1 Exponential

Definition 1.1. *Let $\mathcal{C}$ a category. For any $B, C \in \mathcal{C}$, (C^B, ev) forms an exponential where $C^B \in \mathcal{C}$ and $ev : C^B \times B \to C$, if for any object $A \in \mathcal{C}$ and $f : A \times B \to C$, there is a unique $u : A \to C^B$ such that $f = ev \circ (u \times 1_B)$. In other words, the follow diagram commutes.*

$$\begin{tikzcd}
A \times B \arrow[dd, dashed, "u\times 1_B"] \arrow[dr, "f"] & \\
& C \\
C^B \times B \arrow[ur, "ev"] &
\end{tikzcd}$$

Suppose we are in **Set**, we know for any function $f : A \to B$, there is an element in some set that represents this function. In programming perspect, we know it is an element of type $A \to B$. What if we generialize it to other category? In order to represent an element of some set/type/object, we can use global element, that is, a morphism from terminal object. We denote the object that represents the morphisms from a to b by b^a, then a morphism $f : a \to b$ should be represented by $g : 1 \to b^a$. Furthermore, we can apply the function to some argument, that is, $\epsilon : b^a \times a \to b$, we should require such g respects this behaviour, and we get this diagram commutes:

$$\begin{tikzcd}
1 \times a \arrow[d, "\langle g, \mathrm{id}_a \rangle"'] \arrow[dr, "f"] & \\
b^a \times a \arrow[r, "\epsilon"'] & b
\end{tikzcd}$$

It says, if we take an element g that represents f, then we can apply g to for any "element" of a, it should behave like we apply f to that "element" of a.

We may loose the requirement that 1 exists, we can use any object c:

$$
\begin{tikzcd}
c \times a \arrow[d, "\langle g{,}\mathrm{id}_a \rangle"'] \arrow[dr, "f"] & \\
b^a \times a \arrow[r, "\epsilon"'] & b
\end{tikzcd}
$$

The morphism g corresponds to the *currying* in programming.

Then we can choose some elements of a and c:

$$
\begin{tikzcd}
1 \times 1 \arrow[d, "\langle i{,}j \rangle"'] \arrow[ddr, bend left, "k"] & \\
c \times a \arrow[d, "\langle g{,}\mathrm{id}_a \rangle"'] \arrow[dr, "f"] & \\
b^a \times a \arrow[r, "\epsilon"'] & b
\end{tikzcd}
$$

The diagram says: For a function object $g \circ i$, apply it to element j by ϵ, should equivalent to we apply f to them, and the result is the element k:

$$
\begin{aligned}
& \epsilon \circ \langle g \circ i, j \rangle && \text{(applying through function object)} \\
=& f \circ \langle i, j \rangle && \text{(applying directly)} \\
=& k && \text{(the result)}
\end{aligned}
$$

Furthermore, we can observe that the morphism f and the "curring" morphism g is one-one corresponding. It notices us that there is a underlying adjunction:

$$\mathcal{C}(c \times a, b) \cong \mathcal{C}(c, b^a)$$

It would be more clear if we rewrite them:

$$L_a c = c \times a$$
$$R_a b = b^a$$
$$\mathcal{C}(L_a c, b) \cong \mathcal{C}(c, R_a b)$$

But we need to show that they are functors. L_a is a functor cause $- \times -$ is a functor (see chapter product). For any morphism $h : s \to t$, we need to provide a morphism $h^a : s^a \to t^a$, one reasonable choice is:

$$\begin{tikzcd}
s^a \times a \arrow[r, "\epsilon_s"] \arrow[d, "\langle h^a, \mathrm{id}_a \rangle"'] & s \arrow[d, "h"] \\
t^a \times a \arrow[r, "\epsilon_t"'] & t
\end{tikzcd}$$

It is easy to check the functoriality by the uniqueness of h^a.