Exercise 3.1. *Let $T : V \to W$, the graph of T is a subset of $V \times W$ such that*

$$\textit{graph of } T = \{\ (v, Tv) \mid v \in V\ \}$$

.

Show that T is a linear mapping $\iff$ the graph of T is a subspace.

Proof.

- $(0, T0) \in$ graph of T. $(v, Tv) + (w + Tw) = (v + w, Tv + Tw) = (v + w, T(v + w))$. $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v))$
- $(v, Tv) + (w + Tw) = (v + w, T(v + w))$ since the graph of T is a subspace, therefore $Tv + Tw = T(v + w)$. Similarly, $\lambda Tv = T(\lambda v)$.

□

Exercise 3.3. *Let V_i are vector spaces, show that $\mathcal{L}(V_0 \times \cdots \times V_{m-1}, W) \simeq \mathcal{L}(V_0, W) \times \cdots \times \mathcal{L}(V_{m-1}, W)$.*

Proof. This can be proven by $A \times B$ is a categorical product, so we will show that for any A, B are vector spaces, $A \times B$ is a product.

In order to show that $A \times B$ is a product, or more specifically, $A \times B$ equipped with linear mappings

$$\pi_0(a, b) = a$$
$$\pi_1(a, b) = b$$

is a product, we have to show that for any C, $s \in \mathcal{L}(C, A)$ and $t \in \mathcal{L}(C, B)$, there is a unique $u \in \mathcal{L}(C, A \times B)$ such that $s = \pi_0 \circ u$ and $t = \pi_1 \circ u$.

Define $u(c) = (sc, tc) : C \to A \times B$, we will show that u is a linear mapping.

- For all $v, w \in C$, $u(v) + u(w) = (sv, tv) + (sw, tw) = (sv + sw, tv + tw) = (s(v + w), t(v + w)) = u(v + w)$
- For all $c \in C$ and $\lambda \in F$, $\lambda u(c) = \lambda(sc, tc) = (\lambda sc, \lambda tc) = (s(\lambda c), t(\lambda c)) = u(\lambda c)$.

Then we can see $\pi_0(u(c)) = \pi_0(sc, tc) = sc$ and $\pi_1(u(c)) = \pi_1(sc, tc) = tc$. Now we have to show that u is unique (which is trivial, I don't want to prove this, sorry). □

Exercise 3.5. *Let m a positive number, define $V^m = \underbrace{V \times \cdots \times V}_{m}$, show that $V^m \simeq \mathcal{L}(F^m, V)$.*

Proof. Define $\varphi(v_0, \cdots, v_{m-1}) = i_0, \cdots, i_{m-1} \mapsto i_0 v_0 + \cdots + i_{m-1} v_{m-1}$ which accept a list of vector and a list of coefficients then produce a linear combination.

For any $T \in \mathcal{L}(F^m, V)$, T is completely determined by $T(1, \cdots, 1) = v_0 + \cdots + v_{m-1}$, therefore $\varphi(v_0, \cdots, v_{m-1}) = T$ and thus φ is surjective.

For any $(v_0, \cdots, v_{m-1}), (w_0, \cdots, w_{m-1}) \in V^m$ such that $\varphi(v_0, \cdots, v_{m-1}) = \varphi(w_0, \cdots, w_{m-1})$, then $w_0 = \varphi(v_0, \cdots, v_{m-1})(1, 0, \cdots, 0) = \varphi(v_0, \cdots, v_{m-1})(1, 0, \cdots, 0) = v_0$, same for other v_i and w_i, so $(v_0, \cdots, v_{m-1}) = (w_0, \cdots, w_{m-1})$, therefore φ is injective. □

Exercise 3.6. *Let $v, x \in V$ and $U, W \subseteq V$ are subspaces such that $v + U = x + W$. Show that $U = W$.*

Proof. We know $v = x + w_0$ for some $w_0 \in W$ since $v + U = x + W$ and $v \in v + U$, then for any $u \in U$, we have $v + u = x + w$ for some $w \in W$, then $(x + w_0) + u = x + w$ therefore $u = x + w - x - w_0 = w - w_0 \in W$ thus $U \subseteq W$. Similarly $W \subseteq U$. □

Exercise 3.7. *Let $U = \{ (x, y, z) \in R^3 \mid 2x + 3y + 5z = 0 \}$ and $A \subseteq R^3$. Show that A is a translate of U (that is $A = a + U$) $\iff$ there is c such that $A = \{ (x, y, z) \in R^3 \mid 2x + 3y + 5z = c \}$.*

Proof.

- ($\Rightarrow$) For any $(a_0, a_1, a_2) + (x, y, z) \in a + U$, we have $2(a_0 + x) + 3(a_1 + y) + 5(a_2 + z) = 2a_0 + 3a_1 + 5a_2$, therefore $c = 2a_0 + 3a_1 + 5a_2$.

- ($\Leftarrow$) We can see 2, 3 and 5 are coprime to each other, therefore there is $2a_0 + 3a_1 + 5a_2 = 1$ (I am not sure if this is true in generalized case, I just extends the theorem "$as + bt = 1 \iff$ a coprime to b" to three elements case without checking), in this case we have $2(1) + 3(-2) + 5(1) = 1$, then for any $2x + 3y + 5z = c$, we have $2x + 3y + 5z = 2(ca_0) + 3(ca_1) + 5(ca_2)$, then $2(x - ca_0) + 3(y - ca_1) + 5(z - ca_2) = 0$, therefore $A = ((-c)(a_0, a_1, a_2)) + U$.

□

Exercise 3.8. *Let $T \in \mathcal{L}(V, W)$ and $c \in W$, show that $\{ v \in V \mid Tv = c \}$ is an empty set or a translate of $\operatorname{null} T$. Then explain why the solutions of a system of linear equations is either an empty set or a translate of some subspace of F^n.*

Proof. Let $Ta = c$ for some $a \in V$, if no such a, then $\{ v \in V \mid Tv = c \} = \varnothing$. We claim $\{ v \in V \mid Tv = c \} = a + \operatorname{null} T$. For any $v \in V$ such that $Tv = c$, then $v = a + v - a$ and $T(v - a) = Tv - Ta = c - c = 0$, therefore $v - a \in \operatorname{null} T$, thus $v \in a + \operatorname{null} T$. In another direction, for any $a + v \in a + \operatorname{null} T$, we have $T(a + v) = Ta + Tv = c + 0 = c$. □

Exercise 3.9. *Let $A \subseteq V$ a non-empty subset. Show that A is a translate of some subspace of $V \iff \lambda v + (1 - \lambda)w \in A$ for any $v, w \in A$ and $\lambda \in F$.*

Proof.

- ($\Rightarrow$) Suppose $A = a + U$ for some subspace $U \subseteq V$.
- ($\Leftarrow$) Let $w \in A$, we will show that $(-w) + A$ is a subspace of V.

  For any $a - w, b - w \in (-w) + A$, we need to show that $a - w + b - w = (a + b - w) - w \in (-w) + A$ or equivalently $a + b - w \in A$. We found that the property $\lambda v + (1 - \lambda)w \in A$ gives us the ability to construct something like $v - w$. Since $2v + (1 - 2)w = 2v - w$, we just let $w = v + a$ then $2v - (v + a) = v - a$. Therefore, we let $\lambda = 2$, $v = a + b$ and $w = a + b + w$, and now $2(a + b) - (a + b + w) = a + b - w \in A$, so $a + b - w - w \in (-w) + A$.

  For any $a - w \in (-w) + A$ and $\lambda \in F$, we need to show that $\lambda(a - w) \in (-w) + A$. $\lambda(a - w) = \lambda a - \lambda w = \lambda a - (\lambda - 1)w - w$. We let $\lambda = (-1)(\lambda - 1) = (1 - \lambda)$, $v = w$ and $w = a$ in $\lambda v + (1 - \lambda w) \in A$, then $(1 - \lambda)w + (1 - (1 - \lambda))a = (-1)(\lambda - 1)w + \lambda a = \lambda a - (\lambda - 1)w \in A$, therefore $\lambda a - (\lambda - 1)w - w = \lambda a - \lambda w \in (-w) + A$.

  Therefore $(-w) + A$ is a subapce of V and $w + (-w) + A$ is a translate.

□

Exercise 3.10. *Let $A = a + U$ and $B = b + W$ where $a, b \in V$, $U, W \subseteq V$ are subspaces. Show that $A \cap B$ is either a translate of some subspace of V or an empty space.*

Proof. Suppose $A \cap B \neq \varnothing$, we claim that $A \cap B$ is a translate of $U \cap W$, more specificly, for any $a+u_0 = b+w_0 \in A \cap B$, we claim that $A \cap B = (a+u_0)+U \cap W$.

For any $u = w \in U \cap W$, we have $(a + u_0) + u = a + (u_0 + u) \in a + U$, similarly, we have $(b + w_0) + w = b + (w_0 + w) \in b + W$, therefore $(a + u_0) + (U \cap W)\ subseteq A \cap B$.

For any $a + u = b + w \in A \cap B$, we have $a + u - (a + u_0) = u - u_0 \in U$ and $b + w - (b + w_0) = w - w_0 \in W$, therefore $A \cap B \subseteq (a + u_0) + (U \cap W)$. ☐

Exercise 3.12. *Let $v_0, \cdots, v_{m-1} \in V$ and*

$$A = \{\ \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1} \mid \lambda_i \in F \text{ and } \lambda_0 + \cdots + \lambda_i = 1\ \}$$

1. *Show that A is a translate of a subspace of V.*
2. *If B a translate of a subspace of V such that $v_0, \cdots, v_{m-1} \in B$, show that $A \subseteq B$.*
3. *Base on (1), show that the dimension of such subspace is less then m.*

Proof.

- If A is a translate of a subspace of V, say B, then for any $a \in A$, we have $A = a + B$. Therefore $B = (-a) + A$, we may pick $a = v_0$, we find that for any $b \in B$, it is in form $(-1)(v_0) + \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}$ where $\lambda_0 + \cdots + \lambda_{m-1} = 1$, which implies $(-1) + \lambda_0 + \cdots + \lambda_{m-1} = 0$. Then we claim $B = \{\ \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1} \mid \lambda_0 + \cdots + \lambda_{m-1} = 0\ \}$ is a subspace and $A = v_0 s + B$.
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Exercise 3.16. *Let $\varphi \in \mathcal{L}(V, F)$ where $\varphi \neq 0$, show that* $\dim(V/(\operatorname{null} \varphi)) = 1$.

Proof. For any non-zero $v + \operatorname{null} \varphi, w + \operatorname{null} \varphi \in V/(\operatorname{null} \varphi)$ (existence is guaranteed since $\varphi \neq 0$), since $\varphi(w) \in F$, then there is some λ such that $\lambda\varphi(w) = \varphi(v)$ cause $\varphi(v)$ and $\varphi(w)$ are non-zero, then $\varphi(\lambda w) = \varphi(v)$, which means $v + \operatorname{null} T = (\lambda w) + \operatorname{null} T$, therefore $\dim(V/\operatorname{null} \varphi)$ cause any two (non-zero) vectors are linear dependent. ☐

Exercise 3.17. *Let $U \subseteq V$ a subspace such that* $\dim(V/U) = 1$. *Show that there is $\varphi \in \mathcal{L}(V, F)$ such that* $\operatorname{null} \varphi = U$.

Proof. We know there is an isomorphism $i \in \mathcal{L}(V/U, F)$ since $\dim(V/U) = \dim F = 1$, then $\varphi = i \circ \pi$ where $\pi \in \mathcal{L}(V, V/U)$. Since i is injective, $\operatorname{null} \varphi = \operatorname{null} \pi = U$. $\square$