Exercise 3.4. *Let V a finite vector space with $\dim V > 1$, show that $S = \{\ T \text{ is singular} \mid T \in \mathcal{L}(V)\ \}$ is **NOT** a subspace of $\mathcal{L}(V)$.*

Proof. If S is a subspace of $\mathcal{L}(V)$, then it is an ideal of $\mathcal{L}(V)$ since for any $A \in S$ and $B \in \mathcal{L}(V)$, AB and BA are singular, therefore $AB, BA \in S$. However, we know the only two ideals of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$, none of them is S. □

Exercise 3.11. *Let V finite vector space, and $S, T \in \mathcal{L}(V)$, show that*

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible}$$

Proof.

- ($\Rightarrow$) Suppose $STW = WST = I$, then $S(TW) = (TW)S = I$ since $\dim V = \dim V$, therefore $S^{-1} = TW$, also $(WS)T = T(WS) = I$ since $\dim V = \dim V$, therefore $T^{-1} = WS$.
- ($\Leftarrow$) Trivial.

□

Exercise 3.12. *Let V finite vector space, and $S, T, U \in \mathcal{L}(V)$ such that $STU = I$, Show that $T^{-1} = US$.*

Proof. Since $STU = I$ we know U is invertible (since STU is invertible), then $ST = U^{-1}$. Since U^{-1} is invertible, we know S and T are invertible therefore $T = S^{-1}U^{-1}$ and $T^{-1} = US$. □

Exercise 3.13. *Show that the conclusion of previous exercise can be false if V is not finite.*

Proof. Let $S(x_0, x_1, \dots) = (x_1, \dots)$ the backward-shift mapping and $U(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ the forward-shift mapping and $T = I$ the identity mapping.

We have $SU = I$ and $US \neq I$, T is clearly inveritble with $T^{-1} = I$, but we know $US \neq I$, so $T^{-1} = US \neq I$.

In fact, this also disprove the infinite version of 3.11 since SU is invertible but neither S nor U is invertible. □

Exercise 3.17. *Let V a finite vector space, $S \in \mathcal{L}(V)$, define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$, show that:*

1. $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$

2. $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$

Proof. Since $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$, we know $\dim \mathcal{L}(V) = \dim \operatorname{null} \mathcal{A} + \dim \operatorname{range} \mathcal{A}$, also, $\dim \mathcal{L}(V) = (\dim V)^2$ and $\dim V = \dim \operatorname{null} S + \dim \operatorname{range} S$. Therefore we have $\dim \operatorname{null} A + \dim \operatorname{range} A = (\dim V)(\dim \operatorname{null} S + \dim \operatorname{range} S)$, which means we only need to prove one of (1) and (2).

We will show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$. We found that $\dim \mathcal{L}(V, \operatorname{null} S) = (\dim V)(\dim \operatorname{null} S)$, so it would be nice if $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$. For any $T \in \operatorname{null} \mathcal{A}$, we have $ST = 0$, which means $\operatorname{range} T \subseteq \operatorname{null} S$, therefore $T \in \mathcal{L}(V, \operatorname{null} S)$. For any $T \in \mathcal{L}(V, \operatorname{null} S)$, we have $ST = 0$ since $\operatorname{range} T \subseteq \operatorname{null} S$, so $T \in \operatorname{null} \mathcal{A}$, therefore $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$, thus $\dim \operatorname{null} A = (\dim V)(\dim \operatorname{null} S)$. $\square$

Exercise 3.18. *Show that V and $\mathcal{L}(F, V)$ are isomorphic.*

Proof. This can be proven by $\dim V = \dim \mathcal{L}(F, V) = 1(\dim V)$, but we can find $\varphi(v) = x \mapsto xv$ an isomorphism. For any $T \in \mathcal{L}(F, V)$, T is determined by $T(1)$. $\square$