Exercise 5.1. *Let $T \in \mathcal{L}(V)$. Show that 9 is an eigenvalue of $T^2 \iff 3$ or -3 is an eigenvalue of T.*

Proof.

- ($\Rightarrow$) We have $T^2 - 9I$ is not injective since 9 is an eigenvalue of T^2, then $(T - 3I)(T + 3I) = T^2 - 9I$ is not injective means one of $T - 3I$ and $T + 3I$ is not injective, thus 3 or -3 is an eigenvalue of T.

- ($\Leftarrow$) Similarly, we have $(T - 3I)(T + 3I)v = (T^2 - 9I)v = 0$ (if 3 is an eigenvalue of T) or $(T + 3I)(T - 3I)v = (T^2 - 9I)v = 0$ (if -3 is an eigenvalue of T).

$\square$

Exercise 5.2. *Let V a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$ has no eigenvalue. Show that any subspace of V that is invariant under T is either $\{0\}$ or infinite dimension.*

Proof. Let $U \subseteq V$ a subspace that is invariant under T, and non-zero $u \in U$. We can repeatly apply T to u, say $u, Tu, T^2u, \cdots$. Suppose $k > 0$ is minimum such that $u, Tu, \cdots, T^k u$ is linear dependent, we have $p \in \mathcal{P}(\mathbb{C})$ with $\deg p = k$ such that $p(T) = 0$. Clearly p is not constants, thus it has a zero since p is a polynomial of complex coefficient. Thus such zero is an eigenvalue of T. $\square$

Exercise 5.3. *Let $n > 1$ an integer, and $T \in \mathcal{L}(F^n)$ is defined by:*

$$T(x_0, \cdots, x_{n-1}) = (x_0 + \cdots + x_{n-1}, \cdots, x_0 + \cdots + x_{n-1})$$

- *Find all eigenvalue and eigenvector of T.*

- *Find the minimal polynomial of T.*

Proof.

- Observe that $\operatorname{range} T = \operatorname{span}((1, \cdots, 1))$, thus $T(1, \cdots, 1) = n(1, \cdots, 1)$.

- Observe that $T(x_0, \cdots, x_{n-1}) = (x_0 + \cdots + x_{n-1})(1, \cdots, 1)$ and $T^2(x_0, \cdots, x_{n-1}) = n(x_0 + \cdots + x_{n-1})(1, \cdots, 1)$, thus $p(T) = nT - T^2 = 0$.

$\square$

Exercise 4 is kinda hard, sorry.

Exercise 5.6. *Let $T \in \mathcal{L}(F^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T.*

Proof. Observe that $T^2(w, z) = T(-z, w) = (-w, -z) = (-1)(w, z)$, thus the minimal polynomial of T is $p(T) = I + T^2$. $\square$

Exercise 5.7.
- *Given an example that the minimal polynomial of ST is not equal to TS's.*
- *Suppose V is finite and $S, T \in \mathcal{L}(V)$. Show that the minimal polynomial of ST is equal to TS's if one of S and T is invertible.*

  Hint: Show that S is invertible and $p \in \mathcal{P}(F)$ implies $p(TS) = S^{-1}p(ST)S$.

Proof.

- The idea is to find S, T such that $ST \neq 0$ but $TS = 0$. We can find $S(x, y) = (x, 0)$ and $T(x, y) = (y, 0)$ holds:

  $$\begin{aligned}(ST)(x, y) &= S(y, 0) = (y, 0)\\ (TS)(x, y) &= T(x, 0) = (0, 0)\end{aligned}$$

  Thus the minimal polynomial of ST is not 0 but TS one does.

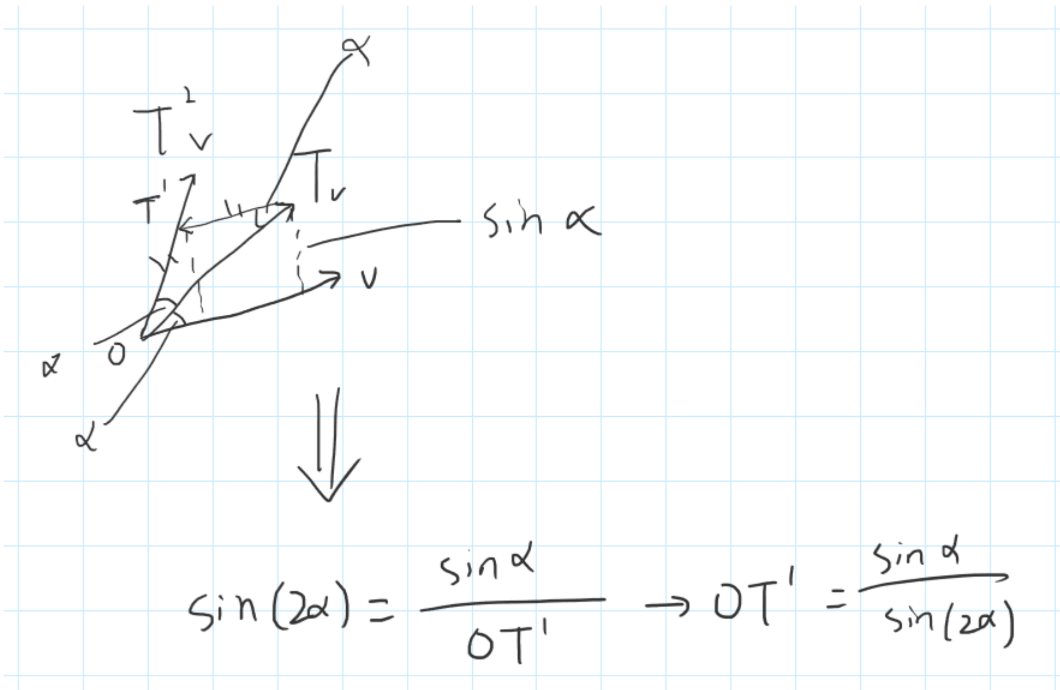
- Suppose S is invertible and $p \in \mathcal{L}(F)$ is the minimal polynomial of TS, then $p(TS) = S^{-1}p(ST)S$ since i-th term of $S^{-1}p(ST)S$ has form $S^{-1}c_i(ST)^iS = c_i(S^{-1}S)(TS)^{i-1}(TS) = c_i(TS)^i$. Thus $S^{-1}p(ST)S = 0$ and then $p(ST) = 0$. We will show that p is the minimal polynomial of ST, suppose $q \in \mathcal{L}(F)$ such that $q(ST) = 0$, then $0 = S^{-1}q(ST)S = q(TS)$, therefore $\deg q = \deg p$. Hence p is the minimal polynomial of ST.

$\square$

Exercise 5.8. *Let $T \in \mathcal{L}(R^2)$ is the opeartor that "rotates 1 degree counter-clockwise", find the minimal polynomial of T.*

*Note that it is **NOT** $x^{180} + 1$ even $T^{180} = -I$.*

Proof. Note that there is some λ such that $Tv - \lambda v = \alpha T^2 v$ (We can show that $\lambda = \alpha$), however the calculation is too complicate.



λ should be $\frac{\sin(1^\circ)}{\sin(2^\circ)}$, thus $p(T) = -\lambda I + T - \lambda T^2$.

We suppose all v below has length 1, thus $v = (\cos\theta, \sin\theta)$, this doesn't lose the generalizability since $p(T)(\alpha v) = \alpha(p(T)v)$.

For the first component of $p(T)v = -\lambda v + Tv - \lambda T^2 v$, we have:

$$
\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - \cos(\theta + 2^\circ)) \\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - (\cos\theta\cos(2^\circ) - \sin\theta\sin(2^\circ))) \\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\cos\theta - \cos\theta\cos(2^\circ)) + \sin\theta\sin(1^\circ) \\
=&\cos\theta\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1 - \cos(2^\circ)) + \sin\theta\sin(1^\circ)
\end{aligned}
$$

where $\sin\theta\sin(1^\circ)$ cancels a part of $(Tv)_1 = \cos(\theta + 1^\circ) = \cos\theta\cos(1^\circ) -$

$\sin\theta\sin(1^\circ)$. Thus we will show that $\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1-\cos(2^\circ)) = -\cos(1^\circ)$.

$$
\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1-\cos(2^\circ))\\
=&\frac{\sin(1^\circ)}{2\sin(1^\circ)\cos(1^\circ)}(-(\cos^2(1^\circ)+\sin^2(1^\circ))-\cos^2(1^\circ)+\sin^2(1^\circ))\\
=&\frac{1}{2\cos(1^\circ)}(-\cos^2(1^\circ)-\cos^2(1^\circ))\\
=&\frac{1}{2\cos(1^\circ)}(-2\cos^2(1^\circ))\\
=&-\cos(1^\circ)
\end{aligned}
$$

For the second component of $p(T)v$, we have:

$$
\begin{aligned}
&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\sin\theta-\sin(\theta+2^\circ))\\
=&\frac{\sin(1^\circ)}{\sin(2^\circ)}(-\sin\theta-\sin\theta\cos(2^\circ)-\cos\theta\sin(2^\circ))\\
=&\sin\theta\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1-\cos(2^\circ))-\cos\theta\sin(1^\circ)
\end{aligned}
$$

similarly, we have $p(T)v_2 = \sin(\theta+1^\circ) = \sin\theta\cos(1^\circ)+\cos\theta\sin(1^\circ)$ we will show that $\frac{\sin(1^\circ)}{\sin(2^\circ)}(-1-\cos(2^\circ)) = -\cos(1^\circ)$, which is proven above. □

Exercise 5.9. *Let $T \in \mathcal{L}(V)$ such that for some basis of V, $\mathcal{M}(T)$ consists of rational numbers. Try to explain why the coefficients of the minimal polynomial of T is rational numbers.*

Proof. I don't know, because $\mathbb{Q}$ is also a field? □

Exercise 5.11. *Let V a vector space and $\dim V = 2$ and $T \in \mathcal{L}(V)$ such that $\mathcal{M}(T)$ for some basis of V is $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Show that:*

- $T^2-(a+d)T+(ad-bc)I = 0$

- *the minimal polynomial of T is:*

$$\begin{cases} z-a & \text{if } b=c=0 \text{ and } a=d \\ z^2-(a+d)z+(ad-bc) & \text{otherwise} \end{cases}$$

Proof.

- $$\begin{aligned} &\mathcal{M}(T^2-(a+d)T+(ad-bc)I) \\ &= \begin{bmatrix} a & c \\ b & d \end{bmatrix}^2 - (a+d)\begin{bmatrix} a & c \\ b & d \end{bmatrix} + (ad-bc)I \\ &= \begin{bmatrix} a^2+bc & ac+bd \\ ab+bd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ac+cd \\ ab+bd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \end{aligned}$$

- If $b=c=0$ and $a=d$, then T is a scalar multiple of identity operator, thus $T=aI$ and $p(T)=-aI+T=0$. Otherwise, $T^2-(a+d)T+(ad-bc)I=0$.

$\square$