1 Monad

A monad is an endofunctor F equipped with:

- identity $\eta : \mathrm{Id} \to F$, which unsually called "pure" or "return".
- multiplication $\mu : F \circ F \to F$, which usually called "join".

and the identity law:

$$\begin{tikzcd}
\mathrm{Id} \circ F \arrow[r, "\eta \cdot F"] \arrow[dr, equal] & F \circ F \arrow[d, "\mu"'] & F \circ \mathrm{Id} \arrow[l, "F \cdot \eta"'] \arrow[dl, equal] \\
& F &
\end{tikzcd}$$

the associative law:

$$\begin{tikzcd}
(F \circ F) \circ F \arrow[rr, equal] \arrow[d, "\mu \cdot F"'] & & F \circ (F \circ F) \arrow[d, "F \cdot \mu"] \\
F \circ F \arrow[r, "\mu"'] & F & F \circ F \arrow[l, "\mu"]
\end{tikzcd}$$

We can easily see that it is a monoid on the category of endofunctor. If you confuse with "what the element of an endofunctor is", try global element, just like η.

Suppose $\alpha, \beta : \mathrm{Id} \to F$ are two "elements" of F, then the multiplication of them will be $\mu \circ (\alpha \cdot \beta) : \mathrm{Id} \to F$, the vertical composition acts the role of function application.

1.1 Free Monad

Just like a list is a free monoid (on set), we also have free monad (free monoid on endofunctor):

List (Free Monoid)	Free Monad
$I : \text{List } A$	$\text{pure} : \mathrm{Id} \to \text{FreeMonad } F$
$(::) : A \otimes (\text{List } A) \to \text{List } A$	$\text{free} : F \circ (\text{FreeMonad } F) \to \text{FreeMonad } F$