

1 Fiber and Fibration

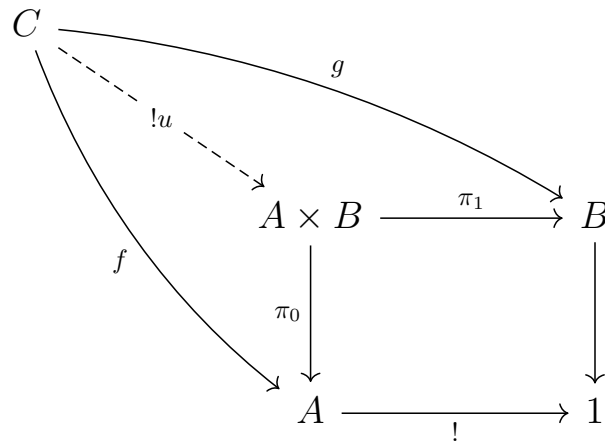
I am trying to understand fiber, fibration and pullback with my stupid brain.

1.1 Fiber

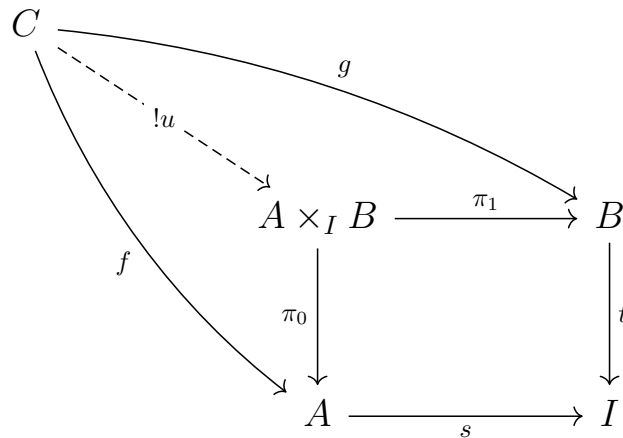
I will use "intuitive" rather than "definition" cause I really don't understand fiber.

Intuitive 1.1 (Fiber). *Suppose we are in a space (i.e. **Set**), and a mapping $f : A \rightarrow B$, then for some point $b \in B$, the inverse image of b , which is exactly $f^{-1}(b)$, is called a fiber.*

We can treat a product as a pullback with apex 1, the terminal object:



We can treat A as the fiber against to the only point in 1, same for B . Now, what if we replace 1 with something else?



For every point $i \in I$, we have fiber $A_i \subseteq A$ and $B_i \subseteq B$, which can form a product $A_i \times B_i$. We may sum all these products, and finally get $A \times_I B$, this is why the pullback is sometimes called *fiber product*.

We can also pick certain fiber from this pullback:

$$\begin{array}{ccc} \varphi & \xrightarrow{i} & A \\ \downarrow ! & & \downarrow f \\ 1 & \xrightarrow{x} & B \end{array}$$

The morphism $x : 1 \rightarrow B$ is a global element, which "pick" an element of B , then i must maps φ to the fiber of f over point x , which should be a injection.

The collection of fiber (the source of the morphism/the domain of the function) is called *fiber bundle*.

1.2 Fibration

Some intuitive comes from this article.

Intuitive 1.2. *A fibration works like an indexed family (i.e. a function $I \rightarrow A$), but do it in fiber way (i.e. a function $A \rightarrow I$).*

1.3 Base-change Functor

These section is related to *The Dao of FP*

We can also treat the morphism on right-hand side as a fibration, and the bottom-left corner a base (the target of a fibration):

$$\begin{array}{ccc} ? & \overset{\text{-----}}{\longrightarrow} & E \\ \downarrow & & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

Then we can treat $E \xrightarrow{p} B$ as an object in the slice category \mathcal{C}/B , similarly, the left-hand side morphism an object in \mathcal{C}/A . Then we can define a base-change functor $f^* : \mathcal{C}/B \rightarrow \mathcal{C}/A$ such that:

$$\begin{array}{ccc} f^*E & \xrightarrow{\quad g \quad} & E \\ \downarrow f^*p & & \downarrow p \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

a pullback.

We denote f^*E as the source of f^*p , it doesn't mean that f^* accept a object in \mathcal{C} .

We need to define the action of base-change functor on the morphism of \mathcal{C}/B :

$$\begin{array}{ccccccc} & & & g' & & & \\ & & & \text{-----} & & & \\ f^*E' & \xleftarrow{\quad ? \quad} & f^*E & \xrightarrow{\quad g \quad} & E & \xrightarrow{\quad e \quad} & E' \\ & \searrow f^*p' & \downarrow f^*p & & \downarrow p & & \swarrow p' \\ & & A & \xrightarrow{\quad f \quad} & B & & \end{array}$$

Two commute triangles are the morphisms in \mathcal{C}/A and \mathcal{C}/B .

Since f^*E' is a pullback of $A \xrightarrow{B} \leftarrow^{E'}$, it tips us that we can find the commute square below to get the morphism we want:

$$\begin{array}{ccc} f^*E & \xrightarrow{\quad \quad} & E' \\ \downarrow & & \downarrow p' \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

If we look the last diagram carefully, we can find this square commutes:

$$\begin{array}{ccccc}
f^*E & \xrightarrow{\quad g \quad} & E & \xrightarrow{\quad e \quad} & E' \\
\downarrow f^*p & & \downarrow p & \swarrow p' & \\
A & \xrightarrow{\quad f \quad} & B & &
\end{array}$$

therefore

$$\begin{array}{ccccc}
f^*E & & & & \\
& \searrow f^*e & & \searrow e \circ g & \\
& & f * E' & \xrightarrow{\quad g' \quad} & E' \\
& \searrow f^*p & \downarrow f^*p' & & \downarrow p' \\
& & A & \xrightarrow{\quad f \quad} & B
\end{array}$$

The functoriality follows the fact that f^*e is unique that makes the diagram commutes.