Definition 0.11. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{null} T = \{\ v \mid Tv = 0\ \}$ *is called the* ***null space*** *of* T*.*

This is also called the **kernal** of T in algebra.

Theorem 0.13. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{null} T$ *is a subspace of* V*.*

Proof.

- We have $0 \in \operatorname{null} T$ since $T0 = 0$, which is the property of linear transformation.
- For any $a, b \in \operatorname{null} T$, we have $0 = Ta + Tb = T(a + b)$, so $a + b \in \operatorname{null} T$.
- For any $Ta \in \operatorname{null} T$ and $\lambda \in F$, we have $\lambda Ta = T(\lambda a)$, so $\lambda a \in \operatorname{null} T$.

□

Definition 0.15. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{range} T = T(V) = \{\ Tv \mid v \in V\ \}$ *is called the* ***range*** *of* T*.*

This is also called the **image** of T in math.

Theorem 0.18. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{range} T$ *is a subsapce of* W*.*

Proof.

- We have $T(0) = 0 \in \operatorname{range} T$.
- For any $Ta, Tb \in \operatorname{range} T$, $Ta + Tb = T(a + b) \in \operatorname{range} T$.
- For any $Ta \in \operatorname{range} T$ and $\lambda \in F$, $\lambda Ta = T(\lambda a) \in \operatorname{range} T$.

□

Theorem 0.21. *Suppose* V *is finite and* $T \in \mathcal{L}(V, W)$*, then* $\operatorname{range} T$ *is finite, and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof. Consider the basis $v_0, \cdots, v_k$ of $\operatorname{null} T$, and the basis $v_0, \cdots, v_n$ of V that expand from $v_0, \cdots, v_k$. We will show that $T(v_{k+1}), \cdots, T(v_n)$ is the basis of $\operatorname{range} T$.

We first show that $T(v_{k+1}), \cdots, T(v_n)$ is linear irrelavent. If it is linear irrelavent, then

$$
\begin{aligned}
&\lambda_1 T(v_{k+1}) + \cdots + \lambda_i T(v_{k+i}) \\
=&T(\lambda_1 v_{k+1} + \cdots + \lambda_i T(v_{k+i})) \\
=&0
\end{aligned}
$$

That means a linear combation of v_{k+i} is in $\operatorname{null} T$, which is $\operatorname{span}(v_0, \cdots, v_k)$, therefore the basis $v_0, \cdots, v_n$ is linear relavent.

Then we show that $T(v_{k+1}), \cdots, T(v_n)$ spans $\operatorname{range} T$. For any $Tv \in \operatorname{range} T$, there must be $v \in V$ such that $Tv = Tv$, then v can be written in form of the linear combination of $v_0, \cdots, v_n$, and then $Tv = T(\lambda_0 v_0 + \cdots + \lambda_n v_n)$. We can drop all terms with v_i where $i \leq k$, since they are in $\operatorname{null} T$, so Tv is now represent by a linear combination of $T(v_{k+i})$ for all $0 < i \leq n - k$, therefore, it is a basis of $\operatorname{range} T$ and $\dim \operatorname{range} T$ is finite.

Finally, $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

Exercise 0.1. *Suppose V is a finite vector space, Show that the only two ideal of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$. A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called an ideal, if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for any $T \in \mathcal{L}(V)$ and $E \in \mathcal{E}$.*

Proof. We will use the concept Matrix. Suppose $\lambda_0 v_0 + \cdots + \lambda_n v_v$ the basis of V. We want to construct T_i that $T(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_i v_i$ for all $0 \leq i < n$, which is a matrix with all zero but 1 at i, i.

For any matrix, we can always select a non-zero value at a, b and place it at i, b, this can be done by left multiply a matrix with 1 at i, a (this produce a vector at line i with values from line a), then right multiply a matrix with 1 at i, b (this produce a vector at column b with values from line i).

Also, we can always select a non-zero value at a, b and place it at a, i, this can be done by right multiply a matrix with 1 at b, i, then left multiply a matrix with 1 at a, i.

By combining these two operations, we ca select a non-zero value at a, b and place it at i, i. Now, consider any non-zero $E \in \mathcal{E}$, we can construct a matrix with non-zero value at i, i for every $0 \leq i < \dim V$. These matrix are in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then we can multiply an apropriate scalar to them so that they are matrices with 1 at i, i. By adds up these matrices, we get I, we know $I \in \mathcal{E}$ since $\mathcal{E}$ is a vector space, and now all $T \in \mathcal{L}(V)$ is also in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then $\mathcal{E} = \mathcal{L}(V)$.

The only exception is $\mathcal{E} = \{0\}$, in this case we can't pick any non-zero element. □

Exercise 0.7. *Suppose vector space V and W are finite* $(2 \leq \dim V \leq \dim W)$, *show that* $\{ \ T \in \mathcal{L}(V, W) \mid T \text{ is not injective } \}$ *is not a subspace.*

Proof. Consider the basis $v_0 + \cdots + v_{(\dim V - 1)} \in V$, and $T(v_0 + \cdots + v_{(\dim V-1)}) = (0 + 1_0 + \cdots + 1_{(\dim V-1)})$ and $T'(v_0 + \cdots + v_{(\dim V-1)}) = (v_0 + 0 + \cdots + v_{(\dim V-1)})$. Then $(T + T')(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_{(\dim V-1)} v_{(\dim V-1)}) = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + 2\lambda_{(\dim V-1)} v_{(\dim V-1)}$, which is obviously injective. □

Exercise 0.11. *Suppose V is finite and $T \in \mathcal{L}(V, W)$, show that there is a subspace $U \subset V$ such that:*

$$U \cap \operatorname{null} T = \{0\} \quad \textit{and} \quad \operatorname{range} T = \{ \ Tu \mid u \in U \ \}$$

Proof. This is similar to the *isomorphism theorems* about groups! This can be done by the similar way we used in proving $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

The next two exercises remind me the categorical injective and surjective, let try them first!

Exercise. *For any $F \in \mathcal{L}(V, W)$, F is injective $\iff$ for any $S, T \in \mathcal{L}(U, V)$, $FS = FT$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(V, W)$ that $FS = FT$, then for any $u \in U$, we have $F(Su) = F(Tu)$, since F is injective, we know $Su = Tu$, so $S = T$.
- ($\Leftarrow$) For any $v, w \in V$ such that $Fv = Fw$. Consider

$$S(\lambda) = \lambda v$$
$$T(\lambda) = \lambda w$$

  in $\mathcal{L}(\mathbb{R}, V)$. Then for any $\lambda \in \mathbb{R}$, we have $FS\lambda = \lambda FS1 = \lambda Fv = \lambda Fw = \lambda FT1 = FT\lambda$. so $FS = FT$ then $S = T$, which means $v = S1 = T1 = w$.

□

Exercise. *Suppose W is finite, then for any $F \in \mathcal{L}(V, W)$, F is surjective $\iff$ for any $S, T \in \mathcal{L}(W, U)$, $SF = TF$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(W, U)$ such that $SF = TF$. For any $w \in W$, there is $v \in V$ such that $Fv = w$ since F is surjective. Then we have $SFv = TFv$ so $Sw = S(Fv) = T(Fv) = Tw$ then $S = T$.

- ($\Leftarrow$) Consider

  $$S = I \quad \text{and} \quad T(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 w_0 + \cdots + \lambda_k w_k$$

  where $w_0, \cdots, w_k$ is the basis of range F and $w_0, \cdots, w_n$ is the basis of W that expand from $w_0, \cdots, w_k$.

  It is easy to show that T is a linear transformation. Then for any $v \in V$, we have $TFv = Fv$ (since T acts like identity transformation on range F) and $SFv = Fv$, so $S = T$ by the property of F. Since range $S = W$, so is range T, that means $w_0, \cdots, w_k$ spans W, so $k = n$, which means range $F = W$, therefore F is surjective.

$\square$

Exercise 0.19. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is injective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $ST = I$.*

Proof.

- ($\Rightarrow$) Consider the basis $v_0, \cdots, v_n$ of V, then $Tv_0, \cdots, Tv_n$ is a basis of range T since T is injective. We denote Tv_i as w_i and $w_0, \cdots, w_m$ as the basis of W which expand from $w_0, \cdots, w_n$. Define $S(\lambda_0 w_0 + \cdots + \lambda_m w_m) = \lambda_0 w_0 + \cdots + \lambda_n w_n$, and then for any $v \in V$, $ST(\lambda_0 v_0 + \cdots + \lambda_n v_n) = S(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 v_0 + \cdots + \lambda_n v_n$, so $ST = I$.

- ($\Leftarrow$) Suppose $A, B \in \mathcal{L}(U, V)$, such that $TA = TB$, we will show that $A = B$. $STA = IA = A$ and $STB = IB = B$ and $STA = STB$ since $TA = TB$. Then we know T is a monomorphism, and then T is injective.

$\square$

Exercise 0.20. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is surjective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $TS = I$.*

Proof. $\square$

Exercise 0.21. *Suppose V is finite, $T \in \mathcal{L}(V, W)$, $U \subseteq W$ a subspace. Show that the inverse image of U: $\{ v \in V \mid Tv \in U \}$ is a subspace of V, and*

$$\dim\{ v \in V \mid Tv \in U \} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$$

Proof. The second part is quite easy, we can restrict the domain of T to $\{ v \in V \mid Tv \in U \}$, say $T' \in \mathcal{L}(\{ v \in V \mid Tv \in U \}, W)$, so that it is in form $\dim\{ v \in V \mid Tv \in U \} = \dim \operatorname{null} T' + \dim \operatorname{range} T'$. Obviously $\operatorname{range} T' = U \cap \operatorname{range} T$ and $\operatorname{null} T' = \operatorname{null} T$.

We will now show that $\{ v \in V \mid Tv \in U \}$ is a subspace of V.

- $T0 \in U$.
- For any $v, w \in V$ such that $Tv, Tw \in U$, we have $T(v+w) = Tv+Tw \in U$.
- For any $v \in V$ such that $Tv \in U$ and $\lambda \in F$, we have $T(\lambda v) = \lambda Tv \in U$.

Therefore it is a subspace. □

Exercise 0.22. *Suppose U and V are finite, $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$, show that*

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$$

Proof. Consider the inverse image of $\operatorname{null} S$ on T: $K = \{ v \in V \mid Tv \in \operatorname{null} S \}$, which dimension: $\dim K = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T)$, where $\dim(\operatorname{null} S \cap \operatorname{range} T)$ caps at $\dim \operatorname{null} S$.

We know show that $\operatorname{null} ST = \operatorname{null} K$. For any $STv = 0$, we know $S(Tv) = 0$, so $Tv \in K$, therefore $\operatorname{null} ST \subseteq \operatorname{null} K$; For any $Tv \in \operatorname{null} S$, that means $S(Tv) = 0$, therefore $v \in \operatorname{null} ST$, therefore $\operatorname{null} ST \supseteq \operatorname{null} K$, and $\operatorname{null} ST = \operatorname{null} K$. □

Exercise 0.25. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that* $\operatorname{null} S \subseteq \operatorname{null} T \iff$ *there is $E \in \mathcal{L}(W)$ such that $T = ES$.*

Proof. We define $E(S(v)) = Tv$ for any $v \in V$, so that $E \in \mathcal{L}(\operatorname{range} S, W)$. We first show that E is a mapping, and also a linear transformation.

Suppose $Sv, Sw \in W$ such that $Sv = Sw$, we need to show that $E(Sv) = E(Sw)$, or normalized $Tv = Tw$. We know $v - w \in \operatorname{null} S$ since $Sv = Sw$, so $v - w \in \operatorname{null} T$ since $\operatorname{null} S \subseteq \operatorname{null} T$, therefore $T(v - w) = 0$, and then $Tv = Tw$, so E is a mapping.

Now we show that E is a linear transformation.

- For any $Sv, Sw \in \operatorname{range} S$, $E(Sv) + E(Sw) = Tv + Tw = T(v + w) = E(S(v + w)) = E(Sv + Sw)$.

- For any $Sv \in \operatorname{range} S$ and $\lambda \in F$, $\lambda E(Sv) = \lambda Tv = T(\lambda v) = E(S(\lambda v) = E(\lambda Sv))$.

therefore E is a linear transformation.

Now we can expand the domain of E to W such that $E'v = Ev$ for any $v \in \operatorname{range} S$ (this is proven in previous exercise). For any $v \in V$, we have $ESv = E(Sv) = Tv$, there fore $T = ES$.

For another direction, for any $v \in \operatorname{null} S$, we have $ESv = E0 = 0 = Tv$, so $v \in \operatorname{null} T$. □

Exercise 0.26. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that* $\operatorname{range} S \subseteq \operatorname{range} T \iff$ *there is $E \in \mathcal{L}(V)$ such that $S = TE$.*

Proof. Consider the inverse image of $\operatorname{range} S$ with basis $w_0, \cdots, w_n$, say $v_0, \cdots, v_n$, it is easy to show $v_0, \cdots, v_n$ is linear irrelavent. Then $E(v) = \lambda_0 v_0 + \cdots + \lambda_n v_n$ where $Sv = \lambda_0 w_0 + \cdots + \lambda_n w_n$. □

Exercise 0.27. *Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$, show that $V = \operatorname{null} P \oplus \operatorname{range} P$.*

Proof. Such element is called *idempotent* in algebra.

We will show $\operatorname{null} P \oplus \operatorname{range} P$ by showing $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. For any $v \in \operatorname{null} P \cap \operatorname{range} P$, we know there is $w \in V$ such that $Pw = v$ since $v \in rangevP$, then $P^2(v) = P(Pv) = P0 = 0$ since $v \in \operatorname{null} P$ and $P^2(v) = P(Pv) = Pw$, so $Pw = 0$ while $Pw = v$ therefore $v = 0$.

Then we have $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$ and $\dim(\operatorname{null} P \oplus \operatorname{range} P) = \dim \operatorname{null} P + \dim \operatorname{range} P - \dim\{0\}$, so $V = \operatorname{null} P \oplus \operatorname{range} P$. □

Exercise 0.28. *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ such that for any non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$, $\deg(Dp) = \deg p - 1$. Show that D is surjective.*

Proof. We induction on n, starts from 1, to show that $D(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_{n-1}(\mathbb{R})$.

- Base: for any $p \in \mathcal{P}(\mathbb{R})$ where $\deg p = 1$, we know $\deg Dp = 0$, so $D(\mathcal{P}_1(\mathbb{R}))$ is a non-zero subspace of $\mathcal{P}_0(\mathbb{R})$, which is $\mathcal{P}_0(\mathbb{R})$.

- Induction: We have induction hypothesis: For any $i \leq n$, we have $D(\mathcal{P}_i(\mathbb{R})) = \mathcal{P}_{i-1}(\mathbb{R})$. We want to show that $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$. For any $p \in \mathcal{P}(\mathbb{R})$ with $\deg p = n+1$, we can write p in form of $p = \lambda x^{n+1} + r$ where $\deg r \leq n$, then $Dp = D(\lambda x^{n+1} + r) = D(\lambda x^{n+1}) + Dr$ where $\deg D(\lambda x^{n+1}) = n$ and $\deg Dr \leq n - 1$. So $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$ since: $\mathcal{P}_n(\mathbb{R}) \subseteq D(\mathcal{P}_{n+1}(\mathbb{R}))$ and $D(\lambda x^{n+1}) \in D(\mathcal{P}_{n+1}(\mathbb{R}))$, it is sufficient to span $\mathcal{P}_n(\mathbb{R})$.

$\square$

Exercise 0.29. *For any $p \in \mathcal{P}(\mathbb{R})$, show that there is $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.*

Proof. We can rewrite the goal as $5DDq + 3Dq = p$ where $D(p) = p'$, then $5DDq + 3Dq = D(5Dq) + D(3q) = D(5Dq + 3q) = p$. We know D is surjective by the previous exercise, the goal is now showing that $5Dq + 3q = r$ where $Dr = p$. Then we continue rewrite the goal $5Dq + 3q = (5D)q + (3I)q = (5D + 3I)q = r$, we will show that $5D + 3I$ is surjective, we use the same method in previous exercise.

We denote $5D+3I$ by F, and induction on $n \in \mathbb{N}$ to show that $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$.

- Base: We should show that $F(\mathcal{P}_0(\mathbb{R})) = \mathcal{P}_0(\mathbb{R})$, for any $p \in \mathcal{P}_0(\mathbb{R})$, we have $Fp = 5Dp + 3p$, where $Dp = 0$ since $\deg p = 0$, so $Fp = 3p$, which means we have $1 \in F(\mathcal{P}_0(\mathbb{R}))$ since p is literally a number and $\frac{1}{3p} Fp = 1$.

- Induction: We have induction hypothesis: $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$, and we want to show $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

  For any $p \in \mathcal{P}_{n+1}(\mathbb{R})$, we have $Fp = 5Dp + 3p$ where $\deg 5Dp = n$ and $\deg 3p = n + 1$, then we can eliminate $5Dp$ and every term in p with degree less then $n + 1$ since $\mathcal{P}_n(\mathbb{R}) \subseteq \operatorname{range} F$, then we get z^{n+1}, thus $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

Therefore there is q such that $(5D + 3I)q = r$ since $5D + 3I$ is surjective.

Another solution from internet: Define $Tq = 5q'' + 3q'$, we can see for any $q \in \mathcal{P}(\mathbb{R})$ we have $\deg Tq = \deg q - 1$, so T is surjective. Then there is q such that $Tq = 5q'' + 3q' = p$. $\square$

Exercise 0.30. *Suppose $\varphi \in \mathcal{L}(V, F)$ not zero, and $u \in V$ that $u \notin \operatorname{null} \varphi$, show that $V = \operatorname{null} \varphi \oplus \{ au \mid a \in F \}$.*

Proof. We can see φ is surjective since $\varphi u \neq 0$, then for any $i \in F$, we have $(i(\varphi u)^{-1})\varphi u = i$.

For any $v \in V$, since φ is surjective (in a particular way), so we have $a\varphi u$ such that $a\varphi u = \varphi v$, then $\varphi(au - v) = 0$ so $au - v \in \operatorname{null} \varphi$. That means $(-1)(au - v) + au = v$ where $(-1)(au - v) \in \operatorname{null} \varphi$ and $au \in \{ au \mid a \in F \}$, so $V = \operatorname{null} \varphi + \{ au \mid a \in F \}$.

Then $\operatorname{null} \varphi \oplus \{ au \mid a \in F \}$ since $u \notin \operatorname{null} \varphi$. $\square$

Exercise 0.31. *Suppose V is finite ($\dim V > 1$), show that if $\varphi : \mathcal{L}(V) \to F$ is a linear mapping with property $\varphi(ST) = \varphi(S)\varphi(T)$ for any $S, T \in \mathcal{L}(V)$, show that $\varphi = 0$.*

Proof. Consider $\operatorname{null}\varphi$, since $\dim V > 1$ while $\dim F = 1$, so φ cannot be injective, therefore $\operatorname{null}\varphi \neq \{0\}$.

For any non-zero $S \in \operatorname{null}\varphi$ and $T \in \mathcal{L}(V)$, we have $\varphi(ST) = \varphi(S)\varphi(T) = 0 = \varphi(T)\varphi(S) = \varphi(TS)$ since $S \in \operatorname{null}\varphi$, thus $ST \in \operatorname{null}\varphi$. We show that $\operatorname{null}\varphi$ is an ideal of $\mathcal{L}(V)$, recall that the property of $\mathcal{L}(V)$, the only ideal of $\mathcal{L}(V)$ is $\{0\}$ and $LT(V)$, so $\operatorname{null}\varphi = \mathcal{L}(V)$, which means $\varphi = 0$. □

Exercise 0.32. *Let V, W are vector spaces and $T \in \mathcal{L}(V, W)$, define $T_C : V_C \to W_C$:*

$$T_C(u + iv) = Tu + iTv$$

for any $u, v \in V$.

1. *Show that T_C is a (complex) linear mapping from V_C to W_C.*
2. *Show that T_C is injective $\iff$ T is injective.*
3. *Show that* $\operatorname{range} T_C = W_C \iff \operatorname{range} T = W$.

Proof.

1. For any $u, v, s, t \in V$ $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
&T((u + iv) + (s + it)) \\
=&T(u + s + i(v + t)) \\
=&T(u + s) + iT(v + t) \\
=&Tu + Ts + iTv + iTt \\
=&T(u + iv) + T(s + it)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda T(u + iv) \\
=&\lambda(Tu + iTv) \\
=&\lambda Tu + \lambda iTv \\
=&T(\lambda u) + iT(\lambda v) \\
=&T(\lambda u + i(\lambda v)) \\
=&T(\lambda u + i\lambda v) \\
=&T(\lambda(u + iv))
\end{aligned}$$

I believe these are trivial, so the future me should be able to prove these without any effort. $\square$

Exercise 0.4. *Suppose* $D(p) = p' : \mathcal{L}(\mathcal{P}_3(\mathbb{R})) \to \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$. *Find a basis of* $\mathcal{L}(\mathcal{P}_3(\mathbb{R}))$ *and a basis of* $\mathcal{L}(\mathcal{P}_2(\mathbb{R}))$, *such that* $\mathcal{M}$

Proof. $\square$