Definition 3.110 (Dual Space). *Let V a vector space, then we denote V' the dual space of V, where*

$$V' = \mathcal{L}(V, F)$$

Theorem 3.111. *Let V a finite vector space, then $\dim V' = \dim V$*

Proof. $\dim V' = \dim \mathcal{L}(V, F) = (\dim V)(\dim F) = \dim V$ □

Definition 3.112 (Dual Basis). *Let $v_0, \cdots, v_{m-1}$ a basis of V, then the dual basis of $v_0, \cdots, v_{m-1}$ is $\varphi_0, \cdots, \varphi_{m-1}$ such that:*

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

holes for any $0 \leq i, j < m$.

We can see that the basis the dual basis extracts the coefficients of any vector in V.

Theorem 3.113. *Let $v_0, \cdots, v_{m-1}$ a basis of V, and dual basis $\varphi_0, \cdots, \varphi_{m-1}$ of which. Then for any $v \in V$,*

$$v = \varphi_0(v)v_0 + \cdots + \varphi_{m-1}(v)v_{m-1}$$

Proof. For any i, $\varphi_i(v) = \varphi_i(\lambda_0 v_0 + \cdots + \lambda_{m-1}v_{m-1}) = \varphi_i(\lambda_i v_i) = \lambda_i \varphi_i(v_i) = \lambda_i \times 1$. □

Theorem 3.116. *Let V a finite space, then the dual basis of basis of V is a basis of V'.*

Proof. Let $v_0, \cdots, v_{m-1}$ a basis of V, then its dual basis has the same length, therefore we only need to show its dual basis is linear independent.

Suppose $\lambda_0\varphi_0 + \cdots + \lambda_{m-1}\varphi_{m-1} = 0$, then for any $0 \leq i < m$, $(\lambda_0\varphi_0 + \cdots + \lambda_{m-1}\varphi_{m-1})(v_i) = \lambda_i = 0$, therefore the dual basis is linear independent. □

Definition 3.118 (Dual Map). *Let $T \in \mathcal{L}(V, W)$. A dual map of T is a linear map $T' \in \mathcal{L}(W', V')$, such that for any $\varphi \in W'$:*

$$T'(\varphi) = \varphi \circ T$$

Theorem 3.128. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that:*

1. $\operatorname{null} T' = (\operatorname{range} T)^0$

2. $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

Proof.

- $\operatorname{null} T'$ is a space $\{ \varphi \in \mathcal{L}(W, F) \mid \varphi \circ T = 0 \}$, which means $\operatorname{range} T \subseteq \operatorname{null} \varphi$. $(\operatorname{range} T)^0$ is a space $\{ \varphi \in \mathcal{L}(W, F) \mid \varphi(\operatorname{range} T) = \{0\} \}$, which means $\operatorname{range} T \subseteq \operatorname{null} \varphi$. Therefore $\operatorname{null} T' = (\operatorname{range} T)^0$
- $\dim(\operatorname{range} T)^0 = \dim W - \dim \operatorname{range} T = \dim W - (\dim V - \dim \operatorname{null} T)$

□

Theorem 3.129. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that*

$$T \text{ is surjective} \iff T' \text{ is injective}$$

Proof.

- Suppose T is surjective, then for any $T'(\varphi) = T'(\psi)$, we have $\varphi \circ T = \psi \circ T$. Since T is surjective, then T is an epimorphism (we proved this in E2B), therefore $\varphi = \psi$.
- Suppose T' is injective, then for any $\varphi, \psi \in \mathcal{L}(W, F)$ such that $\varphi \circ T = \psi \circ T$, we have $\varphi = \psi$ since T' is injective. Therefore T is epimorphism, thus surjective.

□

The last theorem is obviously true in category theory, but we haven't show that T' is a morphism in Vect' where $\mathrm{Vect}' \simeq \mathrm{Vect}^{\mathrm{op}}$.

Theorem 3.130. *Let V, W are finite and $T \in \mathcal{L}(V, W)$, show that:*

1. $\dim \operatorname{range} T' = \dim \operatorname{range} T$

2. $\operatorname{range} T' = (\operatorname{null} T)^0$

Proof.

- $\dim \operatorname{range} T' = \dim W' - \dim \operatorname{null} T' = \dim W' - (\dim \operatorname{null} T + \dim W - \dim V) = \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$

- For any $\varphi \circ T \in \operatorname{range} T'$, $(\varphi \circ T)(\operatorname{null} T) = \varphi(\{0\}) = \{0\}$, therefore $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$. Since $\dim(\operatorname{null} T)^0 = \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T \ dim \operatorname{range} T'$, therefore $\operatorname{range} T' = (\operatorname{null} T)^0$ since both of them are finite and $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$.

$\square$

Theorem 3.131. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show*

$$T \textit{ is injective } \iff T' \textit{ is surjective}$$

Proof.

- $\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim V = \dim V'$ since T is injective, therefore T' is surjective.

- $\dim \operatorname{range} T = \dim \operatorname{range} T' = \dim V' = \dim V$ therefore T is injective.

$\square$