

1 Pullback

Theorem 1.1. *Suppose we have two joined commuting squares like:*

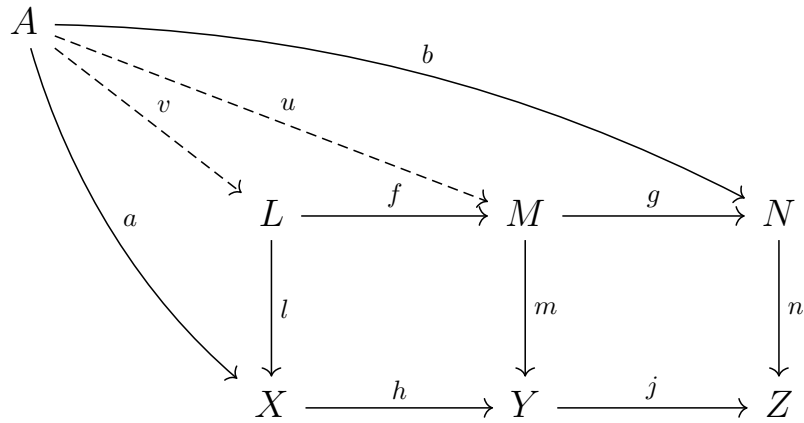
$$\begin{array}{ccccc}
 L & \xrightarrow{f} & M & \xrightarrow{g} & N \\
 \downarrow l & & \downarrow m & & \downarrow n \\
 X & \xrightarrow{h} & Y & \xrightarrow{j} & Z
 \end{array}$$

Then:

1. *The outer rectangle is a pullback square if two inner squares are pullback squares.*
2. *The inner-left square is a pullback square if the outer rectangle and the inner-right square are pullback squares.*

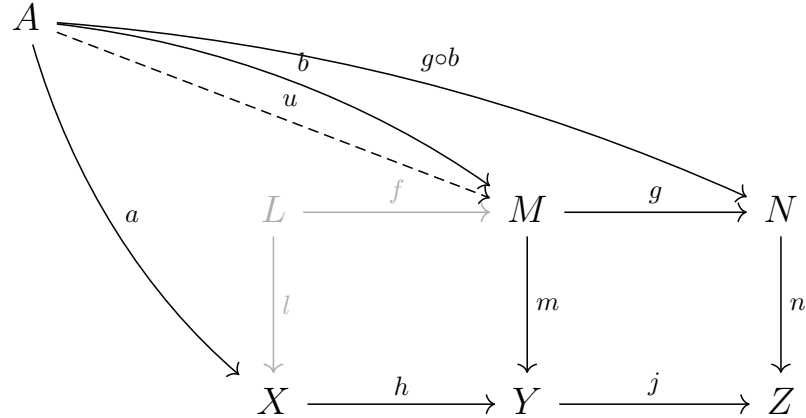
Proof.

1. For any (A, a, b) such that $j \circ h \circ a = n \circ b$, then there is a unique $u : A \rightarrow M$ such that $h \circ a = m \circ u$ and $b = g \circ u$. Then there is a unique $v : A \rightarrow L$ such that $l \circ a = v$ and $f \circ v = u$, which makes (A, a, b) against to the outer rectangle commutes.

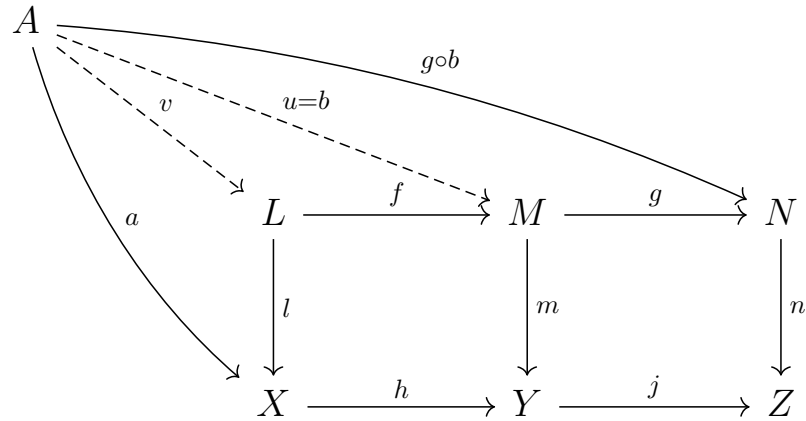


2. For any (A, a, b) such that $h \circ a = m \circ b$, consider the inner-right pullback,

then we have a unique $u : A \rightarrow M$ such that the diagram commutes:



However, if we replace u with b , we have $g \circ b = g \circ b$ and $h \circ a = m \circ b$, that means b can do u 's job, but we know u is unique, so $b = u$. Now consider the outer pullback, we have a unique $v : A \rightarrow L$ such that the diagram commutes:



That is, $l \circ v = a$ and $g \circ f \circ v = g \circ b$, we claim that v is the unique factorization from $(A, a, u = b)$ to (L, l, f) . It is obvious that $l \circ v = a$, we need to show $f \circ v = u = b$. We may use the trick we just used, we can see that $g \circ f \circ v = g \circ u$ and $m \circ f \circ v = h \circ l \circ v = h \circ a$. So $f \circ v$ can do b 's job, so $f \circ v = b$.

For any arrow $w : A \rightarrow L$ such that $l \circ a = w$ and $f \circ w = b$, then we have also $g \circ f \circ w = g \circ b$, which implies w is the unique arrow from $A \rightarrow L$ such that the outer diagram commutes, so $w = v$.

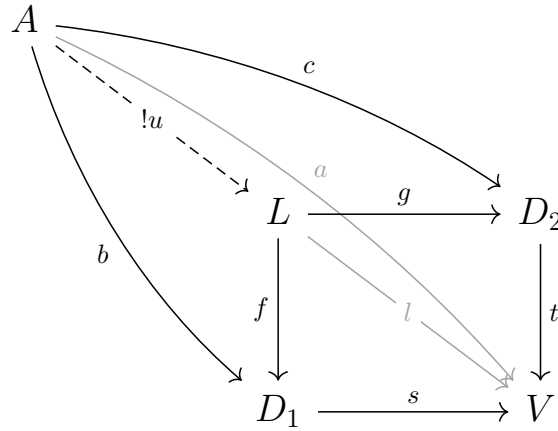
□

Theorem 1.2. *A pullback square for the corner $D_1 \rightarrow V \leftarrow D_2$ is a product of $D_1 \rightarrow V$ and $V \leftarrow D_2$ in the slice category \mathcal{C}/V .*

Proof. Suppose (L, f, g) is the pullback of such corner, then we first need to show that there is an arrow $l : L \rightarrow V$ such that $s \circ f = l$ (therefore a morphism from (L, l) to (D_1, s)) and $t \circ g = l$ (a morphism from (L, l) to (D_2, t)).

Since (L, f, g) makes the pullback square commutes, we know $s \circ f = t \circ g$, therefore we let $l = s \circ f$ (or equivalently $t \circ g$).

We need to show that $((L, l), f, g)$ forms a product of (D_1, s) and (D_2, t) , consider any $((A, a), b, c)$ where $a : A \rightarrow V$ such that $s \circ b = a$ and $t \circ c = a$. Just like l for L , a is redundant, so we may omit it. Now, the diagram looks like:



Since (L, f, g) is a pullback, we know there is a unique $u : A \rightarrow L$ such that two triangle commutes. However, we must first show that u is an arrow from (A, a) to (L, l) , that is, $l \circ u = a$. It is easy to see that $l \circ u = s \circ f \circ u = s \circ b = a$. \square

Theorem 1.3. *if a category has all binary products and all equalizers for every pair of parallel arrows, then it has a pullback for any corners.*

Proof. Suppose $X \rightarrow Z \leftarrow Y$ a corner, then consider the product $X \times Y$:

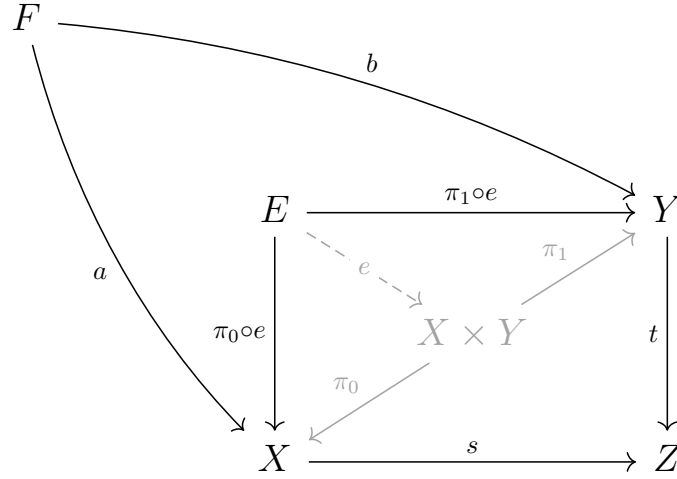
$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_1} & Y \\ \pi_0 \downarrow & & \downarrow t \\ X & \xrightarrow{s} & Z \end{array}$$

Now, consider the equalizer for the parallel arrows $t \circ \pi_1$ and $s \circ \pi_0$:

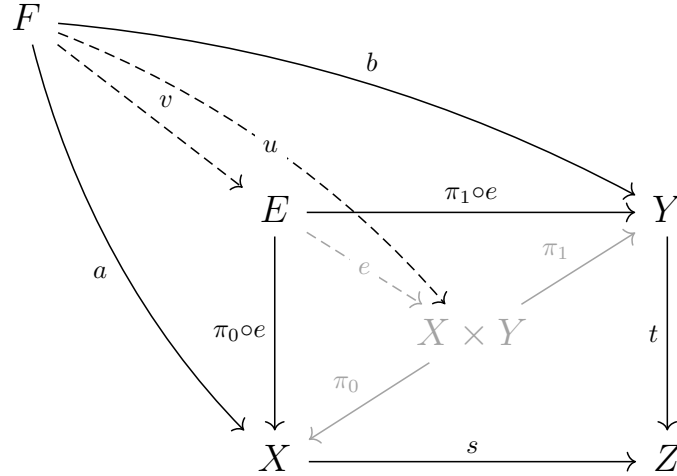
$$E \xrightarrow{e} X \times Y \rightrightarrows Z$$

$\begin{array}{c} s \circ \pi_0 \\ \hline t \circ \pi_1 \end{array}$

We claim $(E, \pi_0 \circ e, \pi_1 \circ e)$ is a pullback of such corner. For any (F, f, g) such that the outer diagram commutes:



it is easy to see that there is a unique arrow $u : F \rightarrow X \times Y$ such that $\pi_0 \circ u = a$ and $\pi_1 \circ u = b$ since $X \times Y$ is a product. Then there is another unique arrow $v : F \rightarrow E$ such that $e \circ v = u$ since E is a equalizer.



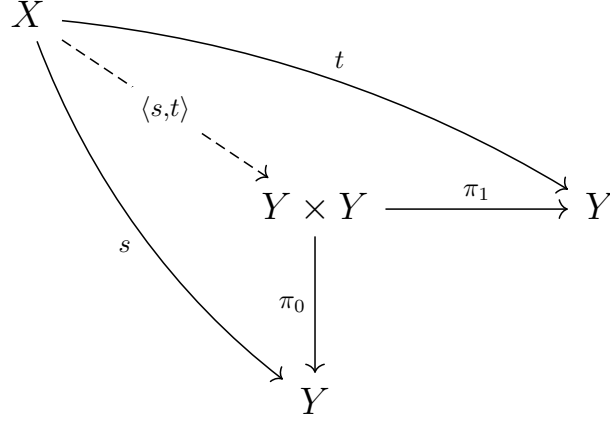
Obviously, (commute) $\pi_0 \circ e \circ v = \pi_0 \circ u = a$ and $\pi_1 \circ e \circ v = \pi_1 \circ u = b$.
(unique) If an arrow $w : F \rightarrow E$ can do the job, then $e \circ w : F \rightarrow X \times Y$ is another factorization from F to the product $X \times Y$, so $e \circ w = u$, but that means w is also a factorization from F to the equalizer E , which means $v = w$.

So $(E, \pi_0 \circ e, \pi_1 \circ e)$ is a pullback of such corner. \square

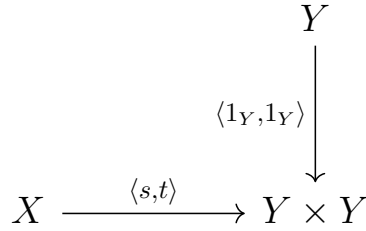
Theorem 1.4. *If a category has a terminal object and has a pullback for every corner, then it has all binary product.*

Theorem 1.5. *If a category has a terminal object and has a pullback for every corner, then it has an equalizer for every parallel arrows.*

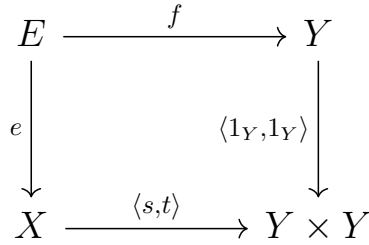
Proof. Suppose $s, t : X \rightarrow Y$ are parallel arrows, then the following diagram commutes:



Note that we have $Y \times Y$ since this category has all binary products. Then consider this corner:



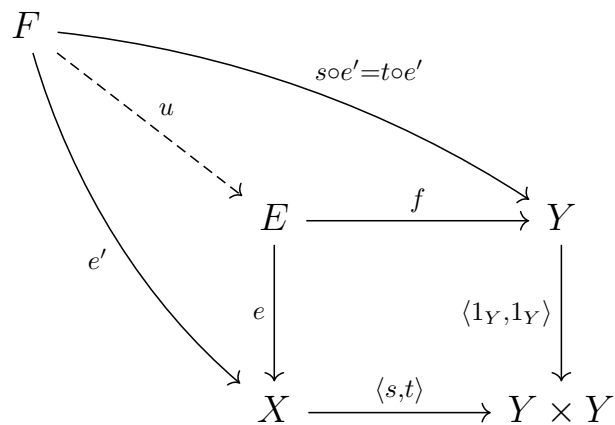
We have an object E , $e : E \rightarrow X$ and $f : E \rightarrow Y$ such that the square commutes:



(Proof comes from textbook until here)

We can see that $\pi_0 \circ \langle s, t \rangle \circ e = s \circ e$ while $\pi_0 \circ \langle 1_Y, 1_Y \rangle \circ f = 1_Y \circ f = f$, therefore $s \circ e = f$, similarly $t \circ e = f$, so $s \circ e = t \circ e$. We claim E is the equalizer for the parallel arrows $s, t : X \rightarrow Y$. For any (F, e') such that $s \circ e' = t \circ e'$, then we have a unique arrow $u : F \rightarrow E$ such that this diagram

commutes:



where $e \circ u = e'$. Suppose $v : F \rightarrow E$ where $e \circ v = e'$, then $f \circ v = s \circ e \circ v = s \circ e'$. \square