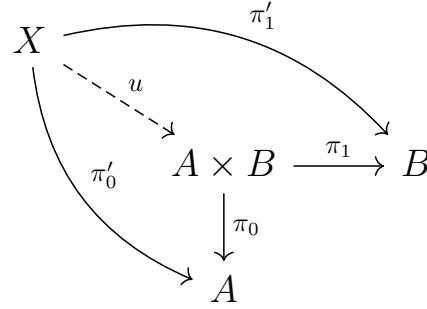


1 Product

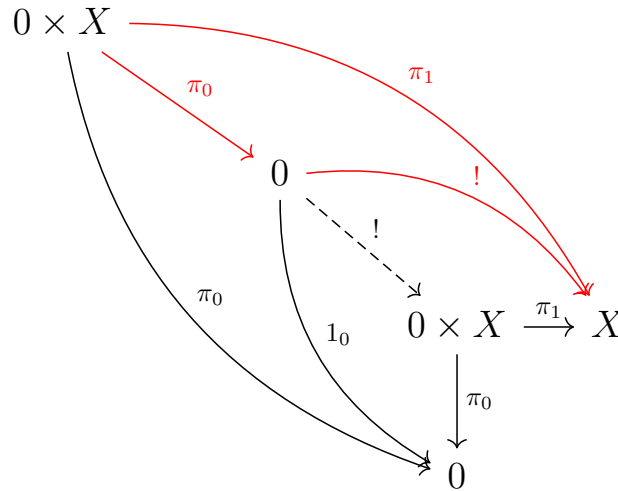
Definition 1.1 (Product). Let \mathcal{C} a category and $A, B \in \mathcal{C}$, $(A \times B, \pi_0, \pi_1)$ forms a product of A and B where $A \times B \in \mathcal{C}$, $\pi_0 : A \times B \rightarrow A$ and $\pi_1 : A \times B \rightarrow B$, if for any $X \in \mathcal{C}$ with $\pi'_0 : X \rightarrow A$ and $\pi'_1 : X \rightarrow B$, there is a unique arrow $u : X \rightarrow A \times B$ such that the following diagram commutes:



Furthermore, a product of A and B is a limit of diagram:

$$(A \quad B)$$

One may trying to show that $0 \times X \simeq 0$ by:



However, the red triangle needs not to commutes, that is, the arrow π_0 from $(0 \times X, \pi_0, \pi_1)$ to $(0, 1_0, !)$ may not exist.

Definition 1.2 (Product of Arrow). Suppose $(A \times B, \pi_0, \pi_1)$ and $(C \times D, \pi_2, \pi_3)$ are two product, and $f : A \rightarrow C$, $g : B \rightarrow D$. The product of arrow $f \times g$ is a

unique arrow from $A \times B$ to $C \times D$ such that the following diagram commutes:

$$\begin{array}{ccccc}
 A & \xleftarrow{\pi_0} & A \times B & \xrightarrow{\pi_1} & B \\
 \downarrow f & & \downarrow f \times g & & \downarrow g \\
 C & \xleftarrow{\pi_2} & C \times D & \xrightarrow{\pi_3} & D
 \end{array}$$

2 Exponential

Definition 2.1. Let \mathcal{C} a category. For any $B, C \in \mathcal{C}$, (C^B, ev) forms an exponential where $C^B \in \mathcal{C}$ and $ev : C^B \times B \rightarrow C$, if for any object $A \in \mathcal{C}$ and $f : A \times B \rightarrow C$, there is a unique $u : A \rightarrow C^B$ such that $f = ev \circ (u \times 1_B)$. In other words, the follow diagram commutes.

$$\begin{array}{ccc}
 A \times B & & \\
 \downarrow u \times 1_B & \searrow f & \\
 C^B \times B & \xrightarrow{ev} & C
 \end{array}$$

3 Pullback

Theorem 3.1. Suppose we have two joined commuting squares like:

$$\begin{array}{ccccc}
 L & \xrightarrow{f} & M & \xrightarrow{g} & N \\
 \downarrow l & & \downarrow m & & \downarrow n \\
 X & \xrightarrow{h} & Y & \xrightarrow{j} & Z
 \end{array}$$

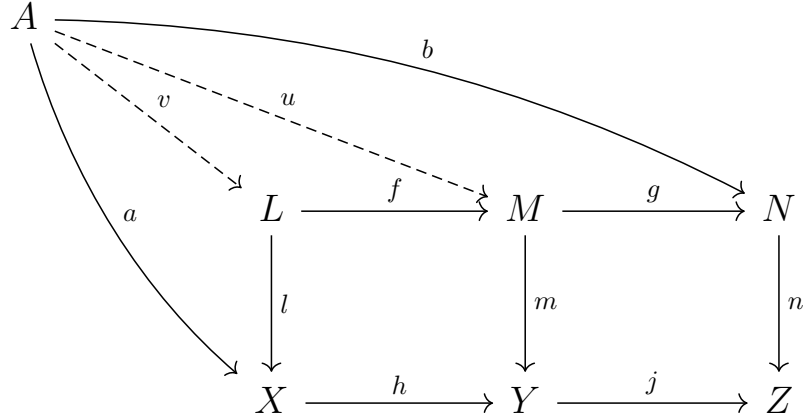
Then:

1. The outer rectangle is a pullback square if two inner squares are pullback squares.

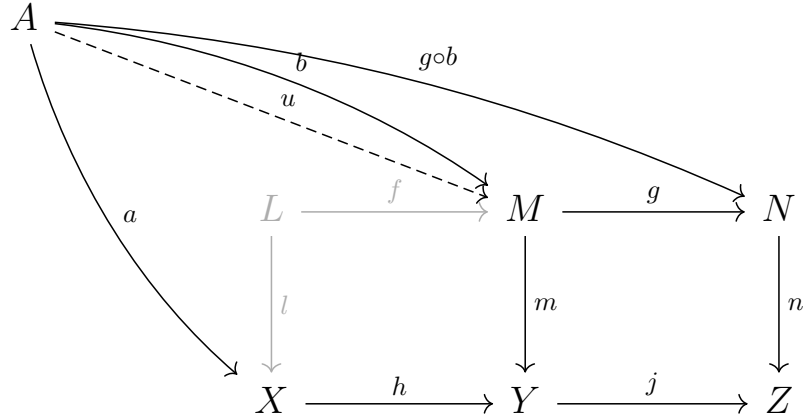
2. The inner-left square is a pullback square if the outer rectangle and the inner-right square are pullback squares.

Proof.

1. For any (A, a, b) such that $j \circ h \circ a = n \circ b$, then there is a unique $u : A \rightarrow M$ such that $h \circ a = m \circ u$ and $b = g \circ u$. Then there is a unique $v : A \rightarrow L$ such that $l \circ a = v$ and $f \circ v = u$, which makes (A, a, b) against to the outer rectangle commutes.

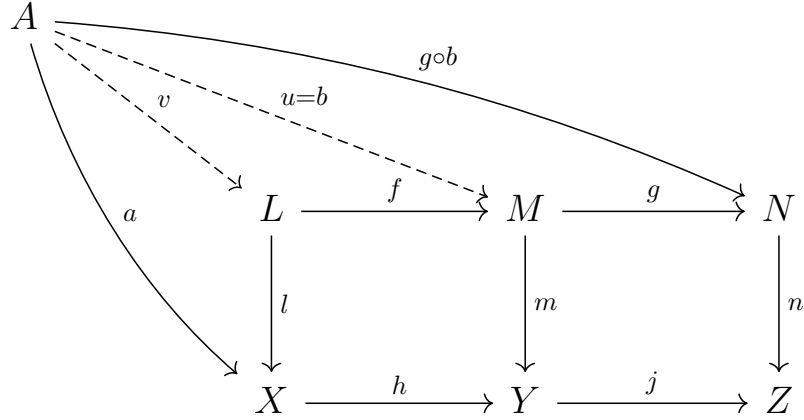


2. For any (A, a, b) such that $h \circ a = m \circ b$, consider the inner-right pullback, then we have a unique $u : A \rightarrow M$ such that the diagram commutes:



However, if we replace u with b , we have $g \circ b = g \circ b$ and $h \circ a = m \circ b$, that means b can do u 's job, but we know u is unique, so $b = u$. Now consider the outer pullback, we have a unique $v : A \rightarrow L$ such that the

diagram commutes:



That is, $l \circ v = a$ and $g \circ f \circ v = g \circ b$, we claim that v is the unique factorization from $(A, a, u = b)$ to (L, l, f) . It is obvious that $l \circ v = a$, we need to show $f \circ v = u = b$. We may use the trick we just used, we can see that $g \circ f \circ v = g \circ u$ and $m \circ f \circ v = h \circ l \circ v = h \circ a$. So $f \circ v$ can do b 's job, so $f \circ v = b$.

For any arrow $w : A \rightarrow L$ such that $l \circ a = w$ and $f \circ w = b$, then we have also $g \circ f \circ w = g \circ b$, which implies w is the unique arrow from $A \rightarrow L$ such that the outer diagram commutes, so $w = v$.

□

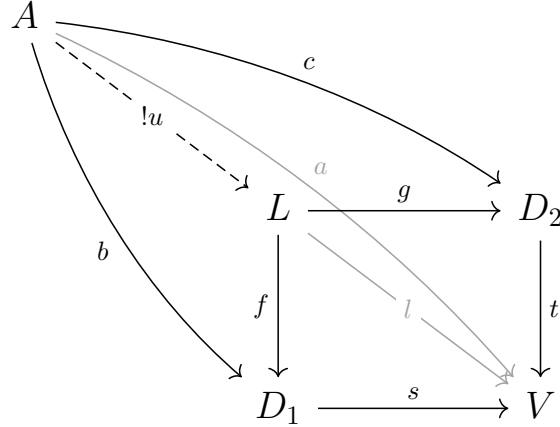
Theorem 3.2. *A pullback square for the corner $D_1 \rightarrow V \leftarrow D_2$ is a product of $D_1 \rightarrow V$ and $V \leftarrow D_2$ in the slice category \mathcal{C}/V .*

Proof. Suppose (L, f, g) is the pullback of such corner, then we first need to show that there is an arrow $l : L \rightarrow V$ such that $s \circ f = l$ (therefore a morphism from (L, l) to (D_1, s)) and $t \circ g = l$ (a morphism from (L, l) to (D_2, t)).

Since (L, f, g) makes the pullback square commutes, we know $s \circ f = t \circ g$, therefore we let $l = s \circ f$ (or equivalently $t \circ g$).

We need to show that $((L, l), f, g)$ forms a product of (D_1, s) and (D_2, t) , consider any $((A, a), b, c)$ where $a : A \rightarrow V$ such that $s \circ b = a$ and $t \circ c = a$. Just like l for L , a is redundant, so we may omit it. Now, the diagram looks

like:



Since (L, f, g) is a pullback, we know there is a unique $u : A \rightarrow L$ such that two triangle commutes. However, we must first show that u is an arrow from (A, a) to (L, l) , that is, $l \circ u = a$. It is easy to see that $l \circ u = s \circ f \circ u = s \circ b = a$. \square

Theorem 3.3. *if a category has all binary products and all equalizers for every pair of parallel arrows, then it has a pullback for any corners.*

Proof. Suppose $X \rightarrow Z \leftarrow Y$ a corner, then consider the product $X \times Y$:

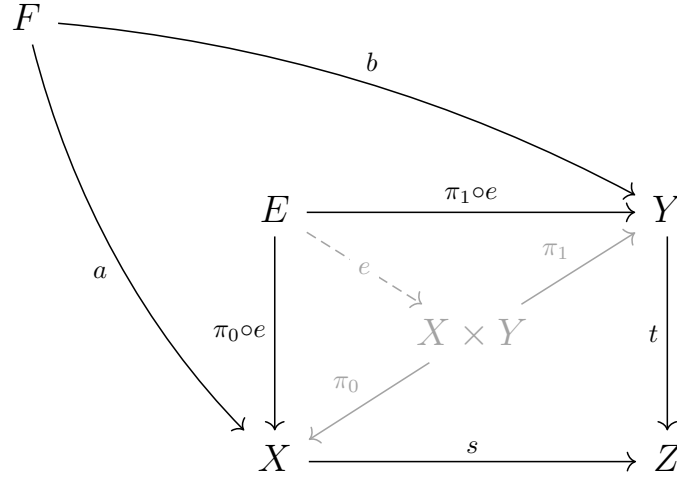
$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_1} & Y \\ \pi_0 \downarrow & & \downarrow t \\ X & \xrightarrow{s} & Z \end{array}$$

Now, consider the equalizer for the parallel arrows $t \circ \pi_1$ and $s \circ \pi_0$:

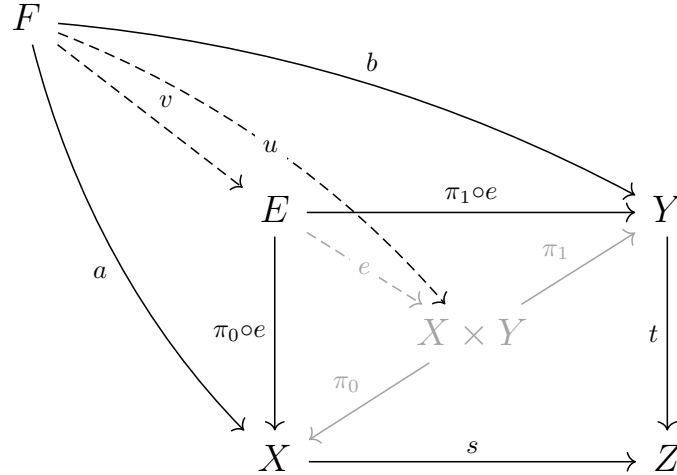
$$E \xrightarrow{e} X \times Y \rightrightarrows Z$$

$\begin{array}{c} s \circ \pi_0 \\ \hline t \circ \pi_1 \end{array}$

We claim $(E, \pi_0 \circ e, \pi_1 \circ e)$ is a pullback of such corner. For any (F, f, g) such that the outer diagram commutes:



it is easy to see that there is a unique arrow $u : F \rightarrow X \times Y$ such that $\pi_0 \circ u = a$ and $\pi_1 \circ u = b$ since $X \times Y$ is a product. Then there is another unique arrow $v : F \rightarrow E$ such that $e \circ v = u$ since E is a equalizer.



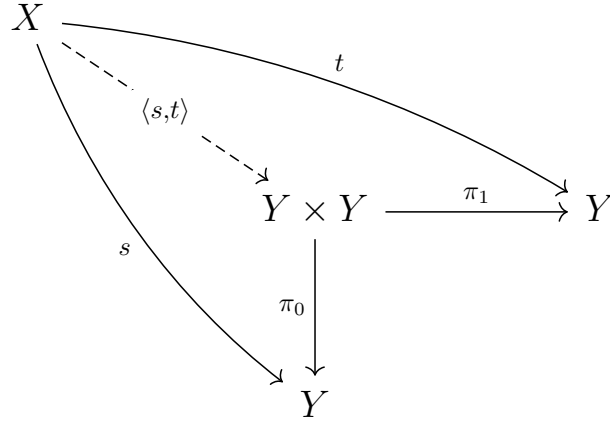
Obviously, (commute) $\pi_0 \circ e \circ v = \pi_0 \circ u = a$ and $\pi_1 \circ e \circ v = \pi_1 \circ u = b$.
 (unique) If an arrow $w : F \rightarrow E$ can do the job, then $e \circ w : F \rightarrow X \times Y$ is another factorization from F to the product $X \times Y$, so $e \circ w = u$, but that means w is also a factorization from F to the equalizer E , which means $v = w$.

So $(E, \pi_0 \circ e, \pi_1 \circ e)$ is a pullback of such corner. \square

Theorem 3.4. *If a category has a terminal object and has a pullback for every corner, then it has all binary product.*

Theorem 3.5. *If a category has a terminal object and has a pullback for every corner, then it has a equalizer for every parallel arrwos.*

Proof. Suppose $s, t : X \rightarrow Y$ are parallel arrows, then the following diagram commutes:



Note that we have $Y \times Y$ since this category has all binary products. Then consider this corner:

$$\begin{array}{ccc}
 & Y & \\
 & \downarrow \scriptstyle \langle 1_Y, 1_Y \rangle & \\
 X & \xrightarrow{\scriptstyle \langle s, t \rangle} & Y \times Y
 \end{array}$$

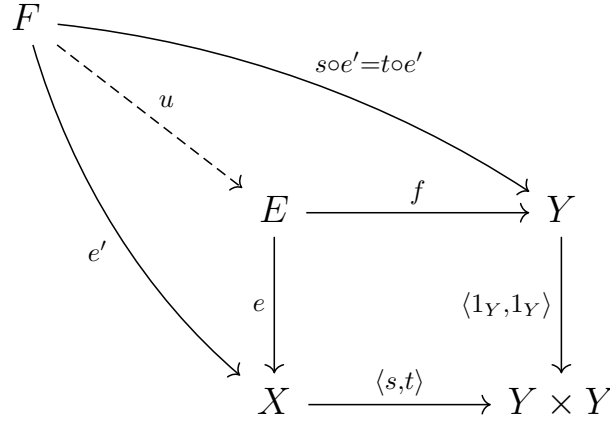
We have an object E , $e : E \rightarrow X$ and $f : E \rightarrow Y$ such that the square commutes:

$$\begin{array}{ccc}
 E & \xrightarrow{f} & Y \\
 \downarrow \scriptstyle e & & \downarrow \scriptstyle \langle 1_Y, 1_Y \rangle \\
 X & \xrightarrow{\scriptstyle \langle s, t \rangle} & Y \times Y
 \end{array}$$

(Proof comes from textbook until here)

We can see that $\pi_0 \circ \langle s, t \rangle \circ e = s \circ e$ while $\pi_0 \circ \langle 1_Y, 1_Y \rangle \circ f = 1_Y \circ f = f$, therefore $s \circ e = f$, similarly $t \circ e = f$, so $s \circ e = t \circ e$. We claim E is the equalizer for the parallel arrows $s, t : X \rightarrow Y$. For any (F, e') such that $s \circ e' = t \circ e'$, then we have a unique arrow $u : F \rightarrow E$ such that this diagram

commutes:



where $e \circ u = e'$. Suppose $v : F \rightarrow E$ where $e \circ v = e'$, then $f \circ v = s \circ e \circ v = s \circ e'$. \square

$$| -|x - z|_X + |y - z|_X |_{\approx, \cap \approx} \geq \epsilon$$

4 Functors

Definition 4.1 (Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called full, if for any $a, b \in \mathcal{C}$, the mapping on morphism $F : \mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$ is surjective.

Definition 4.2 (Faithful). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called faithful, if for any $a, b \in \mathcal{C}$, the mapping on morphism $F : \mathcal{C}(a, b) \rightarrow \mathcal{D}(Fa, Fb)$ is injective.

Definition 4.3 (Essentially Full). A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called Essentially full, if for any $a \in \mathcal{C}$, the mapping on object $F : \mathcal{C} \rightarrow \mathcal{D}$ is surjective.

Theorem 4.1. Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor, and $f : a \rightarrow b$ a morphism in \mathcal{C} . Then f is an isomorphism iff Ff is an isomorphism.

Proof. (\Rightarrow) We claim $F(f^{-1}) : Fb \rightarrow Fa$ is an inverse, we can see that $F(f^{-1} \circ f) = F(id_a) = id_{Fa}$ and $F(f \circ f^{-1}) = F(id_b) = id_{Fb}$.

(\Leftarrow) Suppose Fg is the inverse of Ff , and we can retrieve g from Fg cause F is full faithful. Then $F(g \circ f) = Fg \circ Ff = id_{Fa} = F(id_a)$ therefore $g \circ f = id_a$ since F is full faithful, similar to $F(f \circ g)$, so f is indeed an isomorphism. \square

Corollary 4.1. 123

Proof. \square

5 Yoneda

This chapter combines arguments from some books:

- The Dao of FP
- The Joy of Abstraction

Lemma 5.1. *Any functor preserves isomorphism.*

Proof. Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor and $f : a \rightarrow b$ an isomorphism in \mathcal{C} . We need to show that $Ff : Fa \rightarrow Fb$ is an isomorphism. We claim $F(f^{-1}) : Fb \rightarrow Fa$ is an inverse, we can see that $F(f^{-1} \circ f) = F(id_a) = id_{Fa}$ and $F(f \circ f^{-1}) = F(id_b) = id_{Fb}$. \square

Definition 5.1. $H_x = \mathcal{C}(-, x)$ and $H^x = \mathcal{C}(x, -)$.

Take H^x as an example, it sends \mathcal{C} to **Set**, the interesting part is the mapping on morphism. For any morphism $f : a \rightarrow b$ of \mathcal{C} , H^f must be a mapping $\mathcal{C}(x, a) \rightarrow \mathcal{C}(x, b)$, we can see that $g \mapsto f \circ g$ would be a choice.

We have to show that it satisfies the functoriality:

- $H^{id_a}(g) = id_a \circ g = g$
- $H^{f \circ g}(h) = (f \circ g) \circ h = f \circ (g \circ h) = H^f(g \circ h) = H^f(H^g(h)) = (H^f \circ H^g)(h)$.

Similar to H_x , the only difference is that H_x is a contrafunctor.

Suppose $f : a \rightarrow b$ an isomorphism, we can see that H^x gives an isomorphism between two hom-sets: $\mathcal{C}(x, a)$ and $\mathcal{C}(x, b)$.