Definition 0.11. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{null} T = \{ v \mid Tv = 0 \}$ *is called the* ***null space*** *of* T*.*

This is also called the **kernal** of T in algebra.

Theorem 0.13. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{null} T$ *is a subspace of* V*.*

Proof.

- We have $0 \in \operatorname{null} T$ since $T0 = 0$, which is the property of linear transformation.
- For any $Ta, Tb \in \operatorname{null} T$, we have $0 = Ta + Tb = T(a + b)$, so $a + b \in \operatorname{null} T$.
- For any $Ta \in \operatorname{null} T$ and $\lambda \in F$, we have $\lambda Ta = T(\lambda a)$, so $\lambda a \in \operatorname{null} T$.

□

Definition 0.15. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{range} T = T(V) = \{ Tv \mid v \in V \}$ *is called the* ***range*** *of* T*.*

This is also called the **image** of T in math.

Theorem 0.18. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{range} T$ *is a subsapce of* W*.*

Proof. Consider the basis $_0 \cdots$ □

Exercise 0.1. *Suppose* V *is a finite vector space, Show that the only two ideal of* $\mathcal{L}(V)$ *is* $\{0\}$ *and* $\mathcal{L}(V)$*. A subspace* $\mathcal{E}$ *of* $\mathcal{L}(V)$ *is called an ideal, if* $TE \in \mathcal{E}$ *and* $ET \in \mathcal{E}$ *for any* $T \in \mathcal{L}(V)$ *and* $E \in \mathcal{E}$*.*

Proof. We will use the concept Matrix. Suppose $\lambda_0 v_0 + \cdots + \lambda_n v_v$ the basis of V. We want to construct T_i that $T(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_i v_i$ for all $0 \le i < n$, which is a matrix with all zero but 1 at i, i.

For any matrix, we can always select a non-zero value at a, b and place it at i, b, this can be done by left multiply a matrix with 1 at i, a (this produce a vector at line i with values from line a), then right multiply a matrix with 1 at i, b (this produce a vector at column b with values from line i).

Also, we can always select a non-zero value at a, b and place it at a, i, this can be done by right multiply a matrix with 1 at b, i, then left multiply a matrix with 1 at a, i.

By combining these two operations, we ca select a non-zero value at a, b and place it at i, i. Now, consider any non-zero $E \in \mathcal{E}$, we can construct a matrix with non-zero value at i, i for every $0 \leq i < \dim V$. These matrix are in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then we can multiply an apropriate scalar to them so that they are matrices with 1 at i, i. By adds up these matrices, we get I, we know $I \in \mathcal{E}$ since $\mathcal{E}$ is a vector space, and now all $T \in \mathcal{L}(V)$ is also in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then $\mathcal{E} = \mathcal{L}(V)$.

The only exception is $\mathcal{E} = \{0\}$, in this case we can't pick any non-zero element. $\square$