1 Functors

Definition 1.1 (Full). *A functor $F : \mathcal{C} \to \mathcal{D}$ is called full, if for any $a, b \in \mathcal{C}$, the mapping on morphism $F : \mathcal{C}(a, b) \to \mathcal{D}(Fa, Fb)$ is surjective.*

Definition 1.2 (Faithful). *A functor $F : \mathcal{C} \to \mathcal{D}$ is called faithful, if for any $a, b \in \mathcal{C}$, the mapping on morphism $F : \mathcal{C}(a, b) \to \mathcal{D}(Fa, Fb)$ is injective.*

Definition 1.3 (Essentially Full). *A functor $F : \mathcal{C} \to \mathcal{D}$ is called Essentially full, if for any $a \in \mathcal{C}$, the mapping on object $F : \mathcal{C} \to \mathcal{D}$ is surjective.*

Theorem 1.1. *Suppose $F : \mathcal{C} \to \mathcal{D}$ a functor, and $f : a \to b$ a morphism in $\mathcal{C}$. Then f is an isomorphism iff Ff is an isomorphism.*

Proof. ($\Rightarrow$) We claim $F(f^{-1}) : Fb \to Fa$ is an inverse, we can see that $F(f^{-1} \circ f) = F(id_a) = id_{Fa}$ and $F(f \circ f^{-1}) = F(id_b) = id_{Fb}$.

($\Leftarrow$) Suppose Fg is the inverse of Ff, and we can retrieve g from Fg cause F is full faithful. Then $F(g \circ f) = Fg \circ Ff = id_{Fa} = F(id_a)$ therefore $g \circ f = id_a$ since F is full faithful, similar to $F(f \circ g)$, so f is indeed an isomorphism. □

Corollary 1.1. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, show that F is injective on object.*

Proof. Trivial by previous theorem. □