Exercise 5.9. *Let $P \in \mathcal{L}(V)$, such that $P^2 = P$. Suppose λ an eigenvalue of P, show that $\lambda = 0$ or $\lambda = 1$.*

Proof. Suppose $P(v) = \lambda v$ for some non-zero $v \in V$, then $P(v) = PP(v) = P(\lambda v)$, therefore $P((\lambda - 1)v) = 0$. thus $(\lambda - 1)v \in \operatorname{null} P$. We may suppose $\lambda \neq 1$, then $(\frac{1}{\lambda-1})(\lambda - 1)v = v \in \operatorname{null} P$, therefore $P(v) = 0$, thus $\lambda = 0$ cause $v \neq 0$. $\square$

Exercise 5.10. *Let $T(p) = p' : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$. Find all eigenvalues and eigenvectors of T.*

Proof. Suppose $T(p) = p' = \lambda p$, then $\deg p = 0$, otherwise the degree doesn't match. For any $p \in \mathcal{P}(\mathbb{R})$ such that $\deg p = 0$, we have $p' = 0 = 0p$. $\square$

Exercise 5.12. *Let $V = U \oplus W$ where U and W are non-zero subspaces. Define $P(u + w) = u$ for all $u \in U$ and $w \in W$. Find all eigenvalue and eigenvector of P.*

Proof. We can see $P^2 = P$, since for any $u + w \in V$, we have $P(P(u + w)) = P(u) = u = P(u + w)$, therefore $\lambda = 0$ and $\lambda = 1$ are eigenvalues of P , $P(u) = 1u$ and $P(w) = 0w$ are eigenvectors of P. $\square$

Exercise 5.13. *Let $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$, where S is invertible.*

- *Show that T has the same eigenvalue of $S^{-1}TS$.*
- *What is the relationship between the eigenvector of T and the eigenvector of $S^{-1}TS$.*

Proof.

- For any $T(v) = \lambda v$ where $v \in V$ and $\lambda \in F$, let $S(w) = v$, then $S^{-1}TS(w) = S^{-1}(T(Sw)) = S^{-1}(\lambda v) = \lambda S^{-1}(v) = \lambda w$, thus λ is an eigenvalue of $S^{-1}TS$.
- $S(w) = v$ where v is an eigenvector of T and w is the corresponding eigenvector of $S^{-1}TS$.

$\square$

Exercise 5.15. *Let V finite, $T \in \mathcal{L}(V)$, $\lambda \in F$. Show that λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of T'.*

Proof.

- ($\Rightarrow$) Suppose $Tv = \lambda v$, we will show $T' - \lambda I$ is not surjective (Note that $I \in \mathcal{L}(V')$).

  For any $\varphi \in V'$, we have:

  $$\begin{aligned}&(T' - \lambda I)(\varphi)\\=&T'(\varphi) - \lambda\varphi\\=&\varphi \circ T - \lambda\varphi\end{aligned}$$

  then

  $$\begin{aligned}&(\varphi \circ T - \lambda\varphi)(v)\\=&(\varphi \circ T)(v) - (\lambda\varphi)(v)\\=&\varphi(Tv) - \lambda(\varphi(v))\\=&\varphi(\lambda v) - \lambda(\varphi(v))\\=&0\end{aligned}$$

  This means $\operatorname{range} T' \neq V'$ cause any $\psi \in V'$ where $\psi(v) \neq 0$ is not in $\operatorname{range} T'$. Therefore λ is an eigenvalue of T'.

$\square$

Exercise 5.22. *Let $T \in \mathcal{L}(V)$ and non-zero $v, w \in V$ such that*

$$Tu = 3w \quad \textit{and} \quad Tw = 3u$$

.

Show that 3 or -3 is the eigenvalue of T.

Proof. Since v and w are non-zero, then one of $u + w$ and $u - w$ is non-zero.

We have $T(u + w) = 3w + 3u = 3(u + w)$ and $T(u - w) = 3w - (3u) = (-3)(u - w)$. $\square$

Exercise 5.23. *Let V finite, and $S, T \in \mathcal{L}(V)$, show that ST and TS have the same eigenvalues.*

Proof. For any $ST(v) = \lambda v$ where $v \neq 0$, we have $TST(v) = T(\lambda v)$ then $TS(Tv) = \lambda(Tv)$.

- If $Tv = 0$, then $STv = 0 = \lambda v$, thus $\lambda = 0$ since $v \neq 0$. then λ is an eigenvalue of TS since $\operatorname{null} TS \neq 0$ ($\operatorname{null} T \neq 0$, and S is an operator of V, therefore, S is injective or not doesn't affect our conclusion).

  If $Tv \neq 0$, then $TS(Tv) = \lambda(Tv)$.

- Ditto.

$\square$

Exercise 5.26. *Let $T \in \mathcal{L}(V)$ and any non-zero $v \in V$ we have $Tv = cv$ for some c. Show that $T = \lambda I$.*

Proof. Let non-zero $v, w \in V$, we have $Tv = sv$ and $Tw = tw$, then $T(v+w) = \lambda(v + w) = \lambda v + \lambda w = sv + tw = T(v) + T(w)$. Then $\lambda v + \lambda w - tw = sv$.

- If $w \in \operatorname{span}(v)$, then $w = cv$, therefore $Tw = T(cv) = tcv = cTv = csv$, thus $t = s$.

- If $w \notin \operatorname{span}(v)$, then $\lambda = t$ (otherwise $\lambda v + (\lambda - t)w = sv$), therefore $\lambda v = sv$ and $\lambda = s$, thus $s = t$.

$\square$

Exercise 5.27. *Let V finite and $1 \leq k \leq \dim V - 1$. Let $T \in \mathcal{L}(V)$ such that any subspace of V with k dimension is invariant under T. Show that $T = \lambda I$ for some λ.*

Proof. For any $v \in V$, we have $Tv = w$ where $w \in \operatorname{span}(v)$.

- If $w = 0$, then $Tv = 0v$.

- If $w \neq 0$, then $w = \lambda v$ since $w \in \operatorname{span}(v)$, then $Tv = \lambda v$.

Thus $T = \lambda I$ by the previous exercise. $\square$

Exercise 5.30. *Let $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Show that 2 or 3 or 4 is the eigenvalue of T.*

Proof. Suppose 2 is not an eigenvalue of T, then $(T - 2I)$ is injective, thus $(T - 3T)(T - 4I)$ must map all $v \in V$ to 0. Similarly, we can show that $(T - 4I) = 0$ if 3 is not an eigenvalue of T. $\square$

Exercise 5.31. *Find $T \in \mathcal{L}(\mathcal{R}^2)$ such that $T^4 = -I$.*

Proof. We may treat $-I$ as rotating the vector 180 degrees, then T rotates a vector 45 degrees, which matrix is:

$$\mathcal{M}(T) = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$\square$

Exercise 5.32. *Let $T \in \mathcal{L}(V)$ with no eigenvalue and $T^4 = I$. Show that $T^2 = -I$.*

Proof. We will show that $T^2(v) = -v$ for any $v \in V$. Consider $T^2(T^2(v)+v) = v+T^2(v)$, we will show that $T^2(v)+v = 0$. Suppose $w \in V$ and $T^2(w) = w$, we may let $T(w) = u$, then $T^2(w) = Tu = w$. Consider $T(w+u) = T(w)+T(u) = u + w$, then $w + u = 0$ since T has no eigenvalue, therefore $u = -w$ and $T(w) = -w$. Again $w = 0$ since T has no eigenvalue. Therefore $T^2(w) = w$ implies $w = 0$, thus $T^2(v) + v = 0$ and $T^2(v) = -v$ for any $v \in V$. $\square$

Exercise 5.33. *Let $T \in \mathcal{L}(V)$ and $m \in \mathbb{N}^+$.*

1. *Show that T is injective $\iff$ T^m is injective*
2. *Show that T is surjective $\iff$ T^m is surjective.*

Proof. Recall that $T^0 = I$.

- ($\Rightarrow$) For any $T^m(v) = T^m(w)$ we have $T^{m-1}(v) = T^{m-1}(w)$ and so on, we will get $v = w$.

  ($\Leftarrow$) For any $T(v) = T(w)$, we have $T^{m-1}(T(v)) = T^{m-1}(T(w))$, then $T^m(v) = T^m(w)$ and $v = w$.

- ($\Rightarrow$) For any $w \in V$, we have $T(v) = w$, then we have $T(u) = v$ and now $T(T(u)) = w$, continue this progress until we get $T^m(r) = w$.

  ($\Leftarrow$) For any $w \in V$, we have $T^m(v) = w$, therefore $T(T^{m-1}(v)) = w$.

$\square$

Exercise 5.34. *Let V finite and $v_0, \cdots, v_{m-1} \in V$. Show that $v_0, \cdots, v_{m-1}$ is linear independent $\iff$ there is $T \in \mathcal{L}(V)$ such that $v_0, \cdots, v_{m-1}$ are eigenvectors of distinct eigenvalues of T.*

Proof.

- ($\Rightarrow$) Consider $v_0, \cdots, v_{k-1}$ a basis of V, then $T(\lambda_0 v_0 + \cdots + \lambda_{k-1} v_{k-1}) = 1\lambda_0 v_0 + 2\lambda_1 v_1 + \cdots + m\lambda_{m-1} v_{m-1}$ where $T(v_i) = (i+1)v_i$.

- ($\Leftarrow$) Trivial, since eigenvectors of distinct eigenvalues are linear independent.

$\square$

Exercise 5.37. *Let V finite and $T \in \mathcal{L}(V)$. Define $\mathcal{A}(S) = TS : \mathcal{L}(V) \to \mathcal{L}(V)$. Show that T has the same eigenvalues as $\mathcal{A}$.*

Proof.

- ($\subseteq$) For any eigenvalue λ of T, we can find $S \in \mathcal{L}(V)$ such that range $S = \{ v \in V \mid Tv = \lambda v \}$ (it is easy to show that such set is a subspace). Then for any $v \in V$, $(TS)v = T(Sv) = \lambda(Sv) = (\lambda S)v$ thus $\mathcal{A}(S) = TS = \lambda S$.

- ($\supseteq$) For any eigenvalue λ of $\mathcal{A}$, then we have $\mathcal{A}(S) = \lambda S$ for some non-zero $S \in \mathcal{L}(V)$. Then there is $v \in V$ such that $Sv \neq 0$, and $T(Sv) = (TS)v = (\lambda S)(v) = \lambda(Sv)$, thus λ is an eigenvalue of T.

$\square$

Exercise 5.38. *Let V finite and $T \in \mathcal{L}(V)$ and $U \subseteq V$ is invariant under T. A quotient operator $T/U \in \mathcal{L}(V/U)$ is defined by:*

$$(T/U)(v + U) = Tv + U$$

for any $v \in V$.

1. *Show that T/U is well-defined and T/U is an operator over V/U.*

2. *Show that each eigenvalue of T/U is also an eigenvalue of T.*

Proof.

- Suppose $v+U = w+U$, then $(T/U)(v+U) = Tv+U$ and $(T/U)(w+U) = Tw + U$, we will show that $Tv - Tw \in U$. Note that $v + U = w + U$ implies $v - w \in U$, then $T(v - w) \in U$ since U is invariant under T, that is, for any $u \in U$, $Tu \in U$. Thus $Tv + U = Tw + U$.

Now we will show that T/U is a linear map, we can see:

$$\begin{aligned}&(T/U)(v+U)+(T/U)(w+U)\\=&(Tv+U)+(Tw+U)\\=&(Tv+Tw)+U\\=&T(v+w)+U\\=&(T/U)((v+w)+U)\end{aligned}$$

and

$$\begin{aligned}&\lambda(T/U)(v+U)\\=&\lambda(Tv+U)\\=&(\lambda(Tv))+U\\=&T(\lambda v)+U\\=&(T/U)((\lambda v)+U)\\=&(T/U)(\lambda(v+U))\end{aligned}$$

- Suppose $(T/U)(v+U) = Tv+U = \lambda v+U$ where $v \notin U$, consider $T-\lambda I$, we will show that $T-\lambda I$ is not injective. We can see U is invariant under $T-\lambda I$, $Tu-\lambda u \in U$ cause U is invariant under T. We may suppose T is injective (thus surjective and invertible) on U (in other words, $T(U)=U$), otherwise the proof is complete. Then consider $(T-\lambda I)(v) = Tv-\lambda v \in U$ where $v \notin U$, thus $T-\lambda I$ is not injective.

$\square$

Exercise 5.39. *Let V finite and $T \in \mathcal{L}(V)$. Show that T has an eigenvalue $\iff$ there is a subspace of V with dimension* $\dim V - 1$ *which is invariant under T.*

Proof.

- This part is hinted by AI. Suppose $Tv=\lambda v$, then consider $T-\lambda I$, we know range$(T-\lambda I)$ is invariant under T, since for any $Tw-\lambda w$, we have $T(Tw-\lambda w) = T(Tw)-T(\lambda w) = T(Tw)-\lambda(Tw)$. Then $\dim \operatorname{range}(T-\lambda I) \leq \dim V - 1$ since $w \in \operatorname{null} T-\lambda I$. Then consider $\operatorname{null}(T-\lambda I) = \operatorname{span}(v) \oplus W$, we have $\operatorname{range}(T-\lambda I) \oplus W$ a subspace which is invariant under T.

The key is finding a smaller invariant subspace and expand it with null space, as any vector in null space always maps to 0, thus preserve the property of invariant.

- Suppose U is a subspace of V of dimension $\dim V - 1$ such that U is invariant under T, then $V = U \oplus \operatorname{span}(v)$ for some $v \notin U$. We may suppose T is injective on U, otherwise the proof is complete ($\operatorname{null} T \neq 0$). Consider $T(v)$, there are three cases:
  - $T(v) = \lambda v + 0u$, then the proof is complete.
  - $T(v) = 0v + u$, then T is not injective since there is $Tw = u$ where $w \in U$.
  - $T(v) = \lambda v + u$, then consider $T - \lambda I$. We have U is invariant under $T - \lambda I$ cause $Tu - \lambda u \in U$ by $Tu \in U$. Again, if $T - \lambda I$ is not injective on U, the proof is complete. Then $(T - \lambda I)v = Tv - \lambda v \in U = \lambda v + u - \lambda v = u \in U$, thus $T - \lambda I$ is not injective and λ is an eigenvalue of T.

□

Exercise 5.42. *Let $T \in \mathcal{L}(F^n)$ defined by $T(x_1, x_2, \cdots, x_n) = (x_1, 2x_2, \cdots, nx_n)$.*

1. *Find all eigenvalues and eigenvectors of T.*
2. *Find all subspace of F^n which is invariant under T.*

Proof.

- $1, 2, \cdots, n$ and $(x_1, 0, \cdots), (0, x_2, 0, \cdots), \cdots$
- We claim any subspace that is invariant under T is a direct sum of some spaces that spans by the standard basis, say $\operatorname{span}(x_0) \oplus \cdots \oplus \operatorname{span}(x_k)$.

  Let U a subspace that is invariant under T and $u \in U$, we have $T(u) = T(\lambda_1 x_1, \cdots, \lambda_n x_n) = (\lambda_1 x_1, \cdots, n\lambda_n x_n)$, then $T(u) - iu = ((1-i)(\lambda_1 x_1), (2-i)(\lambda_2 x_2), \cdots, (n-1)(\lambda_i x_i)) \in U$ is a vector that is a linear combination of standard basis except x_i. Repeat this progress by apply $T - jI$ to $(T - iI)(u)$ with a different j, we can finally get a vector that is a scalar multiple of x_k. Thus $x_i \in U$ as long as there is $u \in U$ that the ith scalar of the linear combination of standard basis is not zero.

□