Exercise 3.1. *Explain why a linear functional is either surjective or* 0.

Proof. Cause $\dim F = 1$. $\square$

Exercise 3.6. *Let* $\varphi, \beta \in V'$, *show that* $\operatorname{null}\varphi \subseteq \operatorname{null}\beta \iff \exists c \in F, \beta = c\varphi$.

Proof.

- ($\Rightarrow$) For any $v \notin \operatorname{null}\beta$, we have $\beta(v) = \beta(v)(\varphi(v))^{-1}\varphi(v)$ we claim that $\beta = \beta(v)(\varphi(v))^{-1}\varphi$. We may denote $\beta(v)(\varphi(v))^{-1}$ by c. For any $v, w \notin \operatorname{null}\beta$, we have $\beta(v) = a\varphi(v)$ and $\beta(w) = b\varphi(w)$, we want to show that $a = b$, which can be proven by:

$$
\begin{aligned}
a &= b \\
\frac{\beta(v)}{\varphi(v)} &= \frac{\beta(w)}{\varphi(w)} \\
\beta(v)\varphi(w) &= \beta(w)\varphi(v) \\
\beta(\varphi(w)v) &= \beta(\varphi(v)w)
\end{aligned}
$$

  which is equivalent to $\varphi(w)v - \varphi(v)w \in \operatorname{null}\beta$, then:

$$
\begin{aligned}
&\varphi(\varphi(w)v - \varphi(v)w) \\
=&\varphi(\varphi(w)v) - \varphi(\varphi(v)w) \\
=&\varphi(w)\varphi(v) - \varphi(v)\varphi(w) \\
=&0
\end{aligned}
$$

  therefore $\varphi(w)v - \varphi(v)w \in \operatorname{null}\varphi \subseteq \operatorname{null}\beta$, thus $a = b$.

  The case $v \in \operatorname{null}\beta$ is trivial.

- ($\Leftarrow$) For any $v \in \operatorname{null}\varphi$, $\beta(v) = c\varphi(v) = 0$, therefore $v \in \operatorname{null}\beta$, thus $\operatorname{null}\varphi \subseteq \operatorname{null}\beta$.

$\square$

Exercise 3.7. *Let* $V_0, \cdots, V_{m-1}$ *are vector spaces, show that* $V_0' \times \cdots \times V_{m-1}'$ *and* $(V_0 \times \cdots \times V_{m-1})'$ *are isomorphic.*

Proof. Define $\psi(\varphi) = v_0 \mapsto \varphi(v_0, 0, \cdots), \cdots, v_{m-1} \mapsto \varphi(\cdots, 0, v_{m-1})$ and $\psi^{-1}(\varphi_0, \cdots, \varphi_{m-1}) = (v_0, \cdots, v_{m-1}) \mapsto \varphi_0(v_0) + \cdots + \varphi_{m-1}(v_{m-1})$.

For any $\alpha, \beta \in (V_0 \times \cdots \times V_{m-1})'$ and $\lambda \in F$, we have

$$\begin{aligned}
&\psi(\alpha+\beta)_i \\
=&v_i \mapsto (\alpha+\beta)(\cdots, v_i, \cdots) \\
=&v_i \mapsto \alpha(\cdots, v_i, \cdots) + \beta(\cdots, v_i, \cdots) \\
=&(v_i \mapsto \alpha(\cdots, v_i, \cdots)) + (v_i \mapsto \beta(\cdots, v_i, \cdots)) \\
=&\psi(\alpha)_i + \psi(\beta)_i
\end{aligned}$$

and $(\lambda\psi(\alpha))_i = \lambda(v_i \mapsto \alpha(v_i)) = v_i \mapsto \lambda\alpha(v_i) = \psi(\lambda\alpha)_i$ Therefore ψ is a linear map.

For any $\alpha, \beta \in V_0' \times \cdots \times V_{m-1}'$ and $\lambda \in F$, we have:

$$\begin{aligned}
&\psi^{-1}(\alpha+\beta) \\
=&(v_0, \cdots, v_{m-1}) \mapsto (\alpha+\beta)(v_0) + \cdots + (\alpha+\beta)(v_{m-1}) \\
=&(v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \beta(v_0) + \cdots \\
=&((v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \cdots) + ((v_0, \cdots, v_{m-1}) \mapsto \beta(v_0) + \cdots) \\
=&\psi^{-1}(\alpha) + \psi^{-1}(\beta)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda\psi^{-1}(\alpha) \\
=&\lambda((v_0, \cdots, v_{m-1}) \mapsto \alpha(v_0) + \cdots) \\
=&(v_0, \cdots, v_{m-1}) \mapsto \lambda(\alpha(v_0) + \cdots) \\
=&(v_0, \cdots, v_{m-1}) \mapsto (\lambda\alpha(v_0)) + \cdots \\
=&\psi^{-1}(\lambda\alpha)
\end{aligned}$$

thus ψ^{-1} is a linear map.

We will show that ψ^{-1} is the inverse of ψ then ψ is an isomorphism. For any $\varphi \in (V_0 \times \cdots \times V_{m-1})'$,

$$\begin{aligned}
&\psi^{-1}(\psi(\varphi)) \\
=&v_0, \cdots, v_{m-1} \mapsto \psi(\varphi)_0(v_0) + \cdots \\
=&v_0, \cdots, v_{m-1} \mapsto \varphi(v_0, 0, \cdots) + \cdots + \varphi(\cdots, 0, v_{m-1}) \\
=&v_0, \cdots, v_{m-1} \mapsto \varphi(v_0, \cdots, v_{m-1}) \\
=&\varphi
\end{aligned}$$

and for any $\varphi \in V_0' \times \cdots \times V_{m-1}'$,

$$\begin{aligned}
&\psi(\psi^{-1}(\varphi)) \\
=&v_0 \mapsto \psi^{-1}(\varphi)(v_0, 0, \cdots), \cdots \\
=&v_0 \mapsto \varphi_0(v_0), \cdots \\
=&\varphi_0, \cdots, \varphi_{m-1} \\
=&\varphi
\end{aligned}$$

□

Exercise 3.16. *Let W a finite vector space, $T \in \mathcal{L}(V, W)$, show that*

$$T' = 0 \iff T = 0$$

Proof.

- ($\Rightarrow$) Suppose $T \neq 0$, then we can always find $\varphi \in \mathcal{L}(W, F)$ which $\varphi(\operatorname{range} T) \neq 0$, then $\varphi \circ T \neq 0$.
- ($\Leftarrow$) Trivial.

□

Exercise 3.17. *Let V, W are finite vector spaces, $T \in \mathcal{L}(V, W)$. Show that T is invertible $\iff$ T' is invertible.*

Proof. Since T is invertible, then T is injective, therefore T' is surjective. Similarly, T' is injective since T is surjective. Therefore T' is invertible. □

Exercise 3.18. *Let V, W are finite vector spaces, show that the mapping $\varphi(T) = T'$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathcal{L}(W', V')$.*

Proof. Since V and W are finite, we only need to show that φ is injective or surjective. We will show that φ is injective.

For any $\varphi(T) = T' \in \mathcal{L}(W', V')$, we know $T = 0 \iff T' = 0$, therefore $\operatorname{null} \varphi = \{0\}$, thus φ is injective.

I was wonder if I can prove this by $\varphi(S)(\mathrm{id}) = \varphi(T)(\mathrm{id}) \implies S = T$. This may work if the codomain restriction doesn't lose information, i.e. only restrict to a superset of its range, therefore the restriction is one-to-one. □

Exercise 3.21. *Let V finite and $U, W \subseteq V$ are subspaces.*

1. *Show that* $W^0 \subseteq U^0 \iff U \subseteq W$

2. *Show that* $W^0 = U^0 \iff U = W$

Proof. The second statement can be easy proved by the first one.

- $(\Rightarrow)$ We can always find a $f \in \mathcal{L}(W, F)$ such that $\operatorname{null} f = W$, then $f(U) = \{0\}$ since $f \in W^0 \subseteq U^0$, therefore $U \subseteq \operatorname{null} f = W$.

- $(\Leftarrow)$ For any $\varphi \in W^0$, we know $W \subseteq \operatorname{null} \varphi$, then $U \subseteq W \subseteq \operatorname{null} \varphi$, therefore $\varphi in U^0$, thus $W^0 \subseteq U^0$.

$\square$