Exercise 6.1. *Prove or disprove: If $v_0, \cdots, v_{n-1} \in V$, then:*

$$\sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \langle v_j, v_k \rangle \geq 0$$

Proof. For any j, we have $\sum_{k=0}^{n-1} \langle v_j, v_k \rangle = \langle v_j, v_0 + \cdots + v_{n-1} \rangle$, thus the equation is now $\langle v_0 + \cdots + v_{n-1}, v_0 + \cdots + v_{n-1} \rangle \geq 0$, which is trivial by definition. □

Exercise 6.2. *Let $S \in \mathcal{L}(V)$. Define $\langle \cdot, \cdot \rangle_1$ by*

$$\langle u, v \rangle_1 = \langle Su, Sv \rangle$$

for all $u, v \in V$. Show that $\langle \cdot, \cdot \rangle_1$ is an inner product over $V \iff S$ is injective.

Proof.

- Try to prove $\langle \cdot, \cdot \rangle_1$ is a inner product and see where we stuck without injective. We find that $\langle v, v \rangle_1 = 0 \iff v = 0$ stuck, as if S is not injective, then we have $\langle u - v, u - v \rangle_1 = \langle S(u - v), S(u - v) \rangle = \langle 0, 0 \rangle = 0$ where $u - v \neq 0$ and $Su = Sv$. Therefore S must be injective.

  Although the proof is not *constructive*, it help us to find a constructive proof: Suppose $Su = Sv$, then $\langle u - v, u - v \rangle_1 = \langle Su - Sv, Su - Sv \rangle = 0$, thus $u - v = 0$, therefore $u = v$.

- Positivity, Additivity, Homogeneity holds cause S is a linear map and $\langle \cdot, \cdot \rangle$ is an inner product, and Conjugate Symmetry holds cause $\langle \cdot, \cdot \rangle$ is an inner product. For Definiteness, $\langle v, v \rangle_1 = \langle Sv, Sv \rangle = 0$ implies $v = 0$ cause S injective, and $\langle 0, 0 \rangle_1 = \langle S0, S0 \rangle = \langle 0, 0 \rangle = 0$.

□

Exercise 6.3.

- *Show that $f((a, b), (c, d)) = |ac| + |bd|$ is not an inner product over $\mathbb{R}^2$.*
- *Show that $f((a, b, c), (x, y, z)) = ax + cz$ is not an inner product over $\mathbb{R}^3$.*

Proof.

- $f((1,1),(1,1)) = 1+1$ and $f((-1,-1),(1,1)) = 1+1$, then $f((1,1)+(-1,-1),(1,1)) = f((0,0),(1,1)) = 0+0 = 0$ while $f((1,1)+(-1,-1),(1,1)) = f((1,1),(1,1)) + f((-1,-1),(1,1)) = 2+2 = 4$.

- $f((0,1,0),(0,1,0)) = 0$ but $(0,1,0) \neq 0$.

$\square$

Exercise 6.4. *Let $T \in \mathcal{L}(V)$ and $\|Tv\| \leq \|v\|$ for all $v \in V$. Show that $T - \sqrt{2}I$ is injective.*

Proof. Suppose $T - \sqrt{2}$ is not injective, then $Tv = \sqrt{2}v$ for some v, then $\|Tv\| = \left\|\sqrt{2}v\right\| = |\sqrt{2}| \|v\| \geq \|v\|$. Basically any eigenvalue with absolute value greater than 1 can make it. $\square$

Exercise 6.5. *Let V an inner product space over $\mathbb{R}$.*

- *Show that $\langle u+v, u-v \rangle = \|u\|^2 - \|v\|^2$.*

- *Show that $u+v \perp u-v$ if $\|u\| = \|v\|$.*

- *Use last conclusion to show that the diagonal of 菱形 are orthogonal.*

Proof.

- $\langle u+v, u-v \rangle = \langle u,u \rangle + \langle v,u \rangle - \langle u,v \rangle - \langle v,v \rangle$, note that $\langle v,u \rangle = \langle u,v \rangle$ since V over $\mathbb{R}$, thus $\langle u+v, u-v \rangle = \langle u,u \rangle - \langle v,v \rangle = \|u\|^2 - \|v\|^2$.

- By $\Uparrow$.

- We know 菱形 is a parallelogram that four sides have same length, thus $\|u\| = \|v\|$ and $u+v \perp u-v$, where $u+v$ and $u-v$ are two diagonal of 菱形.

$\square$

Exercise 6.6. *Let $u, v \in V$. Show that $\langle u,v \rangle = 0 \iff \|u\| \leq \|u+av\|$ for any $a \in F$.*

Proof.

- ($\Rightarrow$) We will show that $\|u\| \leq \|u+av\|$ by $\|u\|^2 \leq \|u+av\|^2$ (recall that norm is always non-negative). $\|u+av\|^2 = \|u\|^2 + (a\|v\|)^2$

- ($\Leftarrow$) If $v = 0$, then $0 \perp u$ and the proof is complete, we assume $v \neq 0$. Let $w \in V$ such that $cv + w = u$ and $v \perp w$, then $\|u\|^2 = \|cv + w\|^2 = \|cv\|^2 + \|w\|^2$ (see **??**), thus $\|u\|^2 \geq \|w\|^2$, therefore $\|u\| \geq \|w\|$ where $w = u - cv$, therefore $\|w\| \leq \|w\|$, hence $\|u\| = \|w\|$. Then $\|u\|^2 = \|cv\|^2 = \|w\|^2$ is now $\|cv\|^2 = 0$, therefore $c = 0$ or $v = 0$, but $v \neq 0$, thus $c = \dfrac{\langle u, v \rangle}{\|v\|^2} = 0$ then $\langle u, v \rangle = 0$ and $u \perp v$.

$\square$