

# 1 Functors

**Definition 1.1** (Full). *A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called full, if for any  $a, b \in \mathcal{C}$ , the mapping on morphism  $F : \mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$  is surjective.*

**Definition 1.2** (Faithful). *A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called faithful, if for any  $a, b \in \mathcal{C}$ , the mapping on morphism  $F : \mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$  is injective.*

**Definition 1.3** (Essentially Full). *A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called Essentially full, if for any  $a \in \mathcal{C}$ , the mapping on object  $F : \mathcal{C} \rightarrow \mathcal{D}$  is surjective.*

**Theorem 1.1.** *Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  a functor, and  $f : a \rightarrow b$  a morphism in  $\mathcal{C}$ . Then  $f$  is an isomorphism iff  $Ff$  is an isomorphism.*

*Proof.* ( $\Rightarrow$ ) We claim  $F(f^{-1}) : Fb \rightarrow Fa$  is an inverse, we can see that  $F(f^{-1} \circ f) = F(id_a) = id_{Fa}$  and  $F(f \circ f^{-1}) = F(id_b) = id_{Fb}$ .

( $\Leftarrow$ ) Suppose  $Fg$  is the inverse of  $Ff$ , and we can retrieve  $g$  from  $Fg$  cause  $F$  is full faithful. Then  $F(g \circ f) = Fg \circ Ff = id_{Fa} = F(id_a)$  therefore  $g \circ f = id_a$  since  $F$  is full faithful, similar to  $F(f \circ g)$ , so  $f$  is indeed an isomorphism.  $\square$

**Corollary 1.1.** *Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  is full and faithful, show that  $F$  is injective on object.*

*Proof.* Trivial by previous theorem.  $\square$