

# 1 Fiber and Fibration

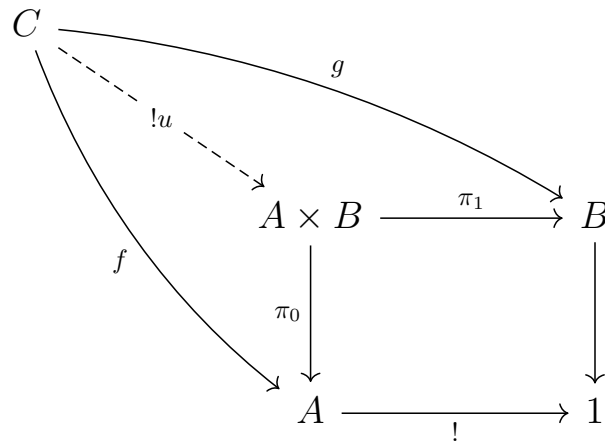
I am trying to understand fiber, fibration and pullback with my stupid brain.

## 1.1 Fiber

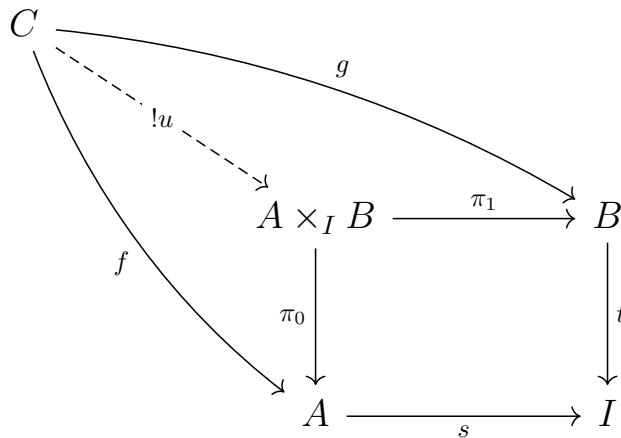
I will use "intuitive" rather than "definition" cause I really don't understand fiber.

**Intuitive 1.1** (Fiber). *Suppose we are in a space (i.e. **Set**), and a mapping  $f : A \rightarrow B$ , then for some point  $b \in B$ , the inverse image of  $b$ , which is exactly  $f^{-1}(b)$ , is called a fiber of  $f$  over  $b$ .*

We can treat a product as a pullback with apex 1, the terminal object:



We can treat  $A$  as the fiber against to the only point in 1, same for  $B$ . Now, what if we replace 1 with something else?



For every point  $i \in I$ , we have fiber  $A_i \subseteq A$  and  $B_i \subseteq B$ , which can form a product  $A_i \times B_i$ . We may sum all these products, and finally get  $A \times_I B$ , this is why the pullback is sometimes called *fiber product*.

We can also pick certain fiber from this pullback:

$$\begin{array}{ccc} \varphi & \xrightarrow{i} & A \\ \downarrow ! & & \downarrow f \\ 1 & \xrightarrow{x} & B \end{array}$$

The morphism  $x : 1 \rightarrow B$  is a global element, which "pick" an element of  $B$ , then  $i$  must maps  $\varphi$  to the fiber of  $f$  over point  $x$ , which should be a injection.

The collection of fiber (the source of the morphism/the domain of the function) is called *fiber bundle*.

## 1.2 Fibration

Some intuitive comes from this article.

**Intuitive 1.2.** *A fibration works like an indexed family (i.e. a function  $I \rightarrow A$ ), but do it in fiber way (i.e. a function  $A \rightarrow I$ ).*

## 1.3 Base-change Functor

These section is related to *The Dao of FP*

We can also treat the morphism on right-hand side as a fibration, and the bottom-left corner a base (the target of a fibration):

$$\begin{array}{ccc} ? & \overset{\text{-----}}{\longrightarrow} & E \\ \downarrow & & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

Then we can treat  $E \xrightarrow{p} B$  as an object in the slice category  $\mathcal{C}/B$ , similarly, the left-hand side morphism an object in  $\mathcal{C}/A$ . Then we can define a base-change functor  $f^* : \mathcal{C}/B \rightarrow \mathcal{C}/A$  such that:

$$\begin{array}{ccc} f^*E & \xrightarrow{\quad g \quad} & E \\ \downarrow f^*p & & \downarrow p \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

a pullback.

We denote  $f^*E$  as the source of  $f^*p$ , it doesn't mean that  $f^*$  accept a object in  $\mathcal{C}$ .

We need to define the action of base-change functor on the morphism of  $\mathcal{C}/B$ :

$$\begin{array}{ccccccc} & & & g' & & & \\ & & & \text{-----} & & & \\ f^*E' & \xleftarrow{\quad ? \quad} & f^*E & \xrightarrow{\quad g \quad} & E & \xrightarrow{\quad e \quad} & E' \\ & \searrow f^*p' & \downarrow f^*p & & \downarrow p & & \swarrow p' \\ & & A & \xrightarrow{\quad f \quad} & B & & \end{array}$$

Two commute triangles are the morphisms in  $\mathcal{C}/A$  and  $\mathcal{C}/B$ .

Since  $f^*E'$  is a pullback of  $A \rightarrow B \leftarrow E'$ , it tips us that we can find the commute square below to get the morphism we want:

$$\begin{array}{ccc} f^*E & \xrightarrow{\quad \quad} & E' \\ \downarrow & & \downarrow p' \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

If we look the last diagram carefully, we can find this square commutes:

$$\begin{array}{ccccc}
f^*E & \xrightarrow{\quad g \quad} & E & \xrightarrow{\quad e \quad} & E' \\
\downarrow f^*p & & \downarrow p & \swarrow p' & \\
A & \xrightarrow{\quad f \quad} & B & & 
\end{array}$$

therefore

$$\begin{array}{ccccc}
f^*E & & & & \\
\downarrow f^*p & \searrow f^*e & & \searrow e \circ g & \\
A & & f * E' & \xrightarrow{\quad g' \quad} & E' \\
& & \downarrow f^*p' & & \downarrow p' \\
& & A & \xrightarrow{\quad f \quad} & B
\end{array}$$

The functoriality follows the fact that  $f^*e$  is unique that makes the diagram commutes.