Definition 0.11. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{null} T = \{ v \mid Tv = 0 \}$ *is called the* ***null space*** *of* T*.*

This is also called the **kernal** of T in algebra.

Theorem 0.13. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{null} T$ *is a subspace of* V*.*

Proof.

- We have $0 \in \operatorname{null} T$ since $T0 = 0$, which is the property of linear transformation.
- For any $a, b \in \operatorname{null} T$, we have $0 = Ta + Tb = T(a + b)$, so $a + b \in \operatorname{null} T$.
- For any $Ta \in \operatorname{null} T$ and $\lambda \in F$, we have $\lambda Ta = T(\lambda a)$, so $\lambda a \in \operatorname{null} T$.

□

Definition 0.15. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{range} T = T(V) = \{ Tv \mid v \in V \}$ *is called the* ***range*** *of* T*.*

This is also called the **image** of T in math.

Theorem 0.18. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{range} T$ *is a subsapce of* W*.*

Proof.

- We have $T(0) = 0 \in \operatorname{range} T$.
- For any $Ta, Tb \in \operatorname{range} T$, $Ta + Tb = T(a + b) \in \operatorname{range} T$.
- For any $Ta \in \operatorname{range} T$ and $\lambda \in F$, $\lambda Ta = T(\lambda a) \in \operatorname{range} T$.

□

Theorem 0.21. *Suppose* V *is finite and* $T \in \mathcal{L}(V, W)$*, then* $\operatorname{range} T$ *is finite, and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof. Consider the basis $v_0, \cdots, v_k$ of $\operatorname{null} T$, and the basis $v_0, \cdots, v_n$ of V that expand from $v_0, \cdots, v_k$. We will show that $T(v_{k+1}), \cdots, T(v_n)$ is the basis of $\operatorname{range} T$.

We first show that $T(v_{k+1}), \cdots, T(v_n)$ is linear irrelavent. If it is linear irrelavent, then

$$
\begin{aligned}
&\lambda_1 T(v_{k+1}) + \cdots + \lambda_i T(v_{k+i}) \\
=&T(\lambda_1 v_{k+1} + \cdots + \lambda_i T(v_{k+i})) \\
=&0
\end{aligned}
$$

That means a linear combation of v_{k+i} is in $\operatorname{null} T$, which is $\operatorname{span}(v_0, \cdots, v_k)$, therefore the basis $v_0, \cdots, v_n$ is linear relavent.

Then we show that $T(v_{k+1}), \cdots, T(v_n)$ spans $\operatorname{range} T$. For any $Tv \in \operatorname{range} T$, there must be $v \in V$ such that $Tv = Tv$, then v can be written in form of the linear combination of $v_0, \cdots, v_n$, and then $Tv = T(\lambda_0 v_0 + \cdots + \lambda_n v_n)$. We can drop all terms with v_i where $i \leq k$, since they are in $\operatorname{null} T$, so Tv is now represent by a linear combination of $T(v_{k+i})$ for all $0 < i \leq n - k$, therefore, it is a basis of $\operatorname{range} T$ and $\dim \operatorname{range} T$ is finite.

Finally, $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

Exercise 0.1. *Suppose V is a finite vector space, Show that the only two ideal of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$. A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called an ideal, if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for any $T \in \mathcal{L}(V)$ and $E \in \mathcal{E}$.*

Proof. We will use the concept Matrix. Suppose $\lambda_0 v_0 + \cdots + \lambda_n v_v$ the basis of V. We want to construct T_i that $T(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_i v_i$ for all $0 \leq i < n$, which is a matrix with all zero but 1 at i, i.

For any matrix, we can always select a non-zero value at a, b and place it at i, b, this can be done by left multiply a matrix with 1 at i, a (this produce a vector at line i with values from line a), then right multiply a matrix with 1 at i, b (this produce a vector at column b with values from line i).

Also, we can always select a non-zero value at a, b and place it at a, i, this can be done by right multiply a matrix with 1 at b, i, then left multiply a matrix with 1 at a, i.

By combining these two operations, we ca select a non-zero value at a, b and place it at i, i. Now, consider any non-zero $E \in \mathcal{E}$, we can construct a matrix with non-zero value at i, i for every $0 \leq i < \dim V$. These matrix are in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then we can multiply an apropriate scalar to them so that they are matrices with 1 at i, i. By adds up these matrices, we get I, we know $I \in \mathcal{E}$ since $\mathcal{E}$ is a vector space, and now all $T \in \mathcal{L}(V)$ is also in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then $\mathcal{E} = \mathcal{L}(V)$.

The only exception is $\mathcal{E} = \{0\}$, in this case we can't pick any non-zero element. □

Exercise 0.7. *Suppose vector space V and W are finite* ($2 \le \dim V \le \dim W$), *show that* $\{ \ T \in \mathcal{L}(V, W) \mid T$ *is not injective* $\}$ *is not a subspace.*

Proof. Consider the basis $v_0 + \cdots + v_{(\dim V - 1)} \in V$, and $T(v_0 + \cdots + v_{(\dim V-1)}) = (0 + 1_0 + \cdots + 1_{(\dim V-1)})$ and $T'(v_0 + \cdots + v_{(\dim V-1)}) = (v_0 + 0 + \cdots + v_{(\dim V-1)})$. Then $(T + T')(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_{(\dim V-1)} v_{(\dim V-1)}) = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + 2\lambda_{(\dim V-1)} v_{(\dim V-1)}$, which is obviously injective. □

Exercise 0.11. *Suppose V is finite and $T \in \mathcal{L}(V, W)$, show that there is a subspace $U \subset V$ such that:*

$$U \cap \operatorname{null} T = \{0\} \quad \textit{and} \quad \operatorname{range} T = \{ \ Tu \mid u \in U \ \}$$

Proof. This is similar to the *isomorphism theorems* about groups! This can be done by the similar way we used in proving $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

The next two exercises remind me the categorical injective and surjective, let try them first!

Exercise. *For any $F \in \mathcal{L}(V, W)$, F is injective $\iff$ for any $S, T \in \mathcal{L}(U, V)$, $FS = FT$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(V, W)$ that $FS = FT$, then for any $u \in U$, we have $F(Su) = F(Tu)$, since F is injective, we know $Su = Tu$, so $S = T$.
- ($\Leftarrow$) For any $v, w \in V$ such that $Fv = Fw$. Consider

$$S(\lambda) = \lambda v$$
$$T(\lambda) = \lambda w$$

  in $\mathcal{L}(\mathbb{R}, V)$. Then for any $\lambda \in \mathbb{R}$, we have $FS\lambda = \lambda FS1 = \lambda Fv = \lambda Fw = \lambda FT1 = FT\lambda$. so $FS = FT$ then $S = T$, which means $v = S1 = T1 = w$.

□

Exercise. *Suppose W is finite, then for any $F \in \mathcal{L}(V, W)$, F is surjective $\iff$ for any $S, T \in \mathcal{L}(W, U)$, $SF = TF$ implies $S = T$.*

Proof.

- (⇒) For any $S, T \in \mathcal{L}(W, U)$ such that $SF = TF$. For any $w \in W$, there is $v \in V$ such that $Fv = w$ since F is surjective. Then we have $SFv = TFv$ so $Sw = S(Fv) = T(Fv) = Tw$ then $S = T$.

- (⇐) Consider

  $$S = I \quad \text{and} \quad T(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 w_0 + \cdots + \lambda_k w_k$$

  where $w_0, \cdots, w_k$ is the basis of range F and $w_0, \cdots, w_n$ is the basis of W that expand from $w_0, \cdots, w_k$.

  It is easy to show that T is a linear transformation. Then for any $v \in V$, we have $TFv = Fv$ (since T acts like identity transformation on range F) and $SFv = Fv$, so $S = T$ by the property of F. Since range $S = W$, so is range T, that means $w_0, \cdots, w_k$ spans W, so $k = n$, which means range $F = W$, therefore F is surjective.

□

Exercise 0.19. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is injective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $ST = I$*

Proof.

- (⇒) Consider the basis $v_0, \cdots, v_n$ of V, then $Tv_0, \cdots, Tv_n$ is a basis of range T since T is injective. We denote Tv_i as w_i and $w_0, \cdots, w_m$ as the basis of W which expand from $w_0, \cdots, w_n$. Define $S(\lambda_0 w_0 + \cdots + \lambda_m w_m) = \lambda_0 w_0 + \cdots + \lambda_n w_n$, and then for any $v \in V$, $ST(\lambda_0 v_0 + \cdots + \lambda_n v_n) = S(\lambda)$

□