \in W$, therefore $A \cap B \subseteq (a + u_0) + (U \cap W)$. □

Exercise 3.12. *Let $v_0, \cdots, v_{m-1} \in V$ and*

$$A = \{\ \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1} \mid \lambda_i \in F \text{ and } \lambda_0 + \cdots + \lambda_i = 1\ \}$$

1. *Show that A is a translate of a subspace of V.*
2. *If B a translate of a subspace of V such that $v_0, \cdots, v_{m-1} \in B$, show that $A \subseteq B$.*
3. *Base on (1), show that the dimension of such subspace is less then m.*

Proof.

- If A is a translate of a subspace of V, say B, then for any $a \in A$, we have $A = a + B$. Therefore $B = (-a) + A$, we may pick $a = v_0$, we find that for any $b \in B$, it is in form $(-1)(v_0) + \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}$ where $\lambda_0 + \cdots + \lambda_{m-1} = 1$, which implies $(-1) + \lambda_0 + \cdots + \lambda_{m-1} = 0$. Then we claim $B = \{\ \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1} \mid \lambda_0 + \cdots + \lambda_{m-1} = 0\ \}$ is a subspace and $A = v_0 s + B$.
-

□

Exercise 3.16. *Let $\varphi \in \mathcal{L}(V, F)$ where $\varphi \neq 0$, show that* $\dim(V/(\operatorname{null}\varphi)) = 1$.

Proof. For any non-zero $v + \operatorname{null}\varphi, w + \operatorname{null}\varphi \in V/(\operatorname{null}\varphi)$ (existence is guaranteed since $\varphi \neq 0$), since $\varphi(w) \in F$, then there is some λ such that $\lambda\varphi(w) = \varphi(v)$ cause $\varphi(v)$ and $\varphi(w)$ are non-zero, then $\varphi(\lambda w) = \varphi(v)$, which means $v + \operatorname{null} T = (\lambda w) + \operatorname{null} T$, therefore $\dim(V/\operatorname{null}\varphi)$ cause any two (non-zero) vectors are linear dependent. □

Exercise 3.17. *Let $U \subseteq V$ a subspace such that* $\dim(V/U) = 1$. *Show that there is $\varphi \in \mathcal{L}(V, F)$ such that* $\operatorname{null}\varphi = U$.

Proof. We know there is an isomorphism $i \in \mathcal{L}(V/U, F)$ since $\dim(V/U) = \dim F = 1$, then $\varphi = i \circ \pi$ where $\pi \in \mathcal{L}(V, V/U)$. Since i is injective, $\operatorname{null}\varphi = \operatorname{null}\pi = U$. □

Exercise 3.18. *Explain why a linear functional is either surjective or* 0.

Proof. Cause $\dim F = 1$. □

Exercise 3.6. *Let $\varphi, \beta \in V'$, show that* $\operatorname{null}\varphi \subseteq \operatorname{null}\beta \iff \exists c \in F, \beta = c\varphi$.

Proof.

- $(\Rightarrow)$ For any $v \notin \operatorname{null}\beta$, we have $\beta(v) = \beta(v)(\varphi(v))^{-1}\varphi(v)$ we claim that $\beta = \beta(v)(\varphi(v))^{-1}\varphi$. We may denote $\beta(v)(\varphi(v))^{-1}$ by c. For any $v, w \notin \operatorname{null}\beta$, we have $\beta(v) = a\varphi(v)$ and $\beta(w) = b\varphi(w)$, we want to show that $a = b$, which can be proven by:

$$\begin{gathered} a = b \\ \frac{\beta(v)}{\varphi(v)} = \frac{\beta(w)}{\varphi(w)} \\ \beta(v)\varphi(w) = \beta(w)\varphi(v) \\ \beta(\varphi(w)v) = \beta(\varphi(v)w) \end{gathered}$$

  which is equivalent to $\varphi(w)v - \varphi(v)w \in \operatorname{null}\beta$, then:

$$\begin{aligned} &\varphi(\varphi(w)v - \varphi(v)w) \\ =&\varphi(\varphi(w)v) - \varphi(\varphi(v)w) \\ =&\varphi(w)\varphi(v) - \varphi(v)\varphi(w) \\ =&0 \end{aligned}$$

therefore $\varphi(w)v - \varphi(v)w \in \operatorname{null}\varphi \subseteq \operatorname{null}\beta$, thus $a = b$.

The case $v \in \operatorname{null}\beta$ is trivial.

- ($\Leftarrow$) For any $v \in \operatorname{null}\varphi$, $\beta(v) = c\varphi(v) = 0$, therefore $v \in \operatorname{null}\beta$, thus $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$.

$\square$

Exercise 3.7. *Let $V_0, \cdots, V_{m-1}$ are vector spaces, show that $V_0' \times \cdots \times V_{m-1}'$ and $(V_0 \times \cdots \times V_{m-1})'$ are isomorphic.*

Proof. Define $\psi(\varphi) = v_0 \mapsto \varphi(v_0, 0, \cdots), \cdots, v_{m-1} \mapsto \varphi(\cdots, 0, v_{m-1})$ and $\psi^{-1}(\varphi_0, \cdots, \varphi_{m-1}) = (v_0, \cdots, v_{m-1}) \mapsto \varphi_0(v_0) + \cdots + \varphi_{m-1}(v_{m-1})$.

For any $\alpha, \beta \in (V_0 \times \cdots \times V_{m-1})'$ and $\lambda \in F$, we have

$$\begin{aligned}
&\psi(\alpha+\beta)_i \\
=&v_i \mapsto (\alpha+\beta)(\cdots, v_i, \cdots) \\
=&v_i \mapsto \alpha(\cdots, v_i, \cdots) + \beta(\cdots, v_i, \cdots) \\
=&(v_i \mapsto \alpha(\cdots, v_i, \cdots)) + (v_i \mapsto \beta(\cdots, v_i, \cdots)) \\
=&\psi(\alpha)_i + \psi(\beta)_i
\end{aligned}$$

and $(\lambda\psi(\alpha))_i = \lambda(v_i \mapsto \alpha(v_i)) = v_i \mapsto \lambda\alpha(v_i) = \psi(\lambda\alpha)_i$ Therefore ψ is a linear map.

For any $\alpha, \beta \in V_0' \times \cdots \times V_{m-1}'$ and $\lambda \in F$, we have:

$$\begin{aligned}
&\psi^{-1}(\alpha+\beta) \\
=&(v_0, \cdots, v_{m-1}) \mapsto (\alpha+\beta)(v_0) + \cdots + (\alpha+\beta)(v_{m-1}) \\
=&(v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \beta(v_0) + \cdots \\
=&((v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \cdots) + ((v_0, \cdots, v_{m-1}) \mapsto \beta(v_0) + \cdots) \\
=&\psi^{-1}(\alpha) + \psi^{-1}(\beta)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda\psi^{-1}(\alpha) \\
=&\lambda((v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \cdots) \\
=&(v_0, \cdots, v_{m-1}) \mapsto \lambda(\alpha(v_0) + \cdots) \\
=&(v_0, \cdots, v_{m-1}) \mapsto (\lambda\alpha(v_0)) + \cdots \\
=&\psi^{-1}(\lambda\alpha)
\end{aligned}$$

thus ψ^{-1} is a linear map.

We will show that ψ^{-1} is the inverse of ψ then ψ is an isomorphism. For any $\varphi \in (V_0 \times \cdots \times V_{m-1})'$,

$$
\begin{aligned}
&\psi^{-1}(\psi(\varphi)) \\
=&v_0, \cdots, v_{m-1} \mapsto \psi(\varphi)_0(v_0) + \cdots \\
=&v_0, \cdots, v_{m-1} \mapsto \varphi(v_0, 0, \cdots) + \cdots + \varphi(\cdots, 0, v_{m-1}) \\
=&v_0, \cdots, v_{m-1} \mapsto \varphi(v_0, \cdots, v_{m-1}) \\
=&\varphi
\end{aligned}
$$

and for any $\varphi \in V_0' \times \cdots \times V_{m-1}'$,

$$
\begin{aligned}
&\psi(\psi^{-1}(\varphi)) \\
=&v_0 \mapsto \psi^{-1}(\varphi)(v_0, 0, \cdots), \cdots \\
=&v_0 \mapsto \varphi_0(v_0), \cdots \\
=&\varphi_0, \cdots, \varphi_{m-1} \\
=&\varphi
\end{aligned}
$$

□

Exercise 3.16. *Let W a finite vector space, $T \in \mathcal{L}(V, W)$, show that*

$$T' = 0 \iff T = 0$$

Proof.

- ($\Rightarrow$) Suppose $T \neq 0$, then we can always find $\varphi \in \mathcal{L}(W, F)$ which $\varphi(\operatorname{range} T) \neq 0$, then $\varphi \circ T \neq 0$.
- ($\Leftarrow$) Trivial.

□

Exercise 3.17. *Let V, W are finite vector spaces, $T \in \mathcal{L}(V, W)$. Show that T is invertible $\iff$ T' is invertible.*

Proof. Since T is invertible, then T is injective, therefore T' is surjective. Similarly, T' is injective since T is surjective. Therefore T' is invertible. □

Exercise 3.18. *Let V, W are finite vector spaces, show that the mapping $\varphi(T) = T'$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathcal{L}(W', V')$.*

Proof. Since V and W are finite, we only need to show that φ is injective or surjective. We will show that φ is injective.

For any $\varphi(T) = T' \in \mathcal{L}(W', V')$, we know $T = 0 \iff T' = 0$, therefore $\operatorname{null} \varphi = \{0\}$, thus φ is injective.

I was wonder if I can prove this by $\varphi(S)(\mathrm{id}) = \varphi(T)(\mathrm{id}) \implies S = T$. This may work if the codomain restriction doesn't lose information, i.e. only restrict to a superset of its range, therefore the restriction is one-to-one. □

Exercise 3.21. *Let V finite and $U, W \subseteq V$ are subspaces.*

1. *Show that $W^0 \subseteq U^0 \iff U \subseteq W$*
2. *Show that $W^0 = U^0 \iff U = W$*

Proof. The second statement can be easy proved by the first one.

- ($\Rightarrow$) We can always find a $f \in \mathcal{L}(W, F)$ such that $\operatorname{null} f = W$, then $f(U) = \{0\}$ since $f \in W^0 \subseteq U^0$, therefore $U \subseteq \operatorname{null} f = W$.
- ($\Leftarrow$) For any $\varphi \in W^0$, we know $W \subseteq \operatorname{null} \varphi$, then $U \subseteq W \subseteq \operatorname{null} \varphi$, therefore $\varphi in U^0$, thus $W^0 \subseteq U^0$.

□

Exercise 3.22. *Let V finite and $U, W \subseteq V$ are subspaces. Show that:*

- $(U + W)^0 = U^0 \cap W^0$
- $(U \cap W)^0 = U^0 + W^0$

Proof.

- For any $\varphi \in (U+W)^0$ we have $U+W \subseteq \operatorname{null} \varphi$, then $U \subseteq U+W \subseteq \operatorname{null} \varphi$ and $W \subseteq U + W$, therefore $\varphi \in U^0 \cap W^0$.

  For any $\varphi \in U^0 \cap W^0$, we have $U \subseteq \operatorname{null} \varphi$ and $W \subseteq \operatorname{null} \varphi$. For any $u + w \in U + W$, we have $\varphi(u+w) = \varphi(u) + \varphi(w) = 0 + 0 = 0$, therefore $U + W \subseteq \operatorname{null} \varphi$, thus $\varphi \in (U + W)^0$.
- For any $su + tw \in U^0 + W^0$, for any $v \in U \cap W$, we have $su(v) + tw(v) = s0 + t0$ since $v \in U$ and $v \in W$. Therefore we have an injective map (also

linear, this map just produce what it receive) from $U^0 + W^0$ to $(U \cap W)^0$. We have:

$$
\begin{aligned}
&\dim(U^0 + W^0) \\
=&\dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
=&\dim V - \dim U + \dim V - \dim W - \dim(U + W)^0 \\
=&\dim V - \dim U + \dim V - \dim W - (\dim V - \dim(U + W)) \\
=&\dim V - \dim U - \dim W + (\dim U + \dim W - \dim(U \cap W)) \\
=&\dim V - \dim(U \cap W) \\
=&\dim(U \cap W)^0
\end{aligned}
$$

therefore $(U \cap W)^0 = U^0 + W^0$.

□

Exercise 3.23. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1} \in V'$. Show that the following sets are equal to each others:*

- $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1})$
- $((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0$
- $\{ \varphi \in V' \mid (\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}) \subseteq \operatorname{null} \varphi \}$

Proof.

- $((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0 = (\operatorname{null} \varphi_0)^0 + \cdots + (\operatorname{null} \varphi_{m-1})^0$, then $\operatorname{span}(\varphi_i) \subseteq (\operatorname{null} \varphi_i)^0$ therefore $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) \subseteq ((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0$.

  For any $\varphi \in \operatorname{span}(\varphi_0, \cdots, \varphi_{m-1})$, we have $\varphi(v) = \varphi_0(v) + \cdots + \varphi_{m-1}(v) = 0 + \cdots + 0 = 0$ for any $v \in (\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1})$, therefore $\varphi \in ((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0$.

- Last two sets are definitional equal.

□

Exercise 3.24. *Let V finite and $v_0, \cdots, v_{m-1} \in V$.*
Define $\Gamma(\varphi) = (\varphi(v_0), \cdots \varphi(v_{m-1})) : V' \to F^m$, show that:

- *$v_0, \cdots, v_{m-1}$ spans $V \iff \Gamma$ is injective.*

- $v_0, \cdots, v_{m-1}$ *is linear independent* $\iff$ Γ *is surjective.*

Proof.

- ($\Rightarrow$) Suppose $\Gamma(\alpha) = \Gamma(\beta)$, then for all $v \in V$ can be factorized into $\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}$, then $\alpha(v) = \alpha(\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}) = \beta(\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}) = \beta(v)$ since $\Gamma(\alpha) = \Gamma(\beta)$ and α and β are linear map, thus $\alpha = \beta$.

  ($\Rightarrow$) We first make $v_0, \cdots, v_{m-1}$ linear independent, say $v_0, \cdots, v_{k-1}$, then for any $w \in V$ such that $v_0, \cdots, v_{k-1}, w$ is linear independent, then we have its dual basis $\varphi_0, \cdots, \varphi_{k-1}, \psi$. Consider $\Gamma(\psi)$, by definition, we know $\Gamma(\psi) = (\psi(v_0), \cdots) = (0, \cdots)$ then $\psi = 0$ since Γ is injective, which contradicts our assumption. Therefore $v_0, \cdots, v_{k-1}$ spans V.

- ($\Rightarrow$) Consider the dual basis of $v_0, \cdots, v_{m-1}$, then Γ is surjective since we have the standard basis of F^m.

  ($\Leftarrow$) Γ is surjective implies we have $\varphi_0, \cdots, \varphi_{m-1}$ such that $\Gamma(\varphi_i) = (\cdots, 1, \cdots)$, which means $v_0, \cdots, v_{m-1}$ is linear independent.

$\square$

Exercise 3.25. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1} \in V'$.*
Define $\Gamma(v) = (\varphi_0(v), \cdots, \varphi_{m-1}(v)) : V \to F^m$. Show that

- $\varphi_0, \cdots, \varphi_{m-1}$ *spans* V' $\iff$ Γ *is injective*

- $\varphi_0, \cdots, \varphi_{m-1}$ *is linear independent* $\iff$ Γ *is surjective*

Proof.

- ($\Rightarrow$) Suppose $\Gamma(v) = \Gamma(w)$, then $\varphi_i(v) = \varphi_i(w)$, which means $\varphi_i(v-w) = 0$ for all i. If $v - w \neq 0$, then $((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0 \neq \{0\}$, thus $\varphi_0, \cdots, \varphi_{m-1}$ doesn't span V'.

  ($\Leftarrow$) $(\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}) = \{0\}$ since Γ is injective. therefore $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) = ((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0 = (\{0\})^0 = V'$

- ($\Rightarrow$) We may treat Γ as the following matrix:

$$\begin{bmatrix} \varphi_0 \\ \vdots \\ \varphi_{m-1} \end{bmatrix}$$

which line rank is m since $\varphi_0, \cdots, \varphi_{m-1}$ is linear independent, therefore its column rank is m, thus $\dim \operatorname{range} \Gamma = m = \dim F^m$, then Γ is surjective.

($\Leftarrow$) It seems the proof of ($\Rightarrow$) also works here.

$\square$

Exercise 3.26. *Let V finite, and $\Omega \subseteq V'$ a subspace. Show that*

$$\Omega = \{\ v \in V \mid \varphi(v) = 0 \quad \forall \varphi \in \Omega\ \}^0$$

Proof. This construction looks like an inverse of $-^0$.

We may rewrite the equation to $\Omega = (\bigcap_{\varphi \in \Omega} \operatorname{null} \varphi)^0$, then $\Omega = \operatorname{span}(\varphi) \forall \varphi \in \Omega$, which is trivial. $\square$

Exercise 3.28. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1}$ is linear independent. Show that*

$$\dim((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1})) = \dim V - m$$

Proof.

$$\begin{aligned} m &= \dim \operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) \\ &= \dim((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0 \\ &= \dim V - \dim((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1})) \end{aligned}$$

$\square$

Exercise 3.30. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1}$ a basis of V'. Show that there is a basis of V which dual basis is $\varphi_0, \cdots, \varphi_{m-1}$.*

Proof. Since $\varphi_0, \cdots, \varphi_{m-1}$ spans V' and linear independent, we know Γ is both injective and surjective. Consider $v_0, \cdots, v_{m-1}$ such that $\Gamma(v_i) = (\cdots, 0, 1, 0, \cdots)$. We claim $v_0, \cdots, v_{m-1}$ is a basis of V and which dual basis if $\varphi_0, \cdots, \varphi_{m-1}$.

The second part is trivial by the way construct them. For the first part, $v_0, \cdots, v_{m-1}$ is linear independent since $(\cdots, 0, 1, 0 \cdots)$ is linear independent, and $v_0, \cdots, v_{m-1}$ spans V since $\dim V = \dim V' = m$. $\square$

Exercise 3.31. *Let $U \subseteq V$ a subspace and $i(u) = u : U \to V$. Then $i' \in \mathcal{L}(V', U')$, show that:*

1. $\operatorname{null} i' = U^0$
2. $\operatorname{range} i' = U'$ *if* V *is finite*
3. $\tilde{i'}$ *is an isomorphism between* V'/U^0 *and* U' *if* V *is finite*

Proof.

- For any $\varphi \in \operatorname{null} i'$, $\varphi \circ i = 0$, therefore $\operatorname{range} i = U \subseteq \operatorname{null} \varphi$, thus $\varphi \in U^0$.

  For any $\varphi \in U^0$, $\varphi \circ i = 0$ since $\operatorname{range} i = U \subseteq \operatorname{null} \varphi$.

- Suppose V is finite, then i' is surjective since i' is injective, therefore $\operatorname{range} i' = U'$.

- $\tilde{i'}(\varphi + U^0) = i'(\varphi)$ is surjective since i' is surjective. Then $\dim(V'/U^0) = \dim V' - \dim U^0 = \dim V - (\dim V - \dim U) = \dim U = \dim U'$, therefore $\tilde{i'}$ is an isomorphism.

$\square$

Exercise 3.32. *We denote* V'' *as the* ***double dual space of*** V*, defined by* $V'' = (V')'$*. Define* $\Lambda(v)(\varphi) = \varphi(v) : V \to V''$

Show that:

1. $\Lambda \in \mathcal{L}(V, V'')$
2. *Let* $V \in \mathcal{L}(V)$*, then* $T'' \circ \Lambda = \Lambda \circ T$ *where* $T'' = (T')'$*.*
3. Λ *is an isomorphism if* V *is finite.*

Proof.

- For any $v, w \in V$ and $\lambda \in F$, we have $(\Lambda(v) + \Lambda(w))(\varphi) = \Lambda(v)(\varphi) + \Lambda(w)(\varphi) = \varphi(v) + \varphi(w) = \varphi(v + w) = \Lambda(v + w)(\varphi)$ and $(\lambda\Lambda(v))(\varphi) = \lambda(\Lambda(v)(\varphi)) = \lambda(\varphi(v)) = \varphi(\lambda v) = \Lambda(\lambda v)(\varphi)$.

- For any $v \in V$,

$$\begin{aligned}
&(T'' \circ \Lambda)(v)(\varphi)\\
=&(T''(\Lambda(v)))(\varphi)\\
=&((\Lambda(v)) \circ T')(\varphi)\\
=&\Lambda(v)(T'(\varphi))\\
=&\Lambda(v)(\varphi \circ T)\\
=&(\varphi \circ T)(v)\\
=&\varphi(T(v))\\
=&\Lambda(T(v))(\varphi)\\
=&(\Lambda \circ T)(v)(\varphi)
\end{aligned}$$

- Suppose $\Lambda(v) = \Lambda(w)$, that is, $\Lambda(v)(\varphi) = \varphi(v) = \varphi(w) = \Lambda(w)(\varphi)$ for all $\varphi \in V'$. Let $\varphi_0, \cdots, \varphi_{m-1}$ the dual basis of some basis of V, then $v = \varphi_0(v)v_0 + \cdots + \varphi_{m-1}(v)v_{m-1} = \varphi_0(w)v_0 + \cdots + \varphi_{m-1}(w)v_{m-1} = w$. Therefore Λ is injective, thus surjective and isomorphism since $\dim V = \dim V''$.

□

Exercise 3.33. *Let $U \subseteq V$ a subspace and $\pi : V \to V/U$ the quotient map, then $\pi' \in \mathcal{L}((V/U)', V')$.*

1. *Show that π' is injective.*
2. *Show that* $\operatorname{range} \pi' = U^0$.
3. *Conclude that π' is an isomorphism between $(V/U)'$ and U^0.*

Proof.

- π is surjective, therefore π' is injective. The statement is true even V or V/U may be infinite, cause the proof about surjective-implies-epimorphism doesn't require that the codomain is finite but epimorphism-implies-surjective does.

  We may prove those theorem again, but with weaker assumption. For any $\pi'(\varphi) = \pi'(\psi)$, we have $\varphi \circ \pi = \psi \circ \pi$. For any $v + U \in V/U$, there is $v \in V$ such that $\pi(v) = v + U$ since π is surjective. Therefore $\varphi(\pi(v)) = \psi(\pi(v))$ for all $\pi(v) = v + U \in V/U$,thus $\varphi = \psi$.

  Therefore π' is injective.

- range $\pi' = (\text{null}\,\pi)^0 = U^0$.
- Trivial.

$\square$

Exercise 3.7. *Let m a non-negative integer and $z_0, \cdots, z_m \in F$ are different to each others, $w_0, \cdots, w_m \in F$, show that there is a unique $p \in \mathcal{P}_m(F)$ such that $p(z_k) = w_k$ holds for all $0 \leq k \leq m$.*

Proof. Define $\Gamma(p) = (p(z_0), \cdots, p(z_m)) : \mathcal{P}_m(F) \to F^{m+1}$, we will show that Γ is injective, therefore an isomorphism.

Suppose $\Gamma(p) = \Gamma(q)$, then $p(z_k) = q(z_k)$ for all k, therefore $(p-q)(z_k) = 0$ for all k. This means $p - q$ has $m + 1$ zeros but $\deg(p - q) \leq m$, therefore $p - q = 0$ and $p = q$.

Then there is a unique $p \in \mathcal{P}_m(F)$ such that $\Gamma(p) = (w_0, \cdots, w_m)$. $\square$

Exercise 3.9. *Let $P \in \mathcal{L}(V)$, such that $P^2 = P$. Suppose λ an eigenvalue of P, show that $\lambda = 0$ or $\lambda = 1$.*

Proof. Suppose $P(v) = \lambda v$ for some non-zero $v \in V$, then $P(v) = PP(v) = P(\lambda v)$, therefore $P((\lambda - 1)v) = 0$. thus $(\lambda - 1)v \in \text{null}\,P$. We may suppose $\lambda \neq 1$, then $(\frac{1}{\lambda-1})(\lambda - 1)v = v \in \text{null}\,P$, therefore $P(v) = 0$, thus $\lambda = 0$ cause $v \neq 0$. $\square$

Exercise 3.10. *Let $T(p) = p' : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$. Find all eigenvalues and eigenvectors of T.*

Proof. Suppose $T(p) = p' = \lambda p$, then $\deg p = 0$, otherwise the degree doesn't match. For any $p \in \mathcal{P}(\mathbb{R})$ such that $\deg p = 0$, we have $p' = 0 = 0p$. $\square$

Exercise 3.12. *Let $V = U \oplus W$ where U and W are non-zero subspaces. Define $P(u + w) = u$ for all $u \in U$ and $w \in W$. Find all eigenvalue and eigenvector of P.*

Proof. We can see $P^2 = P$, since for any $u + w \in V$, we have $P(P(u + w)) = P(u) = u = P(u + w)$, therefore $\lambda = 0$ and $\lambda = 1$ are eigenvalues of P , $P(u) = 1u$ and $P(w) = 0w$ are eigenvectors of P. $\square$

Exercise 3.13. *Let $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$, where S is invertible.*

- *Show that T has the same eigenvalue of $S^{-1}TS$.*

- *What is the relationship between the eigenvector of T and the eigenvector of $S^{-1}TS$.*

Proof.

- For any $T(v) = \lambda v$ where $v \in V$ and $\lambda \in F$, let $S(w) = v$, then $S^{-1}TS(w) = S^{-1}(T(Sw)) = S^{-1}(\lambda v) = \lambda S^{-1}(v) = \lambda w$, thus λ is an eigenvalue of $S^{-1}TS$.

- $S(w) = v$ where v is an eigenvector of T and w is the corresponding eigenvector of $S^{-1}TS$.

$\square$

Exercise 3.15. *Let V finite, $T \in \mathcal{L}(V)$, $\lambda \in F$. Show that λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of T'.*

Proof.

- ($\Rightarrow$) Suppose $Tv = \lambda v$, we will show $T' - \lambda I$ is not surjective (Note that $I \in \mathcal{L}(V')$).

  For any $\varphi \in V'$, we have:

$$
\begin{aligned}
&(T' - \lambda I)(\varphi) \\
=&T'(\varphi) - \lambda\varphi \\
=&\varphi \circ T - \lambda\varphi
\end{aligned}
$$

  then

$$
\begin{aligned}
&(\varphi \circ T - \lambda\varphi)(v) \\
=&(\varphi \circ T)(v) - (\lambda\varphi)(v) \\
=&\varphi(Tv) - \lambda(\varphi(v)) \\
=&\varphi(\lambda v) - \lambda(\varphi(v)) \\
=&0
\end{aligned}
$$

  This means $\operatorname{range} T' \neq V'$ cause any $\psi \in V'$ where $\psi(v) \neq 0$ is not in $\operatorname{range} T'$. Therefore λ is an eigenvalue of T'.

$\square$

Exercise 3.22. *Let $T \in \mathcal{L}(V)$ and non-zero $v, w \in V$ such that*

$$Tu = 3w \quad and \quad Tw = 3u$$

.

Show that 3 or -3 is the eigenvalue of T.

Proof. Since v and w are non-zero, then one of $u + w$ and $u - w$ is non-zero.

We have $T(u + w) = 3w + 3u = 3(u + w)$ and $T(u - w) = 3w - (3u) = (-3)(u - w)$. □

Exercise 3.23. *Let V finite, and $S, T \in \mathcal{L}(V)$, show that ST and TS have the same eigenvalues.*

Proof. For any $ST(v) = \lambda v$ where $v \neq 0$, we have $TST(v) = T(\lambda v)$ then $TS(Tv) = \lambda(Tv)$.

- If $Tv = 0$, then $STv = 0 = \lambda v$, thus $\lambda = 0$ since $v \neq 0$. then λ is an eigenvalue of TS since $\operatorname{null} TS \neq 0$ ($\operatorname{null} T \neq 0$, and S is an operator of V, therefore, S is injective or not doesn't affect our conclusion).

  If $Tv \neq 0$, then $TS(Tv) = \lambda(Tv)$.

- Ditto.

□

Exercise 3.26. *Let $T \in \mathcal{L}(V)$ and any non-zero $v \in V$ we have $Tv = cv$ for some c. Show that $T = \lambda I$.*

Proof. Let non-zero $v, w \in V$, we have $Tv = sv$ and $Tw = tw$, then $T(v+w) = \lambda(v + w) = \lambda v + \lambda w = sv + tw = T(v) + T(w)$. Then $\lambda v + \lambda w - tw = sv$.

- If $w \in \operatorname{span}(v)$, then $w = cv$, therefore $Tw = T(cv) = tcv = cTv = csv$, thus $t = s$.

- If $w \notin \operatorname{span}(v)$, then $\lambda = t$ (otherwise $\lambda v + (\lambda - t)w = sv$), therefore $\lambda v = sv$ and $\lambda = s$, thus $s = t$.

□

Exercise 3.27. *Let V finite and $1 \leq k \leq \dim V - 1$. Let $T \in \mathcal{L}(V)$ such that any subspace of V with k dimension is invariant under T. Show that $T = \lambda I$ for some λ.*

Proof. For any $v \in V$, we have $Tv = w$ where $w \in \text{span}(v)$.

- If $w = 0$, then $Tv = 0v$.
- If $w \neq 0$, then $w = \lambda v$ since $w \in \text{span}(v)$, then $Tv = \lambda v$.

Thus $T = \lambda I$ by the previous exercise. □

Exercise 3.30. *Let $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Show that 2 or 3 or 4 is the eigenvalue of T.*

Proof. Suppose 2 is not an eigenvalue of T, then $(T - 2I)$ is injective, thus $(T - 3T)(T - 4I)$ must map all $v \in V$ to 0. Similarly, we can show that $(T - 4I) = 0$ if 3 is not an eigenvalue of T. □

Exercise 3.31. *Find $T \in \mathcal{L}(\mathcal{R}^2)$ such that $T^4 = -I$.*

Proof. We may treat $-I$ as rotating the vector 180 degrees, then T rotates a vector 45 degrees, which matrix is:

$$\mathcal{M}(T) = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

□

Exercise 3.32. *Let $T \in \mathcal{L}(V)$ with no eigenvalue and $T^4 = I$. Show that $T^2 = -I$.*

Proof. We will show that $T^2(v) = -v$ for any $v \in V$. Consider $T^2(T^2(v)+v) = v+T^2(v)$, we will show that $T^2(v)+v = 0$. Suppose $w \in V$ and $T^2(w) = w$, we may let $T(w) = u$, then $T^2(w) = Tu = w$. Consider $T(w+u) = T(w)+T(u) = u + w$, then $w + u = 0$ since T has no eigenvalue, therefore $u = -w$ and $T(w) = -w$. Again $w = 0$ since T has no eigenvalue. Therefore $T^2(w) = w$ implies $w = 0$, thus $T^2(v) + v = 0$ and $T^2(v) = -v$ for any $v \in V$. □

Exercise 3.33. *Let $T \in \mathcal{L}(V)$ and $m \in \mathbb{N}^+$.*

1. *Show that T is injective $\iff$ T^m is injective*
2. *Show that T is surjective $\iff$ T^m is surjective.*

Proof. Recall that $T^0 = I$.

- ($\Rightarrow$) For any $T^m(v) = T^m(w)$ we have $T^{m-1}(v) = T^{m-1}(w)$ and so on, we will get $v = w$.

  ($\Leftarrow$) For any $T(v) = T(w)$, we have $T^{m-1}(T(v)) = T^{m-1}(T(w))$, then $T^m(v) = T^m(w)$ and $v = w$.

- ($\Rightarrow$) For any $w \in V$, we have $T(v) = w$, then we have $T(u) = v$ and now $T(T(u)) = w$, continue this progress until we get $T^m(r) = w$.

  ($\Leftarrow$) For any $w \in V$, we have $T^m(v) = w$, therefore $T(T^{m-1}(v)) = w$.

□

Exercise 3.34. *Let V finite and $v_0, \cdots, v_{m-1} \in V$. Show that $v_0, \cdots, v_{m-1}$ is linear independent $\iff$ there is $T \in \mathcal{L}(V)$ such that $v_0, \cdots, v_{m-1}$ are eigenvectors of distinct eigenvalues of T.*

Proof.

- ($\Rightarrow$) Consider $v_0, \cdots, v_{k-1}$ a basis of V, then $T(\lambda_0 v_0 + \cdots + \lambda_{k-1} v_{k-1}) = 1\lambda_0 v_0 + 2\lambda_1 v_1 + \cdots + m\lambda_{m-1} v_{m-1}$ where $T(v_i) = (i+1)v_i$.

- ($\Leftarrow$) Trivial, since eigenvectors of distinct eigenvalues are linear independent.

□

Exercise 3.37. *Let V finite and $T \in \mathcal{L}(V)$. Define $\mathcal{A}(S) = TS : \mathcal{L}(V) \to \mathcal{L}(V)$. Show that T has the same eigenvalues as $\mathcal{A}$.*

Proof.

- ($\subseteq$) For any eigenvalue λ of T, we can find $S \in \mathcal{L}(V)$ such that range $S = \{ v \in V \mid Tv = \lambda v \}$ (it is easy to show that such set is a subspace). Then for any $v \in V$, $(TS)v = T(Sv) = \lambda(Sv) = (\lambda S)v$ thus $\mathcal{A}(S) = TS = \lambda S$.

- ($\supseteq$) For any eigenvalue λ of $\mathcal{A}$, then we have $\mathcal{A}(S) = \lambda S$ for some non-zero $S \in \mathcal{L}(V)$. Then there is $v \in V$ such that $Sv \neq 0$, and $T(Sv) = (TS)v = (\lambda S)(v) = \lambda(Sv)$, thus λ is an eigenvalue of T.

□

Exercise 3.38. *Let V finite and $T \in \mathcal{L}(V)$ and $U \subseteq V$ is invariant under T. A quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by:*

$$(T/U)(v+U) = Tv + U$$

for any $v \in V$.

1. *Show that T/U is well-defined and T/U is an operator over V/U.*

2. *Show that each eigenvalue of T/U is also an eigenvalue of T.*

Proof.

- Suppose $v+U = w+U$, then $(T/U)(v+U) = Tv+U$ and $(T/U)(w+U) = Tw + U$, we will show that $Tv - Tw \in U$. Note that $v + U = w + U$ implies $v - w \in U$, then $T(v-w) \in U$ since U is invariant under T, that is, for any $u \in U$, $Tu \in U$. Thus $Tv + U = Tw + U$.

  Now we will show that T/U is a linear map, we can see:

$$\begin{aligned}
&(T/U)(v+U) + (T/U)(w+U) \\
=&(Tv+U) + (Tw+U) \\
=&(Tv+Tw) + U \\
=&T(v+w) + U \\
=&(T/U)((v+w)+U)
\end{aligned}$$

  and

$$\begin{aligned}
&\lambda(T/U)(v+U) \\
=&\lambda(Tv+U) \\
=&(\lambda(Tv)) + U \\
=&T(\lambda v) + U \\
=&(T/U)((\lambda v)+U) \\
=&(T/U)(\lambda(v+U))
\end{aligned}$$

- Suppose $(T/U)(v+U) = Tv + U = \lambda v + U$ where $v \notin U$, consider $T - \lambda I$, we will show that $T - \lambda I$ is not injective. We can see U is invariant under $T - \lambda I$, $Tu - \lambda u \in U$ cause U is invariant under T. We may suppose T is injective (thus surjective and invertible) on U (in other words, $T(U) = U$), otherwise the proof is complete. Then consider $(T - \lambda I)(v) = Tv - \lambda v \in U$ where $v \notin U$, thus $T - \lambda I$ is not injective.

□

Exercise 3.39. *Let V finite and $T \in \mathcal{L}(V)$. Show that T has an eigenvalue $\iff$ there is a subspace of V with dimension* $\dim V - 1$ *which is invariant under T.*

Proof.

- This part is hinted by AI. Suppose $Tv = \lambda v$, then consider $T - \lambda I$, we know range$(T - \lambda I)$ is invariant under T, since for any $Tw - \lambda w$, we have $T(Tw - \lambda w) = T(Tw) - T(\lambda w) = T(Tw) - \lambda(Tw)$. Then $\dim \operatorname{range}(T - \lambda I) \leq \dim V - 1$ since $w \in \operatorname{null} T - \lambda I$. Then consider $\operatorname{null}(T - \lambda I) = \operatorname{span}(v) \oplus W$, we have $\operatorname{range}(T - \lambda I) \oplus W$ a subspace which is invariant under T.

  The key is finding a smaller invariant subspace and expand it with null space, as any vector in null space always maps to 0, thus preserve the property of invariant.

- Suppose U is a subspace of V of dimension $\dim V - 1$ such that U is invariant under T, then $V = U \oplus \operatorname{span}(v)$ for some $v \notin U$. We may suppose T is injective on U, otherwise the proof is complete ($\operatorname{null} T \neq 0$). Consider $T(v)$, there are three cases:

  - $T(v) = \lambda v + 0u$, then the proof is complete.
  - $T(v) = 0v + u$, then T is not injective since there is $Tw = u$ where $w \in U$.
  - $T(v) = \lambda v + u$, then consider $T - \lambda I$. We have U is invariant under $T - \lambda I$ cause $Tu - \lambda u \in U$ by $Tu \in U$. Again, if $T - \lambda I$ is not injective on U, the proof is complete. Then $(T - \lambda I)v = Tv - \lambda v \in U = \lambda v + u - \lambda v = u \in U$, thus $T - \lambda I$ is not injective and λ is an eigenvalue of T.

□

Exercise 3.42. *Let $T \in \mathcal{L}(F^n)$ defined by $T(x_1, x_2, \cdots, x_n) = (x_1, 2x_2, \cdots, nx_n)$.*

1. *Find all eigenvalues and eigenvectors of T.*
2. *Find all subspace of F^n which is invariant under T.*

Proof.

- $1, 2, \cdots, n$ and $(x_1, 0, \cdots), (0, x_2, 0, \cdots), \cdots$
- We claim any subspace that is invariant under T is a direct sum of some spaces that spans by the standard basis, say $\text{span}(x_0) \oplus \cdots \oplus \text{span}(x_k)$.

  Let U a subspace that is invariant under T and $u \in U$, we have $T(u) = T(\lambda_1 x_1, \cdots, \lambda_n x_n) = (\lambda_1 x_1, \cdots, n\lambda_n x_n)$, then $T(u) - iu = ((1-i)(\lambda_1 x_1), (2-i)(\lambda_2 x_2), \cdots, (n-1)(\lambda_i x_i)) \in U$ is a vector that is a linear combination of standard basis except x_i. Repeat this progress by apply $T - jI$ to $(T - iI)(u)$ with a different j, we can finally get a vector that is a scalar multiple of x_k. Thus $x_i \in U$ as long as there is $u \in U$ that the ith scalar of the linear combination of standard basis is not zero.

□

Exercise 3.43. *Let $T \in \mathcal{L}(V)$. Show that 9 is an eigenvalue of $T^2 \iff 3$ or -3 is an eigenvalue of T.*

Proof.

- ($\Rightarrow$) We have $T^2 - 9I$ is not injective since 9 is an eigenvalue of T^2, then $(T - 3I)(T + 3I) = T^2 - 9I$ is not injective means one of $T - 3I$ and $T + 3I$ is not injective, thus 3 or -3 is an eigenvalue of T.
- ($\Leftarrow$) Similarly, we have $(T - 3I)(T + 3I)v = (T^2 - 9I)v = 0$ (if 3 is an eigenvalue of T) or $(T + 3I)(T - 3I)v = (T^2 - 9I)v = 0$ (if -3 is an eigenvalue of T).

□

Exercise 3.44. *Let V a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$ has no eigenvalue. Show that any subspace of V that is invariant under T is either $\{0\}$ or infinite dimension.*

Proof. Let $U \subseteq V$ a subspace that is invariant under T, and non-zero $u \in U$. We can repeatly apply T to u, say $u, Tu, T^2u, \cdots$. Suppose $k > 0$ is minimum such that $u, Tu, \cdots, T^k u$ is linear dependent, we have $p \in \mathcal{P}(\mathbb{C})$ with $\deg p = k$ such that $p(T) = 0$. Clearly p is not constants, thus it has a zero since p is a polynomial of complex coefficient. Thus such zero is an eigenvalue of T. □

Exercise 3.45. *Let $n > 1$ an integer, and $T \in \mathcal{L}(F^n)$ is defined by:*

$$T(x_0, \cdots, x_{n-1}) = (x_0 + \cdots + x_{n-1}, \cdots, x_0 + \cdots + x_{n-1})$$

- *Find all eigenvalue and eigenvector of T.*
- *Find the minimal polynomial of T.*

Proof.

- Observe that range $T = \operatorname{span}((1, \cdots, 1))$, thus $T(1, \cdots, 1) = n(1, \cdots, 1)$.
- Observe that $T(x_0, \cdots, x_{n-1}) = (x_0 + \cdots + x_{n-1})(1, \cdots, 1)$ and $T^2(x_0, \cdots, x_{n-1}) = n(x_0 + \cdots + x_{n-1})(1, \cdots, 1)$, thus $p(T) = nT - T^2 = 0$.

□

Exercise 4 is kinda hard, sorry.

Exercise 3.6. *Let $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T.*

Proof. Observe that $T^2(w, z) = T(-z, w) = (-w, -z) = (-1)(w, z)$, thus the minimal polynomial of T is $p(T) = I + T^2$. □

Exercise 3.7.
- *Given an example that the minimal polynomial of ST is not equal to TS's.*
- *Suppose V is finite and $S, T \in \mathcal{L}(V)$. Show that the minimal polynomial of ST is equal to TS's if one of S and T is invertible.*

  Hint: Show that S is invertible and $p \in \mathcal{P}(F)$ implies $p(TS) = S^{-1}p(ST)S$.

Proof.

- The idea is to find S, T such that $ST \neq 0$ but $TS = 0$. We can find $S(x, y) = (x, 0)$ and $T(x, y) = (y, 0)$ holds:

  $$(ST)(x, y) = S(y, 0) = (y, 0)$$
  $$(TS)(x, y) = T(x, 0) = (0, 0)$$

  Thus the minimal polynomial of ST is not 0 but TS one does.

- Suppose S is invertible and $p \in \mathcal{L}(F)$ is the minimal polynomial of TS, then $p(TS) = S^{-1}p(ST)S$ since i-th term of $S^{-1}p(ST)S$ has form $S^{-1}c_i(ST)^iS = c_i(S^{-1}S)(TS)^{i-1}(TS) = c_i(TS)^i$. Thus $S^{-1}p(ST)S = 0$ and then $p(ST) = 0$. We will show that p is the minimal polynomial of ST, suppose $q \in \mathcal{L}(F)$ such that $q(ST) = 0$, then $0 = S^{-1}q(ST)S = q(TS)$, therefore $\deg q = \deg p$. Hence p is the minimal polynomial of ST.

$\square$

Exercise 3.8. *Let $T \in \mathcal{L}(R^2)$ is the opeartor that "rotates 1 degree counter-clockwise", find the minimal polynomial of T.*

*Note that it is **NOT** $x^{180}+1$ even $T^{180} = -I$.*

Proof. Note that there is some λ such that $Tv - \lambda v = \alpha T^2 v$ (We can show that $\lambda = \alpha$), however the calculation is too complicate.

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λ should be $\frac{\sin(1^\circ)}{\sin(2^\circ)}$, thus $p(T) = -\lambda I + T - \lambda T^2$.

We suppose all v below has length 1, thus $v = (\cos\theta, \sin\theta)$, this doesn't lose the generalizability since $p(T)(\alpha v) = \alpha(p(T)v)$.

For the first component of $p(T)v = -\lambda v + Tv - \lambda T^2 v$, we have:

$$\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - \cos(\theta + 2^\circ)) \\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - (\cos\theta\cos(2^\circ) - \sin\theta\sin(2^\circ))) \\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - \cos\theta\cos(2^\circ)) + \sin\theta\sin(1^\circ) \\
=&\cos\theta\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) + \sin\theta\sin(1^\circ)
\end{aligned}$$

where $\sin\theta\sin(1^\circ)$ cancels a part of $(Tv)_1 = \cos(\theta + 1^\circ) = \cos\theta\cos(1^\circ) - \sin\theta\sin(1^\circ)$. Thus we will show that $\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) = -\cos(1^\circ)$.

$$\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) \\
=&\frac{\sin(1^\circ)}{2\sin(1^\circ)\cos(1^\circ)}(-(\cos^2(1^\circ) + \sin^2(1^\circ)) - \cos^2(1^\circ) + \sin^2(1^\circ)) \\
=&\frac{1}{2\cos(1^\circ)}(-\cos^2(1^\circ) - \cos^2(1^\circ)) \\
=&\frac{1}{2\cos(1^\circ)}(-2\cos^2(1^\circ)) \\
=&-\cos(1^\circ)
\end{aligned}$$

For the second component of $p(T)v$, we have:

$$\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\sin\theta - \sin(\theta + 2^\circ)) \\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\sin\theta - \sin\theta\cos(2^\circ) - \cos\theta\sin(2^\circ)) \\
=&\sin\theta\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) - \cos\theta\sin(1^\circ)
\end{aligned}$$

similarly, we have $p(T)v_2 = \sin(\theta + 1^\circ) = \sin\theta\cos(1^\circ) + \cos\theta\sin(1^\circ)$ we will show that $\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) = -\cos(1^\circ)$, which is proven above. □

Exercise 3.9. *Let $T \in \mathcal{L}(V)$ such that for some basis of V, $\mathcal{M}(T)$ consists of rational numbers. Try to explain why the coefficients of the minimal polynomial of T is rational numbers.*

Proof. I don't know, because $\mathbb{Q}$ is also a field? □

Exercise 3.11. *Let V a vector space and $\dim V = 2$ and $T \in \mathcal{L}(V)$ such that $\mathcal{M}(T)$ for some basis of V is $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Show that:*

- $T^2 - (a + d)T + (ad - bc)I = 0$
- *the minimal polynomial of T is:*

$$\begin{cases} z - a & \text{if } b = c = 0 \text{ and } a = d \\ z^2 - (a + d)z + (ad - bc) & \text{otherwise} \end{cases}$$

Proof.

-

$$\begin{aligned}
&\mathcal{M}(T^2 - (a + d)T + (ad - bc)I) \\
=&\begin{bmatrix} a & c \\ b & d \end{bmatrix}^2 - (a + d)\begin{bmatrix} a & c \\ b & d \end{bmatrix} + (ad - bc)I \\
=&\begin{bmatrix} a^2 + bc & ac + bd \\ ab + bd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ac + cd \\ ab + bd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}
\end{aligned}$$

- If $b = c = 0$ and $a = d$, then T is a scalar multiple of identity operator, thus $T = aI$ and $p(T) = -aI + T = 0$. Otherwise, $T^2 - (a+d)T + (ad - bc)I = 0$.

$\square$

Exercise 3.13. *Let V finite, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(F)$. Show that there is a unique $r \in \mathcal{P}(F)$ such that $p(T) = r(T)$ where* $\deg p$ *is less than the degree of the minimal polynomial of T.*

Proof. Let q the minimal polynomial of T.

If $\deg p < \deg q$, then $r = p$. The uniqueness is guaranteed by $\deg p < \deg q$ (try $(p - s)(T)$ where $p(T) = s(T)$ and $\deg s < \deg q$).

If $\deg p >= \deg q$, then $p = sq + r$ where $s, r \in \mathcal{P}(F)$ with $\deg r < \deg q$.Then $p(T) = s(T)q(T) + r(T) = r(T)$ since $q(T) = 0$. The uniqueness is guaranteed by the property of division. $\square$

Exercise 3.14. *Let V finite, $T \in \mathcal{L}(V)$ with minimal polynomial $p(z) = 4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$. Find the minimal polynomial of T^{-1}.*

Proof. Suppose p is the minimal polynomial of T, then we can repeatly apply T^{-1} to $p(T)$, say $T^{-(\deg p)}(p(T))$, then it should be 0, and the coefficients are reversed, that is, $p_{\deg p}I + p_{\deg p-1}T^{-1} + \cdots + p_0(T^{-1})^{\deg p}$.

So the answer is $1 + 2z^1 - 7z^2 - 6z^3 + 5z^4 + 4z^5$. $\square$

Exercise 3.16. *Let $a_0, \cdots, a_{n-1} \in F$ and T an operator over F^n. Its matrix (about the standard basis) is:*

$$\begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & \ddots & & & -a_2 \\ & & \ddots & & & \vdots \\ & & & 1 & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix}$$

. Show that the minimal polynomial of T is:

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$$

Proof. We first need some property of this matrix, we will see it moves all number to the left when we repeatly self-multily T. We can see the k-th column of T^p is equal to $k+1$-th column of T^{p-1}, thus it is also equal to i-th column if T^j where $1 \leq i, j \leq n$ and $i+j = k+p$. In fact, j can be 0 and we have $T^0 = I$ and the property still holds.

Then, the i-th column of T^n is equal to n-th column (the last one) of T^i, and it is produced by $T^{i-1}\begin{bmatrix} -a_0 \\ -a_1 \\ \vdots \\ -a_{n-1} \end{bmatrix}$, which is equal to

$$T_i^n = -a_0 T_1^{i-1} - a_1 T_2^{i-1} - \cdots - a_{n-1} T_n^{i-1}$$

which is equal to $T^n v_i$ where v_i is i-th standard basis of F^n, that is, $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$. We may rewrite the equation into

$$T^n v = -a_0 T^0 v - a_1 T^1 v - \cdots - a_{n-1} T^{n-1} v$$

where $T^0 v = T_i^0 = T_1^{i-1}$, $T^1 v = T_i^1 = T_2^{i-1}$ and so on.

Thus all v vector in standard basis has $p(T)v = 0$, thus $p(T) = 0$.

For minimal, we can see T is invertible, thus $p(T)v_1 = 0$ (recall that T moves number to the left, thus the first column of T^i is the i-th columns of T). means there is a (non-zero) linear combination of columns of T that is equal to 0. Thus $\deg p \geq n$ since T the columns are linear independent. $\square$

Exercise 3.17. *Let V finite and $T \in \mathcal{L}(V)$ and p is the minimal polynomial of T. Let $\lambda \in F$, show that the minimal polynomial of $T - \lambda I$ is $q(z) = p(z+\lambda)$.*

Proof. $q(T-\lambda I) = p((T-\lambda I) + \lambda I) = p(T) = 0$. Suppose r is the minimal polynomial of $T - \lambda I$, then $s(z) = r(z-\lambda)$ and $s(T) = r(T-\lambda I) = 0$, thus $\deg r = \deg p = \deg q$. $\square$

Exercise 3.18. *Let V finite and $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Let $\lambda \in F$ that $\lambda \neq 0$, show that the minimal polynomial of λT is $q(z) = \lambda^{\deg p} p(\frac{z}{\lambda})$.*

Proof. $q(\lambda T) = \lambda^{\deg p} p(\frac{1}{\lambda}(\lambda T)) = \lambda^{\deg p} p(T) = 0$. $\lambda^{\deg p}$ only makes q a monic polynomial.

Suppose r is the minimal polynomial of λT, then $s(z) = \frac{1}{\lambda^{\deg p}} r(\lambda z)$ and $s(T) = \frac{1}{\lambda^{\deg p}} r(\lambda T) = 0$, thus $\deg s = \deg r = \deg p = \deg q$. □

Exercise 3.19. *Let V finite and $T \in \mathcal{L}(V)$. Let $\mathcal{E} \subseteq \mathcal{L}(V)$ a subspace, defined by*

$$\mathcal{E} = \{\ q(T) \mid q \in \mathcal{P}(F)\ \}$$

.

Show that $\dim \mathcal{E}$ *is equal to the degree of the minimal polynomial of T.*

Proof. We can see $I, T, T^2, \cdots, T^{\deg p - 1}$ is linear independent (since p is the minimal polynomial of T) where p is the minimal polynomial of T. For any $q \in \mathcal{P}(F)$ where $\deg q \geq \deg p$, then $q = sp + r$ where $s, r \in \mathcal{P}(F)$ and $\deg r < \deg p$, therefore $q(T) = s(T)p(T) + r(T) = r(T) \in \operatorname{span}(I, T, T^2, \cdots, T^{\deg p - 1})$. □

Exercise 3.20. *let $T \in \mathcal{L}(F^4)$, which eigenvalues are* $3, 5, 8$. *Show that* $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$.

Proof. Suppose p is the minimal polynomial of T, then $p(z) = c(z-3)(z-5)(z-8)q(z)$ since $3, 5, 8$ are the eigenvalue of T, thus the zeros of p. Note that $\deg q \leq 1$ since $\deg p \leq \dim F^4 = 4$. Since there is no other eigenvalue (thus zero) than $3, 5, 8$, q is either 1 or one of $z - 3$, $z - 5$, $z - 8$, thus $(z - 3)^2(z - 5)^2(z - 8)^2$ is polynomial multiple of p, therefore $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$. □

Exercise 3.21. *Let V finite and $T \in \mathcal{L}(V)$. Show that the degree of the minimal polynomial of T caps at* $1 + \dim \operatorname{range} T$.

Proof. IDk □

Exercise 3.22. *Let V finite and $T \in \mathcal{L}(V)$. Show that T is invertible $\iff$* $I \in \operatorname{span}(T, T^2, \cdots, T^{\dim V})$.

Proof.

- ($\Rightarrow$) Suppose T is invertible, then the minimal polynomial p of T satisfies $p(0) \neq 0$ (since $p(0) = 0$ implies 0 is a eigenvalues of T). We know $\deg p \leq \dim V$, thus there is a linear combination of $T, T^2, \cdots, T^{\dim V}$ that is equal to a scalar multiple of I, therefore $I \in \operatorname{span}(T, T^2, \cdots, T^{\dim V})$.

- ($\Leftarrow$) Suppose $I = \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_{\dim V} T^{\dim V}$, then $I = T(\lambda_1 I + \lambda_2 T + \cdots + \lambda_{\dim V} T^{\dim V - 1}) = (\lambda_1 I + \lambda_2 T + \cdots + \lambda_{\dim V} T^{\dim V-1})T$, thus T is invertible.

$\square$

Exercise. *Let V a vector space and $T \in \mathcal{L}(V)$, $v, Tv, \cdots, T^k v$ a list of linear independent vectors but $v, Tv, \cdots, T^{k+1}v$ isn't. Show that $T^{k+i}v \in$* $\operatorname{span}(v, Tv, \cdots, T^k v)$ *for all $0 < i$.*

Proof. Induction on i.

- Base($i = 1$): By assumption.
- Ind($i = i+1$): $T^{k+i+1}v = T(T^{k+i}v)$, since $T^{k+i}v \in \operatorname{span}(v, Tv, \cdots, T^k v)$, thus it can be write as a linear combination of $v, Tv, \cdots, T^k v$, say $T(\lambda_0 v + \lambda_1 Tv + \cdots + \lambda_k T^k v)$, then $\lambda_0 Tv + \lambda_1 T^2 v + \cdots, + \lambda_k T^{k+1} v \in \operatorname{span}(v, Tv, \cdots, T^k v)$ since $T^{k+1}v \in \operatorname{span}(v, Tv, \cdots, T^k v)$.

$\square$

Exercise 3.23. *Let V finite and $T \in \mathcal{L}(V)$. Let $n = \dim V$, show that for any $v \in V$,* $\operatorname{span}(v, Tv, \cdots, T^{n-1}v)$ *is invariant under T.*

Proof. Note that the list $v, Tv, \cdots, T^{n-1}v$ has length $n = \dim V$, thus for the list $v, Tv, \cdots, T^n v$ is linear dependent, thus $T^n v$ must be a linear combination of $v, Tv, \cdots, ^{n-1} v$.

- If $v, Tv, \cdots, T^{n-1}v$ is linear dependent, then $T^n v \in \operatorname{span}(v, Tv, \cdots, T^{n-1}v)$ (by our lemma exercise).
- Otherwise, the list $v, Tv, \cdots, T^n v$ is linear dependent while $v, Tv, \cdots, T^{n-1}v$ isn't, therefore $T^n v$ is a linear combination of $v, Tv, \cdots, T^{n-1}v$.

$\square$

Theorem 3.29. $q(T) = 0 \iff$ *q is a polynomial multiple of the minimal polynomial of T.*

Proof.
- ($\Rightarrow$) Let p the minimal polynomial of T, consider $q = sp + r$ where $\deg r < \deg p$, we may suppose $r \neq 0$. Then $0 = q(T) = s(T)p(T) + r(T) = r(T)$, which contradict to the assumption that p is the minimal polynomial of T.

- ($\Leftarrow$) Trivial.

$\square$

Exercise 3.25. *Let V finite, $T \in \mathcal{L}(V)$, subspace $U \subseteq V$ is invariant under T.*

- *Show that the minimal polynomial of T is polynomial multiple of the minimal polynomial of T/U.*
- *Show that*

$$(\textit{the minimal polynomial of } T\big|_U) \times (\textit{the minimal polynomial of } T/U)$$

  is a polynomial multiple of the minimal polynomial of T.

Proof.
- Let p the minimal polynomial of T, then $p(T/U)(v+U) = p(T)v+U = 0+U$ for any $v+U \in V/U$, thus $p(T/U) = 0$, therefore p is a polynomial multiple of the minimal polynomial of T/U.
- Let p the minimal polynomial of $T\big|_U$ and q the minimal polynomial of T/U. Then $(pq)(T)v = (p(T)q(T))v = p(T)(q(T)v)$ where $q(T)v \in U$, thus $p(T)(q(T)v) = 0$.

$\square$

Exercise 3.26. *Let V finite, $T \in \mathcal{L}(V)$, U is invariant under T. Show that the set of eigenvalues of T is equal to the union of eigenvalues of $T\big|_U$ and T/U.*

Proof. This theorem separate the eigenvalues into two parts: eigenvectors in U and eigenvectors not in U (may have intersection).

- ($\subseteq$) For any $Tv = \lambda v$ where non-zero $v \in V$. If $v \in U$, then $T\big|_U(v) = Tv = \lambda v$. If $v \notin U$, then $(T/U)(v+U) = Tv+U = \lambda v+U = \lambda(v+U)$.
- ($\supseteq$) For any $T\big|_U(v) = \lambda v$, we have $T\big|_U(v) = Tv = \lambda v$. The case of T/U is proven in Exercise 5.38 of E5A.

$\square$

We will use this conclusion several times, so we prove it first.

Exercise. *Let p, q two non-constant* ***monic*** *polynomial and $p = sq$, $q = tp$ where s, t two non-zero polynomial. Show that $p = q$.*

Proof. We have $p = stp$, thus $st = 1$ and $\deg s = \deg t = 0$. Furthermore, we have $p = sq$ where p and q are monic, thus s must be 1, similar to t, hence $s = t = 1$ and $p = q$. $\square$

Exercise 3.27. *Let $F = R$ and V finite and $T \in \mathcal{L}(V)$. Show that the minimal polynomial of T_C is equal to the T one.*

Proof. Let p the minimal polynomial of T and q the minimal polynomial of T_C. We have:

$$\begin{aligned}
&p(T_C)(v + iu) \\
=&p(T)v + ip(T)u \\
=&0v + i0u \\
=&0
\end{aligned}$$

and

$$\begin{aligned}
&q(T)(v) \\
=&q(T_C)(v + i0) \\
=&0
\end{aligned}$$

thus $p = sq$ and $q = tp$ where s, t are non-zero polynomials, therefore $p = q$. $\square$

Exercise 3.28. *Let V finite and $T \in \mathcal{L}(V)$. Show that the minimal polynomial of $T' \in \mathcal{L}(V')$ is equal to the T one.*

Proof. Let p the minimal polynomial of T and p' the minimal polynomial of T'.

For any $\varphi \in V'$ and $v \in V$, we have $p(T')(\varphi)(v) = \varphi(p(T)v) = \varphi 0 = 0$ (since φ is linear), thus $p(T')(\varphi) = 0$, therefore $p(T') = 0$.

For any $v \in V$, $p'(T)v = \varphi_1(p'(T)v)v_1 + \cdots = p'(T')(\varphi)(v) + \cdots = 0$, where $v_0, \cdots, v_{\dim V - 1}$ is a basis of V and $\varphi_0, \cdots, \varphi_{\dim V-1}$ is a dual basis. Thus $p'(T) = 0$.

Hence, p and p' are polynomial multiple to each other, therefore $p = p'$. $\square$

Exercise 3.29. *Prove or disprove: $T \in \mathcal{L}(V)$ and $\mathcal{M}(T^2)$ is upper-triangular for some basis of V, then $\mathcal{M}(T)$ is upper-triangular for some basis of V (not necessary the same as the $\mathcal{M}(T^2)$ one).*

Proof. WoBuHui. $\square$

Exercise 3.30. *Let A, B are upper-triangular matrices with same size, the diagonal of A is $\alpha_0, \cdots, \alpha_{n-1}$ and the diagonal of B is $\beta_0, \cdots, \beta_{n-1}$. Show that*

- *$A + B$ is upper-triangular and the diagonal is $\alpha_0 + \beta_0, \cdots, \alpha_{n-1} + \beta_{n-1}$.*
- *AB is upper-triangular and the diagonal is $\alpha_0\beta_0, \cdots, \alpha_{n-1}\beta_{n-1}$.*

Proof.

- Trivial.
- Take the standard basis of F^n, we have $Bv_i \in \operatorname{span}(v_0, \cdots, v_i)$ and then $A(Bv_i) \in \operatorname{span}(v_0, \cdots, v_i)$ since both A and B are upper-triangular, thus AB is upper-triangular. For the diagonal, we know $AB_{i,i} = A_{i,-}B_{-,i}$, however, components before i-th of $A_{i,-}$ are 0 and componetns since $i-th$ of $B_{-,i}$ are 0, therefore $AB_{i,i} = A_{i,i}B_{i,i} = \alpha_i\beta_i$.

□

Exercise 3.31. *Let $T \in \mathcal{L}(V)$ invertible, and $\mathcal{M}(T)$ with respect to the basis $v_0, \cdots, v_{n-1}$ of V is upper-triangular, while the diagonal is $\lambda_0, \cdots, \lambda_{n-1}$. Show that $\mathcal{M}(T^{-1})$ with respect to that basis is also upper-triangular, and the diagonal is $\frac{1}{\lambda_0}, \cdots, \frac{1}{\lambda_{n-1}}$.*

Proof. For any $i = 1, \cdots, n$, $\operatorname{span}(v_0, \cdots, v_{i-1})$ is invariant under T, thus it is invariant under T^{-1} since T^{-1} is the inverse of T.

For the diagonal, $TT^{-1} = I$, which diagonal is $1, \cdots, 1$, which is equal to $\lambda_0\beta_0, \cdots, \lambda_{n-1}\beta_{n-1}$ where β_i is the diagonal of T^{-1}. Thus $\beta_i = \frac{1}{\lambda_i}$. □

Exercise 3.32. *Give an example that T an invertible operator, where the diagonal of $\mathcal{M}(T)$ is all 0.*

Proof.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

□

Exercise 3.33. *Give an example that T an singular operator, where the diagonal of $\mathcal{M}(T)$ is all non-zero.*

Proof.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

□

Exercise 3.34. *Let $F = C$ and V finite, and $T \in \mathcal{L}(V)$. Show that $k = 1, \cdots, \dim V$, then there is a k-dimension subspace of V that is invariant under T.*

Proof. If $F = C$, then $\mathcal{M}(T)$ is upper-triangular for some basis of V. Thus $\mathrm{span}(v_0, \cdots, v_{k-1})$ is invariant under T where v_i is such basis. □

Exercise 3.35. *Let V finite and $T \in \mathcal{L}(V)$ and $v \in V$. Show that:*

- *There is a unique monic polymonial p_v with minimal degree such that $p_v(T)v = 0$*
- *Show that the minimal polymonial of T is polymonial multiple of p_v.*

Proof.

- $p(T)v = 0$, therefore we only need to show the uniqueness. Let s, t a monic polymonial with minimal degree such that $s(T)v = t(T)v = 0$, then $(s - t)(T)v = 0$, therefore $s = t$, otherwise there is a polymonial $s - t$ with lower degree such that $(s - t)(T)v = 0$.
- We divide p by p_v, then $p = sp_v + r$ where $s, r \in \mathcal{P}(F)$ and $\deg r < \deg p_v$. Therefore $r = 0$, otherwise r is a lower polymonial such that $r(T)v = 0$, which contradict the property of p_v. Thus $p = sp_v$.

□

Exercise 3.36. *Let V finite and $T \in \mathcal{L}(V)$, and non-zero $v \in V$ such that $T^2 + 2Tv + 2v = 0$. Show that*

- *If $F = R$, then $\mathcal{M}(T)$ is **NOT** upper-triangular for all basis of V.*
- *If $F = C$, then the diagonal of upper-triangular $\mathcal{M}(T)$ contains $-1 + i$ and $-1 - i$.*

Proof.

- Note that $p_v(z) = z^2+2z+2$ is a minimal polymonial of Tv, it is minimal since p_v has no zero, therefore cannot have lower degree.

  Then the minimal polymonial p of T is a polymonial multiple of p_v, thus p is **NOT** in form of $(z-\lambda_0)\cdots(z-\lambda_{n-1})$ since p_v has no zero, thus there is no upper-triangular matrix for T for any basis of V.

- $-1+i$ and $-1-i$ are two zeros of p_v, thus are zeros of p, therefore are in the diagonal.

$\square$

Exercise 3.37. *Let B square matrix with complex elements. Show that there is a square matrix A with complex elements such that $A^{-1}BA$ is a upper-triangular matrix.*

Proof. We can find an operator T such that its matrix is B with respect to the standard basis. Then we can find a basis such that $\mathcal{M}(T)$ with respect to such basis is upper-triangular since B is complex. Then
$A = \mathcal{M}(I, \text{standard basis}, \text{upper-trianguler basis})$, and $A^{-1}BA$ is upper-triangular, this is the change-of-basis formula. $\square$

Exercise 3.38. *Let $T \in \mathcal{L}(V)$ and $v_0, \cdots, v_{n-1}$ a basis of V, show that the following statements are equivalent:*

- *the matrix of T with respect to $v_0, \cdots, v_{n-1}$ is lower-triangular.*
- *For any $k = 1, \cdots, n$, $\operatorname{span}(v_{k-1}, \cdots, v_{n-1})$ is invariant under T.*
- *For any $k = 1, \cdots, n$, $Tv_{k-1} \in \operatorname{span}(v_{k-1}, \cdots, v_{n-1})$.*

Proof. The proof is similar to the upper-triangular one.

- (1) $\Rightarrow$ (2) For any $i \leq j$, $Tv_{j-1} \in \operatorname{span}(v_{j-1}, \cdots, v_{n-1}) \subseteq \operatorname{span}(v_{i-1}, \cdots, v_{n-1})$, thus $\operatorname{span}(v_{k-1}, \cdots, v_{n-1})$ is invariant under T.
- (2) $\Rightarrow$ (3) Tirival.
- (3) $\Rightarrow$ (1) Basically the definition.

$\square$

Exercise 3.39. *Let $F = C$ and V finite. Show that $T \in \mathcal{L}(V)$, then $\mathcal{M}(T)$ is lower-triangular with respect to some basis of V.*

Proof. Consider the dual map T', we know there is a basis of V' such that $\mathcal{M}(T')$ is upper-triangular, then $\mathcal{M}(T') = \mathcal{M}(T)^T$ which means $\mathcal{M}(T)^T$ is a upper-triangular, thus $(\mathcal{M}(T)^T)^T = \mathcal{M}(T)$ is lower-triangular. $\square$

Exercise 3.40. *Let V finite and the matrix of $T \in \mathcal{L}(V)$ is upper-triangular with respect to some basis of V, and $U \subseteq V$ is invariant under T. Show that*

- *The matrix of $T\big|_U$ is upper-triangular with respect to some basis of U.*
- *The matrix of T/U is upper-triangular with respect to some basis of V/U.*

Proof.

- Since $\mathcal{M}(T)$ is upper-triangular, then the minimal polymonial of T is in form of $p(z) = (z - \lambda_0) \cdots (z - \lambda_{n-1})$. Then $p(T\big|_U) = 0$, thus p is polymonial multiple of the minimal polymonial q of $T\big|_U$. therefore q is also in form of $(z - \lambda_0) \cdots (z - \lambda_{k-1})$. Thus there is a basis of U such that the matrix of $T\big|_U$ is upper-triangular.
- Let q the minimal polymonial of T/U, and p the minimal polymonial of T, then p is polymonial multiple of q (see Exercise 5.25 in E5B). Then follow the same step as last proof.

$\square$

Exercise 3.41. *Let V finite, $T \in \mathcal{L}(V)$, $U \subseteq V$ invariant under T, $\mathcal{M}(T\big|_U)$ is upper-triangular for some basis of U, $\mathcal{M}(T/U)$ is upper-triangular for some basis of V/U. Show that $\mathcal{M}(T)$ is upper-triangular for some basis of V.*

Proof. We will use the conclusion of Exercise 5.25 in E5B:

$$st = (\text{the minimal polymonial of } T\big|_U) \times (\text{the minimal polymonial of } T/U)$$

is a polymonial multiple of the minimal polymonial p of T. Thus st is in form of $(z-\lambda_0) \cdots (z-\lambda_{n-1})$ since both $\mathcal{M}(T\big|_U)$ and $\mathcal{M}(T/U)$ are upper-triangular for some basis, therefore p is also in form of $(z-\lambda_0) \cdots (z-\lambda_{k-1})$, hence $\mathcal{M}(T)$ is upper-triangular for some basis of V. $\square$