1 Constructions

Definition 1.1 (Induced). *Let A a subset of a topological space $\mathcal{Y}$. Then*

- *all subsets $V \subseteq A$ such that $V = A \cap W$ for some open W in $\mathcal{Y}$*

forms a topology on A. This topology is called induced topology on A.

Proof. Obviously, $\varnothing = A \cap \varnothing$ and $A = A \cap A$.

For any open set $V_0 = A \cap W_0$ and $V_1 = A \cap W_1$, then it is easy to see that $V_0 \cup V_1 = (A \cap W_0) \cup (A \cap W_1) = A \cup (W_0 \cap W_1)$, therefore for a collection $\{V_\alpha\}$ of open sets in A, $\bigcup_\alpha V_\alpha = A \cap (\bigcup_\alpha W_\alpha)$, and we know $\bigcup_\alpha W_\alpha$ is still a open set in $\mathcal{Y}$, so is $\bigcup_\alpha V_\alpha$.

For any open set $V_0 = A \cap W_0$ and $V_1 = A \cap W_1$, we have $V_0 \cap V_1 = (A \cap W_0) \cap (A \cap W_1) = (A \cap W_0) \cap W_1 = A \cap (W_0 \cap W_1)$, so $V_0 \cap V_1$ is also a open set in A. $\square$

Definition 1.2 (Embedding). *A map $f : \mathcal{X} \to \mathcal{Y}$ is called embedding if f defines a homeomorphism from $\mathcal{X}$ to the subspace $f(\mathcal{X})$ in $\mathcal{Y}$.*

Definition 1.3 (Product). *Let $\mathcal{X}$ and $\mathcal{Y}$ are topology spaces, the product of $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} \times \mathcal{Y}$ is a topology space, which open set is a union of the product of open sets in $\mathcal{X}$ and $\mathcal{Y}$, that is, $\bigcup_i V_i \times W_i$ where V_i is an open set of $\mathcal{X}$ and W_i is an open set of $\mathcal{Y}$ (It won't work if the open set is just a product of open sets from both space).*

Proof. We need to show that $\mathcal{X} \times \mathcal{Y}$ is a topology space.

1. $\mathcal{X} \times \mathcal{Y}$ is an open set since both $\mathcal{X}$ and $\mathcal{Y}$ are open set, similarly, $\varnothing$ is an open set in $\mathcal{X} \times \mathcal{Y}$.

2. Trivial.

3. $$\bigcup_{a,b} (V_a \cap V_b) \times (W_a \cap W_b)$$

$\square$

Exercise 1.1. *Let $\mathscr{B}$ be a collection of open sets in a topological space $\mathcal{X}$. Show that $\mathscr{B}$ is base in $\mathcal{X}$ iff for any point $x \in \mathcal{X}$ and any neighborhood N, there is $B \in \mathscr{B}$ such that $x \in B \subseteq N$.*

Proof.

- ($\Rightarrow$) We know N can be expressed as a union of some open sets in $\mathscr{B}$, that is, $N = B_0 \cup B_1 \cup \dots$. Then there is B_i such that $x \in B_i$, and we know $B_i \subseteq N$, so $x \in B_i \subseteq N$.

- ($\Leftarrow$) For any open set A, consider any point $x \in A$, we know A is a neighborhood of x, so there is $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq A$. Then $\bigcup_{x\in A} B_x$ is a subset of A, but note that $x \in B_x$, so $\bigcup_{x\in A} B_x$ contains all points of A, so $\bigcup_{x\in A} B_x = A$.

$\square$

Theorem 1.1. *Let $\mathscr{B}$ be a set of subsets in some set $\mathcal{X}$. Show that $\mathscr{B}$ is a base of some topology on $\mathcal{X}$ iff it satisfies the following conditions:*

1. *$\mathscr{B}$ coverse $\mathcal{X}$, that is, for any $x \in \mathcal{X}$, there is $B \in \mathscr{B}$ such that $x \in B$.*

2. *For any $B_1, B_2 \in \mathscr{B}$, $x \in B_1 \cap B_2$, then there exists a set $B \in \mathscr{B}$ such that $x \in B \subseteq B_1 \cap B_2$.*

Proof. ($\Rightarrow$) Trivial by the previous exercise.

($\Leftarrow$) Let $\mathscr{O}$ the set of all unions of sets of $\mathscr{B}$. We claim $\mathscr{O}$ is a topology on $\mathcal{X}$. First, $\emptyset \in \mathscr{O}$ since it is the result of union of no set. And $\mathcal{X} \in \mathscr{O}$ since $\mathscr{B}$ covers $\mathcal{X}$. Obviously, the union of any sets of $\mathscr{O}$ is in $\mathscr{O}$.

Let $O_0, O_1 \in \mathscr{O}$, then $O_0 = \bigcup_i B_i$ and $O_1 = \bigcup_j B_j$. For any $x \in O_0 \cap O_1$, we have $x \in B_x \subseteq O_0 \cap O_1$ where $B_x \in \mathscr{O}$ by (2). Then $O_0 \cap O_1 = \bigcup_x = B_x$, therefore $O_0 \cap O_1 \in \mathscr{O}$ cause it is a union of some set B_x. $\square$

Definition 1.4 (Prebase)**.** *Suppose $\mathscr{P}$ is a collection of subsets in $\mathcal{X}$ that covers the whole space. Show that the set of all finite intersections of sets in $\mathscr{P}$ is a base for some topology on $\mathcal{X}$.*

Proof. We denote all finite intersections of sets in $\mathscr{P}$ as P, then $\mathscr{P} \subseteq P$ since $\forall S \in P, S \cap S = S$. So P covers $\mathcal{X}$. For any $B_0, B_1 \in P$ and $x \in B_0 \cap B_1$, it is obvious that $x \in B_0 \cap B_1 \subseteq B_0 \cap B_1 \in P$. So $\mathscr{P}$ is a base for some topology on $\mathcal{X}$ by the previous theorem. $\square$

Theorem 1.2. *Let $\mathscr{P}$ be a prebase for the topology on $\mathcal{Y}$. Show that a map $f : \mathcal{X} \to \mathcal{Y}$ is continuous iff $f^{-1}(P)$ is open for any $P \in \mathscr{P}$.*

Proof. (⇒) Obviously, every set in $\mathscr{P}$ is open.

(⇐) For any open set in $\mathcal{Y}$, it is a union of sets in $\mathscr{P}$, that is, $\bigcup_i P_i$. Then $f^{-1}(\bigcup_i P_i) = \bigcup_i f^{-1}(P_i)$ which is a union of open sets, so it is also open. □

Theorem 1.3. *For any map $f : \mathcal{X} \to \mathcal{Y}$, the map $F : \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$ given by $F(x) = (x, f(x))$. Show that $f(x)$ is continuous iff F is an embedding.*

Proof. (⇒) We first show that $F : \mathcal{X} \to F(\mathcal{X})$ is continuous. For any open set G in the induced topology on $F(\mathcal{X})$, it has form $F(\mathcal{X}) \cap (\bigcup_\alpha V_\alpha \times W_\alpha)$, where $(\bigcup_\alpha V_\alpha \times W_\alpha)$ is the open set in $\mathcal{X} \times \mathcal{Y}$. Clearly, $F^{-1}(G) = G.0 = \{ x \mid \forall (x, y) \in G \} = X \cap (\bigcup_\alpha V_\alpha)$, so it is an open set in $\mathcal{X}$. It is easy to see $F^{-1} : F(\mathcal{X}) \to \mathcal{X}$ given by $F^{-1}(x, y) = x$ is the inverse of F, we need to show that it is continuous, or in other words, show that F sends open to open. For any open set $V \subseteq \mathcal{X}$, $F(V) = F(\mathcal{X}) \cap (V \times \mathcal{Y})$, which is open in $F(\mathcal{X})$. So F is an embedding.

(⇐) For any open set G in $\mathcal{Y}$, we have $H = F(\mathcal{X}) \cap (\mathcal{X} \times G)$ is open in the induced topology on $F(\mathcal{X})$, then $F^{-1}(H)$ is open in $\mathcal{X}$. We claim $f^{-1}(G) = F^{-1}(H)$.

- (⊇) For any $x \in F^{-1}(H)$, we know $f(x) \in f(\mathcal{X}) \cap G$ since $F(x) = (x, f(x)) \in H$, so $x \in f^{-1}(G)$ since $f(x) \in G$.
- (⊆) For any $x \in f^{-1}(G)$, we know $f(x) \in G$, so $(x, f(x)) \in H$ since $x \in \mathcal{X} \cap \mathcal{X}$ and $f(x) \in f(\mathcal{X}) \cap G$, so $x \in F^{-1}(H)$.

Therefore f is continuous. □

2 Compactness

Exercise 2.1. *Let $\{V_\alpha\}$ be an open cover of a topological space $\mathcal{X}$. Show that $W \subseteq \mathcal{X}$ is open iff $W \cap V_\alpha$ is open for any $V_\alpha \in \{V_\alpha\}$.*

Proof. (⇒) is trivial.

(⇐) For any point $x \in W$, we have $x \in V_x$ since $\{V_\alpha\}$ is open cover. Consider $\bigcup_{x \in W}(W \cap V_x)$, it is a union of open sets, and it is a subset of W, and it contains all points of W, so W is open. □

Theorem 2.1. *Show that a space $\mathcal{X}$ is compact iff for any collection of closed sets $\{Q_\alpha\}$ in $\mathcal{X}$ such that*

$$\bigcap_\alpha Q_\alpha = \varnothing$$

Then there is a finite subset of $\{Q_\alpha\}$ such that

$$Q_0 \cap Q_1 \cap \cdots \cap Q_n = \varnothing$$

Proof.

- ($\Rightarrow$) For any collection of closed set $\{Q_\alpha\}$ in $\mathcal{X}$ such that $\bigcap_\alpha Q_\alpha = \varnothing$, we can see $\{\mathcal{X} \setminus Q_\alpha\}$ is a collection of open set and $\bigcup_\alpha (\mathcal{X} \setminus Q_\alpha) = \mathcal{X} \setminus (\bigcap_\alpha Q_\alpha) = \mathcal{X} \setminus \varnothing = \mathcal{X}$, therefore the collection of complements $\{\mathcal{X} \setminus Q_\alpha\}$ is an open cover of $\mathcal{X}$. So there is a finite subset of $\{\mathcal{X} \setminus Q_\alpha\}$ that also open covers $\mathcal{X}$. Then it is also a finite subset of $\{Q_\alpha\}$ since $\mathcal{X} = \bigcup_i (\mathcal{X} \setminus Q_i) = \mathcal{X} \setminus (\bigcap_i Q_i)$ and $Q_i \subseteq \mathcal{X}$ implies $\bigcap_i Q_i = \varnothing$.

- ($\Leftarrow$) For any open cover $\{V_\alpha\}$ of $\mathcal{X}$, consider the collection of complements $\{\mathcal{X} \setminus V_\alpha\}$, we have $\bigcap_\alpha (\mathcal{X} \setminus V_\alpha) = \mathcal{X} \setminus (\bigcup_\alpha V_\alpha) = \varnothing$. So there is a finite subset $\{\mathcal{X} \setminus V_i\}$ such that $\varnothing = \bigcap_i (\mathcal{X} \setminus V_i) = \mathcal{X} \setminus (\bigcup_i V_i)$, therefore $\bigcup_i V_i = \mathcal{X}$, the space $\mathcal{X}$ is compact.

$\square$

Theorem 2.2. *Let $Q_0 \supseteq Q_1 \supseteq \ldots$ be a nested sequence of closed nonempty sets in a compact space $\mathcal{K}$. Show that there is a point $q \in \mathcal{K}$ such that $\forall i, q \in Q_i$.*

Proof. If there is no such point, we know $\bigcap_i Q_i = \varnothing$, which means there is a finite subsequence such that $\bigcap_j Q_j = \varnothing$. Since the sequence Q_i is a nested sequence of nonempty sets, so $\bigcap_j Q_j$ must equal to some "smallest" Q_j, but that means this Q_j is empty set, which contradicts to the assumption. $\square$

Theorem 2.3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous mapping between topological spaces and $\mathcal{K}$ is a compact subset in $\mathcal{X}$. Show that $\mathcal{Q} = f(\mathcal{K})$ is also compact in $\mathcal{Y}$. That is, continuous mapping preserve compactness.*

Proof. For any open cover $\{V_\alpha\}$ of $f(\mathcal{K})$, then the inverse images of $\{V_\alpha\}$ cover are also open since f is continous, and also an open cover of $\mathcal{K}$ since they cover $f(\mathcal{K})$. So there is a finite subset of open cover such that covers $\mathcal{K}$, then map those cover by f, we get a finite subset of open cover of $\{V_\alpha\}$ $\square$

Theorem 2.4. *Any closed set in a compact space is also compact.*

Proof. This proof comes from textbook.

Suppose $\mathcal{X}$ a compact space and $\mathcal{Q}$ a closed set in it. Consider any open cover $\{V_\alpha\}$ of $\mathcal{Q}$ and the complement $\mathcal{C} = \mathcal{X} \setminus \mathcal{Q}$. Obviously, $\mathcal{C}$ is open, and $\{\mathcal{C}\} \cup \{V_\alpha\}$ is an open cover of $\mathcal{X}$, therefore there is a finite open cover on $\mathcal{X}$, that open cover may or may not contains $\mathcal{W}$, but we can always add $\mathcal{W}$ to it, and it is still a finite subcover. Since the finite subcover $\{\mathcal{W}, V_{\alpha_0}, \ldots, V_{\alpha_n}\}$ covers $\mathcal{X}$, then it also covers $\mathcal{Q}$, and we can see that $\mathcal{W}$ contributes nothing for $\mathcal{Q}$, so it is safe to remove it, and $\{V_{\alpha_i}\}$ is still a finite open cover on $\mathcal{Q}$.

Furthermore, the proposition can be *iff*, since the whole space is a closed set. If any closed set in that space is compact, then the whole space is also compact. □

Definition 2.1. *Let $\{V_\alpha\}$ and $\{W_\beta\}$ be two covers of a topological space $\mathcal{X}$. We say $\{V_\alpha\}$ is inscried in $\{W_\beta\}$ if for any α, there is β such that $V_\alpha \subseteq W_\beta$*

Theorem 2.5. *A space $\mathcal{X}$ is compact iff for any cover $\{V_\alpha\}$ of $\mathcal{X}$, there is a finite cover such that inscried in $\{V_\alpha\}$.*

Proof. ($\Rightarrow$) For any cover on $\mathcal{X}$, there is a finite subcover on $\mathcal{X}$, and the finite subcover is an inscried in itself.

($\Leftarrow$) For any finite subcover $\{W_\beta\}$ on $\mathcal{X}$ that is inscried in $\{V_\alpha\}$, since for any W_β there is a V_α such that $W_\beta \subseteq V_\alpha$, we may collect these V_α and they form a cover on $\mathcal{X}$. □

Theorem 2.6. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are compact topological spaces, show that the product $\mathcal{X} \times \mathcal{Y}$ is also compact.*

Proof. □

Theorem 2.7. *Suppose that a product space $\mathcal{X} \times \mathcal{Y}$ is nonempty and compact. Show that $\mathcal{X}$ and $\mathcal{Y}$ are compact.*

Proof. For any open cover $\{V_\alpha\}_{\alpha\in\mathcal{I}}$ on $\mathcal{X}$, consider the open cover $\{V_\alpha \times \mathcal{Y}\}_{\alpha\in\mathcal{I}}$ on $\mathcal{X} \times \mathcal{Y}$, there is a finite subcover $\{V_\alpha \times \mathcal{Y}\}_{\alpha\in\mathcal{J}}$. For any $x \in \mathcal{X}$, take $y \in \mathcal{Y}$ (it is possible cause $\mathcal{X} \times \mathcal{Y}$ is nonempty), we have $(x, y) \in V_\alpha \times \mathcal{Y}$ for some α, so $\{V_\alpha\}_{\alpha\in\mathcal{J}}$ is a open cover on $\mathcal{X}$. □

Definition 2.2. *A topological space $\mathcal{X}$ is called sequentially compact if any point sequence in $\mathcal{X}$ has a converging subsequence.*

Theorem 2.8. *A metric space $\mathcal{M}$ is comapct implies it is sequentially compact.*

Proof. Recall that if a sequence x_n converage to some point p, then for any ϵ, $B(p, \epsilon)$ contains infinite points in x_n.

If any subsequence in x_n is not converging, then for any point $p \in \mathcal{M}$, there is ϵ such that $B(p, \epsilon)$ contains finite points in x_n. Consider the collection of $B(p, \epsilon)$ for every $p \in \mathcal{M}$ is a cover on $\mathcal{M}$, then we have a finite subcover on $\mathcal{M}$, but then the subcover contains finite points in $\mathcal{M}$ while x_n is infinite. □

Theorem 2.9. *Show that the product of two sequentially compact spaces is sequentially compact.*

Proof. TODO □

Definition 2.3. *A sequence x_n of points in a metric space is called Cauchy if for any $\epsilon > 0$ there is n such that $|x_i - x_j| < \epsilon$ for all $i, j > n$.*

Theorem 2.10. *Any converging sequence in a metric space is Cauchy.*

Proof. For any converging sequence, the distance between points becomes smaller and smaller, so for any ϵ, there is n such that $|x_i - x_j| < \epsilon$ for all $i, j > n$. □

Definition 2.4. *A metric space $\mathcal{M}$ is called complete if any Cauchy sequence in $\mathcal{M}$ converage to a point in $\mathcal{M}$.*

Theorem 2.11. *Show that any compact metric space $\mathcal{M}$ is complete.*

Proof. For any Cauchy sequence x_n, suppose it is not converage, that is, for any $p \in \mathcal{M}$, there is ϵ such that $B(p, \epsilon)$ contains finite points of x_n. Consider the cover $\{B(p, \epsilon)\}$ for all $p \in \mathcal{M}$ and corresponding ϵ on $\mathcal{M}$, we know there is a finite subcover on $\mathcal{M}$ since $\mathcal{M}$ is compact. Then these finite subcover contains all points in x_n cause it covers $\mathcal{M}$, and it contains finite points in x_n cause each $B(p, \epsilon)$ contains finite points in x_n, therefore the sequence x_n is finite, which is unacceptible. □

Definition 2.5. *Let $\mathcal{M}$ be a metric space. A subset $A \subseteq \mathcal{M}$ is called ϵ-net of $\mathcal{M}$ if for any $p \in \mathcal{M}$, there is $a \in A$ such that $|p - a|_{\mathcal{M}} < \epsilon$ (or equivalently, $p \in B(a, \epsilon)$).*

Theorem 2.12. *Let $\mathcal{M}$ be a sequentially compact metric space, then for any $\epsilon > 0$, there is a finite ϵ-net of $\mathcal{M}$.*

Proof. This proof comes from textbook.

We may trying to construct an ϵ-net of $\mathcal{M}$. We pick a point in $\mathcal{M}$ randomly, say $x_0 \in \mathcal{M}$, then we pick another point $x_1 \in \mathcal{M}$ such that $x_1 \notin B(x_0, \epsilon)$, and then we pick $x_2 \in \mathcal{M}$ such that $x_2 \notin B(x_0, \epsilon)$ and $x_2 \notin B(x_1, \epsilon)$, for any i, we pick $x_i \in \mathcal{M}$ such that $x_i \notin B(x_j, \epsilon)$ for any $j < i$. If at some point, we can't pick any x_i that satisfies the requirement, then $\{x_0, x_1, \ldots, x_{i-1}\}$ is an ϵ-net of $\mathcal{M}$. If this procedure cannot stop, then we get a sequence x_i where their distance are always greater than ϵ. Since $\mathcal{M}$ is sequentially compact, so there is a converging subsequence, however, the distance of points in the subsequence can not below ϵ, so it can't be converging. $\square$

3 Hausdorff spaces

Definition 3.1. *A topological space $\mathcal{X}$ is called Hausdorff, if for each pair of distinct points $x, y \in \mathcal{X}$ there are disjoint neighborhoods $x \in V$ and $y \in W$.*

Theorem 3.1. *Show that any converging sequence in a Hausdorff space has a unique limit.*

Proof. Suppose a sequence x_n converage to x and y, then there are disjoint neighborhoods $x \in V$ and $y \in W$. Since x is the limit of x_n, so there is i such that for any neighborhood of x, x_j is in that neighborhood for any $j > i$. Similar to y, but V and W are disjoint, and V contains infinite points of x_n from some i, therefore W contains finite points of x_n, which is unacceptible. $\square$

Theorem 3.2. *Show that a topological space $\mathcal{X}$ is Hausdorff iff the diagonal*

$$\Delta = \{\, (x, x) \in \mathcal{X} \times \mathcal{X} \,\}$$

is a closed set in the product space $\mathcal{X} \times \mathcal{X}$

Proof. ($\Rightarrow$) The set $S = \bigcup_{x,y \in \mathcal{X}, x \neq y} V_{(x,y)} \times W_{(x,y)}$ is an open set in $\mathcal{X} \times \mathcal{X}$ where $V_{(x,y)}$ and $W_{(x,y)}$ are disjoint neighborhoods of x and y, respectively. It is easy to see that $(x, x) \notin S$ for any $x \in \mathcal{X}$, otherwise (x, x) must belongs to some $V_{(x,x)} \times W_{(x,x)}$ while $V_{(x,x)}$ and $W_{(x,x)}$ are disjoint and $x \in V_{(x,x)} \cap W_{(x,x)}$. Also, there is no point that is not in S beside the point in Δ. So Δ is the complement of S, and S is open, so Δ is closed.

($\Leftarrow$) For any distinct $x, y \in \mathcal{X}$, we have $(x, y) \in \Delta^C$. We know Δ^C is a union of products of open sets in $\mathcal{X}$, so we may suppose $x \in V$ and $y \in W$ where V and W are open sets in $\mathcal{X}$. Suppose $z \in V \cap W$, then $(z, z) \in \Delta^C$ and then $(z, z) \notin \Delta$, which is unacceptible. $\square$

Corollary. *Any one-point set in a Hausdorff space is closed.*

Proof. For any Hausdorff space $\mathcal{X}$ and $p \in \mathcal{X}$, for any $y \in \mathcal{X}$ that $x \neq y$, we have a pair of disjoint neighborhood V_y of x and W_y of y. Consider the union of these W_y, obviously it is an open set that contains every point in $\mathcal{X}$ beside x, so the one-point set $\{x\}$ is closed. $\square$

Corollary. *Any subspace of Hausdorff space is Hausdorff.*

Proof. Suppose $\mathcal{X}$ is Hausdorff and $S \subseteq \mathcal{X}$ a subspace. Consider $x, y \in S$ where $x \neq y$, we know there are disjoint neighborhood V and W such that $V \cap W = \varnothing$, then $(V \cap S) \cap (W \cap S) = \varnothing$ where $V \cap S$ and $W \cap S$ are open set in S and $x \in V \cap S$ and $y \in W \cap S$. $\square$

Theorem 3.3. *Let $\mathcal{X}$ be a Hausdorff space and $K \subseteq \mathcal{X}$ be a compact subset. Then for any $y \notin K$, there are open sets $K \subseteq V$ and $y \in W$ such that $V \cap W = \varnothing$.*

Proof. For any point $z \in K$, we have disjoint neighborhoods $y \in W_z$ and $z \in V_z$, consider the union of V_z, obviously it is an open cover on K, so there is a finite subcover $\{V_{z_\alpha}\}$, then we consider the intersection of the corresponding W_{z_α}, that is, $\bigcap_\alpha W_{z_\alpha}$, we can do this intersection cause the subcover is finite. Then let $V = \bigcup_\alpha V_{z_\alpha}$ and $W = \bigcap_\alpha W_{z_\alpha}$, obviously $V \cap W = \varnothing$. $\square$

Theorem 3.4. *Any compact subset of Hausdorff is closed.*

Proof. Suppose $\mathcal{X}$ is Hausdorff and $K \subseteq \mathcal{X}$ a compact subset. For any $y \notin K$, we have $K \subseteq V_y$ and $y \in W_y$ such that $V_y \cap W_y = \varnothing$. Consider the union of these W_y, we can see that it is an open set and contains every point in $\mathcal{X} \setminus K$. $\square$

Theorem 3.5. *Let $\mathcal{X}$ be a Hausdorff space and $K, L \subseteq \mathcal{X}$ be two compact subsets that $K \cap L = \varnothing$. Show that there are open sets $K \subseteq V$ and $L \subseteq W$ such that $V \cap W = \varnothing$*

Proof. For any $l \in L$, we have $K \subseteq V_l$ and $l \in W_l$ where $V_l \cap W_l = \varnothing$ by Theorem 3.3. Consider the union of these W_l, it is an open cover on L, therefore there is a finite subcover $\{W_{l_\alpha}\}$. Then we can take the intersection of the corresponding V_l, that is, $V = \bigcap_\alpha V_{l_\alpha}$, and the union of the subcover $W = \bigcup_\alpha W_{l_\alpha}$. They are disjoint, otherwise there is $x \in V_{l_\alpha}$ and $x \in W_{l_\alpha}$ for some α, which contradict to the property from Theorem 3.3. □

4 More Constructions

Definition 4.1 (Initial Topology)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ a mapping between two sets and $\mathcal{Y}$ is equipped with a topology. Then we declare that $V \subseteq \mathcal{X}$ is open if there is an open set $W \subseteq \mathcal{Y}$ such that $V = f^{-1}(W)$. This topology is called initial topology.*

Proof. We need to show that it forms a topology.

- $\varnothing = f^{-1}(\varnothing)$ and $\mathcal{X} = f^{-1}(\mathcal{Y})$.
- For a collection of open set $\{V_\alpha\}$, we claim $\bigcup_\alpha V_\alpha = f^{-1}(\bigcup_\alpha W_\alpha)$. For any $x \in \bigcup_\alpha V_\alpha$, it must belongs to some V_α, therefore $x \in f^{-1}(W_\alpha) \subseteq f^{-1}(\bigcup_\alpha W_\alpha)$. For any $x \in f^{-1}(\bigcup_\alpha W_\alpha)$, we know $f(x)$ must belongs to some W_α, so $x \in f^{-1}(W_\alpha) \subseteq f^{-1}(\bigcup_\alpha W_\alpha)$.
- For two open sets $V_0, V_1 \subseteq \mathcal{X}$, we claim $V_0 \cap V_1 = f^{-1}(W_0 \cap W_1)$. For any $x \in V_0 \cap V_1$, then $f(x) \in W_0$ and $f(x) \in W_1$, so $x \in f^{-1}(W_0 \cap W_1)$. For any $x \in f^{-1}(W_0 \cap W_1)$, we know $f(x) \in W_0 \cap W_1$, therefore $x \in f^{-1}(W_0) = V_0$ and $x \in f^{-1}(W_1) = V_1$, so $x \in V_0 \cap V_1$.

□

Definition 4.2 (Final Topology)**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ a mapping between two sets and $\mathcal{X}$ is equipped with a topology. Then we declare that $W \subseteq \mathcal{Y}$ is open if there is an open set $V \subseteq \mathcal{X}$ such that $V = f^{-1}(W)$. This topology is called final topology*

Proof. Similar to the proof of initial topology. □

We can see that the induced topology makes the mapping open.

Theorem 4.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous mapping between topological spaces, then:*

- *The initial topology on $\mathcal{X}$ is weaker than its own topology.*
- *The final topology on $\mathcal{Y}$ is stronger than its own topology.*

Recall that V is weaker than W if any open set under V is also an open set under W.

Proof.

- For any open sets V in initial topology of $\mathcal{X}$, we know there is $W \subseteq \mathcal{Y}$ such that $V = f^{-1}(W)$. Then V is an open set in the original topology cause f is continuous.
- For any open sets W in the original topology of $\mathcal{Y}$, we know there is $V \subseteq \mathcal{X}$ such that $V = f^{-1}(W)$ since f is continuous, then W is an open set in final topology of $\mathcal{Y}$.

□

Theorem 4.2. *Let $g : \mathcal{X} \to \mathcal{Y}$ be a continuous map.*

- *Suppose $\mathcal{X}$ is equipped with the initial topology induced by g. Show that a map $f : \mathcal{W} \to \mathcal{X}$ is continuous iff $g \circ f : \mathcal{W} \to \mathcal{Y}$ is continuous.*
- *Suppose $\mathcal{Y}$ is equipped with the final topology induced by g. Show that a map $h : \mathcal{Y} \to \mathcal{Z}$ is continuous iff $h \circ g : \mathcal{X} \to \mathcal{Z}$ is continuous.*

Proof. ($\Rightarrow$)s are trivial, we focus on ($\Leftarrow$)s.

- For any open set $W \subseteq \mathcal{Y}$, there is an open set $V \subseteq \mathcal{W}$ such that $V = (g \circ f)^{-1}(W)$ and $S \subseteq \mathcal{X}$ such that $S = g^{-1}(W)$. Then $V = (g \circ f)^{-1}(W) = (f^{-1} \circ g^{-1})(W) = f^{-1}(g^{-1}(W)) = f^{-1}(S)$. Also, every open set in $\mathcal{X}$ is induced by an open sets in $\mathcal{Y}$, so we proved that the inverse image of every open sets in $\mathcal{X}$ is also open in $\mathcal{W}$, that is, f is continuous.
- For any open set $W \subseteq \mathcal{Z}$, there is an open set $V \subseteq \mathcal{X}$ such that $V = (h \circ g)^{-1}(W)$. Then $V = (h \circ g)^{-1}(W) = g^{-1}(h^{-1}(W))$, and we know $h^{-1}(W)$ is open cause $\mathcal{Y}$ is the final topology induced by g, so there must be an open set which inverse image of g is V, and $h^{-1}(W)$ can do the job. So $h^{-1}(W)$ is an open set, therefore h is continuous.

□

Definition 4.3 (Quotient Topology)**.** *Let be an equivalence relation on a topology space $\mathcal{X}$. The set $[x] = \{ y \in \mathcal{X} \mid x \sim y \}$ is called the equivalence class of x. Then the final topology on $\mathcal{X}/\sim$ induced by $f(x) = [x]$ is called quotient topology. The set $\mathcal{X}/\sim$ equipped with a quotient topology is called quotient space.*

Theorem 4.3. *Let $f : \mathcal{K} \to \mathcal{Y}$ be a continuous map. Suppose $\mathcal{K}$ is compact and $\mathcal{Y}$ is Hausdorff, show that f is closed.*

Proof. For any closed set S in $\mathcal{K}$, we know S is compact since $\mathcal{K}$ is compact, then $f(S)$ is also compact in $\mathcal{Y}$ cause f is continuous. Then $f(S)$ is closed cause $\mathcal{Y}$ is Hausdorff. $\square$

Definition 4.4. *Let $\mathcal{X}$ be a topological space and G be a group. Suppose $- \cdot - : G \times \mathcal{X} \to \mathcal{X}$ is a mapping such that:*

1. *For any $x \in \mathcal{X}$, $1 \cdot x = x$. 1 is the identity of G.*
2. *For any $g, h \in G$ and $x \in \mathcal{X}$, $g \cdot (h \cdot x) = (g \cdot h) \cdot x$.*
3. *For any $g \in \mathcal{G}$, the map $x \mapsto g \cdot x$ is continuous.*

Then we say G acts on $\mathcal{X}$, or $\mathcal{X}$ is a G-space. In this case, the set $G \cdot x = \{ g \cdot x \mid \forall g \in G \}$ is called the G-orbit of x.

Theorem 4.4. *Suppose that a group G acts on a topological space $\mathcal{X}$. Show that for any $g \in G$, the map $x \mapsto g \cdot x$ defines a homeomorphism $\mathcal{X} \to \mathcal{X}$.*

Proof. We first show that $f(x) = g \cdot x$ is bijective. (Injective) If $f(a) = f(b)$ for some $a, b \in \mathcal{X}$, then $g \cdot a = g \cdot b$ and then $g^{-1} \cdot g \cdot a = g^{-1} \cdot g \cdot b$, which is eventually $a = b$. (Surjective) For any $x \in \mathcal{X}$, we have $g^{-1} \cdot x \in \mathcal{X}$ that $f(g^{-1} \cdot x) = g \cdot g^{-1} \cdot x = x$. By definition, we know f is continuous, we need to show that f^{-1} is continuous. It is easy to see that $f^{-1}(x) = g^{-1} \cdot x$, which is continuous by definition. So f is a homeomorphism. $\square$

Definition 4.5. *Suppose that a group G acts on a topological space $\mathcal{X}$. Define $x \sim y$ if there is $g \in G$ such that $y = g \cdot x$. We can show that $\sim$ is an equivalence relation, and $\mathcal{X}/\sim$ can be also denoted by $\mathcal{X}/G$. Note that $[x] = G \cdot x$, the orbit of x, therefore $\mathcal{X}/G$ is also called orbit space.*

Proof. We will show that $\sim$ is an equivalence relation.

- (Reflexivity) $x = 1 \cdot x$

- (Symmetry) If $x \sim y$, then $y = g \cdot x$, we have $x = g^{-1} \cdot g \cdot x = g^{-1} \cdot y$, that is, $y \sim x$.

- (Transitivity) if $x \sim y$ and $y \sim z$, then $y = g \cdot x$ and $z = h \cdot y$, we have $z = (h \cdot g) \cdot x$, that is, $z \sim x$.

$\square$

Theorem 4.5. *Suppose a group G acts on a topological space $\mathcal{X}$, and $f : \mathcal{X} \to \mathcal{X}/G$ is the quotient map.*

- *Show that f is open.*

- *Show that f is closed if G is finite.*

Proof.

- We can see that $f^{-1}(f(V))$ for any open set V is the set that contains points that equivalence to the points in V, that is, $G \cdot V$, it is easy to see that $G \cdot V = \bigcup_{g \in G} g \cdot V$ is open set, since every $g \cdot V$ is open while $g \cdot -$ is a homeomorphism by Theorem 4.4. Therefore, $f(V)$ has to be open cause $f^{-1}(f(V))$ is open.

- Similar to the previous answer, the finite condition is used when we are trying to obtain a union of some closed sets.

$\square$

5 Connected Spaces

Definition 5.1. *A subset of a topological space is called clopen if it is open and closed.*

Definition 5.2. *A topological space $\mathcal{X}$ is called connected if it has exactly two clopen sets: $\varnothing$ and $\mathcal{X}$.*

Note that the empty space $\varnothing$ is not connected.

A subset of topological space is called connected or disconnected if so is the corresponding subspace.

Definition 5.3. *A subset S of a topological space is called disconnected if it is empty or there are two open sets V and W such that:*

- $(V \cap S) \cap (W \cap S) = V \cap W \cap S = \varnothing$
- $V \cap S \neq \varnothing$ *and* $W \cap S \neq \varnothing$
- $V \cap W \cap S = S$ *(or equivalently,* $S \subseteq V \cap W$*)*

Otherwise, we say S is connected.

Theorem 5.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a continuous map between topological spaces. Show that f preserves connectness.*

Theorem 5.2. *Suppose $\mathcal{X}$ is a connected space, show that the quotient space $\mathcal{X}/\sim$ is connected for any equivalence relation $\sim$ on $\mathcal{X}$.*

Proof. Consider the quotient map $f : \mathcal{X} \to \mathcal{X}/\sim$ which is onto, then $f(\mathcal{X})$ is connected since $\mathcal{X}$ is connected. □

Theorem 5.3. *Suppose $\{A_\alpha\}_{\alpha \in \mathcal{I}}$ is a collection of connected subsets of a topological space. Suppose that $\bigcap_\alpha A_\alpha \neq \varnothing$, show that $A = \bigcup_\alpha A_\alpha$ is connected.*

Proof. Suppose A is disconnected, then there are two splitting V and W. We take $p \in \bigcap_\alpha A_\alpha \neq \varnothing$, since $A \subseteq V \cup W$, then $p \in V \cup W$, we may suppose $p \in V$. Then for any α, we have $V \cap A_\alpha \neq \varnothing$ cuase $p \in V \cap A_\alpha$, therefore $W \cap A_\alpha = \varnothing$, otherwise A_α is no longer connected. □

Theorem 5.4. *Let A be a connected set in a topological space. Suppose $A \subseteq B \subseteq \bar{A}$, show that B is connected.*

Proof. Suppose B is disconnected and V, W is a splitting of B. We may suppose $A \subseteq V$, otherwise V, W also splits A. Then $W \subseteq \partial A$ while W is open, which means there is a smaller close set $\bar{A} \setminus W$ that contains A, which is unacceptible. □

Definition 5.4. *Suppose $\mathcal{X}$ a topological space and $x \in \mathcal{X}$, the intersection of all clopen neighborhoods of x is called connected component of x. Note that the space $\mathcal{X}$ is connected iff $\mathcal{X}$ is a connected component of some point in $\mathcal{X}$.*

Theorem 5.5. *Show that any connected component is closed. Show that connected component is not necessary open.*

Proof. The intersection of closed sets is closed. However, the infinite intersection of open sets is not necessary open. □

6 Path-connected spaces

Definition 6.1. *Let $\mathcal{X}$ be a topological space. A continuous map $f : [0,1] \to \mathcal{X}$ is called path. If $f(0) = x$ and $f(1) = y$, we say f is a path from x to y.*

Definition 6.2. *A space $\mathcal{X}$ is called path-connected if it is nonempty and any two points in $\mathcal{X}$ can be connected by a path.*

Theorem 6.1. *Any path-connected space is connected.*

Proof. Suppose $\mathcal{X}$ a path-connected space, take $x \in \mathcal{X}$. Then we consider all path from x to y for all $y \in \mathcal{X}$, we can see that all these path is connected cause $[0,1]$ is connected. Then:

$$\bigcup_{y \in \mathcal{X}} f_y([0,1])$$

is connected cause every $f_y([0,1])$ (the path from x to y) is connected, and these path contain x. $\square$

Definition 6.3. *Given a path $f : [0,1] \to \mathcal{X}$, one can consider the time-reversed path $\bar{f}$:*

$$\bar{f}(t) = f(1-t)$$

Note that $\bar{f}$ is continuous cause f is continuous.

Definition 6.4. *let f and g be paths in the topological space $\mathcal{X}$. If $f(1) = g(0)$, we can join these two paths into one $h : [0,1] \to \mathcal{X}$:*

$$h(t) = \begin{cases} f(2t) & \text{if } t \leq \frac{1}{2} \\ g(2t-1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

Theorem 6.2. *Show that $\sim$ is equivalence relation: $x \sim y$ iff there is a path from x to y.*

Proof.

- (Reflexivity) Obviously, there is a path from x to x.
- (Symmetry) Consider the time-reversed path.
- (Transitivity) Consider the concatenation of paths.

$\square$

The equivalence class of point x for the equivalence relation $\sim$ is called path-connected component of x.

Theorem 6.3. *Show that the product of path-connected space is path-connected.*

Proof. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are path-connected spaces, for any $(a,b),(c,d) \in \mathcal{X} \times \mathcal{Y}$, we know there are paths $a \xrightarrow{f} c$ and $b \xrightarrow{g} d$. We claim $h(t) = (f(t), g(t))$ is a path from (a,b) to (c,d). We need to show that h is continuous. For any open sets $\bigcup_\alpha V_\alpha \times W_\alpha$ in $\mathcal{X} \times \mathcal{Y}$ for every α, we claim $h^{-1}(V_\alpha \times W_\alpha) = f^{-1}(V_\alpha) \cap g^{-1}(W_\alpha)$.

($\subseteq$) For any $i \in [0,1]$ such that $h(i) \in V_\alpha \times W_\alpha$, we know $f(i) \in V_\alpha$ and $g(i) \in W_\alpha$, therefore $i \in f^{-1}(V_\alpha) \cap g^{-1}(W_\alpha)$.

($\supseteq$) For any $i \in f^{-1}(V_\alpha) \cap g^{-1}(W_\alpha)$, we know $h(i) = (f(i), g(i)) \in V_\alpha \times W_\alpha$ cause $f(i) \in V_\alpha$ and $g(i) \in W_\alpha$, therefore $i \in h^{-1}(V_\alpha \times W_\alpha)$.

We can see that $h^{-1}(V_\alpha \times W_\alpha)$ is open cause f and g are continuous. Then we claim $h^{-1}(\bigcup V_\alpha \times W_\alpha) = \bigcup h^{-1}(V_\alpha \times W_\alpha)$.

($\subseteq$) For any $i \in [0,1]$ such that $h(i) \in \bigcup V_\alpha \times W_\alpha$, we know $h(i) \in V_\alpha \times W_\alpha$ for some α, therefore $i \in h^{-1}(V_\alpha \times W_\alpha)$.

($\supseteq$) For any $i \in \bigcup h^{-1}(V_\alpha \times W_\alpha)$, we know $i \in h^{-1}(V_\alpha \times W_\alpha)$ for some α, therefore $h(i) \in V_\alpha \times W_\alpha \subseteq \bigcup V_\alpha \times W_\alpha$.

Therefore the inverse image of some open sets in $\mathcal{X} \times \mathcal{Y}$ is open, cause it is a union of some open sets, then h is continuous. $\square$

Hey, we are trying to obtain an arrow $h : [0,1] \to \mathcal{X} \times \mathcal{Y}$ from $f : [0,1] \to \mathcal{X}$ and $g : [0,1] \to \mathcal{Y}$, it is similar to the unique morphism in the product diagram!