1 Yoneda

This chapter combines arguments from some books:

- The Dao of FP
- The Joy of Abstraction

Definition 1.1.
$$H_x = \mathcal{C}(-,x)$$
 and $H^x = \mathcal{C}(x,-)$.

Take H^x as an example, it sends \mathcal{C} to **Set**, the interesting part is the mapping on morphism. For any morphism $f: a \to b$ of \mathcal{C} , H^f must be a mapping $\mathcal{C}(x,a) \to \mathcal{C}(x,b)$, we can see that $g \mapsto f \circ g$ would be a choice.

We have to show that it satisfies the functoriality:

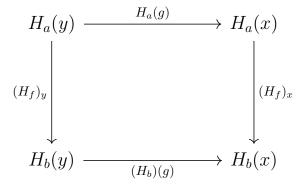
- $H^{id_a}(g) = id_a \circ g = g$
- $H^{f \circ g}(h) = (f \circ g) \circ h = f \circ (g \circ h) = H^f(g \circ h) = H^f(H^g(h)) = (H^f \circ H^g)(h).$

Similar to H_x , the only difference is that H_x is a contrafunctor.

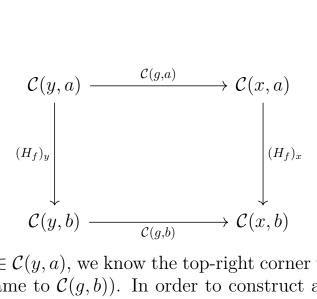
Suppose $f: a \to b$ an isomorphism, we can see that H^x gives an isomorphism between two hom-sets: C(x, a) and C(x, b).

Furthermore, we can no more fix x, that is, make H_{\bullet} (or H^{\bullet}) a functor from \mathcal{C} to $[\mathcal{C}^{op}, \mathbf{Set}]$, a functor to a functor!

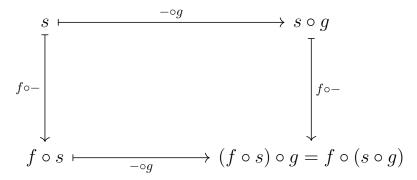
The problem we need to solve is that what should H_{\bullet} do on a morphism $f: a \to b$. Since H_{\bullet} produce a functor, H_f must produce a natural transformation between H_a and H_b . Suppose $x, y \in \mathcal{C}$ and $g: x \to y$, note that H_a and H_b are contrafunctor, so we need to reverse the arrows!



and we can unfold the definitions



and suppose $s \in \mathcal{C}(y, a)$, we know the top-right corner would be $s \circ g$ since $\mathcal{C}(g, a) = -\circ g$ (same to $\mathcal{C}(g, b)$). In order to construct an arrow in $\mathcal{C}(y, b)$, we can pre-compose the arrow $f : a \to b$. Then the bottom-left corner would be $f \circ s$, and the bottom-right corner would be $(f \circ s) \circ g$ (by left-bottom path) and $f \circ (s \circ g)$ (by top-right path), which is exactly the same!



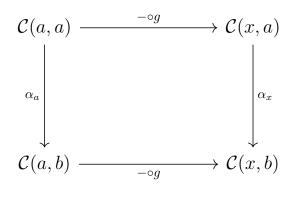
Note that the condition here $(f \circ -) \circ (- \circ g) = (- \circ g) \circ (f \circ -)$ is the naturality condition which is mentioned in *The Dao of FP*. A bijection between two hom-sets $\alpha_x = \mathcal{C}(x,a) \to \mathcal{C}(x,b)$ that satisfies the naturality condition $\alpha_y \circ (- \circ g) = (- \circ g) \circ \alpha_x$ can retrieve the isomorphism between a and b. This will be unsurprised if we notice that such bijection with naturality condition forms a natural transformation, then we can retrieve the morphism (not isomorphism yet) from it. The morphism becomes iso- when we know H_{\bullet} is full and faithful (see below and chapter functor),

Definition 1.2. H_{\bullet} is called Yoneda embedding.

Theorem 1.1. Shows H_{\bullet} is an embedding by showing it is full and faithful.

Proof. (Full) For any $a, b \in \mathcal{C}$, suppose $\alpha : [\mathcal{C}^{op}, \mathbf{Set}](H_a, H_b)$ a morphism (natural transformation). Use the Yoneda trick, we have $\alpha_a : \mathcal{C}(a, a) \to \mathcal{C}(a, b)$ and then $\alpha_a(id_a) : \mathcal{C}(a, b)$. As we see the definition of H_{\bullet} on morphism,

we should expect that α has form $f \circ -$ for some $f : a \to b$. But how coincident, we have a morphism $\alpha_a(id_a): \mathcal{C}(a,b)$. So we claim $H_{\alpha_a(id_a)} = \alpha$. (In the other hand, if α has form $f \circ -$, then $\alpha_a(id_a) = f \circ id_a = f$). For any $x \in \mathcal{C}$, we need to show $(H_{\alpha_a(id_a)})_x = \alpha_x : \mathcal{C}(x,a) \to \mathcal{C}(x,b)$. So we suppose $g \in \mathcal{C}(x,a)$, then the following diagram commutes since α is natural:



Then

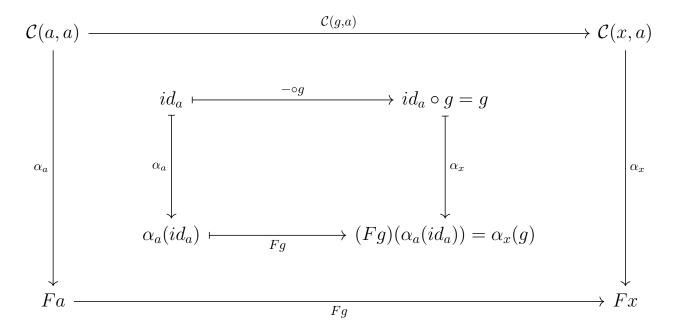
$$id_{a} \longmapsto \begin{array}{c} -\circ g \\ \\ \downarrow \\ \alpha_{a} \end{array} \qquad id_{a} \circ g = g$$

$$\downarrow \alpha_{x} \\ \alpha_{a}(id_{a}) \longmapsto \begin{array}{c} \alpha_{a} \\ \\ -\circ g \end{array} \qquad \alpha_{a}(id_{a}) \circ g = \alpha_{x}(g)$$

is a proof of
$$H_{\alpha_a(id_a)}(g) = \alpha_a(id_a) \circ g = \alpha_x(g)$$
.
(Faithful) Suppose $f \circ -H_f = H_g = g \circ -$, then $f = f \circ id_a = H_f(id_a) = H_g(od_a) = g \circ id_a = g$.

As we see in the proof of H_{\bullet} is a full functor, the natural transformation α at some x (therefore any $x \in \mathcal{C}$) is completely determined by the value $\alpha_a(id_a)$, cause for any g, we have $\alpha_a(id_a) \circ g = \alpha_x(g)$.

We may rename H_b with F, then



It seems that H_b can be replaced with any functor in $[\mathcal{C}^{op}, \mathbf{Set}]$, furthermore, the natural transformation is still determined by $\alpha_a(id_a)$ (and $\alpha_a(id_a)$ is determined by α , trivial though, but it implies that there is a precisely corresponding).

Theorem 1.2 (Yoneda Lemma). Show that the natural transformation between H_a and any functor $F \in [C^{op}, Set]$ correspond precisely to the elements of Fa. In other words,

$$[\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \cong Fa$$

Proof. The arrow $f: [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \to Fa$ is obvious by the Yoneda trick.

$$\alpha \mapsto \alpha_a(id_a) : [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \to Fa$$

For any $x \in Fa$, as we see before, we may expect there is a natural transformation α such that $\alpha_a(id_a) = x$, then the action on other objects is completely determined by $\alpha_a(id_a)$ as we see before.

$$x, g \mapsto (Fg)(x) : Fa \to [\mathcal{C}^{op}, \mathbf{Set}](H_a, F)$$

Note that $(F(id_a))(x) = id_{Fa}(x) = x$.

We need to show that they are inverse to each other. For any natural transformation α :

$$[\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \xrightarrow{f} Fa \xrightarrow{f^{-1}} [\mathcal{C}^{op}, \mathbf{Set}](H_a, F)$$

$$\alpha \longmapsto^{-a(id_a)} \alpha_a(id_a) \longmapsto^{(F-)(-)} (F-)(\alpha_a(id_a))$$

And we can see $(Fg)(\alpha_a(id_a)) = \alpha_x(g)$ for any $x \in \mathcal{C}$ and $g \in \mathcal{C}(x,a)$ by the same way as the proof which shows H_{\bullet} is a full functor.

In another direction, we get:

$$Fa \xrightarrow{f^{-1}} [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \xrightarrow{f} Fa$$

$$x \longmapsto^{(F-)(-)} (F-)(x) \longmapsto^{-a(id_a)} F(id_a)(x)$$

It is obvious that $F(id_a)x = id_{Fa}x = x$.