

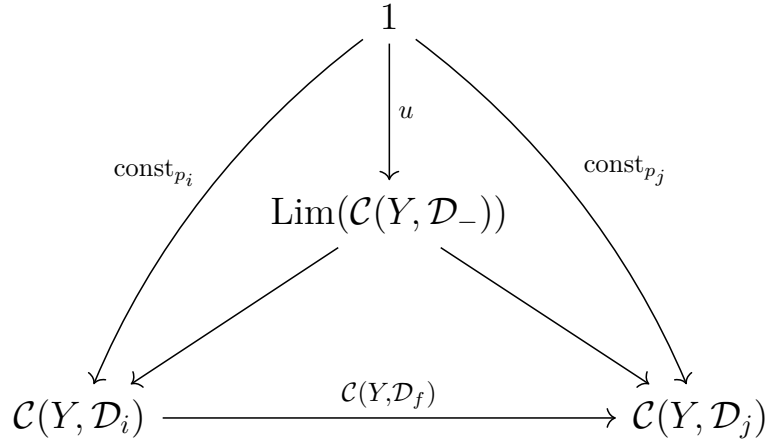
# 1 Adjoint

**Theorem 1.1.** *Show that the hom-functor preserves limit, that is, for any  $Y \in \mathcal{C}$  and diagram  $\mathcal{D}$ , we have:*

$$\text{Lim}(\mathcal{C}(Y, \mathcal{D}_-)) \cong \mathcal{C}(Y, \text{Lim } \mathcal{D})$$

*Proof.* The idea comes from *The Dao of FP* and nlab.

We may consider the cone with singleton set as vertex:



where  $\text{const}_{p_i}$  is the function that takes a morphism  $p_i \in \mathcal{C}(Y, \mathcal{D}_i)$ .

We know there is a one-to-one corresponding between  $u$  and the pair  $\langle \text{const}_{p_i}, \text{const}_{p_j} \rangle$ :

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, \mathcal{C}(Y, \mathcal{D}_-)) \cong \mathbf{Set}(1, \text{Lim}(\mathcal{C}(Y, \mathcal{D}_-)))$$

or we can simplify the equation by defining  $F_j = \mathcal{C}(Y, \mathcal{D}_j) : \mathcal{J} \rightarrow \mathbf{Set}$ .

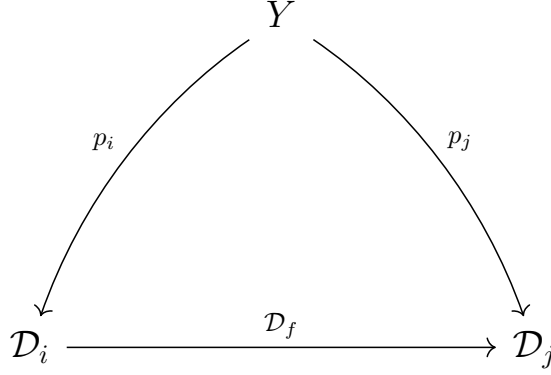
$$[\mathcal{J}, \mathbf{Set}](\Delta_1, F) \cong \mathbf{Set}(1, \text{Lim } F)$$

We may recall that the hom-set maps out from 1 is isomorphic to the target, that is:

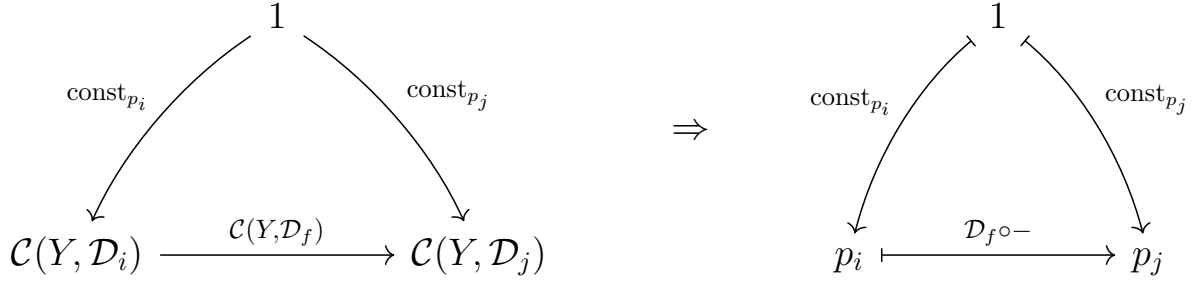
$$\mathbf{Set}(1, \text{Lim } F) \cong \text{Lim } F$$

Similarly, the cone with 1 as vertex is a pair of selection of  $\mathcal{C}(Y, \mathcal{D}_i)$ , it

forms a cone with  $Y$  as vertex:



the diagram is indeed commute since



Also we can make a pair of selection of  $\mathcal{C}(Y, \mathcal{D}_-)$  from a cone with  $Y$  as vertex.

Then the cone of  $\mathcal{C}(Y, \mathcal{D}_-)$  with vertex 1, is isomorphic to the cone of  $\mathcal{D}_-$  with vertex  $Y$ :

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, F) \cong [\mathcal{J}, \mathcal{C}](\Delta_Y, D)$$

While the later one is naturally isomorphic to the limit of  $\mathcal{J}_\bullet$ :

$$[\mathcal{J}, \mathcal{C}](\Delta_Y, D) \cong \mathcal{C}(Y, \text{Lim } D)$$

Finally, we have:

$$\begin{aligned}
 & \text{Lim } \mathcal{C}(Y, \mathcal{D}_-) \\
 & \cong \\
 & \mathbf{Set}(1, \text{Lim}(\mathcal{C}(Y, \mathcal{D}_-))) \\
 & \cong \\
 & [\mathcal{J}, \mathbf{Set}](\Delta_1, \mathcal{C}(Y, \mathcal{D}_-)) \\
 & \cong \\
 & [\mathcal{J}, \mathcal{C}](\Delta_Y, D) \\
 & \cong \\
 & \mathcal{C}(Y, \text{Lim } D)
 \end{aligned}$$

□

Dually, we also have:

$$\mathrm{Lim}(\mathcal{C}(\mathcal{D}_-, Y)) \cong \mathcal{C}(\mathrm{Colim} \mathcal{D}, Y)$$