Exercise 5.1. *Let V a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Show that*

- *$T^4 = I$ implies T is diagonalizable.*
- *$T^4 = T$ implies T is diagonalizable.*
- *Give an example that $T \in \mathcal{L}(C^2)$ such that $T^4 = T^2$ while T is not diagonalizable.*

Proof.

- $T^4 = I$ implies $p(z) = z^4 - 1$ and $p(T) = 0$, then $p(z) = (z+i)(z-i)(z+1)(z-1)$ where $i, -i, 1, -1$ are distinct to each others, and p is polynomial multiple of the minimal polynomial of T, thus T is diagonalizable.
- $T^4 = T$ implies $p(z) = z^4 - z$ and $p(T) = 0$, then $p(z) = (z-0)(z^3-1) = (z-0)(z-(\cos(120°)+i\sin(120°)))(z-(\cos(120°-i\sin(120°))))(z-1)$, thus T is diagonalizable.
- Maybe $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$? I have no idea why C^2.

$\square$

Exercise 5.3. *Let V finite and $T \in \mathcal{L}(V)$ Show that T is diagonalizable, then $V = \operatorname{null} T \oplus \operatorname{range} T$.*

Proof. Let $v \in \operatorname{null} T \cap \operatorname{range} T$, then $Tv = 0$ and $Tv = T(c_0v_0 + \cdots + c_{n-1}v_{n-1}) = \lambda_0c_0v_0 + \cdots + \lambda_{n-1}c_{n-1}v_{n-1}$ where λ_i are the numbers in the diagonal of the matrix of T. Since $v_0, \cdots, v_{n-1}$ is linear independent, then $\lambda_0c_0, \cdots, \lambda_{n-1}c_{n-1}$ is all zero, if $\lambda_0, \cdots, \lambda_{n-1}$ is not all zero, then $v = 0$, which means $\operatorname{null} T + \operatorname{range} T$ is a direct sum and $\dim V = \operatorname{null} T + \operatorname{range} T$, thus $V = \operatorname{null} T \oplus \operatorname{range} T$. If $\lambda_0, \cdots, \lambda_{n-1}$ is all zero, then $T = 0$, thus $V = \operatorname{null} T \oplus \{0\}$. $\square$

Exercise 5.5. *Let V finite vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Show that T is diagonalizable $\iff$ $V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$ for any $\lambda \in \mathbb{C}$.*

Proof.

- For any $\lambda \in \mathbb{C}$, $T - \lambda I$ is also diagonalizable since both T and $-\lambda I$ are diagonalizable, thus $V = \operatorname{null}(T - \lambda I) \oplus \operatorname{range}(T - \lambda I)$.

- We induction on the dimension of V

  Base($\dim V = 1$): Clearly T is diagonalizable.

  Ind($\dim V = n+1$): Since T is an operator of finite complex vector space, then there is $\lambda \in \mathbb{C}$ an eigenvalue of T, thus $\text{null}(T-\lambda I) = E(\lambda, T)$, also we have $\text{range}(T-\lambda I)$ is invariant under T (since $\text{range}\, p(T)$ is invariant under T for any $p \in \mathcal{P}(F)$). We may define $U = \text{range}(T - \lambda I)$, then $T\big|_U$ is an opeartor on U, which has lower dimension. Furthermore, for any $\alpha \in \mathbb{C}$, $\text{null}(T\big|_U - \alpha I) \subseteq \text{null}(T - \alpha I)$ and $\text{range}(T\big|_U - \alpha I) \subseteq \text{range}(T-\alpha I)$, also $\text{null}(T-\alpha I) \cap \text{range}(T-\alpha I) = \{0\}$, thus $\text{null}(T\big|_U - \alpha I) \cap \text{range}(T\big|_U - \alpha I) = \{0\}$, therefore $U = \text{null}(T\big|_U - \alpha I) \oplus \text{range}(T\big|_U - \alpha I)$. Hence $T\big|_U$ is diagonalizable by the induction hypothesis, then $U = E(\lambda_1, T\big|_U) \oplus \cdots \oplus E(\lambda_{n-1}, T\big|_U)$ where $E(\lambda_i, T\big|_U) \subseteq E(\lambda_i, T)$, also, $V = E(\lambda, T) \oplus U$, therefore $E(\lambda_i, T\big|_U) = E(\lambda_i, T)$ otherwise the dimension doesn't match.

  Now, $V = E(\lambda, T) \oplus E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_{n-1}, T)$, thus T is diagonalizable.

□

Exercise 5.6. *Let $T \in \mathcal{L}(F^5)$ and $\dim E(8, T) = 4$, show that $T-2I$ or $T-6I$ is invertible.*

Proof. Basically it says that $\dim F^5 = 5$ and $\dim E(8, T) = 4$, therefore T can only have one another eigenvalue, thus T can not have both eigenvalue 2 and 6, therefore one of $T - 2I$ and $T - 6I$ is invertible. □

Exercise 5.7. *Let $T \in \mathcal{L}(V)$ and T inveritble. Show that*

$$E(\lambda, T) = E(\frac{1}{\lambda}, T^{-1})$$

for all $\lambda \in F$ where $\lambda \neq 0$.

Proof. For any $\lambda \in F$ where $\lambda \neq 0$, if

- λ is an eigenvalue of T, then $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}. For any $v \in E(\lambda, T)$, then $T^{-1}v = T^{-1}(\frac{1}{\lambda}Tv) = \frac{1}{\lambda}T^{-1}(Tv) = \frac{1}{\lambda}v$ and vice versa

- λ is not an eigenvalue of T, then $\frac{1}{\lambda}$ is not an eigenvalue of T^{-1}, therefore $E(\lambda, T) = E(\frac{1}{\lambda}, T^{-1}) = \{0\}$.

$\square$

Exercise 5.10. *Find $R, T \in \mathcal{L}(F^4)$ with eigenvalues $2, 6, 7$ only, that there is no $S \in \mathcal{L}(F^4)$ such that $R = S^{-1}TS$.*

Proof. Let $R \in \mathcal{L}(F^4)$ such that $\dim E(2, R) = 2$ and $R \in \mathcal{L}(F^4)$ such that $\dim E(6, R) = 2$. Then for any $S \in \mathcal{L}(F^4)$ such that $R = S^{-1}TS$, we have $R-2I = S^{-1}TS-2I = S^{-1}TS-2S^{-1}IS = S^{-1}(T-2I)S$. Then null $R-2I = E(2, R) = \operatorname{null}(S^{-1}(T-2I)S)$, it is easy to see that $\dim \operatorname{null}(S^{-1}(T-2I)(S)) = \dim E(2, T)$, however $\dim E(2, T) = 1$ and $\dim E(2, R) = 2$, therefore no such S. $\square$

Exercise 5.11. *Find $T \in \mathcal{L}(\mathbb{C}^3)$ such that $6, 7$ are eigenvalues of T, and T is not diagonalizable.*

Proof. That means we need to find a T such that which minimal polynomial is $p(z) = (z-6)(z-7)^2$ or $(z-6)^2(z-7)$, we will find one for the former one. The formula reminds us that there is $w \in \mathbb{C}^3$ such that $(T-7I)w \neq 0$ but $(T-7I)^2 w = 0$, which means $(T-7I)w \in E(7, T)$. We may let $w = (0, 0, 1)$ and $(T-7I)w = (0, 1, 0)$, which is much simple. Then $T(0, 0, 1) - (0, 0, 7) = (0, 1, 0)$ gives us $T(0, 0, 1) = (0, 1, 7)$ and we can get:

$$\mathcal{M}(T) = \begin{bmatrix} 6 & & \\ & 7 & 1 \\ & & 7 \end{bmatrix}$$

with respect to the standard basis of $\mathbb{C}^3$.

Clearly for $p(z) = (z-6)(z-7)^2$, we have $p(T) = 0$, and $6, 7$ are eigenvalues of T, thus we only need to show that $q(z) = (z-6)(z-7)$ doesn't make $q(T) = 0$ (cause $\deg p = 3$ and $\deg q = 2$, where the minimal polynomial is a polynomial multiple of q). Then $(T-6I)(T-7I)(0, 0, 1) = (T-6I)(0, 1, 0) \neq 0$ since $(0, 1, 0) \notin E(6, T)$. Thus T has eigenvalue $6, 7$ and cannot be diagonalized. $\square$

Exercise 5.12. *Let $T \in \mathcal{L}(\mathbb{C}^3)$ such that $6, 7$ are eigenvalues of T, and T is not diagonalizable. Show that there is $(z_0, z_1, z_2) \in \mathbb{C}^3$ such that $T(z_0, \cdots, z_2) = (6 + 8z_0, 7 + 8z_1, 13 + 8z_2)$.*

Proof. Before proving, we can verify the previous proof, we can see: $T(-3, -7, -20)$ holds (by solving equation like $6 + 8z_0 = 6z_0$).

And I can't prove it (maybe i can, but 2 complicate.) $\square$

Exercise 5.13. *Let A is diagonal matrix with distinct element in diagonal, and matrix B has the same size as A. Show that $AB = BA \iff B$ is diagonal matrix.*

Proof. ($\Leftarrow$) is trivial, since elements in a field is communitive on multiplication.

We can see i-th line of AB is α_i times i-th line of B, and the i-th column of BA is α_i times i-th column of B.

Thus for any $i, j = 0, \cdots, n-1$, we have $(AB)_{i,j} = \alpha_i B_{i,j}$ and $(BA)_{i,j} = \alpha_j B_{i,j}$, thus $\alpha_i B_{i,j} = \alpha_j B_{i,j}$. If $B_{i,j} = 0$, then the proof is complete, otherwise $\alpha_i = \alpha_j$, while the elements in diagonal of A are distinct, thus $i = j$.

Therefore, elements that is not in the diagonal of B is 0, hence B is a diagonal matrix. □

Exercise 5.14.

- *Find V a finite vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$, such that T^2 is diagonalizable but T isn't.*
- *Let $F = \mathbb{C}$ and k a positive integer, show that T is diagonalizable $\iff$ T^k is diagonalizable.*

Proof.

- $V = \mathbb{C}^2$ and $T(x, y) = (y, 0)$, which matrix is (with respect to the standard basis):
  $$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
  and $p(T) = T^2 = 0$ is the minimal polynomial of T.
- ($\Rightarrow$) if T is diagonalizable obviously T^k is diagonalizable.

  ($\Leftarrow$) if T^k is diagonalizable, let p the minimal polynomial of T^k. Then let $q(z) = (z^k - \lambda_0) \cdots (z^k - \lambda_{m-1})$ where $\lambda_0, \cdots, \lambda_{m-1}$ are distinct elements in the diagonal of T^k, therefore $q(T) = (T^k - \lambda_0) \cdots (T^k - \lambda_{m-1}) = 0$, hence q is polynomial multiple of the minimal polynomial of T.

  Note that for complex number c, $z^k - c$ have k different zeros, then $q(z)$ consists of k^k term like $z - \lambda_i$ where λ_i are distinct to each others. Thus the minimal polynomial of T is also in a similar form such that T is diagonalizable.

□

Exercise 5.15. *Let V finite vector space over $\mathbb{C}$, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Show that the following statements are equivalent:*

- *T is diagonalizable*
- *There is no $\lambda \in \mathbb{C}$ such that p is polynomial multiple of $(z-\lambda)^2$.*
- *p and p' share no zero.*
- *The greatest common divisor (gcd) of p and p' is 1 (in other words, they are coprime).*

Proof.

- (1) $\Rightarrow$ (2) Trivial, since p must in form of $(z-\lambda_0)\cdots(z-\lambda_{n-1})$ where $\lambda_0, \cdots, \lambda_{n-1}$ are distinct.
- One idea is that p and p' share the same zero means $z-\lambda$ in both p and p', thus there must be $(z-\lambda)^2$ or higher in p so that $(z-\lambda)$ in p'. But I can't give a formal prove, maybe FIXME later.
- (3) $\Rightarrow$ (2) If p is polynomial multiple of $(z-\lambda)^2$, then $p(z) = (z-\lambda)^2 q(z)$, thus $p'(z) = (z-\lambda)^2 q'(z) + 2(z-\lambda)q(z)$, thus λ is zero of p and p', which contradict our assumption.
- (3) $\Rightarrow$ (4) Let d is the common divisor of p and p'. The zero of d is also the zero of p and p', however, we know there is no zero for d since our assumption (since our vector space is finite and over $\mathbb{C}$). That means $\deg d = 1$, also, p is monic polynomial, and $d(z)$ is the gcd of the first coefficient of p and p', which is $\gcd(1, \lambda) = 1$.

  In fact (3) $\iff$ (4), (3) $\Leftarrow$ (4) is much trivial.

$\square$

Exercise 5.16. *Let $T \in \mathcal{L}(V)$ diagonalizable, $\lambda_0, \cdots, \lambda_{n-1}$ are distinct eigenvalues of T. Show that $U \subseteq V$ a subspace is invariant under $T \iff$ there is $U_0, \cdots, U_{n-1} \subseteq V$ such that $U_{k-1} \subseteq E(\lambda_{k-1}, T)$ and $U = U_0 \oplus \cdots \oplus U_{n-1}$.*

Proof.

- ($\Rightarrow$) We can see $T\big|_U$ is an operator on U, consider the minimal polynomial p of T, then $p(T\big|_U) = 0$, therefore p is polynomial multiple of the minimal polynomial of $T\big|_U$, also p is in form of $(z-\lambda_0)\cdots(z-\lambda_{n-1})$ where λ_i are distinct, thus the minimal polynomial of $T\big|_U$ is in a similar form, therefore $T\big|_U$ is diagonalizable with eigenvalues $\lambda_0, \cdots, \lambda_{m-1}$, note that $T\big|_U$ may have less eigenvalues. Then $U = E(\lambda_0, T\big|_U) \oplus \cdots \oplus E(\lambda_{m-1}, T\big|_U) \oplus E(\lambda_m, T\big|_U) \oplus \cdots \oplus E(\lambda_{n-1}, T\big|_U)$ where $E(\lambda_{i-1}, T\big|_U) \subseteq E(\lambda_{i-1}, T)$ for all $i = 1, \cdots, m$ and $E(\lambda_{m+i-1}, T\big|_U) = \{0\} \subseteq E(\lambda_{m+i-1}, T)$ for all $i = 1, \cdots, n-m$.

- Trivial, for any $u \in U_{k-1} \subseteq E(\lambda_{k-1}, T)$, we have $Tu = \lambda_{k-1}u$, thus U_{k-1} is invariant under T, therefore U is invariant under T, where $U = U_0 \oplus \cdots \oplus U_{n-1}$

□

Exercise 5.17. *Let V finite, show that there is a basis which consists of diagonalizable operators for $\mathcal{L}(V)$.*

Proof. Let $v_0, \cdots, v_{n-1}$ a basis of V, then define $T_{i-1}(v) = \varphi_i(v)v_i$ where φ_i is dual basis of $v_0, \cdots, v_{n-1}$. Then $T_0, \cdots, T_{n-1}$ is linear independent with length n, therefore it is a basis of $\mathcal{L}(V)$.

T_i is diagonalizable since $p(z) = (z-0)(z-1)$ and $p(T) = 0$. □

Exercise 5.18. *Let $T \in \mathcal{L}(V)$ is diagonalizable, and $U \subseteq V$ is invariant under T. Show that T/U is diagonalizable.*

Proof. The minimal polynomial of T is polynomial multiple of the minimal polynomial of T/U. □

Exercise 5.19. *Prove or disprove: Let $T \in \mathcal{L}(V)$ and U is invariant under T, $T\big|_U$ and T/U is diagonalizable show that T is diagonalizable.*

Proof. Unlike a similar statement about upper-triangular (see Exercise 5.13 in E5C), diagonalizable matrix requires that which minimal polynomial has no factor like $(z-\lambda)^2$.

Consider $T(x, y) = (x, x + y)$, which matrix is:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

.

Let $U = \text{span}((1,0)) = E(1,T)$, then $T\big|_U$ and T/U is diagonalizable since U and V/U have dimension 1, however T is not diagonalizable since $p(z) = (z-1)^2$ is the minimal polynomial of T, we have $(T-I)(0,1) = (1,1)-(0,1) = (1,0)$. □

Exercise 5.20. *Let V finite and $T \in \mathcal{L}(V)$. Show that T is diagonalizable $\iff$ T' is diagonalizable*

Proof. T and T' have the same minimal polynomial. □

Exercise 5.21. *A fibonacci sequence $F_0, F_1, F_2, \cdots$ is defined by:*

$$F_0 = 0, F_1 = 1 \quad \text{and} \quad F_n = F_{n-2} + F_{n-1} \quad \forall n \geq 2.$$

.

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x,y) = (y, x+y)$.

1. *Show that $T^n(0,1) = (F_n, F_{n+1})$ for any non-negative n.*
2. *Find the eigenvalues of T.*
3. *Find a basis of R^2 which consists of eigenvectors of T.*
4. *Use (3) to calculate $T^n(0,1)$ and conclude that:*

$$F_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big]$$

   for any non-negative n.

5. *Use (4) to conclude that F^n is an integer that nearest to $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$ for any non-positive n.*

Proof.

- Induction on n.

  Base($n=0$): $T^0(0,1) = (0,1) = (F_0, F_1)$.

  Base($n=1$): $T^1(0,1) = (1,1) = (F_1, F_2)$.

  Ind($n = n+1$): $T^{n+1}(0,1) = T(T^n(0,1)) = T(F_n, F_{n+1}) = (F_{n+1}, F_n + F_{n+1}) = (F_{n+1}, F_{n+2})$.

- Suppose $F(x,y) = (y, x+y) = \lambda(x,y)$ where $(x,y) \neq 0$, then:

$$\lambda x = y$$
$$\lambda y = x + y$$

therefore

$$\lambda^2 x = x + \lambda x$$

we may suppose $x \neq 0$, otherwise $x = y = 0$ then $(x,y) = 0$ gives us $\perp$, therefore

$$\lambda^2 = \lambda + 1$$

has solution $\lambda = \frac{1 \pm \sqrt{5}}{2}$.

- We denote $\frac{1+\sqrt{5}}{2}$ by λ_0 and $\frac{1-\sqrt{5}}{2}$ by λ_1. Thus we have $E(\lambda_0, T) = \text{span}((1, \lambda_0), T)$ and $E(\lambda_1, T) = \text{span}((1, \lambda_1))$.

- First, by the change-of-basis formula, we have:

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 1 \\ \lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_0 & \\ & \lambda_1 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & 1 \\ -\lambda_1 & -1 \end{bmatrix}$$

therefore

$$\begin{aligned} \mathcal{M}(T^n(0,1)) &= \begin{bmatrix} 1 & 1 \\ \lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix}^n \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & 1 \\ -\lambda_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ \lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_0^n & 0 \\ 0 & \lambda_1^n \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_0^n \\ -\lambda_1^n \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_0^n - \lambda_1^n \\ \lambda_0^{n+1} - \lambda_1^{n+1} \end{bmatrix} \end{aligned}$$

where the first component is F_n and the second component is F_{n-1}, therefore:

$$F^n = \frac{1}{\sqrt{5}} (\lambda_0^n - \lambda_1^n)$$

where $\lambda = \frac{1 \pm \sqrt{5}}{2}$.

- We want to show that F^n is the nearest integer to $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$, that means we need to show:

$$|\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n| \leq \frac{1}{2}$$

Then $\left|\frac{1-\sqrt{5}}{2}\right| \leq 1$, thus we only need to show:

$$\begin{aligned}|\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})| &\leq \frac{1}{2} \\ |\frac{1-\sqrt{5}}{\sqrt{5}}| &\leq 1 \\ \left|1-\sqrt{5}\right| &\leq \sqrt{5}\end{aligned}$$

Obviously $\sqrt{5} > 2$ and $|1-\sqrt{5}| < 2$.

□

Exercise 5.22. *Let $T \in \mathcal{L}(V)$ and A is an $n \times n$ matrix of T with respect to some basis of V. Show that*

$$|A_{j,j}| > \sum_{\substack{k=1 \\ k\neq j}}^{n} |A_{j,k}|$$

implies T is invertible.

Proof. Gershgroin disk theorem says that any eigenvalue of T is in some gershgroin disk, therefore we can see 0 is not an eigenvalue of T, otherwise the inequality above becomes $\geq$. □