Exercise 3.1. *Explain why a linear functional is either surjective or* 0.

Proof. Cause $\dim F = 1$. $\square$

Exercise 3.6. *Let* $\varphi, \beta \in V'$*, show that* $\operatorname{null}\varphi \subseteq \operatorname{null}\beta \iff \exists c \in F, \beta = c\varphi$.

Proof.

- ($\Rightarrow$) For any $v \notin \operatorname{null}\beta$, we have $\beta(v) = \beta(v)(\varphi(v))^{-1}\varphi(v)$ we claim that $\beta = \beta(v)(\varphi(v))^{-1}\varphi$. We may denote $\beta(v)(\varphi(v))^{-1}$ by c. For any $v, w \notin \operatorname{null}\beta$, we have $\beta(v) = a\varphi(v)$ and $\beta(w) = b\varphi(w)$, we want to show that $a = b$, which can be proven by:

$$\begin{aligned} a &= b \\ \frac{\beta(v)}{\varphi(v)} &= \frac{\beta(w)}{\varphi(w)} \\ \beta(v)\varphi(w) &= \beta(w)\varphi(v) \\ \beta(\varphi(w)v) &= \beta(\varphi(v)w) \end{aligned}$$

  which is equivalent to $\varphi(w)v - \varphi(v)w \in \operatorname{null}\beta$, then:

$$\begin{aligned} &\varphi(\varphi(w)v - \varphi(v)w) \\ =&\varphi(\varphi(w)v) - \varphi(\varphi(v)w) \\ =&\varphi(w)\varphi(v) - \varphi(v)\varphi(w) \\ =&0 \end{aligned}$$

  therefore $\varphi(w)v - \varphi(v)w \in \operatorname{null}\varphi \subseteq \operatorname{null}\beta$, thus $a = b$.

  The case $v \in \operatorname{null}\beta$ is trivial.

- ($\Leftarrow$) For any $v \in \operatorname{null}\varphi$, $\beta(v) = c\varphi(v) = 0$, therefore $v \in \operatorname{null}\beta$, thus $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$.

$\square$

Exercise 3.7. *Let* $V_0, \cdots, V_{m-1}$ *are vector spaces, show that* $V_0' \times \cdots \times V_{m-1}'$ *and* $(V_0 \times \cdots \times V_{m-1})'$ *are isomorphic.*

Proof. Define $\psi(\varphi) = v_0 \mapsto \varphi(v_0, 0, \cdots), \cdots, v_{m-1} \mapsto \varphi(\cdots, 0, v_{m-1})$ and $\psi^{-1}(\varphi_0, \cdots, \varphi_{m-1}) = (v_0, \cdots, v_{m-1}) \mapsto \varphi_0(v_0) + \cdots + \varphi_{m-1}(v_{m-1})$.

For any $\alpha, \beta \in (V_0 \times \cdots \times V_{m-1})'$ and $\lambda \in F$, we have

$$\begin{aligned}
&\psi(\alpha+\beta)_i \\
=&v_i \mapsto (\alpha+\beta)(\cdots, v_i, \cdots) \\
=&v_i \mapsto \alpha(\cdots, v_i, \cdots) + \beta(\cdots, v_i, \cdots) \\
=&(v_i \mapsto \alpha(\cdots, v_i, \cdots)) + (v_i \mapsto \beta(\cdots, v_i, \cdots)) \\
=&\psi(\alpha)_i + \psi(\beta)_i
\end{aligned}$$

and $(\lambda\psi(\alpha))_i = \lambda(v_i \mapsto \alpha(v_i)) = v_i \mapsto \lambda\alpha(v_i) = \psi(\lambda\alpha)_i$ Therefore ψ is a linear map.

For any $\alpha, \beta \in V_0' \times \cdots \times V_{m-1}'$ and $\lambda \in F$, we have:

$$\begin{aligned}
&\psi^{-1}(\alpha+\beta) \\
=&(v_0, \cdots, v_{m-1}) \mapsto (\alpha+\beta)(v_0) + \cdots + (\alpha+\beta)(v_{m-1}) \\
=&(v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \beta(v_0) + \cdots \\
=&((v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \cdots) + ((v_0, \cdots, v_{m-1}) \mapsto \beta(v_0) + \cdots) \\
=&\psi^{-1}(\alpha) + \psi^{-1}(\beta)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda\psi^{-1}(\alpha) \\
=&\lambda((v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \cdots) \\
=&(v_0, \cdots, v_{m-1}) \mapsto \lambda(\alpha(v_0) + \cdots) \\
=&(v_0, \cdots, v_{m-1}) \mapsto (\lambda\alpha(v_0)) + \cdots \\
=&\psi^{-1}(\lambda\alpha)
\end{aligned}$$

thus ψ^{-1} is a linear map.

We will show that ψ^{-1} is the inverse of ψ then ψ is an isomorphism. For any $\varphi \in (V_0 \times \cdots \times V_{m-1})'$,

$$\begin{aligned}
&\psi^{-1}(\psi(\varphi)) \\
=&v_0, \cdots, v_{m-1} \mapsto \psi(\varphi)_0(v_0) + \cdots \\
=&v_0, \cdots, v_{m-1} \mapsto \varphi(v_0, 0, \cdots) + \cdots + \varphi(\cdots, 0, v_{m-1}) \\
=&v_0, \cdots, v_{m-1} \mapsto \varphi(v_0, \cdots, v_{m-1}) \\
=&\varphi
\end{aligned}$$

and for any $\varphi \in V_0' \times \cdots \times V_{m-1}'$,

$$\begin{aligned}
&\psi(\psi^{-1}(\varphi)) \\
=&v_0 \mapsto \psi^{-1}(\varphi)(v_0, 0, \cdots), \cdots \\
=&v_0 \mapsto \varphi_0(v_0), \cdots \\
=&\varphi_0, \cdots, \varphi_{m-1} \\
=&\varphi
\end{aligned}$$

□

Exercise 3.16. *Let W a finite vector space, $T \in \mathcal{L}(V, W)$, show that*

$$T' = 0 \iff T = 0$$

Proof.

- ($\Rightarrow$) Suppose $T \neq 0$, then we can always find $\varphi \in \mathcal{L}(W, F)$ which $\varphi(\operatorname{range} T) \neq 0$, then $\varphi \circ T \neq 0$.
- ($\Leftarrow$) Trivial.

□

Exercise 3.17. *Let V, W are finite vector spaces, $T \in \mathcal{L}(V, W)$. Show that T is invertible $\iff$ T' is invertible.*

Proof. Since T is invertible, then T is injective, therefore T' is surjective. Similarly, T' is injective since T is surjective. Therefore T' is invertible. □

Exercise 3.18. *Let V, W are finite vector spaces, show that the mapping $\varphi(T) = T'$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathcal{L}(W', V')$.*

Proof. Since V and W are finite, we only need to show that φ is injective or surjective. We will show that φ is injective.

For any $\varphi(T) = T' \in \mathcal{L}(W', V')$, we know $T = 0 \iff T' = 0$, therefore $\operatorname{null} \varphi = \{0\}$, thus φ is injective.

I was wonder if I can prove this by $\varphi(S)(\mathrm{id}) = \varphi(T)(\mathrm{id}) \implies S = T$. This may work if the codomain restriction doesn't lose information, i.e. only restrict to a superset of its range, therefore the restriction is one-to-one. □

Exercise 3.21. *Let V finite and $U, W \subseteq V$ are subspaces.*

1. *Show that* $W^0 \subseteq U^0 \iff U \subseteq W$

2. *Show that* $W^0 = U^0 \iff U = W$

Proof. The second statement can be easy proved by the first one.

- ($\Rightarrow$) We can always find a $f \in \mathcal{L}(W, F)$ such that $\operatorname{null} f = W$, then $f(U) = \{0\}$ since $f \in W^0 \subseteq U^0$, therefore $U \subseteq \operatorname{null} f = W$.

- ($\Leftarrow$) For any $\varphi \in W^0$, we know $W \subseteq \operatorname{null} \varphi$, then $U \subseteq W \subseteq \operatorname{null} \varphi$, therefore $\varphi in U^0$, thus $W^0 \subseteq U^0$.

□

Exercise 3.22. *Let V finite and $U, W \subseteq V$ are subspaces. Show that:*

- $(U + W)^0 = U^0 \cap W^0$

- $(U \cap W)^0 = U^0 + W^0$

Proof.

- For any $\varphi \in (U + W)^0$ we have $U + W \subseteq \operatorname{null} \varphi$, then $U \subseteq U + W \subseteq \operatorname{null} \varphi$ and $W \subseteq U + W$, therefore $\varphi \in U^0 \cap W^0$.

  For any $\varphi \in U^0 \cap W^0$, we have $U \subseteq \operatorname{null} \varphi$ and $W \subseteq \operatorname{null} \varphi$. For any $u + w \in U + W$, we have $\varphi(u + w) = \varphi(u) + \varphi(w) = 0 + 0 = 0$, therefore $U + W \subseteq \operatorname{null} \varphi$, thus $\varphi \in (U + W)^0$.

- For any $su + tw \in U^0 + W^0$, for any $v \in U \cap W$, we have $su(v) + tw(v) = s0 + t0$ since $v \in U$ and $v \in W$. Therefore we have an injective map (also linear, this map just produce what it receive) from $U^0 + W^0$ to $(U \cap W)^0$. We have:

$$
\begin{aligned}
&\dim(U^0 + W^0) \\
=&\dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
=&\dim V - \dim U + \dim V - \dim W - \dim(U + W)^0 \\
=&\dim V - \dim U + \dim V - \dim W - (\dim V - \dim(U + W)) \\
=&\dim V - \dim U - \dim W + (\dim U + \dim W - \dim(U \cap W)) \\
=&\dim V - \dim(U \cap W) \\
=&\dim(U \cap W)^0
\end{aligned}
$$

  therefore $(U \cap W)^0 = U^0 + W^0$.

$\square$

Exercise 3.23. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1} \in V'$. Show that the following sets are equal to each others:*

- $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1})$
- $((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0$
- $\{\ \varphi \in V' \mid (\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}) \subseteq \operatorname{null}\varphi\ \}$

Proof.

- $((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0 = (\operatorname{null}\varphi_0)^0 + \cdots + (\operatorname{null}\varphi_{m-1})^0$, then $\operatorname{span}(\varphi_i) \subseteq (\operatorname{null}\varphi_i)^0$ therefore $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) \subseteq ((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0$.

  For any $\varphi \in \operatorname{span}(\varphi_0, \cdots, \varphi_{m-1})$, we have $\varphi(v) = \varphi_0(v) + \cdots + \varphi_{m-1}(v) = 0 + \cdots + 0 = 0$ for any $v \in (\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1})$, therefore $\varphi \in ((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0$.

- Last two sets are definitional equal.

$\square$

Exercise 3.24. *Let V finite and $v_0, \cdots, v_{m-1} \in V$.*
Define $\Gamma(\varphi) = (\varphi(v_0), \cdots \varphi(v_{m-1})) : V' \to F^m$, show that:

- $v_0, \cdots, v_{m-1}$ *spans* $V \iff \Gamma$ *is injective.*
- $v_0, \cdots, v_{m-1}$ *is linear independent* $\iff \Gamma$ *is surjective.*

Proof.

- ($\Rightarrow$) Suppose $\Gamma(\alpha) = \Gamma(\beta)$, then for all $v \in V$ can be factorized into $\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}$, then $\alpha(v) = \alpha(\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}) = \beta(\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}) = \beta(v)$ since $\Gamma(\alpha) = \Gamma(\beta)$ and α and β are linear map, thus $\alpha = \beta$.

  ($\Rightarrow$) We first make $v_0, \cdots, v_{m-1}$ linear independent, say $v_0, \cdots, v_{k-1}$, then for any $w \in V$ such that $v_0, \cdots, v_{k-1}, w$ is linear independent, then we have its dual basis $\varphi_0, \cdots, \varphi_{k-1}, \psi$. Consider $\Gamma(\psi)$, by definition, we know $\Gamma(\psi) = (\psi(v_0), \cdots) = (0, \cdots)$ then $\psi = 0$ since Γ is injective, which contradicts our assumption. Therefore $v_0, \cdots, v_{k-1}$ spans V.

- ($\Rightarrow$) Consider the dual basis of $v_0, \cdots, v_{m-1}$, then Γ is surjective since we have the standard basis of F^m.

  ($\Leftarrow$) Γ is surjective implies we have $\varphi_0, \cdots, \varphi_{m-1}$ such that $\Gamma(\varphi_i) = (\cdots, 1, \cdots)$, which means $v_0, \cdots, v_{m-1}$ is linear independent.

$\square$

Exercise 3.25. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1} \in V'$.*
Define $\Gamma(v) = (\varphi_0(v), \cdots, \varphi_{m-1}(v)) : V \to F^m$. Show that

- *$\varphi_0, \cdots, \varphi_{m-1}$ spans V' $\iff$ Γ is injective*
- *$\varphi_0, \cdots, \varphi_{m-1}$ is linear independent $\iff$ Γ is surjective*

Proof.

- ($\Rightarrow$) Suppose $\Gamma(v) = \Gamma(w)$, then $\varphi_i(v) = \varphi_i(w)$, which means $\varphi_i(v-w) = 0$ for all i. If $v - w \neq 0$, then $((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0 \neq \{0\}$, thus $\varphi_0, \cdots, \varphi_{m-1}$ doesn't span V'.

  ($\Leftarrow$) $(\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}) = \{0\}$ since Γ is injective. therefore $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) = ((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0 = (\{0\})^0 = V'$

- ($\Rightarrow$) We may treat Γ as the following matrix:

$$\begin{bmatrix} \varphi_0 \\ \vdots \\ \varphi_{m-1} \end{bmatrix}$$

  which line rank is m since $\varphi_0, \cdots, \varphi_{m-1}$ is linear independent, therefore its column rank is m, thus $\dim \operatorname{range}\Gamma = m = \dim F^m$, then Γ is surjective.

  ($\Leftarrow$) It seems the proof of ($\Rightarrow$) also works here.

$\square$

Exercise 3.26. *Let V finite, and $\Omega \subseteq V'$ a subspace. Show that*

$$\Omega = \{\ v \in V \mid \varphi(v) = 0 \quad \forall \varphi \in \Omega\ \}^0$$

Proof. This construction looks like an inverse of $-^0$.

We may rewrite the equation to $\Omega = (\bigcap_{\varphi\in\Omega} \operatorname{null}\varphi)^0$, then $\Omega = \operatorname{span}(\varphi) \forall \varphi \in \Omega$, which is trivial. $\square$

Exercise 3.28. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1}$ is linear independent. Show that*

$$\dim((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1})) = \dim V - m$$

Proof.

$$\begin{aligned} m &= \dim \operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) \\ &= \dim((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0 \\ &= \dim V - \dim((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1})) \end{aligned}$$

□

Exercise 3.30. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1}$ a basis of V'. Show that there is a basis of V which dual basis is $\varphi_0, \cdots, \varphi_{m-1}$.*

Proof. Since $\varphi_0, \cdots, \varphi_{m-1}$ spans V' and linear independent, we know Γ is both injective and surjective. Consider $v_0, \cdots, v_{m-1}$ such that $\Gamma(v_i) = (\cdots, 0, 1, 0, \cdots)$. We claim $v_0, \cdots, v_{m-1}$ is a basis of V and which dual basis if $\varphi_0, \cdots, \varphi_{m-1}$.

The second part is trivial by the way construct them. For the first part, $v_0, \cdots, v_{m-1}$ is linear independent since $(\cdots, 0, 1, 0 \cdots)$ is linear independent, and $v_0, \cdots, v_{m-1}$ spans V since $\dim V = \dim V' = m$. □

Exercise 3.31. *Let $U \subseteq V$ a subspace and $i(u) = u : U \to V$. Then $i' \in \mathcal{L}(V', U')$, show that:*

1. $\operatorname{null} i' = U^0$
2. $\operatorname{range} i' = U'$ *if V is finite*
3. $\tilde{i}'$ *is an isomorphism between V'/U^0 and U' if V is finite*

Proof.

- For any $\varphi \in \operatorname{null} i'$, $\varphi \circ i = 0$, therefore $\operatorname{range} i = U \subseteq \operatorname{null}\varphi$, thus $\varphi \in U^0$.

  For any $\varphi \in U^0$, $\varphi \circ i = 0$ since $\operatorname{range} i = U \subseteq \operatorname{null}\varphi$.

- Suppose V is finite, then i' is surjective since i' is injective, therefore $\operatorname{range} i' = U'$.

- $\tilde{i}'(\varphi + U^0) = i'(\varphi)$ is surjective since i' is surjective. Then $\dim(V'/U^0) = \dim V' - \dim U^0 = \dim V - (\dim V - \dim U) = \dim U = \dim U'$, therefore $\tilde{i}'$ is an isomorphism.

□

Exercise 3.32. *We denote V'' as the* ***double dual space of*** V*, defined by* $V'' = (V')'$*. Define* $\Lambda(v)(\varphi) = \varphi(v) : V \to V''$

Show that:

1. $\Lambda \in \mathcal{L}(V, V'')$
2. *Let* $V \in \mathcal{L}(V)$*, then* $T'' \circ \Lambda = \Lambda \circ T$ *where* $T'' = (T')'$*.*
3. Λ *is an isomorphism if* V *is finite.*

Proof.

- For any $v, w \in V$ and $\lambda \in F$, we have $(\Lambda(v) + \Lambda(w))(\varphi) = \Lambda(v)(\varphi) + \Lambda(w)(\varphi) = \varphi(v) + \varphi(w) = \varphi(v + w) = \Lambda(v + w)(\varphi)$ and $(\lambda\Lambda(v))(\varphi) = \lambda(\Lambda(v)(\varphi)) = \lambda(\varphi(v)) = \varphi(\lambda v) = \Lambda(\lambda v)(\varphi)$.
- For any $v \in V$,

$$\begin{aligned}
&(T'' \circ \Lambda)(v)(\varphi) \\
=&(T''(\Lambda(v)))(\varphi) \\
=&((\Lambda(v)) \circ T')(\varphi) \\
=&\Lambda(v)(T'(\varphi)) \\
=&\Lambda(v)(\varphi \circ T) \\
=&(\varphi \circ T)(v) \\
=&\varphi(T(v)) \\
=&\Lambda(T(v))(\varphi) \\
=&(\Lambda \circ T)(v)(\varphi)
\end{aligned}$$

- Suppose $\Lambda(v) = \Lambda(w)$, that is, $\Lambda(v)(\varphi) = \varphi(v) = \varphi(w) = \Lambda(w)(\varphi)$ for all $\varphi \in V'$. Let $\varphi_0, \cdots, \varphi_{m-1}$ the dual basis of some basis of V, then $v = \varphi_0(v)v_0 + \cdots + \varphi_{m-1}(v)v_{m-1} = \varphi_0(w)v_0 + \cdots + \varphi_{m-1}(w)v_{m-1} = w$. Therefore Λ is injective, thus surjective and isomorphism since $\dim V = \dim V''$.

□

Exercise 3.33. *Let* $U \subseteq V$ *a subspace and* $\pi : V \to V/U$ *the quotient map, then* $\pi' \in \mathcal{L}((V/U)', V')$*.*

1. *Show that* π' *is injective.*

2. *Show that* $\operatorname{range}\pi' = U^0$.

3. *Conclude that* π' *is an isomorphism between* $(V/U)'$ *and* U^0.

Proof.

- π is surjective, therefore π' is injective. The statement is true even V or V/U may be infinite, cause the proof about surjective-implies-epimorphism doesn't require that the codomain is finite but epimorphism-implies-surjective does.

  We may prove those theorem again, but with weaker assumption. For any $\pi'(\varphi) = \pi'(\psi)$, we have $\varphi \circ \pi = \psi \circ \pi$. For any $v + U \in V/U$, there is $v \in V$ such that $\pi(v) = v + U$ since π is surjective. Therefore $\varphi(\pi(v)) = \psi(\pi(v))$ for all $\pi(v) = v + U \in V/U$,thus $\varphi = \psi$.

  Therefore π' is injective.

- $\operatorname{range}\pi' = (\operatorname{null}\pi)^0 = U^0$.

- Trivial.

□