1 Hausdorff spaces

Definition 1.1. *A topological space $\mathcal{X}$ is called Hausdorff, if for each pair of distinct points $x, y \in \mathcal{X}$ there are disjoint neighborhoods $x \in V$ and $y \in W$.*

Theorem 1.1. *Show that any converging sequence in a Hausdorff space has a unique limit.*

Proof. Suppose a sequence x_n converage to x and y, then there are disjoint neighborhoods $x \in V$ and $y \in W$. Since x is the limit of x_n, so there is i such that for any neighborhood of x, x_j is in that neighborhood for any $j > i$. Similar to y, but V and W are disjoint, and V contains infinite points of x_n from some i, therefore W contains finite points of x_n, which is unacceptible. $\square$

Theorem 1.2. *Show that a topological space $\mathcal{X}$ is Hausdorff iff the diagonal*

$$\Delta = \{ (x, x) \in \mathcal{X} \times \mathcal{X} \}$$

is a closed set in the product space $\mathcal{X} \times \mathcal{X}$

Proof. ($\Rightarrow$) The set $S = \bigcup_{x,y \in \mathcal{X}, x \neq y} V_{(x,y)} \times W_{(x,y)}$ is an open set in $\mathcal{X} \times \mathcal{X}$ where $V_{(x,y)}$ and $W_{(x,y)}$ are disjoint neighborhoods of x and y, respectively. It is easy to see that $(x, x) \notin S$ for any $x \in \mathcal{X}$, otherwise (x, x) must belongs to some $V_{(x,x)} \times W_{(x,x)}$ while $V_{(x,x)}$ and $W_{(x,x)}$ are disjoint and $x \in V_{(x,x)} \cap W_{(x,x)}$. Also, there is no point that is not in S beside the point in Δ. So Δ is the complement of S, and S is open, so Δ is closed.

($\Leftarrow$) For any distinct $x, y \in \mathcal{X}$, we have $(x, y) \in \Delta^C$. We know Δ^C is a union of products of open sets in $\mathcal{X}$, so we may suppose $x \in V$ and $y \in W$ where V and W are open sets in $\mathcal{X}$. Suppose $z \in V \cap W$, then $(z, z) \in \Delta^C$ and then $(z, z) \notin \Delta$, which is unacceptible. $\square$

Corollary. *Any one-point set in a Hausdorff space is closed.*

Proof. For any Hausdorff space $\mathcal{X}$ and $p \in \mathcal{X}$, for any $y \in \mathcal{X}$ that $x \neq y$, we have a pair of disjoint neighborhood V_y of x and W_y of y. Consider the union of these W_y, obviously it is an open set that contains every point in $\mathcal{X}$ beside x, so the one-point set $\{x\}$ is closed. $\square$

Corollary. *Any subspace of Hausdorff space is Hausdorff.*

Proof. Suppose $\mathcal{X}$ is Hausdorff and $S \subseteq \mathcal{X}$ a subspace. Consider $x, y \in S$ where $x \neq y$, we know there are disjoint neighborhood V and W such that $V \cap W = \varnothing$, then $(V \cap S) \cap (W \cap S) = \varnothing$ where $V \cap S$ and $W \cap S$ are open set in S and $x \in V \cap S$ and $y \in W \cap S$. □

Theorem 1.3. *Let $\mathcal{X}$ be a Hausdorff space and $K \subseteq \mathcal{X}$ be a compact subset. Then for any $y \notin K$, there are open sets $K \subseteq V$ and $y \in W$ such that $V \cap W = \varnothing$.*

Proof. For any point $z \in K$, we have disjoint neighborhoods $y \in W_z$ and $z \in V_z$, consider the union of V_z, obviously it is an open cover on K, so there is a finite subcover $\{V_{z_\alpha}\}$, then we consider the intersection of the corresponding W_{z_α}, that is, $\bigcap_\alpha W_{z_\alpha}$, we can do this intersection cause the subcover is finite. Then let $V = \bigcup_\alpha V_{z_\alpha}$ and $W = \bigcap_\alpha W_{z_\alpha}$, obviously $V \cap W = \varnothing$. □

Theorem 1.4. *Any compact subset of Hausdorff is closed.*

Proof. Suppose $\mathcal{X}$ is Hausdorff and $K \subseteq \mathcal{X}$ a compact subset. For any $y \notin K$, we have $K \subseteq V_y$ and $y \in W_y$ such that $V_y \cap W_y = \varnothing$. Consider the union of these W_y, we can see that it is an open set and contains every point in $\mathcal{X} \setminus K$. □

Theorem 1.5. *Let $\mathcal{X}$ be a Hausdorff space and $K, L \subseteq \mathcal{X}$ be two compact subsets that $K \cap L = \varnothing$. Show that there are open sets $K \subseteq V$ and $L \subseteq W$ such that $V \cap W = \varnothing$*

Proof. For any $l \in L$, we have $K \subseteq V_l$ and $l \in W_l$ where $V_l \cap W_l = \varnothing$ by Theorem 1.3. Consider the union of these W_l, it is an open cover on L, therefore there is a finite subcover $\{W_{l_\alpha}\}$. Then we can take the intersection of the corresponding V_l, that is, $V = \bigcap_\alpha V_{l_\alpha}$, and the union of the subcover $W = \bigcup_\alpha W_{l_\alpha}$. They are disjoint, otherwise there is $x \in V_{l_\alpha}$ and $x \in W_{l_\alpha}$ for some α, which contradict to the property from Theorem 1.3. □