1 Constructions

Definition 1.1 (Product)**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ are topology spaces, the product of $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} \times \mathcal{Y}$ is a topology space, which open set is a union of the product of open sets in $\mathcal{X}$ and $\mathcal{Y}$, that is, $\bigcup_i V_i \times W_i$ where V_i is an open set of $\mathcal{X}$ and W_i is an open set of $\mathcal{Y}$ (It won't work if the open set is just a product of open sets from both space).*

Proof. We need to show that $\mathcal{X} \times \mathcal{Y}$ is a topology space.

1. $\mathcal{X} \times \mathcal{Y}$ is an open set since both $\mathcal{X}$ and $\mathcal{Y}$ are open set, similarly, $\varnothing$ is an open set in $\mathcal{X} \times \mathcal{Y}$.

2. Trivial.

3. $$\bigcup_{a,b} (V_a \cap V_b) \times (W_a \cap W_b)$$

□

Exercise 1.1. *Let $\mathscr{B}$ be a collection of open sets in a topological space $\mathcal{X}$. Show that $\mathscr{B}$ is base in $\mathcal{X}$ iff for any point $x \in \mathcal{X}$ and any neighborhood N, there is $B \in \mathscr{B}$ such that $x \in B \subseteq N$.*

Proof.

- ($\Rightarrow$) We know N can be expressed as a union of some open sets in $\mathscr{B}$, that is, $N = B_0 \cup B_1 \cup \dots$. Then there is B_i such that $x \in B_i$, and we know $B_i \subseteq N$, so $x \in B_i \subseteq N$.

- ($\Leftarrow$) For any open set A, consider any point $x \in A$, we know A is a neighborhood of x, so there is $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq A$. Then $\bigcup_{x \in A} B_x$ is a subset of A, but note that $x \in B_x$, so $\bigcup_{x \in A} B_x$ contains all points of A, so $\bigcup_{x \in A} B_x = A$.

□

Theorem 1.1. *Let $\mathscr{B}$ be a set of subsets in some set $\mathcal{X}$. Show that $\mathscr{B}$ is a base of some topology on $\mathcal{X}$ iff it satisfies the following conditions:*

1. *$\mathscr{B}$ coverse $\mathcal{X}$, that is, for any $x \in \mathcal{X}$, there is $B \in \mathscr{B}$ such that $x \in B$.*

2. *For any $B_1, B_2 \in \mathscr{B}$, $x \in B_1 \cap B_2$, then there exists a set $B \in \mathscr{B}$ such that $x \in B \subseteq B_1 \cap B_2$.*

Proof. ($\Rightarrow$) Trivial by the previous exercise.

($\Leftarrow$) Let $\mathscr{O}$ the set of all unions of sets of $\mathscr{B}$. We claim $\mathscr{O}$ is a topology on $\mathcal{X}$. First, $\emptyset \in \mathscr{O}$ since it is the result of union of no set. And $\mathcal{X} \in \mathscr{O}$ since $\mathscr{B}$ covers $\mathcal{X}$. Obviously, the union of any sets of $\mathscr{O}$ is in $\mathscr{O}$.

Let $O_0, O_1 \in \mathscr{O}$, then $O_0 = \bigcup_i B_i$ and $O_1 = \bigcup_j B_j$. For any $x \in O_0 \cap O_1$, we have $x \in B_x \subseteq O_0 \cap O_1$ where $B_x \in \mathscr{O}$ by (2). Then $O_0 \cap O_1 = \bigcup_x = B_x$, therefore $O_0 \cap O_1 \in \mathscr{O}$ cause it is a union of some set B_x. □

Definition 1.2 (Prebase)**.** *Suppose $\mathscr{P}$ is a collection of subsets in $\mathcal{X}$ that covers the whole space. Show that the set of all finite intersections of sets in $\mathscr{P}$ is a base for some topology on $\mathcal{X}$.*

Proof. We denote all finite intersections of sets in $\mathscr{P}$ as P, then $\mathscr{P} \subseteq P$ since $\forall S \in P, S \cap S = S$. So P covers $\mathcal{X}$. For any $B_0, B_1 \in P$ and $x \in B_0 \cap B_1$, it is obvious that $x \in B_0 \cap B_1 \subseteq B_0 \cap B_1 \in P$. So $\mathscr{P}$ is a base for some topology on $\mathcal{X}$ by the previous theorem. □

Theorem 1.2. *Let $\mathscr{P}$ be a prebase for the topology on $\mathcal{Y}$. Show that a map $f : \mathcal{X} \to \mathcal{Y}$ is continuous iff $f^{-1}(P)$ is open for any $P \in \mathscr{P}$.*

Proof. ($\Rightarrow$) Obviously, every set in $\mathscr{P}$ is open.

($\Leftarrow$) For any open set in $\mathcal{Y}$, it is a union of sets in $\mathscr{P}$, that is, $\bigcup_i P_i$. Then $f^{-1}(\bigcup_i P_i) = \bigcup_i f^{-1}(P_i)$ which is a union of open sets, so it is also open. □

2 Compactness

Exercise 2.1. *Let $\left\{_{V_\alpha}\right\}$ be an open cover of a topological space $\mathcal{X}$. Show that $W \subseteq \mathcal{X}$ is open iff $W \cap V_\alpha$ is open for any $V_\alpha \in \langle V_\alpha \rangle$.*