1 More Constructions

Definition 1.1 (Initial Topology)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ a mapping between two sets and $\mathcal{Y}$ is equipped with a topology. Then we declare that $V \subseteq \mathcal{X}$ is open if there is an open set $W \subseteq \mathcal{Y}$ such that $V = f^{-1}(W)$. This topology is called initial topology.*

Proof. We need to show that it forms a topology.

- $\varnothing = f^{-1}(\varnothing)$ and $\mathcal{X} = f^{-1}(\mathcal{Y})$.

- For a collection of open set $\{V_\alpha\}$, we claim $\bigcup_\alpha V_\alpha = f^{-1}(\bigcup_\alpha W_\alpha)$. For any $x \in \bigcup_\alpha V_\alpha$, it must belongs to some V_α, therefore $x \in f^{-1}(W_\alpha) \subseteq f^{-1}(\bigcup_\alpha W_\alpha)$. For any $x \in f^{-1}(\bigcup_\alpha W_\alpha)$, we know $f(x)$ must belongs to some W_α, so $x \in f^{-1}(W_\alpha) \subseteq f^{-1}(\bigcup_\alpha W_\alpha)$.

- For two open sets $V_0, V_1 \subseteq \mathcal{X}$, we claim $V_0 \cap V_1 = f^{-1}(W_0 \cap W_1)$. For any $x \in V_0 \cap V_1$, then $f(x) \in W_0$ and $f(x) \in W_1$, so $x \in f^{-1}(W_0 \cap W_1)$. For any $x \in f^{-1}(W_0 \cap W_1)$, we know $f(x) \in W_0 \cap W_1$, therefore $x \in f^{-1}(W_0) = V_0$ and $x \in f^{-1}(W_1) = V_1$, so $x \in V_0 \cap V_1$.

□

Definition 1.2 (Final Topology)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ a mapping between two sets and $\mathcal{X}$ is equipped with a topology. Then we declare that $W \subseteq \mathcal{Y}$ is open if there is an open set $V \subseteq \mathcal{X}$ such that $V = f^{-1}(W)$. This topology is called final topology*

Proof. Similar to the proof of initial topology. □

We can see that the induced topology makes the mapping open.

Theorem 1.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous mapping between topological spaces, then:*

- *The initial topology on $\mathcal{X}$ is weaker than its own topology.*

- *The final topology on $\mathcal{Y}$ is stronger than its own topology.*

Recall that V is weaker than W if any open set under V is also an open set under W.

Proof.

- For any open sets V in initial topology of $\mathcal{X}$, we know there is $W \subseteq \mathcal{Y}$ such that $V = f^{-1}(W)$. Then V is an open set in the original topology cause f is continuous.

- For any open sets W in the original topology of $\mathcal{Y}$, we know there is $V \subseteq \mathcal{X}$ such that $V = f^{-1}(W)$ since f is continuous, then W is an open set in final topology of $\mathcal{Y}$.

$\square$

Theorem 1.2. *Let $g : \mathcal{X} \to \mathcal{Y}$ be a continuous map.*

- *Suppose $\mathcal{X}$ is equipped with the initial topology induced by g. Show that a map $f : \mathcal{W} \to \mathcal{X}$ is continuous iff $g \circ f : \mathcal{W} \to \mathcal{Y}$ is continuous.*

- *Suppose $\mathcal{Y}$ is equipped with the final topology induced by g. Show that a map $h : \mathcal{Y} \to \mathcal{Z}$ is continuous iff $h \circ g : \mathcal{X} \to \mathcal{Z}$ is continuous.*

Proof. ($\Rightarrow$)s are trivial, we focus on ($\Leftarrow$)s.

- For any open set $W \subseteq \mathcal{Y}$, there is an open set $V \subseteq \mathcal{W}$ such that $V = (g \circ f)^{-1}(W)$ and $S \subseteq \mathcal{X}$ such that $S = g^{-1}(W)$. Then $V = (g \circ f)^{-1}(W) = (f^{-1} \circ g^{-1})(W) = f^{-1}(g^{-1}(W)) = f^{-1}(S)$. Also, every open set in $\mathcal{X}$ is induced by an open sets in $\mathcal{Y}$, so we proved that the inverse image of every open sets in $\mathcal{X}$ is also open in $\mathcal{W}$, that is, f is continuous.

- For any open set $W \subseteq \mathcal{Z}$, there is an open set $V \subseteq \mathcal{X}$ such that $V = (h \circ g)^{-1}(W)$. Then $V = (h \circ g)^{-1}(W) = g^{-1}(h^{-1}(W))$, and we know $h^{-1}(W)$ is open cause $\mathcal{Y}$ is the final topology induced by g, so there must be an open set which inverse image of g is V, and $h^{-1}(W)$ can do the job. So $h^{-1}(W)$ is an open set, therefore h is continuous.

$\square$

Definition 1.3 (Quotient Topology)**.** *Let be an equivalence relation on a topology space $\mathcal{X}$. The set $[x] = \{\, y \in \mathcal{X} \mid x \sim y \,\}$ is called the equivalence class of x. Then the final topology on $\mathcal{X}/\sim$ induced by $f(x) = [x]$ is called quotient topology. The set $\mathcal{X}/\sim$ equipped with a quotient topology is called quotient space.*

Theorem 1.3. *Let $f : \mathcal{K} \to \mathcal{Y}$ be a continuous map. Suppose $\mathcal{K}$ is compact and $\mathcal{Y}$ is Hausdorff, show that f is closed.*

Proof. For any closed set S in $\mathcal{K}$, we know S is compact since $\mathcal{K}$ is compact, then $f(S)$ is also compact in $\mathcal{Y}$ cause f is continuous. Then $f(S)$ is closed cause $\mathcal{Y}$ is Hausdorff. $\square$

Definition 1.4. *Let $\mathcal{X}$ be a topological space and G be a group. Suppose $- \cdot - : G \times \mathcal{X} \to \mathcal{X}$ is a mapping such that:*

1. *For any $x \in \mathcal{X}$, $1 \cdot x = x$. 1 is the identity of G.*

2. *For any $g, h \in G$ and $x \in \mathcal{X}$, $g \cdot (h \cdot x) = (g \cdot h) \cdot x$.*

3. *For any $g \in \mathcal{G}$, the map $x \mapsto g \cdot x$ is continuous.*

Then we say G acts on $\mathcal{X}$, or $\mathcal{X}$ is a G-space. In this case, the set $G \cdot x = \{ g \cdot x \mid \forall g \in G \}$ is called the G-orbit of x.

Theorem 1.4. *Suppose that a group G acts on a topological space $\mathcal{X}$. Show that for any $g \in G$, the map $x \mapsto g \cdot x$ defines a homeomorphism $\mathcal{X} \to \mathcal{X}$.*

Proof. We first show that $f(x) = g \cdot x$ is bijective. (Injective) If $f(a) = f(b)$ for some $a, b \in \mathcal{X}$, then $g \cdot a = g \cdot b$ and then $g^{-1} \cdot g \cdot a = g^{-1} \cdot g \cdot b$, which is eventually $a = b$. (Surjective) For any $x \in \mathcal{X}$, we have $g^{-1} \cdot x \in \mathcal{X}$ that $f(g^{-1} \cdot x) = g \cdot g^{-1} \cdot x = x$. By definition, we know f is continuous, we need to show that f^{-1} is continuous. It is easy to see that $f^{-1}(x) = g^{-1} \cdot x$, which is continuous by definition. So f is a homeomorphism. $\square$

Definition 1.5. *Suppose that a group G acts on a topological space $\mathcal{X}$. Define $x \sim y$ if there is $g \in G$ such that $y = g \cdot x$. We can show that $\sim$ is an equivalence relation, and $\mathcal{X}/\sim$ can be also denoted by $\mathcal{X}/G$. Note that $[x] = G \cdot x$, the orbit of x, therefore $\mathcal{X}/G$ is also called orbit space.*

Proof. We will show that $\sim$ is an equivalence relation.

- (Reflexivity) $x = 1 \cdot x$

- (Symmetry) If $x \sim y$, then $y = g \cdot x$, we have $x = g^{-1} \cdot g \cdot x = g^{-1} \cdot y$, that is, $y \sim x$.

- (Transitivity) if $x \sim y$ and $y \sim z$, then $y = g \cdot x$ and $z = h \cdot y$, we have $z = (h \cdot g) \cdot x$, that is, $z \sim x$.

$\square$

Theorem 1.5. *Suppose a group G acts on a topological space $\mathcal{X}$, and $f : \mathcal{X} \to \mathcal{X}/G$ is the quotient map.*

- *Show that f is open.*
- *Show that f is closed if G is finite.*

Proof.

- We can see that $f^{-1}(f(V))$ for any open set V is the set that contains points that equivalence to the points in V, that is, $G \cdot V$, it is easy to see that $G \cdot V = \bigcup_{g \in G} g \cdot V$ is open set, since every $g \cdot V$ is open while $g \cdot -$ is a homeomorphism by Theorem 1.4. Therefore, $f(V)$ has to be open cause $f^{-1}(f(V))$ is open.
- Similar to the previous answer, the finite condition is used when we are trying to obtain a union of some closed sets.

$\square$