1 Adjoint

Theorem 1.1. *Show that the hom-functor preserves limit, that is, for any $Y \in \mathcal{C}$ and diagram $\mathcal{D}$, we have:*

$$\mathrm{Lim}(\mathcal{C}(Y, \mathcal{D}_-)) \cong \mathcal{C}(Y, \mathrm{Lim}\,\mathcal{D})$$

Proof. The idea comes from *The Dao of FP* and nlab.

We may consider the cone with singleton set as vertex:

$$\begin{tikzcd}
& 1 \arrow[d, "u"] \arrow[ddl, bend right, "\mathrm{const}_{p_i}"'] \arrow[ddr, bend left, "\mathrm{const}_{p_j}"] & \\
& \mathrm{Lim}(\mathcal{C}(Y, \mathcal{D}_-)) \arrow[dl] \arrow[dr] & \\
\mathcal{C}(Y, \mathcal{D}_i) \arrow[rr, "\mathcal{C}(Y, \mathcal{D}_f)"] & & \mathcal{C}(Y, \mathcal{D}_j)
\end{tikzcd}$$

where const_{p_i} is the function that takes a morphism $p_i \in \mathcal{C}(Y, \mathcal{D}_i)$.

We know there is a one-to-one corresponding between u and the pair $\langle \mathrm{const}_{p_i}, \mathrm{const}_{p_j} \rangle$:

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, \mathcal{C}(Y, \mathcal{D}_-)) \cong \mathbf{Set}(1, \mathrm{Lim}(\mathcal{C}(Y, \mathcal{D}_-)))$$

or we can simplify the equation by defining $F_j = \mathcal{C}(Y, \mathcal{D}_j) : \mathcal{J} \to \mathbf{Set}$.

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, F) \cong \mathbf{Set}(1, \mathrm{Lim}\,F)$$

We may recall that the hom-set maps out from 1 is isomorphic to the target, that is:

$$\mathbf{Set}(1, \mathrm{Lim}\,F) \cong \mathrm{Lim}\,F$$

Similarly, the cone with 1 as vertex is a pair of selection of $\mathcal{C}(Y, \mathcal{D}_i)$, it forms a cone with Y as vertex:

$$\begin{tikzcd}
 & Y \arrow[dl, bend right, "p_i"'] \arrow[dr, bend left, "p_j"] & \\
\mathcal{D}_i \arrow[rr, "\mathcal{D}_f"] & & \mathcal{D}_j
\end{tikzcd}$$

the diagram is indeed commute since

$$\begin{tikzcd}
 & 1 \arrow[dl, bend right, "\mathrm{const}_{p_i}"'] \arrow[dr, bend left, "\mathrm{const}_{p_j}"] & \\
\mathcal{C}(Y, \mathcal{D}_i) \arrow[rr, "\mathcal{C}(Y, \mathcal{D}_f)"] & & \mathcal{C}(Y, \mathcal{D}_j)
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
 & 1 \arrow[dl, mapsto, bend right, "\mathrm{const}_{p_i}"'] \arrow[dr, mapsto, bend left, "\mathrm{const}_{p_j}"] & \\
p_i \arrow[rr, mapsto, "\mathcal{D}_f \circ -"] & & p_j
\end{tikzcd}$$

Also we can make a pair of selection of $\mathcal{C}(Y, \mathcal{D}_-)$ from a cone with Y as vertex.

Then the cone of $\mathcal{C}(Y, \mathcal{D}_-)$ with vertex 1, is isomorphic to the cone of $\mathcal{D}_-$ with vertex Y:

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, F) \cong [\mathcal{J}, \mathcal{C}](\Delta_Y, D)$$

While the later one is naturally isomorphic to the limit of $\mathcal{J}_\bullet$:

$$[\mathcal{J}, \mathcal{C}](\Delta_Y, D) \cong \mathcal{C}(Y, \operatorname{Lim} D)$$

Finally, we have:

$$\begin{gathered}
\operatorname{Lim} \mathcal{C}(Y, \mathcal{D}_-) \\
\cong \\
\mathbf{Set}(1, \operatorname{Lim}(\mathcal{C}(Y, \mathcal{D}_-))) \\
\cong \\
[\mathcal{J}, \mathbf{Set}](\Delta_1, \mathcal{C}(Y, \mathcal{D}_-)) \\
\cong \\
[\mathcal{J}, \mathcal{C}](\Delta_Y, D) \\
\cong \\
\mathcal{C}(Y, \operatorname{Lim} D)
\end{gathered}$$

$\square$

Dually, we also have:

$$\mathrm{Lim}(\mathcal{C}(\mathcal{D}_-, Y)) \cong \mathcal{C}(\mathrm{Colim}\, \mathcal{D}, Y)$$