

Exercise 3.7. Suppose vector space V and W are finite ($2 \leq \dim V \leq \dim W$), show that $\{ T \in \mathcal{L}(V, W) \mid T \text{ is not injective} \}$ is not a subspace.

Proof. Consider the basis $v_0 + \cdots + v_{(\dim V - 1)} \in V$, and $T(v_0 + \cdots + v_{(\dim V - 1)}) = (0 + 1_0 + \cdots + 1_{(\dim V - 1)})$ and $T'(v_0 + \cdots + v_{(\dim V - 1)}) = (v_0 + 0 + \cdots + v_{(\dim V - 1)})$. Then $(T + T')(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_{(\dim V - 1)} v_{(\dim V - 1)}) = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + 2\lambda_{(\dim V - 1)} v_{(\dim V - 1)}$, which is obviously injective. \square

Exercise 3.11. Suppose V is finite and $T \in \mathcal{L}(V, W)$, show that there is a subspace $U \subset V$ such that:

$$U \cap \text{null } T = \{0\} \quad \text{and} \quad \text{range } T = \{ Tu \mid u \in U \}$$

Proof. This is similar to the *isomorphism theorems* about groups! This can be done by the similar way we used in proving $\dim V = \dim \text{null } T + \dim \text{range } T$. \square

The next two exercises remind me the categorical injective and surjective, let try them first!

Exercise. For any $F \in \mathcal{L}(V, W)$, F is injective \iff for any $S, T \in \mathcal{L}(U, V)$, $FS = FT$ implies $S = T$.

Proof.

- (\Rightarrow) For any $S, T \in \mathcal{L}(V, W)$ that $FS = FT$, then for any $u \in U$, we have $F(Su) = F(Tu)$, since F is injective, we know $Su = Tu$, so $S = T$.
- (\Leftarrow) For any $v, w \in V$ such that $Fv = Fw$. Consider

$$\begin{aligned} S(\lambda) &= \lambda v \\ T(\lambda) &= \lambda w \end{aligned}$$

in $\mathcal{L}(\mathbb{R}, V)$. Then for any $\lambda \in \mathbb{R}$, we have $FS\lambda = \lambda FS1 = \lambda Fv = \lambda Fw = \lambda FT1 = FT\lambda$. so $FS = FT$ then $S = T$, which means $v = S1 = T1 = w$. \square

Exercise. Suppose W is finite, then for any $F \in \mathcal{L}(V, W)$, F is surjective \iff for any $S, T \in \mathcal{L}(W, U)$, $SF = TF$ implies $S = T$.

Proof.

- (\Rightarrow) For any $S, T \in \mathcal{L}(W, U)$ such that $SF = TF$. For any $w \in W$, there is $v \in V$ such that $Fv = w$ since F is surjective. Then we have $SFv = TFv$ so $Sw = S(Fv) = T(Fv) = Tw$ then $S = T$.
- (\Leftarrow) Consider

$$S = I \quad \text{and} \quad T(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 w_0 + \cdots + \lambda_k w_k$$

where w_0, \dots, w_k is the basis of $\text{range } F$ and w_0, \dots, w_n is the basis of W that expand from w_0, \dots, w_k .

(If we can use another way to construct T , then W is not need to be finite, for example, $W = \text{range } T \oplus W_0$).

It is easy to show that T is a linear transformation. Then for any $v \in V$, we have $TFv = Fv$ (since T acts like identity transformation on $\text{range } F$) and $SFv = Fv$, so $S = T$ by the property of F . Since $\text{range } S = W$, so is $\text{range } T$, that means w_0, \dots, w_k spans W , so $k = n$, which means $\text{range } F = W$, therefore F is surjective.

□

Exercise 3.19. Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is injective \iff there is $S \in \mathcal{L}(W, V)$ such that $ST = I$.

Proof.

- (\Rightarrow) Consider the basis v_0, \dots, v_n of V , then Tv_0, \dots, Tv_n is a basis of $\text{range } T$ since T is injective. We denote Tv_i as w_i and w_0, \dots, w_m as the basis of W which expand from w_0, \dots, w_n . Define $S(\lambda_0 w_0 + \cdots + \lambda_m w_m) = \lambda_0 w_0 + \cdots + \lambda_n w_n$, and then for any $v \in V$, $ST(\lambda_0 v_0 + \cdots + \lambda_n v_n) = S(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 v_0 + \cdots + \lambda_n v_n$, so $ST = I$.
- (\Leftarrow) Suppose $A, B \in \mathcal{L}(U, V)$, such that $TA = TB$, we will show that $A = B$. $STA = IA = A$ and $STB = IB = B$ and $STA = STB$ since $TA = TB$. Then we know T is a monomorphism, and then T is injective.

□

Exercise 3.20. Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is surjective \iff there is $S \in \mathcal{L}(W, V)$ such that $TS = I$.

Proof.

□

Exercise 3.21. Suppose V is finite, $T \in \mathcal{L}(V, W)$, $U \subseteq W$ a subspace. Show that the inverse image of U : $\{ v \in V \mid Tv \in U \}$ is a subspace of V , and

$$\dim\{ v \in V \mid Tv \in U \} = \dim \text{null } T + \dim(U \cap \text{range } T)$$

Proof. The second part is quite easy, we can restrict the domain of T to $\{ v \in V \mid Tv \in U \}$, say $T' \in \mathcal{L}(\{ v \in V \mid Tv \in U \}, W)$, so that it is in form $\dim\{ v \in V \mid Tv \in U \} = \dim \text{null } T' + \dim \text{range } T'$. Obviously $\text{range } T' = U \cap \text{range } T$ and $\text{null } T' = \text{null } T$.

We will now show that $\{ v \in V \mid Tv \in U \}$ is a subspace of V .

- $T0 \in U$.
- For any $v, w \in V$ such that $Tv, Tw \in U$, we have $T(v+w) = Tv + Tw \in U$.
- For any $v \in V$ such that $Tv \in U$ and $\lambda \in F$, we have $T(\lambda v) = \lambda Tv \in U$.

Therefore it is a subspace. □

Exercise 3.22. Suppose U and V are finite, $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$, show that

$$\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$$

Proof. Consider the inverse image of $\text{null } S$ on T : $K = \{ v \in V \mid Tv \in \text{null } S \}$, which dimension: $\dim K = \dim \text{null } T + \dim(\text{null } S \cap \text{range } T)$, where $\dim(\text{null } S \cap \text{range } T)$ caps at $\dim \text{null } S$.

We know show that $\text{null } ST = \text{null } K$. For any $STv = 0$, we know $S(Tv) = 0$, so $Tv \in K$, therefore $\text{null } ST \subseteq \text{null } K$; For any $Tv \in \text{null } S$, that means $S(Tv) = 0$, therefore $v \in \text{null } ST$, therefore $\text{null } ST \supseteq \text{null } K$, and $\text{null } ST = \text{null } K$. □

Exercise 3.25. Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that $\text{null } S \subseteq \text{null } T \iff$ there is $E \in \mathcal{L}(W)$ such that $T = ES$.

Proof. We define $E(S(v)) = Tv$ for any $v \in V$, so that $E \in \mathcal{L}(\text{range } S, W)$. We first show that E is a mapping, and also a linear transformation.

Suppose $Sv, Sw \in W$ such that $Sv = Sw$, we need to show that $E(Sv) = E(Sw)$, or normalized $Tv = Tw$. We know $v - w \in \text{null } S$ since $Sv = Sw$, so $v - w \in \text{null } T$ since $\text{null } S \subseteq \text{null } T$, therefore $T(v - w) = 0$, and then $Tv = Tw$, so E is a mapping.

Now we show that E is a linear transformation.

- For any $Sv, Sw \in \text{range } S$, $E(Sv) + E(Sw) = Tv + Tw = T(v + w) = E(S(v + w)) = E(Sv + Sw)$.
- For any $Sv \in \text{range } S$ and $\lambda \in F$, $\lambda E(Sv) = \lambda Tv = T(\lambda v) = E(S(\lambda v)) = E(\lambda Sv)$.

therefore E is a linear transformation.

Now we can expand the domain of E to W such that $E'v = Ev$ for any $v \in \text{range } S$ (this is proven in previous exercise). For any $v \in V$, we have $ESv = E(Sv) = Tv$, therefore $T = ES$.

For another direction, for any $v \in \text{null } S$, we have $ESv = E0 = 0 = Tv$, so $v \in \text{null } T$. \square

Exercise 3.26. Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that $\text{range } S \subseteq \text{range } T \iff$ there is $E \in \mathcal{L}(V)$ such that $S = TE$.

Proof. Consider the inverse image of $\text{range } S$ with basis w_0, \dots, w_n , say v_0, \dots, v_n , it is easy to show v_0, \dots, v_n is linear independent. Then $E(v) = \lambda_0 v_0 + \dots + \lambda_n v_n$ where $Sv = \lambda_0 w_0 + \dots + \lambda_n w_n$. \square

Exercise 3.27. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$, show that $V = \text{null } P \oplus \text{range } P$.

Proof. Such element is called *idempotent* in algebra.

We will show $\text{null } P \oplus \text{range } P$ by showing $\text{null } P \cap \text{range } P = \{0\}$. For any $v \in \text{null } P \cap \text{range } P$, we know there is $w \in V$ such that $Pw = v$ since $v \in \text{range } P$, then $P^2(v) = P(Pv) = P0 = 0$ since $v \in \text{null } P$ and $P^2(v) = P(Pv) = Pw$, so $Pw = 0$ while $Pw = v$ therefore $v = 0$.

Then we have $\dim V = \dim \text{null } P + \dim \text{range } P$ and $\dim(\text{null } P \oplus \text{range } P) = \dim \text{null } P + \dim \text{range } P - \dim \{0\}$, so $V = \text{null } P \oplus \text{range } P$. \square

Exercise 3.28. Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ such that for any non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$, $\deg(Dp) = \deg p - 1$. Show that D is surjective.

Proof. We induction on n , starts from 1, to show that $D(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_{n-1}(\mathbb{R})$.

- Base: for any $p \in \mathcal{P}(\mathbb{R})$ where $\deg p = 1$, we know $\deg Dp = 0$, so $D(\mathcal{P}_1(\mathbb{R}))$ is a non-zero subspace of $\mathcal{P}_0(\mathbb{R})$, which is $\mathcal{P}_0(\mathbb{R})$.
- Induction: We have induction hypothesis: For any $i \leq n$, we have $D(\mathcal{P}_i(\mathbb{R})) = \mathcal{P}_{i-1}(\mathbb{R})$. We want to show that $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$. For

any $p \in \mathcal{P}(\mathbb{R})$ with $\deg p = n+1$, we can write p in form of $p = \lambda x^{n+1} + r$ where $\deg r \leq n$, then $Dp = D(\lambda x^{n+1} + r) = D(\lambda x^{n+1}) + Dr$ where $\deg D(\lambda x^{n+1}) = n$ and $\deg Dr \leq n-1$. So $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$ since: $\mathcal{P}_n(\mathbb{R}) \subseteq D(\mathcal{P}_{n+1}(\mathbb{R}))$ and $D(\lambda x^{n+1}) \in D(\mathcal{P}_{n+1}(\mathbb{R}))$, it is sufficient to span $\mathcal{P}_n(\mathbb{R})$.

□

Exercise 3.29. For any $p \in \mathcal{P}(\mathbb{R})$, show that there is $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.

Proof. We can rewrite the goal as $5DDq + 3Dq = p$ where $D(p) = p'$, then $5DDq + 3Dq = D(5Dq) + D(3q) = D(5Dq + 3q) = p$. We know D is surjective by the previous exercise, the goal is now showing that $5Dq + 3q = r$ where $Dr = p$. Then we continue rewrite the goal $5Dq + 3q = (5D)q + (3I)q = (5D + 3I)q = r$, we will show that $5D + 3I$ is surjective, we use the same method in previous exercise.

We denote $5D + 3I$ by F , and induction on $n \in \mathbb{N}$ to show that $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$.

- Base: We should show that $F(\mathcal{P}_0(\mathbb{R})) = \mathcal{P}_0(\mathbb{R})$, for any $p \in \mathcal{P}_0(\mathbb{R})$, we have $Fp = 5Dp + 3p$, where $Dp = 0$ since $\deg p = 0$, so $Fp = 3p$, which means we have $1 \in F(\mathcal{P}_0(\mathbb{R}))$ since p is literally a number and $\frac{1}{3p}Fp = 1$.
- Induction: We have induction hypothesis: $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$, and we want to show $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

For any $p \in \mathcal{P}_{n+1}(\mathbb{R})$, we have $Fp = 5Dp + 3p$ where $\deg 5Dp = n$ and $\deg 3p = n+1$, then we can eliminate $5Dp$ and every term in p with degree less than $n+1$ since $\mathcal{P}_n(\mathbb{R}) \subseteq \text{range } F$, then we get z^{n+1} , thus $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

Therefore there is q such that $(5D + 3I)q = r$ since $5D + 3I$ is surjective.

Another solution from internet: Define $Tq = 5q'' + 3q'$, we can see for any $q \in \mathcal{P}(\mathbb{R})$ we have $\deg Tq = \deg q - 1$, so T is surjective. Then there is q such that $Tq = 5q'' + 3q' = p$. □

Exercise 3.30. Suppose $\varphi \in \mathcal{L}(V, F)$ not zero, and $u \in V$ that $u \notin \text{null } \varphi$, show that $V = \text{null } \varphi \oplus \{ au \mid a \in F \}$.

Proof. We can see φ is surjective since $\varphi u \neq 0$, then for any $i \in F$, we have $(i(\varphi u)^{-1})\varphi u = i$.

For any $v \in V$, since φ is surjective (in a particular way), so we have $a\varphi u$ such that $a\varphi u = \varphi v$, then $\varphi(au - v) = 0$ so $au - v \in \text{null } \varphi$. That means $(-1)(au - v) + au = v$ where $(-1)(au - v) \in \text{null } \varphi$ and $au \in \{ au \mid a \in F \}$, so $V = \text{null } \varphi + \{ au \mid a \in F \}$.

Then $\text{null } \varphi \oplus \{ au \mid a \in F \}$ since $u \notin \text{null } \varphi$. \square

Exercise 3.31. Suppose V is finite ($\dim V > 1$), show that if $\varphi : \mathcal{L}(V) \rightarrow F$ is a linear mapping with property $\varphi(ST) = \varphi(S)\varphi(T)$ for any $S, T \in \mathcal{L}(V)$, show that $\varphi = 0$.

Proof. Consider $\text{null } \varphi$, since $\dim V > 1$ while $\dim F = 1$, so φ cannot be injective, therefore $\text{null } \varphi \neq \{0\}$.

For any non-zero $S \in \text{null } \varphi$ and $T \in \mathcal{L}(V)$, we have $\varphi(ST) = \varphi(S)\varphi(T) = 0 = \varphi(T)\varphi(S) = \varphi(TS)$ since $S \in \text{null } \varphi$, thus $ST \in \text{null } \varphi$. We show that $\text{null } \varphi$ is an ideal of $\mathcal{L}(V)$, recall that the property of $\mathcal{L}(V)$, the only ideal of $\mathcal{L}(V)$ is $\{0\}$ and $LT(V)$, so $\text{null } \varphi = \mathcal{L}(V)$, which means $\varphi = 0$. \square

Exercise 3.32. Let V, W are vector spaces and $T \in \mathcal{L}(V, W)$, define $T_C : V_C \rightarrow W_C$:

$$T_C(u + iv) = Tu + iTv$$

for any $u, v \in V$.

1. Show that T_C is a (complex) linear mapping from V_C to W_C .
2. Show that T_C is injective $\iff T$ is injective.
3. Show that $\text{range } T_C = W_C \iff \text{range } T = W$.

Proof.

1. For any $u, v, s, t \in V$ $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} & T((u + iv) + (s + it)) \\ &= T(u + s + i(v + t)) \\ &= T(u + s) + iT(v + t) \\ &= Tu + Ts + iTv + iTt \\ &= T(u + iv) + T(s + it) \end{aligned}$$

and

$$\begin{aligned} & \lambda T(u + iv) \\ &= \lambda(Tu + iTv) \\ &= \lambda Tu + \lambda iTv \\ &= T(\lambda u) + iT(\lambda v) \\ &= T(\lambda u + i(\lambda v)) \\ &= T(\lambda u + i\lambda v) \\ &= T(\lambda(u + iv)) \end{aligned}$$

I believe these are trivial, so the future me should be able to prove these without any effort. □