# 1 Yoneda

This chapter combines arguments from some books:

- The Dao of FP
- The Joy of Abstraction

**Definition 1.1.** *$H_x = \mathcal{C}(-, x)$ and $H^x = \mathcal{C}(x, -)$.*

Take $H^x$ as an example, it sends $\mathcal{C}$ to **Set**, the interesting part is the mapping on morphism. For any morphism $f : a \to b$ of $\mathcal{C}$, $H^f$ must be a mapping $\mathcal{C}(x, a) \to \mathcal{C}(x, b)$, we can see that $g \mapsto f \circ g$ would be a choice.

We have to show that it satifies the functoriality:

- $H^{id_a}(g) = id_a \circ g = g$
- $H^{f \circ g}(h) = (f \circ g) \circ h = f \circ (g \circ h) = H^f(g \circ h) = H^f(H^g(h)) = (H^f \circ H^g)(h)$.

Similar to $H_x$, the only difference is that $H_x$ is a contrafunctor.

Suppose $f : a \to b$ an isomorphism, we can see that $H^x$ gives an isomorphism between two hom-sets: $\mathcal{C}(x, a)$ and $\mathcal{C}(x, b)$.

Furthermore, we can no more fix $x$, that is, make $H_\bullet$ (or $H^\bullet$) a functor from $\mathcal{C}$ to $[\mathcal{C}^{op}, \mathbf{Set}]$, a functor to a functor!

The problem we need to solve is that what should $H_\bullet$ do on a morphism $f : a \to b$. Since $H_\bullet$ produce a functor, $H_f$ must produce a natural transformation between $H_a$ and $H_b$. Suppose $x, y \in \mathcal{C}$ and $g : x \to y$, note that $H_a$ and $H_b$ are contrafunctor, so we need to reverse the arrows!

$$\begin{tikzcd}
H_a(y) \arrow[r, "H_a(g)"] \arrow[d, "(H_f)_y"'] & H_a(x) \arrow[d, "(H_f)_x"] \\
H_b(y) \arrow[r, "(H_b)(g)"'] & H_b(x)
\end{tikzcd}$$

and we can unfold the definitions

$$\begin{tikzcd}
\mathcal{C}(y,a) \arrow[r, "\mathcal{C}(g,a)"] \arrow[d, "(H_f)_y"'] & \mathcal{C}(x,a) \arrow[d, "(H_f)_x"] \\
\mathcal{C}(y,b) \arrow[r, "\mathcal{C}(g,b)"'] & \mathcal{C}(x,b)
\end{tikzcd}$$

and suppose $s \in \mathcal{C}(y,a)$, we know the top-right corner would be $s \circ g$ since $\mathcal{C}(g,a) = - \circ g$ (same to $\mathcal{C}(g,b)$). In order to construct an arrow in $\mathcal{C}(y,b)$, we can pre-compose the arrow $f : a \to b$. Then the bottom-left corner would be $f \circ s$, and the bottom-right corner would be $(f \circ s) \circ g$ (by left-bottom path) and $f \circ (s \circ g)$ (by top-right path), which is exactly the same!

$$\begin{tikzcd}
s \arrow[r, mapsto, "- \circ g"] \arrow[d, mapsto, "f \circ -"'] & s \circ g \arrow[d, mapsto, "f \circ -"] \\
f \circ s \arrow[r, mapsto, "- \circ g"'] & (f \circ s) \circ g = f \circ (s \circ g)
\end{tikzcd}$$

Note that the condition here $(f \circ -) \circ (- \circ g) = (- \circ g) \circ (f \circ -)$ is the naturality condition which is mentioned in *The Dao of FP*. A bijection between two hom-sets $\alpha_x = \mathcal{C}(x,a) \to \mathcal{C}(x,b)$ that satifies the naturality condition $\alpha_y \circ (- \circ g) = (- \circ g) \circ \alpha_x$ can retrieve the isomorphism between $a$ and $b$. This will be unsurprised if we notice that such bijection with naturality condition forms a natural transformation, then we can retrieve the morphism (not isomorphism yet) from it. The morphism becomes iso- when we know $H_\bullet$ is full and faithful (see below and chapter *functor*),

**Definition 1.2.** *$H_\bullet$ is called Yoneda embedding.*

**Theorem 1.1.** *Shows $H_\bullet$ is an embedding by showing it is full and faithful.*

*Proof.* (Full) For any $a, b \in \mathcal{C}$, suppose $\alpha : [\mathcal{C}^{op}, \mathbf{Set}](H_a, H_b)$ a morphism (natural transformation). Use the Yoneda trick, we have $\alpha_a : \mathcal{C}(a,a) \to \mathcal{C}(a,b)$ and then $\alpha_a(id_a) : \mathcal{C}(a,b)$. As we see the definition of $H_\bullet$ on morphism, we should expect that $\alpha$ has form $f \circ -$ for some $f : a \to b$. But how coincident, we have a morphism $\alpha_a(id_a) : \mathcal{C}(a, b)$. So we claim $H_{\alpha_a(id_a)} = \alpha$. (In the other hand, if $\alpha$ has form $f \circ -$, then $\alpha_a(id_a) = f \circ id_a = f$). For any $x \in \mathcal{C}$, we need to show $(H_{\alpha_a(id_a)})_x = \alpha_x : \mathcal{C}(x, a) \to \mathcal{C}(x, b)$. So we suppose $g \in \mathcal{C}(x, a)$, then the following diagram commutes since $\alpha$ is natural:

$$\begin{tikzcd}
\mathcal{C}(a,a) \arrow[r, "-\circ g"] \arrow[d, "\alpha_a"'] & \mathcal{C}(x,a) \arrow[d, "\alpha_x"] \\
\mathcal{C}(a,b) \arrow[r, "-\circ g"'] & \mathcal{C}(x,b)
\end{tikzcd}$$

Then

$$\begin{tikzcd}
id_a \arrow[r, mapsto, "-\circ g"] \arrow[d, mapsto, "\alpha_a"'] & id_a \circ g = g \arrow[d, mapsto, "\alpha_x"] \\
\alpha_a(id_a) \arrow[r, mapsto, "-\circ g"'] & \alpha_a(id_a) \circ g = \alpha_x(g)
\end{tikzcd}$$

is a proof of $H_{\alpha_a(id_a)}(g) = \alpha_a(id_a) \circ g = \alpha_x(g)$.

(Faithful) Suppose $f \circ - H_f = H_g = g \circ -$, then $f = f \circ id_a = H_f(id_a) = H_g(od_a) = g \circ id_a = g$. $\square$

As we see in the proof of $H_\bullet$ is a full functor, the natural transformation $\alpha$ at some $x$ (therefore any $x \in \mathcal{C}$) is completely determined by the value $\alpha_a(id_a)$, cause for any $g$, we have $\alpha_a(id_a) \circ g = \alpha_x(g)$.

We may rename $H_b$ with $F$, then

$$\begin{tikzcd}
\mathcal{C}(a,a) \arrow[rrr, "\mathcal{C}(g,a)"] \arrow[ddd, "\alpha_a"'] & & & \mathcal{C}(x,a) \arrow[ddd, "\alpha_x"] \\
& id_a \arrow[r, mapsto, "-\circ g"] \arrow[d, mapsto, "\alpha_a"'] & id_a \circ g = g \arrow[d, mapsto, "\alpha_x"] & \\
& \alpha_a(id_a) \arrow[r, mapsto, "Fg"'] & (Fg)(\alpha_a(id_a)) = \alpha_x(g) & \\
Fa \arrow[rrr, "Fg"'] & & & Fx
\end{tikzcd}$$

It seems that $H_b$ can be replaced with any functor in $[\mathcal{C}^{op}, \mathbf{Set}]$, furthermore, the natural transformation is still determined by $\alpha_a(id_a)$ (and $\alpha_a(id_a)$ is determined by $\alpha$, trivial though, but it implies that there is a precisely corresponding).

**Theorem 1.2** (Yoneda Lemma)**.** *Show that the natural transformation between $H_a$ and any functor $F \in [\mathcal{C}^{op}, \boldsymbol{Set}]$ correspond precisely to the elements of $Fa$. In other words,*

$$[\mathcal{C}^{op}, \boldsymbol{Set}](H_a, F) \cong Fa$$

*Proof.* The arrow $f : [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \to Fa$ is obvious by the Yoneda trick.

$$\alpha \mapsto \alpha_a(id_a) : [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \to Fa$$

For any $x \in Fa$, as we see before, we may expect there is a natural transformation $\alpha$ such that $\alpha_a(id_a) = x$, then the action on other objects is completely determined by $\alpha_a(id_a)$ as we see before.

$$x, g \mapsto (Fg)(x) : Fa \to [\mathcal{C}^{op}, \mathbf{Set}](H_a, F)$$

Note that $(F(id_a))(x) = id_{Fa}(x) = x$.

We need to show that they are inverse to each other. For any natural transformation $\alpha$:

$$[\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \xrightarrow{\quad f \quad} Fa \xrightarrow{\quad f^{-1} \quad} [\mathcal{C}^{op}, \mathbf{Set}](H_a, F)$$

$$\alpha \xmapsto{\quad -_a(id_a) \quad} \alpha_a(id_a) \xmapsto{\quad (F-)(-) \quad} (F-)(\alpha_a(id_a))$$

And we can see $(Fg)(\alpha_a(id_a)) = \alpha_x(g)$ for any $x \in \mathcal{C}$ and $g \in \mathcal{C}(x, a)$ by the same way as the proof which shows $H_\bullet$ is a full functor.

In another direction, we get:

$$Fa \xrightarrow{\quad f^{-1} \quad} [\mathcal{C}^{op}, \mathbf{Set}](H_a, F) \xrightarrow{\quad f \quad} Fa$$

$$x \xmapsto{\quad (F-)(-) \quad} (F-)(x) \xmapsto{\quad -_a(id_a) \quad} F(id_a)(x)$$

It is obvious that $F(id_a)x = id_{Fa}x = x$. $\square$