Exercise 3.7. *Suppose vector space V and W are finite $(2 \le \dim V \le \dim W)$, show that $\{\ T \in \mathcal{L}(V, W) \mid T \text{ is not injective } \}$ is not a subspace.*

Proof. Consider the basis $v_0 + \cdots + v_{(\dim V - 1)} \in V$, and $T(v_0 + \cdots + v_{(\dim V-1)}) = (0 + 1_0 + \cdots + 1_{(\dim V-1)})$ and $T'(v_0 + \cdots + v_{(\dim V-1)}) = (v_0 + 0 + \cdots + v_{(\dim V-1)})$. Then $(T + T')(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_{(\dim V-1)} v_{(\dim V-1)}) = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + 2\lambda_{(\dim V-1)} v_{(\dim V-1)}$, which is obviously injective. □

Exercise 3.11. *Suppose V is finite and $T \in \mathcal{L}(V, W)$, show that there is a subspace $U \subset V$ such that:*

$$U \cap \operatorname{null} T = \{0\} \quad \textit{and} \quad \operatorname{range} T = \{\ Tu \mid u \in U\ \}$$

Proof. This is similar to the *isomorphism theorems* about groups! This can be done by the similar way we used in proving $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

The next two exercises remind me the categorical injective and surjective, let try them first!

Exercise. *For any $F \in \mathcal{L}(V, W)$, F is injective $\iff$ for any $S, T \in \mathcal{L}(U, V)$, $FS = FT$ implies $S = T$.*

Proof.

- $(\Rightarrow)$ For any $S, T \in \mathcal{L}(V, W)$ that $FS = FT$, then for any $u \in U$, we have $F(Su) = F(Tu)$, since F is injective, we know $Su = Tu$, so $S = T$.
- $(\Leftarrow)$ For any $v, w \in V$ such that $Fv = Fw$. Consider

$$S(\lambda) = \lambda v$$
$$T(\lambda) = \lambda w$$

in $\mathcal{L}(\mathbb{R}, V)$. Then for any $\lambda \in \mathbb{R}$, we have $FS\lambda = \lambda FS1 = \lambda Fv = \lambda Fw = \lambda FT1 = FT\lambda$. so $FS = FT$ then $S = T$, which means $v = S1 = T1 = w$.

□

Exercise. *Suppose W is finite, then for any $F \in \mathcal{L}(V, W)$, F is surjective $\iff$ for any $S, T \in \mathcal{L}(W, U)$, $SF = TF$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(W, U)$ such that $SF = TF$. For any $w \in W$, there is $v \in V$ such that $Fv = w$ since F is surjective. Then we have $SFv = TFv$ so $Sw = S(Fv) = T(Fv) = Tw$ then $S = T$.

- ($\Leftarrow$) Consider

  $$S = I \quad \text{and} \quad T(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 w_0 + \cdots + \lambda_k w_k$$

  where $w_0, \cdots, w_k$ is the basis of $\operatorname{range} F$ and $w_0, \cdots, w_n$ is the basis of W that expand from $w_0, \cdots, w_k$.

  It is easy to show that T is a linear transformation. Then for any $v \in V$, we have $TFv = Fv$ (since T acts like identity transformation on $\operatorname{range} F$) and $SFv = Fv$, so $S = T$ by the property of F. Since $\operatorname{range} S = W$, so is $\operatorname{range} T$, that means $w_0, \cdots, w_k$ spans W, so $k = n$, which means $\operatorname{range} F = W$, therefore F is surjective.

$\square$

Exercise 3.19. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is injective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $ST = I$.*

Proof.

- ($\Rightarrow$) Consider the basis $v_0, \cdots, v_n$ of V, then $Tv_0, \cdots, Tv_n$ is a basis of $\operatorname{range} T$ since T is injective. We denote Tv_i as w_i and $w_0, \cdots, w_m$ as the basis of W which expand from $w_0, \cdots, w_n$. Define $S(\lambda_0 w_0 + \cdots + \lambda_m w_m) = \lambda_0 w_0 + \cdots + \lambda_n w_n$, and then for any $v \in V$, $ST(\lambda_0 v_0 + \cdots + \lambda_n v_n) = S(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 v_0 + \cdots + \lambda_n v_n$, so $ST = I$.

- ($\Leftarrow$) Suppose $A, B \in \mathcal{L}(U, V)$, such that $TA = TB$, we will show that $A = B$. $STA = IA = A$ and $STB = IB = B$ and $STA = STB$ since $TA = TB$. Then we know T is a monomorphism, and then T is injective.

$\square$

Exercise 3.20. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is surjective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $TS = I$.*

Proof. $\square$