Definition 0.11. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{null} T = \{ v \mid Tv = 0 \}$ *is called the* ***null space*** *of* T*.*

This is also called the **kernal** of T in algebra.

Theorem 0.13. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{null} T$ *is a subspace of* V*.*

Proof.

- We have $0 \in \operatorname{null} T$ since $T0 = 0$, which is the property of linear transformation.
- For any $a, b \in \operatorname{null} T$, we have $0 = Ta + Tb = T(a + b)$, so $a + b \in \operatorname{null} T$.
- For any $Ta \in \operatorname{null} T$ and $\lambda \in F$, we have $\lambda Ta = T(\lambda a)$, so $\lambda a \in \operatorname{null} T$.

□

Definition 0.15. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{range} T = T(V) = \{ Tv \mid v \in V \}$ *is called the* ***range*** *of* T*.*

This is also called the **image** of T in math.

Theorem 0.18. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{range} T$ *is a subsapce of* W*.*

Proof.

- We have $T(0) = 0 \in \operatorname{range} T$.
- For any $Ta, Tb \in \operatorname{range} T$, $Ta + Tb = T(a + b) \in \operatorname{range} T$.
- For any $Ta \in \operatorname{range} T$ and $\lambda \in F$, $\lambda Ta = T(\lambda a) \in \operatorname{range} T$.

□

Theorem 0.21. *Suppose* V *is finite and* $T \in \mathcal{L}(V, W)$*, then* $\operatorname{range} T$ *is finite, and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof. Consider the basis $v_0, \cdots, v_k$ of $\operatorname{null} T$, and the basis $v_0, \cdots, v_n$ of V that expand from $v_0, \cdots, v_k$. We will show that $T(v_{k+1}), \cdots, T(v_n)$ is the basis of $\operatorname{range} T$.

We first show that $T(v_{k+1}), \cdots, T(v_n)$ is linear independent. If it is linear independent, then

$$\begin{aligned}&\lambda_1 T(v_{k+1}) + \cdots + \lambda_i T(v_{k+i})\\ =&T(\lambda_1 v_{k+1} + \cdots + \lambda_i T(v_{k+i}))\\ =&0\end{aligned}$$

That means a linear combation of v_{k+i} is in $\operatorname{null} T$, which is $\operatorname{span}(v_0, \cdots, v_k)$, therefore the basis $v_0, \cdots, v_n$ is linear dependent.

Then we show that $T(v_{k+1}), \cdots, T(v_n)$ spans $\operatorname{range} T$. For any $Tv \in \operatorname{range} T$, there must be $v \in V$ such that $Tv = Tv$, then v can be written in form of the linear combination of $v_0, \cdots, v_n$, and then $Tv = T(\lambda_0 v_0 + \cdots + \lambda_n v_n)$. We can drop all terms with v_i where $i \leq k$, since they are in $\operatorname{null} T$, so Tv is now represent by a linear combination of $T(v_{k+i})$ for all $0 < i \leq n - k$, therefore, it is a basis of $\operatorname{range} T$ and $\dim \operatorname{range} T$ is finite.

Finally, $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

Definition 3.16 (Notation: $v + U$)**.** *Let $v \in V$ and $U \subseteq V$, then $v + U = \{ v + u \mid u \in U \}$.*

Such sets also called *coset* in group theory.

Definition 3.97 (Translate)**.** *Let $v \in V$ and $U \subseteq V$, we say $v + U$ is a translate of U.*

Definition 3.98 (Quotient Space)**.** *Let $U \subseteq V$ a subspace, then the quotient space V/U is a set with translates of U, that is:*

$$V/U = \{ v + U \mid v \in V \}$$

Theorem 3.101. *Let $U \subseteq V$ a subspace and $v, w \in V$, then the following statements are equivalent.*

1. $v - w \in U$
2. $v + U = w + U$
3. $(v + U) \cap (w + U) \neq \varnothing$

Proof.

- If $v - w \in U$, for any $v + u \in v + U$, we have $v + u = v + (v - w) - (v - w) + u = v - w + w + u = w + (v - w) + u \in w + U$ since $v - w \in U$. Similarly, for any $w + u \in w + U$, we have $w + u = w + (v - w) - (v - w) + u = v - v + w + u = v - (v - w) + u = v + (-(v - w) + u) \in v + U$.

- If $v+U = w+U$, then $v = w+u$ since $v \in v+U$, therefore $v-w = u \in U$.
- if $v + U = w + U$, then $(v + U) \cap (w + U) = v + U = w + U \neq \varnothing$
- If $(v+U)\cap(w+U) \neq \varnothing$, then for any $v+u_0 = w+u_1 \in (v+U)\cap(w+U)$, we have $(v - w) + (u0 - u_1) = 0$ and then $v - w = u_1 - u_0 \in U$, so $v + U = w + U$.

$\square$

Definition 3.102. *Let $U \subseteq V$, then addition and scalar multiplication on V/U is defined by:*

$$(v + U) + (w + U) = (v + w) + U$$
$$\lambda(v + U) = (\lambda v) + U$$

Theorem 3.103. *Let $U \subseteq V$ a subspace, then V/U is a vector space with addition and scalar multiplication we defined in previous definition.*

Proof. We must first show that the addition and the sclar multiplication we introduce are functions.

For any $a, b, c, d \in V$, we will show $(a+b)+U = (c+d)+U$ if $a+U = c+U$ and $b + U = d + U$. We can show $(a + b) - (c + d) \in U$ by $a - c \in U$ and $b - d \in U$.

For any $v, w \in V$ and $\lambda \in F$, we will show $(\lambda v) + U = (\lambda w) + U$ if $v+U = w+U$. We know $v-w \in U$, then $\lambda(v-w) = \lambda v - \lambda w \in U$, therefore $(\lambda v) + U = (\lambda w) + U$.

We have identity of addition $0 + U$ and inverse of addition $(-v) + U$ for all $v \in V$. $\square$

Definition 3.104. *Let $U \subseteq V$ a subspace, the quotient map $\pi : V \to V/U$ is a linear mapping defined by:*

$$\pi(v) = v + U$$

Proof. We will show π is a linear mapping, $\pi(v + w) = (v + w) + U = v + U + w + U = \pi(v) + \pi(w)$ and $\lambda\pi(v) = \lambda(v + U) = (\lambda v) + U = \pi(\lambda v)$. $\square$

Theorem 3.105. *Let V finite and $U \subseteq V$ a subspace, show that* $\dim(V/U) = \dim V - \dim U$.

Proof. We can rewrite the equation as $\dim V = \dim(V/U) + \dim U$, and it is easy to see that $\operatorname{range} \pi = \dim(V/U)$ and $\operatorname{null} \pi = \dim U$. $\square$

Definition 3.106. *Let $T \in \mathcal{L}(V, W)$, define $\tilde{T} : V/(\operatorname{null} T) \to W$ by $\tilde{T}(v + \operatorname{null} T) = Tv$.*

Theorem 3.107. *Let $T \in \mathcal{L}(V, W)$, then:*

1. $\tilde{T} \circ \pi = T$
2. $\tilde{T}$ *is injective*
3. $\operatorname{range} \tilde{T} = \operatorname{range} T$
4. $V/(\operatorname{null} T) \cong \operatorname{range} T$

Proof.

1. For all $v \in V$, $\tilde{T}(\pi(v)) = \tilde{T}(v + \operatorname{null} T) = Tv$
2. If $\tilde{T}(v + \operatorname{null} T) = \tilde{T}(w + \operatorname{null} T)$, then $T(v - w) = 0$, which means $v - w \in \operatorname{null} T$, therefore $v + \operatorname{null} T = w + \operatorname{null} T$.
3. For any $Tv \in \operatorname{range} T$, we have $\tilde{T}(v + \operatorname{null} T) \in \operatorname{range} \tilde{T}$. For any $\tilde{T}(v + \operatorname{null} T) = Tv \in \operatorname{range} \tilde{T}$, we have $Tv \in \operatorname{range} T$.
4. Restrict the range of $\tilde{T}$ on $\operatorname{range} T$, say $\varphi(v + \operatorname{null} T) = \tilde{T}(v + \operatorname{null} T) : V/(\operatorname{null} T) \to \operatorname{range} T$, then φ is injective since (2) and surjective since (3), therefore φ is an isomorphism, thus $V/(\operatorname{null} T) \simeq \operatorname{range} T$.

$\square$

Definition 3.110 (Dual Space)**.** *Let V a vector space, then we denote V' the dual space of V, where*

$$V' = \mathcal{L}(V, F)$$

Theorem 3.111. *Let V a finite vector space, then* $\dim V' = \dim V$

Proof. $\dim V' = \dim \mathcal{L}(V, F) = (\dim V)(\dim F) = \dim V$ $\square$

Definition 3.112 (Dual Basis)**.** *Let $v_0, \cdots, v_{m-1}$ a basis of V, then the dual basis of $v_0, \cdots, v_{m-1}$ is $\varphi_0, \cdots, \varphi_{m-1}$ such that:*

$$\varphi_i(v_j) = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{otherwise} \end{cases}$$

holes for any $0 \leq i, j < m$.

We can see that the basis the dual basis extracts the coefficients of any vector in V.

Theorem 3.113. *Let $v_0, \cdots, v_{m-1}$ a basis of V, and dual basis $\varphi_0, \cdots, \varphi_{m-1}$ of which. Then for any $v \in V$,*

$$v = \varphi_0(v)v_0 + \cdots + \varphi_{m-1}(v)v_{m-1}$$

Proof. For any i, $\varphi_i(v) = \varphi_i(\lambda_0 v_0 + \cdots + \lambda_{m-1}v_{m-1}) = \varphi_i(\lambda_i v_i) = \lambda_i \varphi_i(v_i) = \lambda_i \times 1$. $\square$

Theorem 3.116. *Let V a finite space, then the dual basis of basis of V is a basis of V'.*

Proof. Let $v_0, \cdots, v_{m-1}$ a basis of V, then its dual basis has the same length, therefore we only need to show its dual basis is linear independent.

Suppose $\lambda_0\varphi_0 + \cdots + \lambda_{m-1}\varphi_{m-1} = 0$, then for any $0 \leq i < m$, $(\lambda_0\varphi_0 + \cdots + \lambda_{m-1}\varphi_{m-1})(v_i) = \lambda_i = 0$, therefore the dual basis is linear independent. $\square$

Definition 3.118 (Dual Map)**.** *Let $T \in \mathcal{L}(V, W)$. A dual map of T is a linear map $T' \in \mathcal{L}(W', V')$, such that for any $\varphi \in W'$:*

$$T'(\varphi) = \varphi \circ T$$

Theorem 3.128. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that:*

1. $\operatorname{null} T' = (\operatorname{range} T)^0$
2. $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

Proof.

- $\operatorname{null} T'$ is a space $\{\ \varphi \in \mathcal{L}(W, F) \mid \varphi \circ T = 0\ \}$, which means $\operatorname{range} T \subseteq \operatorname{null} \varphi$. $(\operatorname{range} T)^0$ is a space $\{\ \varphi \in \mathcal{L}(W, F) \mid \varphi(\operatorname{range} T) = \{0\}\ \}$, which means $\operatorname{range} T \subseteq \operatorname{null} \varphi$. Therefore $\operatorname{null} T' = (\operatorname{range} T)^0$
- $\dim(\operatorname{range} T)^0 = \dim W - \dim \operatorname{range} T = \dim W - (\dim V - \dim \operatorname{null} T)$

$\square$

Theorem 3.129. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that*

$$T \textit{ is surjective} \iff T' \textit{ is injective}$$

Proof.

- Suppose T is surjective, then for any $T'(\varphi) = T'(\psi)$, we have $\varphi \circ T = \psi \circ T$. Since T is surjective, then T is an epimorphism (we proved this in E3B), therefore $\varphi = \psi$.

- Suppose T' is injective, then for any $\varphi, \psi \in \mathcal{L}(W, F)$ such that $\varphi \circ T = \psi \circ T$, we have $\varphi = \psi$ since T' is injective. Therefore T is epimorphism, thus surjective.

□

The last theorem is obviously true in category theory, but we haven't show that T' is a morphism in Vect' where $\mathrm{Vect}' \simeq \mathrm{Vect}^{\mathrm{op}}$.

Theorem 3.130. *Let V, W are finite and $T \in \mathcal{L}(V, W)$, show that:*

1. $\dim \operatorname{range} T' = \dim \operatorname{range} T$

2. $\operatorname{range} T' = (\operatorname{null} T)^0$

Proof.

- $\dim \operatorname{range} T' = \dim W' - \dim \operatorname{null} T' = \dim W' - (\dim \operatorname{null} T + \dim W - \dim V) = \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$

- For any $\varphi \circ T \in \operatorname{range} T'$, $(\varphi \circ T)(\operatorname{null} T) = \varphi(\{0\}) = \{0\}$, therefore $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$. Since $\dim (\operatorname{null} T)^0 = \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T \ \mathit{dim} \operatorname{range} T'$, therefore $\operatorname{range} T' = (\operatorname{null} T)^0$ since both of them are finite and $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$.

□

Theorem 3.131. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show*

$$T \text{ is injective} \iff T' \text{ is surjective}$$

Proof.

- $\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim V = \dim V'$ since T is injective, therefore T' is surjective.

- $\dim \operatorname{range} T = \dim \operatorname{range} T' = \dim V' = \dim V$ therefore T is injective.

□

As

Exercise 3.1. *Suppose V is a finite vector space, Show that the only two ideal of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$. A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called an ideal, if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for any $T \in \mathcal{L}(V)$ and $E \in \mathcal{E}$.*

Proof. We will use the concept Matrix. Suppose $\lambda_0 v_0 + \cdots + \lambda_n v_v$ the basis of V. We want to construct T_i that $T(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_i v_i$ for all $0 \leq i < n$, which is a matrix with all zero but 1 at i, i.

For any matrix, we can always select a non-zero value at a, b and place it at i, b, this can be done by left multiply a matrix with 1 at i, a (this produce a vector at line i with values from line a), then right multiply a matrix with 1 at i, b (this produce a vector at column b with values from line i).

Also, we can always select a non-zero value at a, b and place it at a, i, this can be done by right multiply a matrix with 1 at b, i, then left multiply a matrix with 1 at a, i.

By combining these two operations, we ca select a non-zero value at a, b and place it at i, i. Now, consider any non-zero $E \in \mathcal{E}$, we can construct a matrix with non-zero value at i, i for every $0 \leq i < \dim V$. These matrix are in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then we can multiply an apropriate scalar to them so that they are matrices with 1 at i, i. By adds up these matrices, we get I, we know $I \in \mathcal{E}$ since $\mathcal{E}$ is a vector space, and now all $T \in \mathcal{L}(V)$ is also in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then $\mathcal{E} = \mathcal{L}(V)$.

The only exception is $\mathcal{E} = \{0\}$, in this case we can't pick any non-zero element.

Another solution, hope this one is more simple.

Suppose $\mathcal{E}$ an ideal of $\mathcal{L}(V)$ and non-zero, non-surjective $E \in \mathcal{E}$. Let $v_0, \cdots, v_{k-1}$ a basis of $\operatorname{null} E$ and $v_k, \cdots, v_{k+n}$ such that Tv_{k+i} is a basis of $\operatorname{range} E$, then we have $n \neq 0$ and $k \neq 0$.

Define A a linear transformation which maps v_i to v_{k+i} for $0 \leq i < \min\{k, n\}$ and maps others to 0, then $\dim \operatorname{range} EA = \min\{k, n\}$.

Expand the basis $w_i = Ev_{k+i}$ of $\operatorname{range} E$ to a basis of V, say $w_0, \cdots, w_{m-1}$, define B maps Ev_{k+i} to $w_{\min\{k,n\}+i}$, we always have enough $w_{\min\{k,n\}+i}$ since $m - 1 = \dim V = \dim \operatorname{null} E + \dim \operatorname{range} E$ while $\min k, n \leq \dim \operatorname{null} E$, then $\dim \operatorname{range} BE = \operatorname{range} E$ since we just re-map the range E.

Now consider $S = EA + BE$, we have $Sv_i = EAv_i = Ev_{k+i} = w_i \in \operatorname{range} E$ for all $0 \leq i < \min\{k, n\}$ and $Sv_{\min\{k,n\}+i} = BEv_{\min\{k,n\}+i} = w_{\min\{k,n\}+i} \in$

range BE for all $0 \leq i < \dim \operatorname{range} E$. We can see range $EA \cap$ range $BE = \{0\}$ and $\dim \operatorname{range}(EA + BE) = \operatorname{range} E + \min\{k, n\}$, where $k = \operatorname{null} E$ and $n = \operatorname{range} E$, the range of $EA + BE$ gets larger and $EA + BE \in \mathcal{E}$ since $EA, BE \in \mathcal{E}$, if $k > n$ (this is the only case that $EA + BE$ is not surjective), then we continue this process with $E = EA + BE$, the procedure will finally terminate since $\mathcal{L}(V)$ is finite (cause V is finite).

Now we show that any $\mathcal{E}$ with non-zero, non-surjective $E \in \mathcal{E}$ implies a surjective (thus injective and invertible) $T \in \mathcal{E}$.

For any ideal with an invertible element $E \in \mathcal{E}$, we have $E^{-1}E = I \in \mathcal{E}$, which causes $\mathcal{E} = \mathcal{L}(V)$ since $IT = T$ for all $T \in \mathcal{L}(V)$.

Therefore, only $\{0\}$ and $\mathcal{L}(V)$ are ideals of $\mathcal{L}(V)$. □

Exercise 3.7. *Suppose vector space V and W are finite ($2 \leq \dim V \leq \dim W$), show that $\{\ T \in \mathcal{L}(V, W) \mid T$ is not injective $\}$ is not a subspace.*

Proof. Consider the basis $v_0 + \cdots + v_{(\dim V - 1)} \in V$, and $T(v_0 + \cdots + v_{(\dim V-1)}) = (0 + 1_0 + \cdots + 1_{(\dim V - 1)})$ and $T'(v_0 + \cdots + v_{(\dim V - 1)}) = (v_0 + 0 + \cdots + v_{(\dim V - 1)})$. Then $(T + T')(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_{(\dim V - 1)} v_{(\dim V - 1)}) = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + 2\lambda_{(\dim V - 1)} v_{(\dim V - 1)}$, which is obviously injective. □

Exercise 3.11. *Suppose V is finite and $T \in \mathcal{L}(V, W)$, show that there is a subspace $U \subset V$ such that:*

$$U \cap \operatorname{null} T = \{0\} \quad \textit{and} \quad \operatorname{range} T = \{\ Tu \mid u \in U\ \}$$

Proof. This is similar to the *isomorphism theorems* about groups! This can be done by the similar way we used in proving $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

The next two exercises remind me the categorical injective and surjective, let try them first!

Exercise. *For any $F \in \mathcal{L}(V, W)$, F is injective $\iff$ for any $S, T \in \mathcal{L}(U, V)$, $FS = FT$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(V, W)$ that $FS = FT$, then for any $u \in U$, we have $F(Su) = F(Tu)$, since F is injective, we know $Su = Tu$, so $S = T$.

- ($\Leftarrow$) For any $v, w \in V$ such that $Fv = Fw$. Consider

  $$S(\lambda) = \lambda v$$
  $$T(\lambda) = \lambda w$$

  in $\mathcal{L}(\mathbb{R}, V)$. Then for any $\lambda \in \mathbb{R}$, we have $FS\lambda = \lambda FS1 = \lambda Fv = \lambda Fw = \lambda FT1 = FT\lambda$. so $FS = FT$ then $S = T$, which means $v = S1 = T1 = w$.

$\square$

Exercise. *Suppose W is finite, then for any $F \in \mathcal{L}(V, W)$, F is surjective $\iff$ for any $S, T \in \mathcal{L}(W, U)$, $SF = TF$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(W, U)$ such that $SF = TF$. For any $w \in W$, there is $v \in V$ such that $Fv = w$ since F is surjective. Then we have $SFv = TFv$ so $Sw = S(Fv) = T(Fv) = Tw$ then $S = T$.

- ($\Leftarrow$) Consider

  $$S = I \quad \text{and} \quad T(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 w_0 + \cdots + \lambda_k w_k$$

  where $w_0, \cdots, w_k$ is the basis of range F and $w_0, \cdots, w_n$ is the basis of W that expand from $w_0, \cdots, w_k$.

  (If we can use another way to construct T, then W is not need to be finite, for example, $W = \operatorname{range} T \oplus W_0$).

  It is easy to show that T is a linear transformation. Then for any $v \in V$, we have $TFv = Fv$ (since T acts like identity transformation on range F) and $SFv = Fv$, so $S = T$ by the property of F. Since range $S = W$, so is range T, that means $w_0, \cdots, w_k$ spans W, so $k = n$, which means range $F = W$, therefore F is surjective.

$\square$

Exercise 3.19. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is injective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $ST = I$.*

Proof.

- ($\Rightarrow$) Consider the basis $v_0, \cdots, v_n$ of V, then $Tv_0, \cdots, Tv_n$ is a basis of $\operatorname{range} T$ since T is injective. We denote Tv_i as w_i and $w_0, \cdots, w_m$ as the basis of W which expand from $w_0, \cdots, w_n$. Define $S(\lambda_0 w_0 + \cdots + \lambda_m w_m) = \lambda_0 w_0 + \cdots + \lambda_n w_n$, and then for any $v \in V$, $ST(\lambda_0 v_0 + \cdots + \lambda_n v_n) = S(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 v_0 + \cdots + \lambda_n v_n$, so $ST = I$.

- ($\Leftarrow$) Suppose $A, B \in \mathcal{L}(U, V)$, such that $TA = TB$, we will show that $A = B$. $STA = IA = A$ and $STB = IB = B$ and $STA = STB$ since $TA = TB$. Then we know T is a monomorphism, and then T is injective.

□

Exercise 3.20. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is surjective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $TS = I$.*

Proof. □

Exercise 3.21. *Suppose V is finite, $T \in \mathcal{L}(V, W)$, $U \subseteq W$ a subspace. Show that the inverse image of U: $\{ v \in V \mid Tv \in U \}$ is a subspace of V, and*

$$\dim\{ v \in V \mid Tv \in U \} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$$

Proof. The second part is quite easy, we can restrict the domain of T to $\{ v \in V \mid Tv \in U \}$, say $T' \in \mathcal{L}(\{ v \in V \mid Tv \in U \}, W)$, so that it is in form $\dim\{ v \in V \mid Tv \in U \} = \dim \operatorname{null} T' + \dim \operatorname{range} T'$. Obviously $\operatorname{range} T' = U \cap \operatorname{range} T$ and $\operatorname{null} T' = \operatorname{null} T$.

We will now show that $\{ v \in V \mid Tv \in U \}$ is a subspace of V.

- $T0 \in U$.

- For any $v, w \in V$ such that $Tv, Tw \in U$, we have $T(v+w) = Tv + Tw \in U$.

- For any $v \in V$ such that $Tv \in U$ and $\lambda \in F$, we have $T(\lambda v) = \lambda Tv \in U$.

Therefore it is a subspace. □

Exercise 3.22. *Suppose U and V are finite, $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$, show that*

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$$

Proof. Consider the inverse image of $\operatorname{null} S$ on T: $K = \{ v \in V \mid Tv \in \operatorname{null} S \}$, which dimension: $\dim K = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T)$, where $\dim(\operatorname{null} S \cap \operatorname{range} T)$ caps at $\dim \operatorname{null} S$.

We know show that $\operatorname{null} ST = \operatorname{null} K$. For any $STv = 0$, we know $S(Tv) = 0$, so $Tv \in K$, therefore $\operatorname{null} ST \subseteq \operatorname{null} K$; For any $Tv \in \operatorname{null} S$, that means $S(Tv) = 0$, therefore $v \in \operatorname{null} ST$, therefore $\operatorname{null} ST \supseteq \operatorname{null} K$, and $\operatorname{null} ST = \operatorname{null} K$. □

Exercise 3.25. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that* $\operatorname{null} S \subseteq \operatorname{null} T \iff$ *there is $E \in \mathcal{L}(W)$ such that $T = ES$.*

Proof. We define $E(S(v)) = Tv$ for any $v \in V$, so that $E \in \mathcal{L}(\operatorname{range} S, W)$. We first show that E is a mapping, and also a linear transformation.

Suppose $Sv, Sw \in W$ such that $Sv = Sw$, we need to show that $E(Sv) = E(Sw)$, or normalized $Tv = Tw$. We know $v - w \in \operatorname{null} S$ since $Sv = Sw$, so $v - w \in \operatorname{null} T$ since $\operatorname{null} S \subseteq \operatorname{null} T$, therefore $T(v - w) = 0$, and then $Tv = Tw$, so E is a mapping.

Now we show that E is a linear transformation.

- For any $Sv, Sw \in \operatorname{range} S$, $E(Sv) + E(Sw) = Tv + Tw = T(v + w) = E(S(v + w)) = E(Sv + Sw)$.
- For any $Sv \in \operatorname{range} S$ and $\lambda \in F$, $\lambda E(Sv) = \lambda Tv = T(\lambda v) = E(S(\lambda v) = E(\lambda Sv))$.

therefore E is a linear transformation.

Now we can expand the domain of E to W such that $E'v = Ev$ for any $v \in \operatorname{range} S$ (this is proven in previous exercise). For any $v \in V$, we have $ESv = E(Sv) = Tv$, there fore $T = ES$.

For another direction, for any $v \in \operatorname{null} S$, we have $ESv = E0 = 0 = Tv$, so $v \in \operatorname{null} T$. □

Exercise 3.26. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that* $\operatorname{range} S \subseteq \operatorname{range} T \iff$ *there is $E \in \mathcal{L}(V)$ such that $S = TE$.*

Proof. Consider the inverse image of $\operatorname{range} S$ with basis $w_0, \cdots, w_n$, say $v_0, \cdots, v_n$, it is easy to show $v_0, \cdots, v_n$ is linear independent. Then $E(v) = \lambda_0 v_0 + \cdots + \lambda_n v_n$ where $Sv = \lambda_0 w_0 + \cdots + \lambda_n w_n$. □

Exercise 3.27. *Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$, show that $V = \operatorname{null} P \oplus \operatorname{range} P$.*

Proof. Such element is called *idempotent* in algebra.

We will show $\operatorname{null} P \oplus \operatorname{range} P$ by showing $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. For any $v \in \operatorname{null} P \cap \operatorname{range} P$, we know there is $w \in V$ such that $Pw = v$ since $v \in rangevP$, then $P^2(v) = P(Pv) = P0 = 0$ since $v \in \operatorname{null} P$ and $P^2(v) = P(Pv) = Pw$, so $Pw = 0$ while $Pw = v$ therefore $v = 0$.

Then we have $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$ and $\dim(\operatorname{null} P \oplus \operatorname{range} P) = \dim \operatorname{null} P + \dim \operatorname{range} P - \dim\{0\}$, so $V = \operatorname{null} P \oplus \operatorname{range} P$. □

Exercise 3.28. *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ such that for any non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$, $\deg(Dp) = \deg p - 1$. Show that D is surjective.*

Proof. We induction on n, starts from 1, to show that $D(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_{n-1}(\mathbb{R})$.

- Base: for any $p \in \mathcal{P}(\mathbb{R})$ where $\deg p = 1$, we know $\deg Dp = 0$, so $D(\mathcal{P}_1(\mathbb{R}))$ is a non-zero subspace of $\mathcal{P}_0(\mathbb{R})$, which is $\mathcal{P}_0(\mathbb{R})$.

- Induction: We have induction hypothesis: For any $i \le n$, we have $D(\mathcal{P}_i(\mathbb{R})) = \mathcal{P}_{i-1}(\mathbb{R})$. We want to show that $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$. For any $p \in \mathcal{P}(\mathbb{R})$ with $\deg p = n+1$, we can write p in form of $p = \lambda x^{n+1} + r$ where $\deg r \le n$, then $Dp = D(\lambda x^{n+1} + r) = D(\lambda x^{n+1}) + Dr$ where $\deg D(\lambda x^{n+1}) = n$ and $\deg Dr \le n-1$. So $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$ since: $\mathcal{P}_n(\mathbb{R}) \subseteq D(\mathcal{P}_{n+1}(\mathbb{R}))$ and $D(\lambda x^{n+1}) \in D(\mathcal{P}_{n+1}(\mathbb{R}))$, it is sufficient to span $\mathcal{P}_n(\mathbb{R})$.

□

Exercise 3.29. *For any $p \in \mathcal{P}(\mathbb{R})$, show that there is $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.*

Proof. We can rewrite the goal as $5DDq + 3Dq = p$ where $D(p) = p'$, then $5DDq + 3Dq = D(5Dq) + D(3q) = D(5Dq + 3q) = p$. We know D is surjective by the previous exercise, the goal is now showing that $5Dq + 3q = r$ where $Dr = p$. Then we continue rewrite the goal $5Dq + 3q = (5D)q + (3I)q = (5D + 3I)q = r$, we will show that $5D + 3I$ is surjective, we use the same method in previous exercise.

We denote $5D + 3I$ by F, and induction on $n \in \mathbb{N}$ to show that $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$.

- Base: We should show that $F(\mathcal{P}_0(\mathbb{R})) = \mathcal{P}_0(\mathbb{R})$, for any $p \in \mathcal{P}_0(\mathbb{R})$, we have $Fp = 5Dp + 3p$, where $Dp = 0$ since $\deg p = 0$, so $Fp = 3p$, which means we have $1 \in F(\mathcal{P}_0(\mathbb{R}))$ since p is literally a number and $\frac{1}{3p}Fp = 1$.

- Induction: We have induction hypothesis: $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$, and we want to show $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

  For any $p \in \mathcal{P}_{n+1}(\mathbb{R})$, we have $Fp = 5Dp + 3p$ where $\deg 5Dp = n$ and $\deg 3p = n+1$, then we can eliminate $5Dp$ and every term in p with degree less then $n+1$ since $\mathcal{P}_n(\mathbb{R}) \subseteq \operatorname{range} F$, then we get z^{n+1}, thus $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

Therefore there is q such that $(5D + 3I)q = r$ since $5D + 3I$ is surjective.

Another solution from internet: Define $Tq = 5q'' + 3q'$, we can see for any $q \in \mathcal{P}(\mathbb{R})$ we have $\deg Tq = \deg q - 1$, so T is surjective. Then there is q such that $Tq = 5q'' + 3q' = p$. □

Exercise 3.30. *Suppose $\varphi \in \mathcal{L}(V, F)$ not zero, and $u \in V$ that $u \notin \operatorname{null} \varphi$, show that $V = \operatorname{null} \varphi \oplus \{ \ au \mid a \in F \ \}$.*

Proof. We can see φ is surjective since $\varphi u \neq 0$, then for any $i \in F$, we have $(i(\varphi u)^{-1})\varphi u = i$.

For any $v \in V$, since φ is surjective (in a particular way), so we have $a\varphi u$ such that $a\varphi u = \varphi v$, then $\varphi(au - v) = 0$ so $au - v \in \operatorname{null} \varphi$. That means $(-1)(au - v) + au = v$ where $(-1)(au - v) \in \operatorname{null} \varphi$ and $au \in \{ \ au \mid a \in F \ \}$, so $V = \operatorname{null} \varphi + \{ \ au \mid a \in F \ \}$.

Then $\operatorname{null} \varphi \oplus \{ \ au \mid a \in F \ \}$ since $u \notin \operatorname{null} \varphi$. □

Exercise 3.31. *Suppose V is finite ($\dim V > 1$), show that if $\varphi : \mathcal{L}(V) \to F$ is a linear mapping with property $\varphi(ST) = \varphi(S)\varphi(T)$ for any $S, T \in \mathcal{L}(V)$, show that $\varphi = 0$.*

Proof. Consider $\operatorname{null} \varphi$, since $\dim V > 1$ while $\dim F = 1$, so φ cannot be injective, therefore $\operatorname{null} \varphi \neq \{0\}$.

For any non-zero $S \in \operatorname{null} \varphi$ and $T \in \mathcal{L}(V)$, we have $\varphi(ST) = \varphi(S)\varphi(T) = 0 = \varphi(T)\varphi(S) = \varphi(TS)$ since $S \in \operatorname{null} \varphi$, thus $ST \in \operatorname{null} \varphi$. We show that $\operatorname{null} \varphi$ is an ideal of $\mathcal{L}(V)$, recall that the property of $\mathcal{L}(V)$, the only ideal of $\mathcal{L}(V)$ is $\{0\}$ and $LT(V)$, so $\operatorname{null} \varphi = \mathcal{L}(V)$, which means $\varphi = 0$. □

Exercise 3.32. *Let V, W are vector spaces and $T \in \mathcal{L}(V, W)$, define $T_C : V_C \to W_C$:*

$$T_C(u + iv) = Tu + iTv$$

for any $u, v \in V$.

1. *Show that T_C is a (complex) linear mapping from V_C to W_C.*

2. *Show that* T_C *is injective* $\iff$ T *is injective.*

3. *Show that* $\operatorname{range} T_C = W_C \iff \operatorname{range} T = W$.

Proof.

1. For any $u, v, s, t \in V$ $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} & T((u+iv)+(s+it)) \\ =& T(u+s+i(v+t)) \\ =& T(u+s)+iT(v+t) \\ =& Tu+Ts+iTv+iTt \\ =& T(u+iv)+T(s+it) \end{aligned}$$

and

$$\begin{aligned} & \lambda T(u+iv) \\ =& \lambda(Tu+iTv) \\ =& \lambda Tu+\lambda iTv \\ =& T(\lambda u)+iT(\lambda v) \\ =& T(\lambda u+i(\lambda v)) \\ =& T(\lambda u+i\lambda v) \\ =& T(\lambda(u+iv)) \end{aligned}$$

I believe these are trivial, so the future me should be able to prove these without any effort. □

Exercise 3.4. *Suppose* $D(p) = p' : \mathcal{L}(\mathcal{P}_3(\mathbb{R})) \to \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$. *Find a basis of* $\mathcal{L}(\mathcal{P}_3(\mathbb{R}))$ *and a basis of* $\mathcal{L}(\mathcal{P}_2(\mathbb{R}))$, *such that* $\mathcal{M}(D)$ *about these basis is:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Proof. Consider $x, x^2, x^3, 1$ the basis of $\mathcal{P}_3(\mathbb{R})$ and $1, x, 2x^2$. □

Exercise 3.5. *Suppose* V *and* W *are finite and* $T \in \mathcal{L}(V, W)$. *Show that there are basis of* V *and* W *respectively, such that* $\mathcal{M}(T, \textit{those basis})$ *is all zero except* 1 *at* $k, k \quad (1 \le k \le \dim \operatorname{range} T)$.

Proof. Consider the basis $w_0, \cdots, w_{k-1}$ of range T and the basis $w_0, \cdots, w_{m-1}$ of W which expands from $w_0, \cdots, w_{k-1}$. Then there must be $v_0, \cdots, v_{k-1}$ such that $Tv_i = w_i$ for all $0 \leq i < k$, we know $v_0, \cdots, v_{k-1}$ is linear independent since $w_0, \cdots, w_{k-1}$ is linear independent, so we can expand it to a basis of V, say $v_0, \cdots, v_{n-1}$.

We claim that $\mathcal{M}(T, v_0, \cdots, v_{n-1}, w_0, \cdots, w_{m-1})$ is a matrix with all zero but 1 at $k, k \quad (1 \leq k < \operatorname{range} T)$. For any $\lambda_0 v_0 + \cdots + \lambda_{n-1} v_{n-1} \in V$, we have $T(\lambda_0 v_0 + \cdots + \lambda_{n-1} v_{n-1}) = \lambda_0 w_0 + \cdots + \lambda_{k-1} w_{k-1}$, note that all v_i where $i \geq k$ disappear, since they maps to 0. Therefore $\mathcal{M}(T)$ is all zero but 1 at k, k (since $\lambda_i w_i$ in the last equation). □

Exercise 3.6. *Show that* $-^T : F^{m,n} \to F^{n,m}$ *is a linear mapping.*

Proof. Trivial, sorry. □

Exercise 3.7. *Show that* $(AB)^T = B^T A^T$.

Proof. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix, then for any $i \in [1, m]$ and $j \in [1, p]$, we have $(AB)^T_{i,j} = (AB)_{j,i} = \displaystyle\sum_{r=1}^{n} A_{j,r} B_{r,i} = \sum_{r=1}^{n} B^T_{i,r} A^T_{r,j} = (B^T A^T)_{i,j}$. □

Exercise 3.8. *Let A a $m \times n$ matrix, show that the rank of A is 1 $\iff$ there is $c_0, \cdots, c_{m-1} \in F^m$ and $d_0, \cdots, d_{n-1} \in F^n$ such that $A_{j,k} = c_j d_k$ for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, n-1$.*

Proof. The right hand side is actually the external product of vectors, that is vw^T.

($\Rightarrow$) is easy since we can use the theorem that any $m \times n$ matrix A can be expressed by CR where C is a $m \times r$ matrix, R is a $r \times n$ matrix, r is the rank of A. In this case, $r = 1$, so C and R are just vectors.

($\Leftarrow$) is also easy since other column is a scalar multiple of the first column, therefore the rank of A is 1. □

Exercise 3.9. *Let $T \in \mathcal{L}(V)$, $u_0, \cdots, u_{n-1}$ and $v_0, \cdots, v_{n-1}$ are the bases of V, show that the following statements are equivalent:*

1. *T is injective*

2. *The columns of $\mathcal{M}(T)$ is linear independent*

3. *The columns of $\mathcal{M}(T)$ spans $F^{n,1}$*

4. *The lines of $\mathcal{M}(T)$ is linear independent*

5. *The lines of $\mathcal{M}(T)$ spans $F^{1,n}$*

Proof. (2), (3) are obviously equivalent and (4), (5) too.

Although I want to make an arrow loop, but the arrow between (1) and (4), (5) is too hard, so I will show that (1) $\iff$ (2), (3) and (2), (3) $\iff$ (4), (5).

- ($\Rightarrow$) Let $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1} = [0, \ldots, 0]$, then $T(\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}) = 0$, so λ_i are 0 since T is injective, which means $\operatorname{null} T = \{0\}$.

  ($\Leftarrow$) For any $T(\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}) = 0$, we have the linear combination of v_i is 0 where the coefficients come from $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1}$ (w_i are the columns of $\mathcal{M}(T)$), therefore the coefficients are all 0 since v_i is linear independent, thus $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1} = 0$, which means λ_i are all 0 since w_i is linear independent.

- For any matrix, its line rank is equal to its column rank, so columns independent $\iff$ lines independent.

□

Exercise 3.4. *Let V a finite vector space with* $\dim V > 1$*, show that* $S = \{\ T \textit{ is singular} \mid T \in \mathcal{L}(V)\ \}$ *is* ***NOT*** *a subspace of* $\mathcal{L}(V)$.

Proof. If S is a subspace of $\mathcal{L}(V)$, then it is an ideal of $\mathcal{L}(V)$ since for any $A \in S$ and $B \in \mathcal{L}(V)$, AB and BA are singular, therefore $AB, BA \in S$. However, we know the only two ideals of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$, none of them is S. □

Exercise 3.11. *Let V finite vector space, and $S, T \in \mathcal{L}(V)$, show that*

$$ST \textit{ is invertible} \iff S \textit{ and } T \textit{ are invertible}$$

Proof.

- ($\Rightarrow$) Suppose $STW = WST = I$, then $S(TW) = (TW)S = I$ since $\dim V = \dim V$, therefore $S^{-1} = TW$, also $(WS)T = T(WS) = I$ since $\dim V = \dim V$, therefore $T^{-1} = WS$.

- ($\Leftarrow$) Trivial.

$\square$

Exercise 3.12. *Let V finite vector space, and $S, T, U \in \mathcal{L}(V)$ such that $STU = I$, Show that $T^{-1} = US$.*

Proof. Since $STU = I$ we know U is invertible (since STU is invertible), then $ST = U^{-1}$. Since U^{-1} is invertible, we know S and T are invertible therefore $T = S^{-1}U^{-1}$ and $T^{-1} = US$. $\square$

Exercise 3.13. *Show that the conclusion of previous exercise can be false if V is not finite.*

Proof. Let $S(x_0, x_1, \dots) = (x_1, \dots)$ the backward-shift mapping and $U(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ the forward-shift mapping and $T = I$ the identity mapping.

We have $SU = I$ and $US \neq I$, T is clearly inveritble with $T^{-1} = I$, but we know $US \neq I$, so $T^{-1} = US \neq I$.

In fact, this also disprove the infinite version of 3.11 since SU is invertible but neither S nor U is invertible. $\square$

Exercise 3.17. *Let V a finite vector space, $S \in \mathcal{L}(V)$, define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$, show that:*

1. $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$
2. $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$

Proof. Since $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$, we know $\dim \mathcal{L}(V) = \dim \operatorname{null} \mathcal{A} + \dim \operatorname{range} \mathcal{A}$, also, $\dim \mathcal{L}(V) = (\dim V)^2$ and $\dim V = \dim \operatorname{null} S + \dim \operatorname{range} S$. Therefore we have $\dim \operatorname{null} A + \dim \operatorname{range} A = (\dim V)(\dim \operatorname{null} S + \dim \operatorname{range} S)$, which means we only need to prove one of (1) and (2).

We will show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$. We found that $\dim \mathcal{L}(V, \operatorname{null} S) = (\dim V)(\dim \operatorname{null} S)$, so it would be nice if $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$. For any $T \in \operatorname{null} \mathcal{A}$, we have $ST = 0$, which means $\operatorname{range} T \subseteq \operatorname{null} S$, therefore $T \in \mathcal{L}(V, \operatorname{null} S)$. For any $T \in \mathcal{L}(V, \operatorname{null} S)$, we have $ST = 0$ since $\operatorname{range} T \subseteq \operatorname{null} S$, so $T \in \operatorname{null} \mathcal{A}$, therefore $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$, thus $\dim \operatorname{null} A = (\dim V)(\dim \operatorname{null} S)$. $\square$

Exercise 3.18. *Show that V and $\mathcal{L}(F, V)$ are isomorphic.*

Proof. This can be proven by $\dim V = \dim \mathcal{L}(F, V) = 1(\dim V)$, but we can find $\varphi(v) = x \mapsto xv$ an isomorphism. For any $T \in \mathcal{L}(F, V)$, T is determined by $T(1)$. □

Exercise 3.1. *Let $T : V \to W$, the graph of T is a subset of $V \times W$ such that*

$$\textit{graph of } T = \{\ (v, Tv) \mid v \in V\ \}$$

.

Show that T is a linear mapping $\iff$ the graph of T is a subspace.

Proof.

- $(0, T0) \in$ graph of T. $(v, Tv) + (w + Tw) = (v + w, Tv + Tw) = (v + w, T(v + w))$. $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v))$
- $(v, Tv) + (w + Tw) = (v + w, T(v + w))$ since the graph of T is a subspace, therefore $Tv + Tw = T(v + w)$. Similarly, $\lambda Tv = T(\lambda v)$.

□

Exercise 3.3. *Let V_i are vector spaces, show that $\mathcal{L}(V_0 \times \cdots \times V_{m-1}, W) \simeq \mathcal{L}(V_0, W) \times \cdots \times \mathcal{L}(V_{m-1}, W)$.*

Proof. This can be proven by $A \times B$ is a categorical product, so we will show that for any A, B are vector spaces, $A \times B$ is a product.

In order to show that $A \times B$ is a product, or more specifically, $A \times B$ equipped with linear mappings

$$\pi_0(a, b) = a$$
$$\pi_1(a, b) = b$$

is a product, we have to show that for any C, $s \in \mathcal{L}(C, A)$ and $t \in \mathcal{L}(C, B)$, there is a unique $u \in \mathcal{L}(C, A \times B)$ such that $s = \pi_0 \circ u$ and $t = \pi_1 \circ u$.

Define $u(c) = (sc, tc) : C \to A \times B$, we will show that u is a linear mapping.

- For all $v, w \in C$, $u(v) + u(w) = (sv, tv) + (sw, tw) = (sv + sw, tv + tw) = (s(v + w), t(v + w)) = u(v + w)$
- For all $c \in C$ and $\lambda \in F$, $\lambda u(c) = \lambda(sc, tc) = (\lambda sc, \lambda tc) = (s(\lambda c), t(\lambda c)) = u(\lambda c)$.

Then we can see $\pi_0(u(c)) = \pi_0(sc, tc) = sc$ and $\pi_1(u(c)) = \pi_1(sc, tc) = tc$. Now we have to show that u is unique (which is trivial, I don't want to prove this, sorry). $\square$

Exercise 3.5. *Let m a positive number, define $V^m = \underbrace{V \times \cdots \times V}_{m}$, show that $V^m \simeq \mathcal{L}(F^m, V)$.*

Proof. Define $\varphi(v_0, \cdots, v_{m-1}) = i_0, \cdots, i_{m-1} \mapsto i_0 v_0 + \cdots + i_{m-1} v_{m-1}$ which accept a list of vector and a list of coefficients then produce a linear combination.

For any $T \in \mathcal{L}(F^m, V)$, T is completely determined by $T(1, \cdots, 1) = v_0 + \cdots + v_{m-1}$, therefore $\varphi(v_0, \cdots, v_{m-1}) = T$ and thus φ is surjective.

For any $(v_0, \cdots, v_{m-1}), (w_0, \cdots, w_{m-1}) \in V^m$ such that $\varphi(v_0, \cdots, v_{m-1}) = \varphi(w_0, \cdots, w_{m-1})$, then $w_0 = \varphi(v_0, \cdots, v_{m-1})(1, 0, \cdots, 0) = \varphi(v_0, \cdots, v_{m-1})(1, 0, \cdots, 0) = v_0$, same for other v_i and w_i, so $(v_0, \cdots, v_{m-1}) = (w_0, \cdots, w_{m-1})$, therefore φ is injective. $\square$

Exercise 3.6. *Let $v, x \in V$ and $U, W \subseteq V$ are subspaces such that $v + U = x + W$. Show that $U = W$.*

Proof. We know $v = x + w_0$ for some $w_0 \in W$ since $v + U = x + W$ and $v \in v + U$, then for any $u \in U$, we have $v + u = x + w$ for some $w \in W$, then $(x + w_0) + u = x + w$ therefore $u = x + w - x - w_0 = w - w_0 \in W$ thus $U \subseteq W$. Similarly $W \subseteq U$. $\square$

Exercise 3.7. *Let $U = \{ (x, y, z) \in R^3 \mid 2x + 3y + 5z = 0 \}$ and $A \subseteq R^3$. Show that A is a translate of U (that is $A = a + U$) $\iff$ there is c such that $A = \{ (x, y, z) \in R^3 \mid 2x + 3y + 5z = c \}$.*

Proof.

- $(\Rightarrow)$ For any $(a_0, a_1, a_2) + (x, y, z) \in a + U$, we have $2(a_0 + x) + 3(a_1 + y) + 5(a_2 + z) = 2a_0 + 3a_1 + 5a_2$, therefore $c = 2a_0 + 3a_1 + 5a_2$.

- $(\Leftarrow)$ We can see 2, 3 and 5 are coprime to each other, therefore there is $2a_0 + 3a_1 + 5a_2 = 1$ (I am not sure if this is true in generalized case, I just extends the theorem "$as + bt = 1 \iff$ a coprime to b" to three elements case without checking), in this case we have $2(1) + 3(-2) + 5(1) = 1$, then for any $2x + 3y + 5z = c$, we have $2x + 3y + 5z = 2(ca_0) + 3(ca_1) + 5(ca_2)$, then $2(x - ca_0) + 3(y - ca_1) + 5(z - ca_2) = 0$, therefore $A = ((-c)(a_0, a_1, a_2)) + U$.

$\square$

Exercise 3.8. *Let $T \in \mathcal{L}(V, W)$ and $c \in W$, show that $\{ v \in V \mid Tv = c \}$ is an empty set or a translate of* $\operatorname{null} T$. *Then explain why the solutions of a system of linear equations is either an empty set or a translate of some subspace of F^n.*

Proof. Let $Ta = c$ for some $a \in V$, if no such a, then $\{ v \in V \mid Tv = c \} = \varnothing$. We claim $\{ v \in V \mid Tv = c \} = a + \operatorname{null} T$. For any $v \in V$ such that $Tv = c$, then $v = a + v - a$ and $T(v-a) = Tv - Ta = c - c = 0$, therefore $v - a \in \operatorname{null} T$, thus $v \in a + \operatorname{null} T$. In another direction, for any $a + v \in a + \operatorname{null} T$, we have $T(a+v) = Ta + Tv = c + 0 = c$. $\square$

Exercise 3.9. *Let $A \subseteq V$ a non-empty subset. Show that A is a translate of some subspace of $V \iff \lambda v + (1-\lambda)w \in A$ for any $v, w \in A$ and $\lambda \in F$.*

Proof.

- ($\Rightarrow$) Suppose $A = a + U$ for some subspace $U \subseteq V$.
- ($\Leftarrow$) Let $w \in A$, we will show that $(-w) + A$ is a subspace of V.

  For any $a - w, b - w \in (-w) + A$, we need to show that $a - w + b - w = (a + b - w) - w \in (-w) + A$ or equivalently $a + b - w \in A$. We found that the property $\lambda v + (1-\lambda)w \in A$ gives us the ability to construct something like $v - w$. Since $2v + (1-2)w = 2v - w$, we just let $w = v + a$ then $2v - (v + a) = v - a$. Therefore, we let $\lambda = 2$, $v = a + b$ and $w = a + b + w$, and now $2(a+b) - (a+b+w) = a + b - w \in A$, so $a + b - w - w \in (-w) + A$.

  For any $a - w \in (-w) + A$ and $\lambda \in F$, we need to show that $\lambda(a - w) \in (-w) + A$. $\lambda(a-w) = \lambda a - \lambda w = \lambda a - (\lambda - 1)w - w$. We let $\lambda = (-1)(\lambda - 1) = (1 - \lambda)$, $v = w$ and $w = a$ in $\lambda v + (1 - \lambda w) \in A$, then $(1-\lambda)w + (1 - (1-\lambda))a = (-1)(\lambda-1)w + \lambda a = \lambda a - (\lambda - 1)w \in A$, therefore $\lambda a - (\lambda - 1)w - w = \lambda a - \lambda w \in (-w) + A$.

  Therefore $(-w) + A$ is a subapce of V and $w + (-w) + A$ is a translate.

$\square$

Exercise 3.10. *Let $A = a + U$ and $B = b + W$ where $a, b \in V$, $U, W \subseteq V$ are subspaces. Show that $A \cap B$ is either a translate of some subspace of V or an empty space.*

Proof. Suppose $A \cap B \neq \varnothing$, we claim that $A \cap B$ is a translate of $U \cap W$, more specificly, for any $a + u_0 = b + w_0 \in A \cap B$, we claim that $A \cap B = (a + u_0) + U \cap W$.

For any $u = w \in U \cap W$, we have $(a + u_0) + u = a + (u_0 + u) \in a + U$, similarly, we have $(b + w_0) + w = b + (w_0 + w) \in b + W$, therefore $(a + u_0) + (U \cap W)\ subseteq A \cap B$.

For any $a + u = b + w \in A \cap B$, we have $a + u - (a + u_0) = u - u_0 \in U$ and $b + w - (b + w_0) = w - w_0 \in W$, therefore $A \cap B \subseteq (a + u_0) + (U \cap W)$. □

Exercise 3.12. *Let $v_0, \cdots, v_{m-1} \in V$ and*

$$A = \{\ \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1} \mid \lambda_i \in F \text{ and } \lambda_0 + \cdots + \lambda_i = 1\ \}$$

1. *Show that A is a translate of a subspace of V.*
2. *If B a translate of a subspace of V such that $v_0, \cdots, v_{m-1} \in B$, show that $A \subseteq B$.*
3. *Base on (1), show that the dimension of such subspace is less then m.*

Proof.

- If A is a translate of a subspace of V, say B, then for any $a \in A$, we have $A = a + B$. Therefore $B = (-a) + A$, we may pick $a = v_0$, we find that for any $b \in B$, it is in form $(-1)(v_0) + \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}$ where $\lambda_0 + \cdots + \lambda_{m-1} = 1$, which implies $(-1) + \lambda_0 + \cdots + \lambda_{m-1} = 0$. Then we claim $B = \{\ \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1} \mid \lambda_0 + \cdots + \lambda_{m-1} = 0\ \}$ is a subspace and $A = v_0 s + B$.
-

□

Exercise 3.16. *Let $\varphi \in \mathcal{L}(V, F)$ where $\varphi \neq 0$, show that* $\dim(V/(\operatorname{null}\varphi)) = 1$.

Proof. For any non-zero $v + \operatorname{null}\varphi, w + \operatorname{null}\varphi \in V/(\operatorname{null}\varphi)$ (existence is guaranteed since $\varphi \neq 0$), since $\varphi(w) \in F$, then there is some λ such that $\lambda\varphi(w) = \varphi(v)$ cause $\varphi(v)$ and $\varphi(w)$ are non-zero, then $\varphi(\lambda w) = \varphi(v)$, which means $v + \operatorname{null} T = (\lambda w) + \operatorname{null} T$, therefore $\dim(V/\operatorname{null}\varphi)$ cause any two (non-zero) vectors are linear dependent. □

Exercise 3.17. *Let $U \subseteq V$ a subspace such that* $\dim(V/U) = 1$. *Show that there is $\varphi \in \mathcal{L}(V, F)$ such that* $\operatorname{null}\varphi = U$.

Proof. We know there is an isomorphism $i \in \mathcal{L}(V/U, F)$ since $\dim(V/U) = \dim F = 1$, then $\varphi = i \circ \pi$ where $\pi \in \mathcal{L}(V, V/U)$. Since i is injective, $\operatorname{null}\varphi = \operatorname{null}\pi = U$. $\square$

Exercise 3.18. *Explain why a linear functional is either surjective or* 0.

Proof. Cause $\dim F = 1$. $\square$

Exercise 3.6. *Let $\varphi, \beta \in V'$, show that* $\operatorname{null}\varphi \subseteq \operatorname{null}\beta \iff \exists c \in F, \beta = c\varphi$.

Proof.

- $(\Rightarrow)$ For any $v \notin \operatorname{null}\beta$, we have $\beta(v) = \beta(v)(\varphi(v))^{-1}\varphi(v)$ we claim that $\beta = \beta(v)(\varphi(v))^{-1}\varphi$. We may denote $\beta(v)(\varphi(v))^{-1}$ by c. For any $v, w \notin \operatorname{null}\beta$, we have $\beta(v) = a\varphi(v)$ and $\beta(w) = b\varphi(w)$, we want to show that $a = b$, which can be proven by:

$$\begin{aligned} a &= b \\ \frac{\beta(v)}{\varphi(v)} &= \frac{\beta(w)}{\varphi(w)} \\ \beta(v)\varphi(w) &= \beta(w)\varphi(v) \\ \beta(\varphi(w)v) &= \beta(\varphi(v)w) \end{aligned}$$

  which is equivalent to $\varphi(w)v - \varphi(v)w \in \operatorname{null}\beta$, then:

$$\begin{aligned} &\varphi(\varphi(w)v - \varphi(v)w) \\ =&\varphi(\varphi(w)v) - \varphi(\varphi(v)w) \\ =&\varphi(w)\varphi(v) - \varphi(v)\varphi(w) \\ =&0 \end{aligned}$$

  therefore $\varphi(w)v - \varphi(v)w \in \operatorname{null}\varphi \subseteq \operatorname{null}\beta$, thus $a = b$.

  The case $v \in \operatorname{null}\beta$ is trivial.

- $(\Leftarrow)$ For any $v \in \operatorname{null}\varphi$, $\beta(v) = c\varphi(v) = 0$, therefore $v \in \operatorname{null}\beta$, thus $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$.

$\square$

Exercise 3.7. *Let $V_0, \cdots, V_{m-1}$ are vector spaces, show that $V_0' \times \cdots \times V_{m-1}'$ and $(V_0 \times \cdots \times V_{m-1})'$ are isomorphic.*

Proof. Define $\psi(\varphi) = v_0 \mapsto \varphi(v_0, 0, \cdots), \cdots, v_{m-1} \mapsto \varphi(\cdots, 0, v_{m-1})$ and $\psi^{-1}(\varphi_0, \cdots, \varphi_{m-1}) = (v_0, \cdots, v_{m-1}) \mapsto \varphi_0(v_0) + \cdots + \varphi_{m-1}(v_{m-1})$.

For any $\alpha, \beta \in (V_0 \times \cdots \times V_{m-1})'$ and $\lambda \in F$, we have

$$\begin{aligned}
&\psi(\alpha+\beta)_i \\
=&v_i \mapsto (\alpha+\beta)(\cdots, v_i, \cdots) \\
=&v_i \mapsto \alpha(\cdots, v_i, \cdots) + \beta(\cdots, v_i, \cdots) \\
=&(v_i \mapsto \alpha(\cdots, v_i, \cdots)) + (v_i \mapsto \beta(\cdots, v_i, \cdots)) \\
=&\psi(\alpha)_i + \psi(\beta)_i
\end{aligned}$$

and $(\lambda\psi(\alpha))_i = \lambda(v_i \mapsto \alpha(v_i)) = v_i \mapsto \lambda\alpha(v_i) = \psi(\lambda\alpha)_i$ Therefore ψ is a linear map.

For any $\alpha, \beta \in V_0' \times \cdots \times V_{m-1}'$ and $\lambda \in F$, we have:

$$\begin{aligned}
&\psi^{-1}(\alpha+\beta) \\
=&(v_0, \cdots, v_{m-1}) \mapsto (\alpha+\beta)(v_0) + \cdots + (\alpha+\beta)(v_{m-1}) \\
=&(v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \beta(v_0) + \cdots \\
=&((v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \cdots) + ((v_0, \cdots, v_{m-1}) \mapsto \beta(v_0) + \cdots) \\
=&\psi^{-1}(\alpha) + \psi^{-1}(\beta)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda\psi^{-1}(\alpha) \\
=&\lambda((v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \cdots) \\
=&(v_0, \cdots, v_{m-1}) \mapsto \lambda(\alpha(v_0) + \cdots) \\
=&(v_0, \cdots, v_{m-1}) \mapsto (\lambda\alpha(v_0)) + \cdots \\
=&\psi^{-1}(\lambda\alpha)
\end{aligned}$$

thus ψ^{-1} is a linear map.

We will show that ψ^{-1} is the inverse of ψ then ψ is an isomorphism. For any $\varphi \in (V_0 \times \cdots \times V_{m-1})'$,

$$\begin{aligned}
&\psi^{-1}(\psi(\varphi)) \\
=&v_0, \cdots, v_{m-1} \mapsto \psi(\varphi)_0(v_0) + \cdots \\
=&v_0, \cdots, v_{m-1} \mapsto \varphi(v_0, 0, \cdots) + \cdots + \varphi(\cdots, 0, v_{m-1}) \\
=&v_0, \cdots, v_{m-1} \mapsto \varphi(v_0, \cdots, v_{m-1}) \\
=&\varphi
\end{aligned}$$

and for any $\varphi \in V_0' \times \cdots \times V_{m-1}'$,

$$\begin{aligned}
&\psi(\psi^{-1}(\varphi)) \\
=&v_0 \mapsto \psi^{-1}(\varphi)(v_0, 0, \cdots), \cdots \\
=&v_0 \mapsto \varphi_0(v_0), \cdots \\
=&\varphi_0, \cdots, \varphi_{m-1} \\
=&\varphi
\end{aligned}$$

□

Exercise 3.16. *Let W a finite vector space, $T \in \mathcal{L}(V, W)$, show that*

$$T' = 0 \iff T = 0$$

Proof.

- ($\Rightarrow$) Suppose $T \neq 0$, then we can always find $\varphi \in \mathcal{L}(W, F)$ which $\varphi(\operatorname{range} T) \neq 0$, then $\varphi \circ T \neq 0$.
- ($\Leftarrow$) Trivial.

□

Exercise 3.17. *Let V, W are finite vector spaces, $T \in \mathcal{L}(V, W)$. Show that T is invertible $\iff$ T' is invertible.*

Proof. Since T is invertible, then T is injective, therefore T' is surjective. Similarly, T' is injective since T is surjective. Therefore T' is invertible. □

Exercise 3.18. *Let V, W are finite vector spaces, show that the mapping $\varphi(T) = T'$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathcal{L}(W', V')$.*

Proof. Since V and W are finite, we only need to show that φ is injective or surjective. We will show that φ is injective.

For any $\varphi(T) = T' \in \mathcal{L}(W', V')$, we know $T = 0 \iff T' = 0$, therefore $\operatorname{null} \varphi = \{0\}$, thus φ is injective.

I was wonder if I can prove this by $\varphi(S)(\mathrm{id}) = \varphi(T)(\mathrm{id}) \implies S = T$. This may work if the codomain restriction doesn't lose information, i.e. only restrict to a superset of its range, therefore the restriction is one-to-one. □

Exercise 3.21. *Let V finite and $U, W \subseteq V$ are subspaces.*

1. *Show that* $W^0 \subseteq U^0 \iff U \subseteq W$

2. *Show that* $W^0 = U^0 \iff U = W$

Proof. The second statement can be easy proved by the first one.

- ($\Rightarrow$) We can always find a $f \in \mathcal{L}(W, F)$ such that $\operatorname{null} f = W$, then $f(U) = \{0\}$ since $f \in W^0 \subseteq U^0$, therefore $U \subseteq \operatorname{null} f = W$.

- ($\Leftarrow$) For any $\varphi \in W^0$, we know $W \subseteq \operatorname{null} \varphi$, then $U \subseteq W \subseteq \operatorname{null} \varphi$, therefore $\varphi in U^0$, thus $W^0 \subseteq U^0$.

$\square$

Exercise 3.22. *Let V finite and $U, W \subseteq V$ are subspaces. Show that:*

- $(U + W)^0 = U^0 \cap W^0$
- $(U \cap W)^0 = U^0 + W^0$

Proof.

- For any $\varphi \in (U+W)^0$ we have $U+W \subseteq \operatorname{null} \varphi$, then $U \subseteq U+W \subseteq \operatorname{null} \varphi$ and $W \subseteq U + W$, therefore $\varphi \in U^0 \cap W^0$.

  For any $\varphi \in U^0 \cap W^0$, we have $U \subseteq \operatorname{null} \varphi$ and $W \subseteq \operatorname{null} \varphi$. For any $u + w \in U + W$, we have $\varphi(u + w) = \varphi(u) + \varphi(w) = 0 + 0 = 0$, therefore $U + W \subseteq \operatorname{null} \varphi$, thus $\varphi \in (U + W)^0$.

- For any $su + tw \in U^0 + W^0$, for any $v \in U \cap W$, we have $su(v) + tw(v) = s0 + t0$ since $v \in U$ and $v \in W$. Therefore we have an injective map (also linear, this map just produce what it receive) from $U^0 + W^0$ to $(U \cap W)^0$. We have:

$$
\begin{aligned}
&\dim(U^0 + W^0) \\
=&\dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
=&\dim V - \dim U + \dim V - \dim W - \dim(U + W)^0 \\
=&\dim V - \dim U + \dim V - \dim W - (\dim V - \dim(U + W)) \\
=&\dim V - \dim U - \dim W + (\dim U + \dim W - \dim(U \cap W)) \\
=&\dim V - \dim(U \cap W) \\
=&\dim(U \cap W)^0
\end{aligned}
$$

  therefore $(U \cap W)^0 = U^0 + W^0$.

$\square$

Exercise 3.23. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1} \in V'$. Show that the following sets are equal to each others:*

- $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1})$
- $((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0$
- $\{\ \varphi \in V' \mid (\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}) \subseteq \operatorname{null} \varphi\ \}$

Proof.

- $((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0 = (\operatorname{null} \varphi_0)^0 + \cdots + (\operatorname{null} \varphi_{m-1})^0$, then $\operatorname{span}(\varphi_i) \subseteq (\operatorname{null} \varphi_i)^0$ therefore $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) \subseteq ((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0$.

  For any $\varphi \in \operatorname{span}(\varphi_0, \cdots, \varphi_{m-1})$, we have $\varphi(v) = \varphi_0(v) + \cdots + \varphi_{m-1}(v) = 0 + \cdots + 0 = 0$ for any $v \in (\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1})$, therefore $\varphi \in ((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0$.

- Last two sets are definitional equal.

$\square$

Exercise 3.24. *Let V finite and $v_0, \cdots, v_{m-1} \in V$.*
Define $\Gamma(\varphi) = (\varphi(v_0), \cdots \varphi(v_{m-1})) : V' \to F^m$, show that:

- *$v_0, \cdots, v_{m-1}$ spans $V \iff \Gamma$ is injective.*
- *$v_0, \cdots, v_{m-1}$ is linear independent $\iff \Gamma$ is surjective.*

Proof.

- ($\Rightarrow$) Suppose $\Gamma(\alpha) = \Gamma(\beta)$, then for all $v \in V$ can be factorized into $\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}$, then $\alpha(v) = \alpha(\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}) = \beta(\lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}) = \beta(v)$ since $\Gamma(\alpha) = \Gamma(\beta)$ and α and β are linear map, thus $\alpha = \beta$.

  ($\Rightarrow$) We first make $v_0, \cdots, v_{m-1}$ linear independent, say $v_0, \cdots, v_{k-1}$, then for any $w \in V$ such that $v_0, \cdots, v_{k-1}, w$ is linear independent, then we have its dual basis $\varphi_0, \cdots, \varphi_{k-1}, \psi$. Consider $\Gamma(\psi)$, by definition, we know $\Gamma(\psi) = (\psi(v_0), \cdots) = (0, \cdots)$ then $\psi = 0$ since Γ is injective, which contradicts our assumption. Therefore $v_0, \cdots, v_{k-1}$ spans V.

- ($\Rightarrow$) Consider the dual basis of $v_0, \cdots, v_{m-1}$, then Γ is surjective since we have the standard basis of F^m.

  ($\Leftarrow$) Γ is surjective implies we have $\varphi_0, \cdots, \varphi_{m-1}$ such that $\Gamma(\varphi_i) = (\cdots, 1, \cdots)$, which means $v_0, \cdots, v_{m-1}$ is linear independent.

$\square$

Exercise 3.25. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1} \in V'$.*
Define $\Gamma(v) = (\varphi_0(v), \cdots, \varphi_{m-1}(v)) : V \to F^m$. Show that

- *$\varphi_0, \cdots, \varphi_{m-1}$ spans $V' \iff \Gamma$ is injective*
- *$\varphi_0, \cdots, \varphi_{m-1}$ is linear independent $\iff \Gamma$ is surjective*

Proof.

- ($\Rightarrow$) Suppose $\Gamma(v) = \Gamma(w)$, then $\varphi_i(v) = \varphi_i(w)$, which means $\varphi_i(v-w) = 0$ for all i. If $v - w \neq 0$, then $((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0 \neq \{0\}$, thus $\varphi_0, \cdots, \varphi_{m-1}$ doesn't span V'.

  ($\Leftarrow$) $(\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}) = \{0\}$ since Γ is injective. therefore $\operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) = ((\operatorname{null} \varphi_0) \cap \cdots \cap (\operatorname{null} \varphi_{m-1}))^0 = (\{0\})^0 = V'$

- ($\Rightarrow$) We may treat Γ as the following matrix:

$$\begin{bmatrix} \varphi_0 \\ \vdots \\ \varphi_{m-1} \end{bmatrix}$$

  which line rank is m since $\varphi_0, \cdots, \varphi_{m-1}$ is linear independent, therefore its column rank is m, thus $\dim \operatorname{range} \Gamma = m = \dim F^m$, then Γ is surjective.

  ($\Leftarrow$) It seems the proof of ($\Rightarrow$) also works here.

$\square$

Exercise 3.26. *Let V finite, and $\Omega \subseteq V'$ a subspace. Show that*

$$\Omega = \{\ v \in V \mid \varphi(v) = 0 \quad \forall \varphi \in \Omega\ \}^0$$

Proof. This construction looks like an inverse of $-^0$.

We may rewrite the equation to $\Omega = (\bigcap_{\varphi \in \Omega} \operatorname{null} \varphi)^0$, then $\Omega = \operatorname{span}(\varphi) \forall \varphi \in \Omega$, which is trivial. $\square$

Exercise 3.28. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1}$ is linear independent. Show that*

$$\dim((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1})) = \dim V - m$$

Proof.

$$\begin{aligned} m &= \dim \operatorname{span}(\varphi_0, \cdots, \varphi_{m-1}) \\ &= \dim((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1}))^0 \\ &= \dim V - \dim((\operatorname{null}\varphi_0) \cap \cdots \cap (\operatorname{null}\varphi_{m-1})) \end{aligned}$$

□

Exercise 3.30. *Let V finite and $\varphi_0, \cdots, \varphi_{m-1}$ a basis of V'. Show that there is a basis of V which dual basis is $\varphi_0, \cdots, \varphi_{m-1}$.*

Proof. Since $\varphi_0, \cdots, \varphi_{m-1}$ spans V' and linear independent, we know Γ is both injective and surjective. Consider $v_0, \cdots, v_{m-1}$ such that $\Gamma(v_i) = (\cdots, 0, 1, 0, \cdots)$. We claim $v_0, \cdots, v_{m-1}$ is a basis of V and which dual basis if $\varphi_0, \cdots, \varphi_{m-1}$.

The second part is trivial by the way construct them. For the first part, $v_0, \cdots, v_{m-1}$ is linear independent since $(\cdots, 0, 1, 0 \cdots)$ is linear independent, and $v_0, \cdots, v_{m-1}$ spans V since $\dim V = \dim V' = m$. □

Exercise 3.31. *Let $U \subseteq V$ a subspace and $i(u) = u : U \to V$. Then $i' \in \mathcal{L}(V', U')$, show that:*

1. $\operatorname{null} i' = U^0$
2. $\operatorname{range} i' = U'$ *if V is finite*
3. $\tilde{i}'$ *is an isomorphism between V'/U^0 and U' if V is finite*

Proof.

- For any $\varphi \in \operatorname{null} i'$, $\varphi \circ i = 0$, therefore $\operatorname{range} i = U \subseteq \operatorname{null}\varphi$, thus $\varphi \in U^0$.

  For any $\varphi \in U^0$, $\varphi \circ i = 0$ since $\operatorname{range} i = U \subseteq \operatorname{null}\varphi$.

- Suppose V is finite, then i' is surjective since i' is injective, therefore $\operatorname{range} i' = U'$.

- $\tilde{i}'(\varphi + U^0) = i'(\varphi)$ is surjective since i' is surjective. Then $\dim(V'/U^0) = \dim V' - \dim U^0 = \dim V - (\dim V - \dim U) = \dim U = \dim U'$, therefore $\tilde{i}'$ is an isomorphism.

□

Exercise 3.32. *We denote V'' as the* ***double dual space of*** *V, defined by $V'' = (V')'$. Define $\Lambda(v)(\varphi) = \varphi(v) : V \to V''$*

Show that:

1. $\Lambda \in \mathcal{L}(V, V'')$
2. *Let $V \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$ where $T'' = (T')'$.*
3. *Λ is an isomorphism if V is finite.*

Proof.

- For any $v, w \in V$ and $\lambda \in F$, we have $(\Lambda(v) + \Lambda(w))(\varphi) = \Lambda(v)(\varphi) + \Lambda(w)(\varphi) = \varphi(v) + \varphi(w) = \varphi(v + w) = \Lambda(v + w)(\varphi)$ and $(\lambda\Lambda(v))(\varphi) = \lambda(\Lambda(v)(\varphi)) = \lambda(\varphi(v)) = \varphi(\lambda v) = \Lambda(\lambda v)(\varphi)$.
- For any $v \in V$,

$$\begin{aligned}
&(T'' \circ \Lambda)(v)(\varphi) \\
=&(T''(\Lambda(v)))(\varphi) \\
=&((\Lambda(v)) \circ T')(\varphi) \\
=&\Lambda(v)(T'(\varphi)) \\
=&\Lambda(v)(\varphi \circ T) \\
=&(\varphi \circ T)(v) \\
=&\varphi(T(v)) \\
=&\Lambda(T(v))(\varphi) \\
=&(\Lambda \circ T)(v)(\varphi)
\end{aligned}$$

- Suppose $\Lambda(v) = \Lambda(w)$, that is, $\Lambda(v)(\varphi) = \varphi(v) = \varphi(w) = \Lambda(w)(\varphi)$ for all $\varphi \in V'$. Let $\varphi_0, \cdots, \varphi_{m-1}$ the dual basis of some basis of V, then $v = \varphi_0(v)v_0 + \cdots + \varphi_{m-1}(v)v_{m-1} = \varphi_0(w)v_0 + \cdots + \varphi_{m-1}(w)v_{m-1} = w$. Therefore Λ is injective, thus surjective and isomorphism since $\dim V = \dim V''$.

□

Exercise 3.33. *Let $U \subseteq V$ a subspace and $\pi : V \to V/U$ the quotient map, then $\pi' \in \mathcal{L}((V/U)', V')$.*

1. *Show that* π' *is injective.*
2. *Show that* $\operatorname{range}\pi' = U^0$.
3. *Conclude that* π' *is an isomorphism between* $(V/U)'$ *and* U^0.

Proof.

- π is surjective, therefore π' is injective. The statement is true even V or V/U may be infinite, cause the proof about surjective-implies-epimorphism doesn't require that the codomain is finite but epimorphism-implies-surjective does.

  We may prove those theorem again, but with weaker assumption. For any $\pi'(\varphi) = \pi'(\psi)$, we have $\varphi \circ \pi = \psi \circ \pi$. For any $v + U \in V/U$, there is $v \in V$ such that $\pi(v) = v + U$ since π is surjective. Therefore $\varphi(\pi(v)) = \psi(\pi(v))$ for all $\pi(v) = v + U \in V/U$,thus $\varphi = \psi$.

  Therefore π' is injective.

- $\operatorname{range}\pi' = (\operatorname{null}\pi)^0 = U^0$.

- Trivial.

□

Exercise 3.7. *Let* m *a non-negative integer and* $z_0, \cdots, z_m \in F$ *are different to each others,* $w_0, \cdots, w_m \in F$, *show that there is a unique* $p \in \mathcal{P}_m(F)$ *such that* $p(z_k) = w_k$ *holds for all* $0 \leq k \leq m$.

Proof. Define $\Gamma(p) = (p(z_0), \cdots, p(z_m)) : \mathcal{P}_m(F) \to F^{m+1}$, we will show that Γ is injective, therefore an isomorphism.

Suppose $\Gamma(p) = \Gamma(q)$, then $p(z_k) = q(z_k)$ for all k, therefore $(p-q)(z_k) = 0$ for all k. This means $p - q$ has $m + 1$ zeros but $\deg(p - q) \leq m$, therefore $p - q = 0$ and $p = q$.

Then there is a unique $p \in \mathcal{P}_m(F)$ such that $\Gamma(p) = (w_0, \cdots, w_m)$. □

Exercise 3.9. *Let* $P \in \mathcal{L}(V)$, *such that* $P^2 = P$. *Suppose* λ *an eigenvalue of* P, *show that* $\lambda = 0$ *or* $\lambda = 1$.

Proof. Suppose $P(v) = \lambda v$ for some non-zero $v \in V$, then $P(v) = PP(v) = P(\lambda v)$, therefore $P((\lambda - 1)v) = 0$. thus $(\lambda - 1)v \in \operatorname{null} P$. We may suppose $\lambda \neq 1$, then $(\frac{1}{\lambda-1})(\lambda - 1)v = v \in \operatorname{null} P$, therefore $P(v) = 0$, thus $\lambda = 0$ cause $v \neq 0$. □

Exercise 3.10. *Let $T(p) = p' : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$. Find all eigenvalues and eigenvectors of T.*

Proof. Suppose $T(p) = p' = \lambda p$, then $\deg p = 0$, otherwise the degree doesn't match. For any $p \in \mathcal{P}(\mathbb{R})$ such that $\deg p = 0$, we have $p' = 0 = 0p$. $\square$

Exercise 3.12. *Let $V = U \oplus W$ where U and W are non-zero subspaces. Define $P(u + w) = u$ for all $u \in U$ and $w \in W$. Find all eigenvalue and eigenvector of P.*

Proof. We can see $P^2 = P$, since for any $u + w \in V$, we have $P(P(u + w)) = P(u) = u = P(u + w)$, therefore $\lambda = 0$ and $\lambda = 1$ are eigenvalues of P , $P(u) = 1u$ and $P(w) = 0w$ are eigenvectors of P. $\square$

Exercise 3.13. *Let $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$, where S is invertible.*

- *Show that T has the same eigenvalue of $S^{-1}TS$.*
- *What is the relationship between the eigenvector of T and the eigenvector of $S^{-1}TS$.*

Proof.

- For any $T(v) = \lambda v$ where $v \in V$ and $\lambda \in F$, let $S(w) = v$, then $S^{-1}TS(w) = S^{-1}(T(Sw)) = S^{-1}(\lambda v) = \lambda S^{-1}(v) = \lambda w$, thus λ is an eigenvalue of $S^{-1}TS$.
- $S(w) = v$ where v is an eigenvector of T and w is the corresponding eigenvector of $S^{-1}TS$.

$\square$

Exercise 3.15. *Let V finite, $T \in \mathcal{L}(V)$, $\lambda \in F$. Show that λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of T'.*

Proof.

- ($\Rightarrow$) Suppose $Tv = \lambda v$, we will show $T' - \lambda I$ is not surjective (Note that $I \in \mathcal{L}(V')$).

  For any $\varphi \in V'$, we have:

$$
\begin{aligned}
&(T' - \lambda I)(\varphi) \\
=&T'(\varphi) - \lambda\varphi \\
=&\varphi \circ T - \lambda\varphi
\end{aligned}
$$

then

$$\begin{aligned} &(\varphi \circ T - \lambda\varphi)(v) \\ =&(\varphi \circ T)(v) - (\lambda\varphi)(v) \\ =&\varphi(Tv) - \lambda(\varphi(v)) \\ =&\varphi(\lambda v) - \lambda(\varphi(v)) \\ =&0 \end{aligned}$$

This means $\operatorname{range} T' \neq V'$ cause any $\psi \in V'$ where $\psi(v) \neq 0$ is not in $\operatorname{range} T'$. Therefore λ is an eigenvalue of T'.

□

Exercise 3.22. *Let $T \in \mathcal{L}(V)$ and non-zero $v, w \in V$ such that*

$$Tu = 3w \quad \textit{and} \quad Tw = 3u$$

.

Show that 3 or -3 is the eigenvalue of T.

Proof. Since v and w are non-zero, then one of $u + w$ and $u - w$ is non-zero.

We have $T(u + w) = 3w + 3u = 3(u + w)$ and $T(u - w) = 3w - (3u) = (-3)(u - w)$. □

Exercise 3.23. *Let V finite, and $S, T \in \mathcal{L}(V)$, show that ST and TS have the same eigenvalues.*

Proof. For any $ST(v) = \lambda v$ where $v \neq 0$, we have $TST(v) = T(\lambda v)$ then $TS(Tv) = \lambda(Tv)$.

- If $Tv = 0$, then $STv = 0 = \lambda v$, thus $\lambda = 0$ since $v \neq 0$. then λ is an eigenvalue of TS since $\operatorname{null} TS \neq 0$ ($\operatorname{null} T \neq 0$, and S is an operator of V, therefore, S is injective or not doesn't affect our conclusion).

  If $Tv \neq 0$, then $TS(Tv) = \lambda(Tv)$.

- Ditto.

□

Exercise 3.26. *Let $T \in \mathcal{L}(V)$ and any non-zero $v \in V$ we have $Tv = cv$ for some c. Show that $T = \lambda I$.*

Proof. Let non-zero $v, w \in V$, we have $Tv = sv$ and $Tw = tw$, then $T(v+w) = \lambda(v + w) = \lambda v + \lambda w = sv + tw = T(v) + T(w)$. Then $\lambda v + \lambda w - tw = sv$.

- If $w \in \text{span}(v)$, then $w = cv$, therefore $Tw = T(cv) = tcv = cTv = csv$, thus $t = s$.

- If $w \notin \text{span}(v)$, then $\lambda = t$ (otherwise $\lambda v + (\lambda - t)w = sv$), therefore $\lambda v = sv$ and $\lambda = s$, thus $s = t$.

□

Exercise 3.27. *Let V finite and $1 \leq k \leq \dim V - 1$. Let $T \in \mathcal{L}(V)$ such that any subspace of V with k dimension is invariant under T. Show that $T = \lambda I$ for some λ.*

Proof. For any $v \in V$, we have $Tv = w$ where $w \in \text{span}(v)$.

- If $w = 0$, then $Tv = 0v$.

- If $w \neq 0$, then $w = \lambda v$ since $w \in \text{span}(v)$, then $Tv = \lambda v$.

Thus $T = \lambda I$ by the previous exercise. □

Exercise 3.30. *Let $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Show that 2 or 3 or 4 is the eigenvalue of T.*

Proof. Suppose 2 is not an eigenvalue of T, then $(T - 2I)$ is injective, thus $(T - 3T)(T - 4I)$ must map all $v \in V$ to 0. Similarly, we can show that $(T - 4I) = 0$ if 3 is not an eigenvalue of T. □

Exercise 3.31. *Find $T \in \mathcal{L}(\mathcal{R}^2)$ such that $T^4 = -I$.*

Proof. We may treat $-I$ as rotating the vector 180 degrees, then T rotates a vector 45 degrees, which matrix is:

$$\mathcal{M}(T) = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

□

Exercise 3.32. *Let $T \in \mathcal{L}(V)$ with no eigenvalue and $T^4 = I$. Show that $T^2 = -I$.*

Proof. We will show that $T^2(v) = -v$ for any $v \in V$. Consider $T^2(T^2(v)+v) = v+T^2(v)$, we will show that $T^2(v)+v = 0$. Suppose $w \in V$ and $T^2(w) = w$, we may let $T(w) = u$, then $T^2(w) = Tu = w$. Consider $T(w+u) = T(w)+T(u) = u + w$, then $w + u = 0$ since T has no eigenvalue, therefore $u = -w$ and $T(w) = -w$. Again $w = 0$ since T has no eigenvalue. Therefore $T^2(w) = w$ implies $w = 0$, thus $T^2(v) + v = 0$ and $T^2(v) = -v$ for any $v \in V$. $\square$

Exercise 3.33. *Let $T \in \mathcal{L}(V)$ and $m \in \mathbb{N}^+$.*

1. *Show that T is injective $\iff$ T^m is injective*

2. *Show that T is surjective $\iff$ T^m is surjective.*

Proof. Recall that $T^0 = I$.

- ($\Rightarrow$) For any $T^m(v) = T^m(w)$ we have $T^{m-1}(v) = T^{m-1}(w)$ and so on, we will get $v = w$.

  ($\Leftarrow$) For any $T(v) = T(w)$, we have $T^{m-1}(T(v)) = T^{m-1}(T(w))$, then $T^m(v) = T^m(w)$ and $v = w$.

- ($\Rightarrow$) For any $w \in V$, we have $T(v) = w$, then we have $T(u) = v$ and now $T(T(u)) = w$, continue this progress until we get $T^m(r) = w$.

  ($\Leftarrow$) For any $w \in V$, we have $T^m(v) = w$, therefore $T(T^{m-1}(v)) = w$.

$\square$

Exercise 3.34. *Let V finite and $v_0, \cdots, v_{m-1} \in V$. Show that $v_0, \cdots, v_{m-1}$ is linear independent $\iff$ there is $T \in \mathcal{L}(V)$ such that $v_0, \cdots, v_{m-1}$ are eigenvectors of distinct eigenvalues of T.*

Proof.

- ($\Rightarrow$) Consider $v_0, \cdots, v_{k-1}$ a basis of V, then $T(\lambda_0 v_0 + \cdots + \lambda_{k-1} v_{k-1}) = 1\lambda_0 v_0 + 2\lambda_1 v_1 + \cdots + m\lambda_{m-1} v_{m-1}$ where $T(v_i) = (i + 1)v_i$.

- ($\Leftarrow$) Trivial, since eigenvectors of distinct eigenvalues are linear independent.

$\square$

Exercise 3.37. *Let V finite and $T \in \mathcal{L}(V)$. Define $\mathcal{A}(S) = TS : \mathcal{L}(V) \to \mathcal{L}(V)$. Show that T has the same eigenvalues as $\mathcal{A}$.*

Proof.

- ($\subseteq$) For any eigenvalue λ of T, we can find $S \in \mathcal{L}(V)$ such that range $S = \{ v \in V \mid Tv = \lambda v \}$ (it is easy to show that such set is a subspace). Then for any $v \in V$, $(TS)v = T(Sv) = \lambda(Sv) = (\lambda S)v$ thus $\mathcal{A}(S) = TS = \lambda S$.

- ($\supseteq$) For any eigenvalue λ of $\mathcal{A}$, then we have $\mathcal{A}(S) = \lambda S$ for some non-zero $S \in \mathcal{L}(V)$. Then there is $v \in V$ such that $Sv \neq 0$, and $T(Sv) = (TS)v = (\lambda S)(v) = \lambda(Sv)$, thus λ is an eigenvalue of T.

$\square$

Exercise 3.38. *Let V finite and $T \in \mathcal{L}(V)$ and $U \subseteq V$ is invariant under T. A quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by:*

$$(T/U)(v + U) = Tv + U$$

for any $v \in V$.

1. *Show that T/U is well-defined and T/U is an operator over V/U.*

2. *Show that each eigenvalue of T/U is also an eigenvalue of T.*

Proof.

- Suppose $v+U = w+U$, then $(T/U)(v+U) = Tv+U$ and $(T/U)(w+U) = Tw + U$, we will show that $Tv - Tw \in U$. Note that $v + U = w + U$ implies $v - w \in U$, then $T(v - w) \in U$ since U is invariant under T, that is, for any $u \in U$, $Tu \in U$. Thus $Tv + U = Tw + U$.

  Now we will show that T/U is a linear map, we can see:

$$\begin{aligned}
&(T/U)(v + U) + (T/U)(w + U) \\
=&(Tv + U) + (Tw + U) \\
=&(Tv + Tw) + U \\
=&T(v + w) + U \\
=&(T/U)((v + w) + U)
\end{aligned}$$

and

$$
\begin{aligned}
&\lambda(T/U)(v+U)\\
=&\lambda(Tv+U)\\
=&(\lambda(Tv))+U\\
=&T(\lambda v)+U\\
=&(T/U)((\lambda v)+U)\\
=&(T/U)(\lambda(v+U))
\end{aligned}
$$

- Suppose $(T/U)(v+U) = Tv+U = \lambda v + U$ where $v \notin U$, consider $T-\lambda I$, we will show that $T-\lambda I$ is not injective. We can see U is invariant under $T-\lambda I$, $Tu-\lambda u \in U$ cause U is invariant under T. We may suppose T is injective (thus surjective and invertible) on U (in other words, $T(U)=U$), otherwise the proof is complete. Then consider $(T-\lambda I)(v) = Tv-\lambda v \in U$ where $v \notin U$, thus $T-\lambda I$ is not injective.

□

Exercise 3.39. *Let V finite and $T \in \mathcal{L}(V)$. Show that T has an eigenvalue $\iff$ there is a subspace of V with dimension* $\dim V - 1$ *which is invariant under T.*

Proof.

- This part is hinted by AI. Suppose $Tv = \lambda v$, then consider $T-\lambda I$, we know range$(T-\lambda I)$ is invariant under T, since for any $Tw-\lambda w$, we have $T(Tw-\lambda w) = T(Tw)-T(\lambda w) = T(Tw)-\lambda(Tw)$. Then $\dim \operatorname{range}(T-\lambda I) \le \dim V - 1$ since $w \in \operatorname{null} T-\lambda I$. Then consider $\operatorname{null}(T-\lambda I) = \operatorname{span}(v) \oplus W$, we have range$(T-\lambda I) \oplus W$ a subspace which is invariant under T.

  The key is finding a smaller invariant subspace and expand it with null space, as any vector in null space always maps to 0, thus preserve the property of invariant.

- Suppose U is a subspace of V of dimension $\dim V - 1$ such that U is invariant under T, then $V = U \oplus \operatorname{span}(v)$ for some $v \notin U$. We may suppose T is injective on U, otherwise the proof is complete ($\operatorname{null} T \neq 0$). Consider $T(v)$, there are three cases:

- $T(v) = \lambda v + 0u$, then the proof is complete.
- $T(v) = 0v + u$, then T is not injective since there is $Tw = u$ where $w \in U$.
- $T(v) = \lambda v + u$, then consider $T - \lambda I$. We have U is invariant under $T - \lambda I$ cause $Tu - \lambda u \in U$ by $Tu \in U$. Again, if $T - \lambda I$ is not injective on U, the proof is complete. Then $(T - \lambda I)v = Tv - \lambda v \in U = \lambda v + u - \lambda v = u \in U$, thus $T - \lambda I$ is not injective and λ is an eigenvalue of T.

□

Exercise 3.42. *Let $T \in \mathcal{L}(F^n)$ defined by $T(x_1, x_2, \cdots, x_n) = (x_1, 2x_2, \cdots, nx_n)$.*

1. *Find all eigenvalues and eigenvectors of T.*
2. *Find all subspace of F^n which is invariant under T.*

Proof.

- $1, 2, \cdots, n$ and $(x_1, 0, \cdots), (0, x_2, 0, \cdots), \cdots$
- We claim any subspace that is invariant under T is a direct sum of some spaces that spans by the standard basis, say $\operatorname{span}(x_0) \oplus \cdots \oplus \operatorname{span}(x_k)$.

  Let U a subspace that is invariant under T and $u \in U$, we have $T(u) = T(\lambda_1 x_1, \cdots, \lambda_n x_n) = (\lambda_1 x_1, \cdots, n\lambda_n x_n)$, then $T(u) - iu = ((1-i)(\lambda_1 x_1), (2-i)(\lambda_2 x_2), \cdots, (n-1)(\lambda_i x_i)) \in U$ is a vector that is a linear combination of standard basis except x_i. Repeat this progress by apply $T - jI$ to $(T - iI)(u)$ with a different j, we can finally get a vector that is a scalar multiple of x_k. Thus $x_i \in U$ as long as there is $u \in U$ that the ith scalar of the linear combination of standard basis is not zero.

□

Exercise 3.43. *Let $T \in \mathcal{L}(V)$. Show that 9 is an eigenvalue of $T^2 \iff 3$ or -3 is an eigenvalue of T.*

Proof.

- $(\Rightarrow)$ We have $T^2 - 9I$ is not injective since 9 is an eigenvalue of T^2, then $(T - 3I)(T + 3I) = T^2 - 9I$ is not injective means one of $T - 3I$ and $T + 3I$ is not injective, thus 3 or -3 is an eigenvalue of T.

- ($\Leftarrow$) Similarly, we have $(T-3I)(T+3I)v = (T^2-9I)v = 0$ (if 3 is an eigenvalue of T) or $(T+3I)(T-3I)v = (T^2-9I)v = 0$ (if -3 is an eigenvalue of T).

$\square$

Exercise 3.44. *Let V a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$ has no eigenvalue. Show that any subspace of V that is invariant under T is either $\{0\}$ or infinite dimension.*

Proof. Let $U \subseteq V$ a subspace that is invariant under T, and non-zero $u \in U$. We can repeatly apply T to u, say $u, Tu, T^2u, \cdots$. Suppose $k > 0$ is minimum such that $u, Tu, \cdots, T^k u$ is linear dependent, we have $p \in \mathcal{P}(\mathbb{C})$ with $\deg p = k$ such that $p(T) = 0$. Clearly p is not constants, thus it has a zero since p is a polynomial of complex coefficient. Thus such zero is an eigenvalue of T. $\square$

Exercise 3.45. *Let $n > 1$ an integer, and $T \in \mathcal{L}(F^n)$ is defined by:*

$$T(x_0, \cdots, x_{n-1}) = (x_0 + \cdots + x_{n-1}, \cdots, x_0 + \cdots + x_{n-1})$$

- *Find all eigenvalue and eigenvector of T.*
- *Find the minimal polynomial of T.*

Proof.

- Observe that $\operatorname{range} T = \operatorname{span}((1, \cdots, 1))$, thus $T(1, \cdots, 1) = n(1, \cdots, 1)$.
- Observe that $T(x_0, \cdots, x_{n-1}) = (x_0 + \cdots + x_{n-1})(1, \cdots, 1)$ and $T^2(x_0, \cdots, x_{n-1}) = n(x_0 + \cdots + x_{n-1})(1, \cdots, 1)$, thus $p(T) = nT - T^2 = 0$.

$\square$

Exercise 4 is kinda hard, sorry.

Exercise 3.6. *Let $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T.*

Proof. Observe that $T^2(w, z) = T(-z, w) = (-w, -z) = (-1)(w, z)$, thus the minimal polynomial of T is $p(T) = I + T^2$. $\square$

Exercise 3.7.
- *Given an example that the minimal polynomial of ST is not equal to TS's.*

- *Suppose V is finite and $S, T \in \mathcal{L}(V)$. Show that the minimal polynomial of ST is equal to TS's if one of S and T is invertible.*

  Hint: Show that S is invertible and $p \in \mathcal{P}(F)$ implies $p(TS) = S^{-1}p(ST)S$.

Proof.

- The idea is to find S, T such that $ST \neq 0$ but $TS = 0$. We can find $S(x, y) = (x, 0)$ and $T(x, y) = (y, 0)$ holds:

  $$(ST)(x, y) = S(y, 0) = (y, 0)$$
  $$(TS)(x, y) = T(x, 0) = (0, 0)$$

  Thus the minimal polynomial of ST is not 0 but TS one does.

- Suppose S is invertible and $p \in \mathcal{L}(F)$ is the minimal polynomial of TS, then $p(TS) = S^{-1}p(ST)S$ since i-th term of $S^{-1}p(ST)S$ has form $S^{-1}c_i(ST)^iS = c_i(S^{-1}S)(TS)^{i-1}(TS) = c_i(TS)^i$. Thus $S^{-1}p(ST)S = 0$ and then $p(ST) = 0$. We will show that p is the minimal polynomial of ST, suppose $q \in \mathcal{L}(F)$ such that $q(ST) = 0$, then $0 = S^{-1}q(ST)S = q(TS)$, therefore $\deg q = \deg p$. Hence p is the minimal polynomial of ST.

□

Exercise 3.8. *Let $T \in \mathcal{L}(R^2)$ is the opeartor that "rotates 1 degree counter-clockwise", find the minimal polynomial of T.*

*Note that it is **NOT** $x^{180} + 1$ even $T^{180} = -I$.*

Proof. Note that there is some λ such that $Tv - \lambda v = \alpha T^2 v$ (We can show that $\lambda = \alpha$), However the calculation is too complicate. □