This chapter is much like a note, I will record some idea about polynomial and linear algebra.

One relationship is that a polynomial is a linear combination of the standard basis of $\mathcal{P}(F)$, that is, $1, x, x^2, \cdots$. This is important when we apply p to an operator of a vector space, say $T \in \mathcal{L}(V)$ and $p(T) = c_0 I + c_1 T + c_2 T^2 + \cdots$. If we apply $p(T)$ to some $v \in V$, it becomes a linear combination of $v, Tv, T^2 v, \cdots$.

Theorem 4.16. *Let $p \in \mathcal{P}(\mathbb{R})$ is not constant, then p can be factorized into:*

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M)$$

where $c, \lambda_1, \cdots, \lambda_m, b_1, \cdots, b_M, c_1, \cdots, c_M \in \mathbb{R}$ and for any $1 \leq k \leq M$, $b_k^2 < 4c_k$.

I won't paste the proof here, but the statement can be considered as: any non-constant, real p can be factorized into:

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x - \lambda_{m+1}) \cdots (x - \lambda_{m+M})$$

where $c, \lambda_1, \cdots, \lambda_1 \in \mathbb{R}$ and $\lambda_{m+1}, \cdots, \lambda_{m+M} \in \mathbb{C}$. Those λ are zeros of p, however some are real, some are complex. This re-expression makes the statement more understandable. Note that λ_{m+k} is paired, since both λ_{m+k} and $\bar{\lambda}_{m+k}$ are zeros of p.