Exercise 3.4. *Suppose $D(p) = p' : \mathcal{L}(\mathcal{P}_3(\mathbb{R})) \to \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$. Find a basis of $\mathcal{L}(\mathcal{P}_3(\mathbb{R}))$ and a basis of $\mathcal{L}(\mathcal{P}_2(\mathbb{R}))$, such that $\mathcal{M}(D)$ about these basis is:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Proof. Consider $x, x^2, x^3, 1$ the basis of $\mathcal{P}_3(\mathbb{R})$ and $1, x, 2x^2$. □

Exercise 3.5. *Suppose V and W are finite and $T \in \mathcal{L}(V, W)$. Show that there are basis of V and W respectively, such that $\mathcal{M}(T, \text{those basis})$ is all zero except 1 at $k, k \quad (1 \leq k \leq \dim \operatorname{range} T)$.*

Proof. Consider the basis $w_0, \cdots, w_{k-1}$ of $\operatorname{range} T$ and the basis $w_0, \cdots, w_{m-1}$ of W which expands from $w_0, \cdots, w_{k-1}$. Then there must be $v_0, \cdots, v_{k-1}$ such that $Tv_i = w_i$ for all $0 \leq i < k$, we know $v_0, \cdots, v_{k-1}$ is linear independent since $w_0, \cdots, w_{k-1}$ is linear independent, so we can expand it to a basis of V, say $v_0, \cdots, v_{n-1}$.

We claim that $\mathcal{M}(T, v_0, \cdots, v_{n-1}, w_0, \cdots, w_{m-1})$ is a matrix with all zero but 1 at $k, k \quad (1 \leq k < \operatorname{range} T)$. For any $\lambda_0 v_0 + \cdots + \lambda_{n-1} v_{n-1} \in V$, we have $T(\lambda_0 v_0 + \cdots + \lambda_{n-1} v_{n-1}) = \lambda_0 w_0 + \cdots + \lambda_{k-1} w_{k-1}$, note that all v_i where $i \geq k$ disappear, since they maps to 0. Therefore $\mathcal{M}(T)$ is all zero but 1 at k, k (since $\lambda_i w_i$ in the last equation). □

Exercise 3.6. *Show that $-^T : F^{m,n} \to F^{n,m}$ is a linear mapping.*

Proof. Trivial, sorry. □

Exercise 3.7. *Show that $(AB)^T = B^T A^T$.*

Proof. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix, then for any $i \in [1, m]$ and $j \in [1, p]$, we have $(AB)^T_{i,j} = (AB)_{j,i} = \sum_{r=1}^{n} A_{j,r} B_{r,i} = \sum_{r=1}^{n} B^T_{i,r} A^T_{r,j} = (B^T A^T)_{i,j}$. □

Exercise 3.8. *Let A a $m \times n$ matrix, show that the rank of A is $1 \iff$ there is $c_0, \cdots, c_{m-1} \in F^m$ and $d_0, \cdots, d_{n-1} \in F^n$ such that $A_{j,k} = c_j d_k$ for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, n-1$.*

Proof. The right hand side is actually the external product of vectors, that is vw^T.

($\Rightarrow$) is easy since we can use the theorem that any $m \times n$ matrix A can be expressed by CR where C is a $m \times r$ matrix, R is a $r \times n$ matrix, r is the rank of A. In this case, $r = 1$, so C and R are just vectors.

($\Leftarrow$) is also easy since other column is a scalar multiple of the first column, therefore the rank of A is 1. $\square$

Exercise 3.9. *Let $T \in \mathcal{L}(V)$, $u_0, \cdots, u_{n-1}$ and $v_0, \cdots, v_{n-1}$ are the bases of V, show that the following statements are equivalent:*

1. *T is injective*
2. *The columns of $\mathcal{M}(T)$ is linear independent*
3. *The columns of $\mathcal{M}(T)$ spans $F^{n,1}$*
4. *The lines of $\mathcal{M}(T)$ is linear independent*
5. *The lines of $\mathcal{M}(T)$ spans $F^{1,n}$*

Proof. (2), (3) are obviously equivalent and (4), (5) too.

Although I want to make an arrow loop, but the arrow between (1) and (4), (5) is too hard, so I will show that (1) $\iff$ (2), (3) and (2), (3) $\iff$ (4), (5).

- ($\Rightarrow$) Let $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1} = [0, \ldots, 0]$, then $T(\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}) = 0$, so λ_i are 0 since T is injective, which means $\operatorname{null} T = \{0\}$.

  ($\Leftarrow$) For any $T(\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}) = 0$, we have the linear combination of v_i is 0 where the coefficients come from $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1}$ (w_i are the columns of $\mathcal{M}(T)$), therefore the coefficients are all 0 since v_i is linear independent, thus $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1} = 0$, which means λ_i are all 0 since w_i is linear independent.

- For any matrix, its line rank is equal to its column rank, so columns independent $\iff$ lines independent.

$\square$