Definition 0.11. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{null} T = \{\ v \mid Tv = 0\ \}$ *is called the* ***null space*** *of* T*.*

This is also called the **kernal** of T in algebra.

Theorem 0.13. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{null} T$ *is a subspace of* V*.*

Proof.

- We have $0 \in \operatorname{null} T$ since $T0 = 0$, which is the property of linear transformation.
- For any $a, b \in \operatorname{null} T$, we have $0 = Ta + Tb = T(a + b)$, so $a + b \in \operatorname{null} T$.
- For any $Ta \in \operatorname{null} T$ and $\lambda \in F$, we have $\lambda Ta = T(\lambda a)$, so $\lambda a \in \operatorname{null} T$.

□

Definition 0.15. *For any* $T \in \mathcal{L}(V, W)$*, set* $\operatorname{range} T = T(V) = \{\ Tv \mid v \in V\ \}$ *is called the* ***range*** *of* T*.*

This is also called the **image** of T in math.

Theorem 0.18. *For any* $T \in \mathcal{L}(V, W)$*,* $\operatorname{range} T$ *is a subsapce of* W*.*

Proof.

- We have $T(0) = 0 \in \operatorname{range} T$.
- For any $Ta, Tb \in \operatorname{range} T$, $Ta + Tb = T(a + b) \in \operatorname{range} T$.
- For any $Ta \in \operatorname{range} T$ and $\lambda \in F$, $\lambda Ta = T(\lambda a) \in \operatorname{range} T$.

□

Theorem 0.21. *Suppose* V *is finite and* $T \in \mathcal{L}(V, W)$*, then* $\operatorname{range} T$ *is finite, and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof. Consider the basis $v_0, \cdots, v_k$ of $\operatorname{null} T$, and the basis $v_0, \cdots, v_n$ of V that expand from $v_0, \cdots, v_k$. We will show that $T(v_{k+1}), \cdots, T(v_n)$ is the basis of $\operatorname{range} T$.

We first show that $T(v_{k+1}), \cdots, T(v_n)$ is linear independent. If it is linear independent, then

$$
\begin{aligned}
&\lambda_1 T(v_{k+1}) + \cdots + \lambda_i T(v_{k+i}) \\
=&T(\lambda_1 v_{k+1} + \cdots + \lambda_i T(v_{k+i})) \\
=&0
\end{aligned}
$$

That means a linear combation of v_{k+i} is in $\operatorname{null} T$, which is $\operatorname{span}(v_0, \cdots, v_k)$, therefore the basis $v_0, \cdots, v_n$ is linear dependent.

Then we show that $T(v_{k+1}), \cdots, T(v_n)$ spans $\operatorname{range} T$. For any $Tv \in \operatorname{range} T$, there must be $v \in V$ such that $Tv = Tv$, then v can be written in form of the linear combination of $v_0, \cdots, v_n$, and then $Tv = T(\lambda_0 v_0 + \cdots + \lambda_n v_n)$. We can drop all terms with v_i where $i \leq k$, since they are in $\operatorname{null} T$, so Tv is now represent by a linear combination of $T(v_{k+i})$ for all $0 < i \leq n - k$, therefore, it is a basis of $\operatorname{range} T$ and $\dim \operatorname{range} T$ is finite.

Finally, $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

Definition 3.16 (Notation: $v + U$)**.** *Let $v \in V$ and $U \subseteq V$, then $v + U = \{ v + u \mid u \in U \}$.*

Such sets also called *coset* in group theory.

Definition 3.97 (Translate)**.** *Let $v \in V$ and $U \subseteq V$, we say $v + U$ is a translate of U.*

Definition 3.98 (Quotient Space)**.** *Let $U \subseteq V$ a subspace, then the quotient space V/U is a set with translates of U, that is:*

$$V/U = \{ v + U \mid v \in V \}$$

Theorem 3.101. *Let $U \subseteq V$ a subspace and $v, w \in V$, then the following statements are equivalent.*

1. $v - w \in U$
2. $v + U = w + U$
3. $(v + U) \cap (w + U) \neq \varnothing$

Proof.

- If $v - w \in U$, for any $v + u \in v + U$, we have $v + u = v + (v - w) - (v - w) + u = v - w + w + u = w + (v - w) + u \in w + U$ since $v - w \in U$. Similarly, for any $w + u \in w + U$, we have $w + u = w + (v - w) - (v - w) + u = v - v + w + u = v - (v - w) + u = v + (-(v - w) + u) \in v + U$.

- If $v+U = w+U$, then $v = w+u$ since $v \in v+U$, therefore $v-w = u \in U$.
- if $v+U = w+U$, then $(v+U) \cap (w+U) = v+U = w+U \neq \varnothing$
- If $(v+U) \cap (w+U) \neq \varnothing$, then for any $v+u_0 = w+u_1 \in (v+U) \cap (w+U)$, we have $(v-w)+(u0-u_1) = 0$ and then $v-w = u_1 - u_0 \in U$, so $v+U = w+U$.

$\square$

Definition 3.102. *Let $U \subseteq V$, then addition and scalar multiplication on V/U is defined by:*

$$(v+U)+(w+U) = (v+w)+U$$
$$\lambda(v+U) = (\lambda v)+U$$

Theorem 3.103. *Let $U \subseteq V$ a subspace, then V/U is a vector space with addition and scalar multiplication we defined in previous definition.*

Proof. We must first show that the addition and the sclar multiplication we introduce are functions.

For any $a, b, c, d \in V$, we will show $(a+b)+U = (c+d)+U$ if $a+U = c+U$ and $b+U = d+U$. We can show $(a+b)-(c+d) \in U$ by $a-c \in U$ and $b-d \in U$.

For any $v, w \in V$ and $\lambda \in F$, we will show $(\lambda v)+U = (\lambda w)+U$ if $v+U = w+U$. We know $v-w \in U$, then $\lambda(v-w) = \lambda v - \lambda w \in U$, therefore $(\lambda v)+U = (\lambda w)+U$.

We have identity of addition $0+U$ and inverse of addition $(-v)+U$ for all $v \in V$. $\square$

Definition 3.104. *Let $U \subseteq V$ a subspace, the quotient map $\pi : V \to V/U$ is a linear mapping defined by:*

$$\pi(v) = v+U$$

Proof. We will show π is a linear mapping, $\pi(v+w) = (v+w)+U = v+U+w+U = \pi(v)+\pi(w)$ and $\lambda\pi(v) = \lambda(v+U) = (\lambda v)+U = \pi(\lambda v)$. $\square$

Theorem 3.105. *Let V finite and $U \subseteq V$ a subspace, show that* $\dim(V/U) = \dim V - \dim U$.

Proof. We can rewrite the equation as $\dim V = \dim(V/U) + \dim U$, and it is easy to see that $\operatorname{range}\pi = \dim(V/U)$ and $\operatorname{null}\pi = \dim U$. □

Definition 3.106. *Let $T \in \mathcal{L}(V, W)$, define $\tilde{T} : V/(\operatorname{null} T) \to W$ by $\tilde{T}(v + \operatorname{null} T) = Tv$.*

Theorem 3.107. *Let $T \in \mathcal{L}(V, W)$, then:*

1. $\tilde{T} \circ \pi = T$
2. *$\tilde{T}$ is injective*
3. $\operatorname{range}\tilde{T} = \operatorname{range} T$
4. $V/(\operatorname{null} T) \cong \operatorname{range} T$

Proof.

1. For all $v \in V$, $\tilde{T}(\pi(v)) = \tilde{T}(v + \operatorname{null} T) = Tv$
2. If $\tilde{T}(v + \operatorname{null} T) = \tilde{T}(w + \operatorname{null} T)$, then $T(v - w) = 0$, which means $v - w \in \operatorname{null} T$, therefore $v + \operatorname{null} T = w + \operatorname{null} T$.
3. For any $Tv \in \operatorname{range} T$, we have $\tilde{T}(v + \operatorname{null} T) \in \operatorname{range}\tilde{T}$. For any $\tilde{T}(v + \operatorname{null} T) = Tv \in \operatorname{range}\tilde{T}$, we have $Tv \in \operatorname{range} T$.
4. Restrict the range of $\tilde{T}$ on $\operatorname{range} T$, say $\varphi(v + \operatorname{null} T) = \tilde{T}(v + \operatorname{null} T) : V/(\operatorname{null} T) \to \operatorname{range} T$, then φ is injective since (2) and surjective since (3), therefore φ is an isomorphism, thus $V/(\operatorname{null} T) \simeq \operatorname{range} T$.

□

Definition 3.110 (Dual Space)**.** *Let V a vector space, then we denote V' the dual space of V, where*

$$V' = \mathcal{L}(V, F)$$

Theorem 3.111. *Let V a finite vector space, then $\dim V' = \dim V$*

Proof. $\dim V' = \dim \mathcal{L}(V, F) = (\dim V)(\dim F) = \dim V$ □

Definition 3.112 (Dual Basis)**.** *Let $v_0, \cdots, v_{m-1}$ a basis of V, then the dual basis of $v_0, \cdots, v_{m-1}$ is $\varphi_0, \cdots, \varphi_{m-1}$ such that:*

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

holes for any $0 \le i, j < m$.

We can see that the basis the dual basis extracts the coefficients of any vector in V.

Theorem 3.113. *Let $v_0, \cdots, v_{m-1}$ a basis of V, and dual basis $\varphi_0, \cdots, \varphi_{m-1}$ of which. Then for any $v \in V$,*

$$v = \varphi_0(v)v_0 + \cdots + \varphi_{m-1}(v)v_{m-1}$$

Proof. For any i, $\varphi_i(v) = \varphi_i(\lambda_0 v_0 + \cdots + \lambda_{m-1}v_{m-1}) = \varphi_i(\lambda_i v_i) = \lambda_i \varphi_i(v_i) = \lambda_i \times 1$. □

Theorem 3.116. *Let V a finite space, then the dual basis of basis of V is a basis of V'.*

Proof. Let $v_0, \cdots, v_{m-1}$ a basis of V, then its dual basis has the same length, therefore we only need to show its dual basis is linear independent.

Suppose $\lambda_0\varphi_0 + \cdots + \lambda_{m-1}\varphi_{m-1} = 0$, then for any $0 \leq i < m$, $(\lambda_0\varphi_0 + \cdots + \lambda_{m-1}\varphi_{m-1})(v_i) = \lambda_i = 0$, therefore the dual basis is linear independent. □

Definition 3.118 (Dual Map)**.** *Let $T \in \mathcal{L}(V, W)$. A dual map of T is a linear map $T' \in \mathcal{L}(W', V')$, such that for any $\varphi \in W'$:*

$$T'(\varphi) = \varphi \circ T$$

Theorem 3.128. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that:*

1. $\operatorname{null} T' = (\operatorname{range} T)^0$
2. $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim W - \dim V$

Proof.

- $\operatorname{null} T'$ is a space $\{\ \varphi \in \mathcal{L}(W, F) \mid \varphi \circ T = 0\ \}$, which means $\operatorname{range} T \subseteq \operatorname{null} \varphi$. $(\operatorname{range} T)^0$ is a space $\{\ \varphi \in \mathcal{L}(W, F) \mid \varphi(\operatorname{range} T) = \{0\}\ \}$, which means $\operatorname{range} T \subseteq \operatorname{null} \varphi$. Therefore $\operatorname{null} T' = (\operatorname{range} T)^0$
- $\dim(\operatorname{range} T)^0 = \dim W - \dim \operatorname{range} T = \dim W - (\dim V - \dim \operatorname{null} T)$

□

Theorem 3.129. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show that*

$$T \text{ is surjective} \iff T' \text{ is injective}$$

Proof.

- Suppose T is surjective, then for any $T'(\varphi) = T'(\psi)$, we have $\varphi \circ T = \psi \circ T$. Since T is surjective, then T is an epimorphism (we proved this in E2B), therefore $\varphi = \psi$.

- Suppose T' is injective, then for any $\varphi, \psi \in \mathcal{L}(W, F)$ such that $\varphi \circ T = \psi \circ T$, we have $\varphi = \psi$ since T' is injective. Therefore T is epimorphism, thus surjective.

□

The last theorem is obviously true in category theory, but we haven't show that T' is a morphism in Vect' where $\mathrm{Vect}' \simeq \mathrm{Vect}^{\mathrm{op}}$.

Theorem 3.130. *Let V, W are finite and $T \in \mathcal{L}(V, W)$, show that:*

1. $\dim \operatorname{range} T' = \dim \operatorname{range} T$

2. $\operatorname{range} T' = (\operatorname{null} T)^0$

Proof.

- $\dim \operatorname{range} T' = \dim W' - \dim \operatorname{null} T' = \dim W' - (\dim \operatorname{null} T + \dim W - \dim V) = \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$

- For any $\varphi \circ T \in \operatorname{range} T'$, $(\varphi \circ T)(\operatorname{null} T) = \varphi(\{0\}) = \{0\}$, therefore $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$. Since $\dim(\operatorname{null} T)^0 = \dim V - \dim \operatorname{null} T = \dim \operatorname{range} T$ *dim* $\operatorname{range} T'$, therefore $\operatorname{range} T' = (\operatorname{null} T)^0$ since both of them are finite and $\operatorname{range} T' \subseteq (\operatorname{null} T)^0$.

□

Theorem 3.131. *Let V, W are finite spaces and $T \in \mathcal{L}(V, W)$. Show*

$$T \text{ is injective} \iff T' \text{ is surjective}$$

Proof.

- $\dim \operatorname{range} T' = \dim \operatorname{range} T = \dim V = \dim V'$ since T is injective, therefore T' is surjective.

- $\dim \operatorname{range} T = \dim \operatorname{range} T' = \dim V' = \dim V$ therefore T is injective.

□

Exercise 3.1. *Suppose V is a finite vector space, Show that the only two ideal of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$. A subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called an ideal, if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for any $T \in \mathcal{L}(V)$ and $E \in \mathcal{E}$.*

Proof. We will use the concept Matrix. Suppose $\lambda_0 v_0 + \cdots + \lambda_n v_v$ the basis of V. We want to construct T_i that $T(\lambda_0 v_0 + \cdots + \lambda_n v_n) = \lambda_i v_i$ for all $0 \le i < n$, which is a matrix with all zero but 1 at i, i.

For any matrix, we can always select a non-zero value at a, b and place it at i, b, this can be done by left multiply a matrix with 1 at i, a (this produce a vector at line i with values from line a), then right multiply a matrix with 1 at i, b (this produce a vector at column b with values from line i).

Also, we can always select a non-zero value at a, b and place it at a, i, this can be done by right multiply a matrix with 1 at b, i, then left multiply a matrix with 1 at a, i.

By combining these two operations, we ca select a non-zero value at a, b and place it at i, i. Now, consider any non-zero $E \in \mathcal{E}$, we can construct a matrix with non-zero value at i, i for every $0 \le i < \dim V$. These matrix are in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then we can multiply an apropriate scalar to them so that they are matrices with 1 at i, i. By adds up these matrices, we get I, we know $I \in \mathcal{E}$ since $\mathcal{E}$ is a vector space, and now all $T \in \mathcal{L}(V)$ is also in $\mathcal{E}$ since $\mathcal{E}$ is an ideal, then $\mathcal{E} = \mathcal{L}(V)$.

The only exception is $\mathcal{E} = \{0\}$, in this case we can't pick any non-zero element.

Another solution, hope this one is more simple.

Suppose $\mathcal{E}$ an ideal of $\mathcal{L}(V)$ and non-zero, non-surjective $E \in \mathcal{E}$. Let $v_0, \cdots, v_{k-1}$ a basis of $\operatorname{null} E$ and $v_k, \cdots, v_{k+n}$ such that Tv_{k+i} is a basis of $\operatorname{range} E$, then we have $n \neq 0$ and $k \neq 0$.

Define A a linear transformation which maps v_i to v_{k+i} for $0 \le i < \min\{k, n\}$ and maps others to 0, then $\dim \operatorname{range} EA = \min\{k, n\}$.

Expand the basis $w_i = Ev_{k+i}$ of $\operatorname{range} E$ to a basis of V, say $w_0, \cdots, w_{m-1}$, define B maps Ev_{k+i} to $w_{\min\{k,n\}+i}$, we always have enough $w_{\min\{k,n\}+i}$ since $m - 1 = \dim V = \dim \operatorname{null} E + \dim \operatorname{range} E$ while $\min k, n \le \dim \operatorname{null} E$, then $\dim \operatorname{range} BE = \operatorname{range} E$ since we just re-map the range E.

Now consider $S = EA + BE$, we have $Sv_i = EAv_i = Ev_{k+i} = w_i \in \operatorname{range} E$ for all $0 \le i < \min\{k, n\}$ and $Sv_{\min\{k,n\}+i} = BEv_{\min\{k,n\}+i} = w_{\min\{k,n\}+i} \in \operatorname{range} BE$ for all $0 \le i < \dim \operatorname{range} E$. We can see $\operatorname{range} EA \cap \operatorname{range} BE = \{0\}$ and $\dim \operatorname{range}(EA + BE) = \operatorname{range} E + \min\{k, n\}$, where $k = \operatorname{null} E$ and

$n = \operatorname{range} E$, the range of $EA + BE$ gets larger and $EA + BE \in \mathcal{E}$ since $EA, BE \in \mathcal{E}$, if $k > n$ (this is the only case that $EA + BE$ is not surjective), then we continue this process with $E = EA + BE$, the procedure will finally terminate since $\mathcal{L}(V)$ is finite (cause V is finite).

Now we show that any $\mathcal{E}$ with non-zero, non-surjective $E \in \mathcal{E}$ implies a surjective (thus injective and invertible) $T \in \mathcal{E}$.

For any ideal with an invertible element $E \in \mathcal{E}$, we have $E^{-1}E = I \in \mathcal{E}$, which causes $\mathcal{E} = \mathcal{L}(V)$ since $IT = T$ for all $T \in \mathcal{L}(V)$.

Therefore, only $\{0\}$ and $\mathcal{L}(V)$ are ideals of $\mathcal{L}(V)$. □

Exercise 3.7. *Suppose vector space V and W are finite* $(2 \leq \dim V \leq \dim W)$, *show that $\{\ T \in \mathcal{L}(V, W) \mid T \text{ is not injective}\ \}$ is not a subspace.*

Proof. Consider the basis $v_0 + \cdots + v_{(\dim V - 1)} \in V$, and $T(v_0 + \cdots + v_{(\dim V-1)}) = (0 + 1_0 + \cdots + 1_{(\dim V-1)})$ and $T'(v_0 + \cdots + v_{(\dim V-1)}) = (v_0 + 0 + \cdots + v_{(\dim V-1)})$. Then $(T + T')(\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_{(\dim V-1)} v_{(\dim V-1)}) = \lambda_0 v_0 + \lambda_1 v_1 + \cdots + 2\lambda_{(\dim V-1)} v_{(\dim V-1)}$, which is obviously injective. □

Exercise 3.11. *Suppose V is finite and $T \in \mathcal{L}(V, W)$, show that there is a subspace $U \subset V$ such that:*

$$U \cap \operatorname{null} T = \{0\} \quad \textit{and} \quad \operatorname{range} T = \{\ Tu \mid u \in U\ \}$$

Proof. This is similar to the *isomorphism theorems* about groups! This can be done by the similar way we used in proving $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. □

The next two exercises remind me the categorical injective and surjective, let try them first!

Exercise. *For any $F \in \mathcal{L}(V, W)$, F is injective $\iff$ for any $S, T \in \mathcal{L}(U, V)$, $FS = FT$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(V, W)$ that $FS = FT$, then for any $u \in U$, we have $F(Su) = F(Tu)$, since F is injective, we know $Su = Tu$, so $S = T$.

- ($\Leftarrow$) For any $v, w \in V$ such that $Fv = Fw$. Consider

$$S(\lambda) = \lambda v$$
$$T(\lambda) = \lambda w$$

in $\mathcal{L}(\mathbb{R}, V)$. Then for any $\lambda \in \mathbb{R}$, we have $FS\lambda = \lambda FS1 = \lambda Fv = \lambda Fw = \lambda FT1 = FT\lambda$. so $FS = FT$ then $S = T$, which means $v = S1 = T1 = w$.

$\square$

Exercise. *Suppose W is finite, then for any $F \in \mathcal{L}(V, W)$, F is surjective $\iff$ for any $S, T \in \mathcal{L}(W, U)$, $SF = TF$ implies $S = T$.*

Proof.

- ($\Rightarrow$) For any $S, T \in \mathcal{L}(W, U)$ such that $SF = TF$. For any $w \in W$, there is $v \in V$ such that $Fv = w$ since F is surjective. Then we have $SFv = TFv$ so $Sw = S(Fv) = T(Fv) = Tw$ then $S = T$.

- ($\Leftarrow$) Consider

$$S = I \quad \text{and} \quad T(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 w_0 + \cdots + \lambda_k w_k$$

  where $w_0, \cdots, w_k$ is the basis of $\operatorname{range} F$ and $w_0, \cdots, w_n$ is the basis of W that expand from $w_0, \cdots, w_k$.

  (If we can use another way to construct T, then W is not need to be finite, for example, $W = \operatorname{range} T \oplus W_0$).

  It is easy to show that T is a linear transformation. Then for any $v \in V$, we have $TFv = Fv$ (since T acts like identity transformation on $\operatorname{range} F$) and $SFv = Fv$, so $S = T$ by the property of F. Since $\operatorname{range} S = W$, so is $\operatorname{range} T$, that means $w_0, \cdots, w_k$ spans W, so $k = n$, which means $\operatorname{range} F = W$, therefore F is surjective.

$\square$

Exercise 3.19. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is injective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $ST = I$.*

Proof.

- ($\Rightarrow$) Consider the basis $v_0, \cdots, v_n$ of V, then $Tv_0, \cdots, Tv_n$ is a basis of $\operatorname{range} T$ since T is injective. We denote Tv_i as w_i and $w_0, \cdots, w_m$ as the basis of W which expand from $w_0, \cdots, w_n$. Define $S(\lambda_0 w_0 + \cdots + \lambda_m w_m) = \lambda_0 w_0 + \cdots + \lambda_n w_n$, and then for any $v \in V$, $ST(\lambda_0 v_0 + \cdots + \lambda_n v_n) = S(\lambda_0 w_0 + \cdots + \lambda_n w_n) = \lambda_0 v_0 + \cdots + \lambda_n v_n$, so $ST = I$.

- ($\Leftarrow$) Suppose $A, B \in \mathcal{L}(U, V)$, such that $TA = TB$, we will show that $A = B$. $STA = IA = A$ and $STB = IB = B$ and $STA = STB$ since $TA = TB$. Then we know T is a monomorphism, and then T is injective.

$\square$

Exercise 3.20. *Suppose W is finite, then for any $T \in \mathcal{L}(V, W)$, show that T is surjective $\iff$ there is $S \in \mathcal{L}(W, V)$ such that $TS = I$.*

Proof. $\square$

Exercise 3.21. *Suppose V is finite, $T \in \mathcal{L}(V, W)$, $U \subseteq W$ a subspace. Show that the inverse image of U: $\{ v \in V \mid Tv \in U \}$ is a subspace of V, and*

$$\dim\{ v \in V \mid Tv \in U \} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T)$$

Proof. The second part is quite easy, we can restrict the domain of T to $\{ v \in V \mid Tv \in U \}$, say $T' \in \mathcal{L}(\{ v \in V \mid Tv \in U \}, W)$, so that it is in form $\dim\{ v \in V \mid Tv \in U \} = \dim \operatorname{null} T' + \dim \operatorname{range} T'$. Obviously $\operatorname{range} T' = U \cap \operatorname{range} T$ and $\operatorname{null} T' = \operatorname{null} T$.

We will now show that $\{ v \in V \mid Tv \in U \}$ is a subspace of V.

- $T0 \in U$.
- For any $v, w \in V$ such that $Tv, Tw \in U$, we have $T(v+w) = Tv + Tw \in U$.
- For any $v \in V$ such that $Tv \in U$ and $\lambda \in F$, we have $T(\lambda v) = \lambda Tv \in U$.

Therefore it is a subspace. $\square$

Exercise 3.22. *Suppose U and V are finite, $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$, show that*

$$\dim \operatorname{null} ST \le \dim \operatorname{null} S + \dim \operatorname{null} T$$

Proof. Consider the inverse image of $\operatorname{null} S$ on T: $K = \{ v \in V \mid Tv \in \operatorname{null} S \}$, which dimension: $\dim K = \dim \operatorname{null} T + \dim(\operatorname{null} S \cap \operatorname{range} T)$, where $\dim(\operatorname{null} S \cap \operatorname{range} T)$ caps at $\dim \operatorname{null} S$.

We know show that $\operatorname{null} ST = \operatorname{null} K$. For any $STv = 0$, we know $S(Tv) = 0$, so $Tv \in K$, therefore $\operatorname{null} ST \subseteq \operatorname{null} K$; For any $Tv \in \operatorname{null} S$, that means $S(Tv) = 0$, therefore $v \in \operatorname{null} ST$, therefore $\operatorname{null} ST \supseteq \operatorname{null} K$, and $\operatorname{null} ST = \operatorname{null} K$. $\square$

Exercise 3.25. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that* $\operatorname{null} S \subseteq \operatorname{null} T \iff$ *there is $E \in \mathcal{L}(W)$ such that $T = ES$.*

Proof. We define $E(S(v)) = Tv$ for any $v \in V$, so that $E \in \mathcal{L}(\operatorname{range} S, W)$. We first show that E is a mapping, and also a linear transformation.

Suppose $Sv, Sw \in W$ such that $Sv = Sw$, we need to show that $E(Sv) = E(Sw)$, or normalized $Tv = Tw$. We know $v - w \in \operatorname{null} S$ since $Sv = Sw$, so $v - w \in \operatorname{null} T$ since $\operatorname{null} S \subseteq \operatorname{null} T$, therefore $T(v - w) = 0$, and then $Tv = Tw$, so E is a mapping.

Now we show that E is a linear transformation.

- For any $Sv, Sw \in \operatorname{range} S$, $E(Sv) + E(Sw) = Tv + Tw = T(v + w) = E(S(v + w)) = E(Sv + Sw)$.

- For any $Sv \in \operatorname{range} S$ and $\lambda \in F$, $\lambda E(Sv) = \lambda Tv = T(\lambda v) = E(S(\lambda v) = E(\lambda Sv))$.

therefore E is a linear transformation.

Now we can expand the domain of E to W such that $E'v = Ev$ for any $v \in \operatorname{range} S$ (this is proven in previous exercise). For any $v \in V$, we have $ESv = E(Sv) = Tv$, there fore $T = ES$.

For another direction, for any $v \in \operatorname{null} S$, we have $ESv = E0 = 0 = Tv$, so $v \in \operatorname{null} T$. □

Exercise 3.26. *Suppose V is finite, $S, T \in \mathcal{L}(V, W)$. Show that* $\operatorname{range} S \subseteq \operatorname{range} T \iff$ *there is $E \in \mathcal{L}(V)$ such that $S = TE$.*

Proof. Consider the inverse image of $\operatorname{range} S$ with basis $w_0, \cdots, w_n$, say $v_0, \cdots, v_n$, it is easy to show $v_0, \cdots, v_n$ is linear independent. Then $E(v) = \lambda_0 v_0 + \cdots + \lambda_n v_n$ where $Sv = \lambda_0 w_0 + \cdots + \lambda_n w_n$. □

Exercise 3.27. *Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$, show that $V =$* $\operatorname{null} P \oplus \operatorname{range} P$.

Proof. Such element is called *idempotent* in algebra.

We will show $\operatorname{null} P \oplus \operatorname{range} P$ by showing $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. For any $v \in \operatorname{null} P \cap \operatorname{range} P$, we know there is $w \in V$ such that $Pw = v$ since $v \in rangevP$, then $P^2(v) = P(Pv) = P0 = 0$ since $v \in \operatorname{null} P$ and $P^2(v) = P(Pv) = Pw$, so $Pw = 0$ while $Pw = v$ therefore $v = 0$.

Then we have $\dim V = \dim \operatorname{null} P + \dim \operatorname{range} P$ and $\dim(\operatorname{null} P \oplus \operatorname{range} P) = \dim \operatorname{null} P + \dim \operatorname{range} P - \dim\{0\}$, so $V = \operatorname{null} P \oplus \operatorname{range} P$. □

Exercise 3.28. *Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ such that for any non-constant polynomial $p \in \mathcal{P}(\mathbb{R})$, $\deg(Dp) = \deg p - 1$. Show that D is surjective.*

Proof. We induction on n, starts from 1, to show that $D(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_{n-1}(\mathbb{R})$.

- Base: for any $p \in \mathcal{P}(\mathbb{R})$ where $\deg p = 1$, we know $\deg Dp = 0$, so $D(\mathcal{P}_1(\mathbb{R}))$ is a non-zero subspace of $\mathcal{P}_0(\mathbb{R})$, which is $\mathcal{P}_0(\mathbb{R})$.

- Induction: We have induction hypothesis: For any $i \leq n$, we have $D(\mathcal{P}_i(\mathbb{R})) = \mathcal{P}_{i-1}(\mathbb{R})$. We want to show that $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$. For any $p \in \mathcal{P}(\mathbb{R})$ with $\deg p = n+1$, we can write p in form of $p = \lambda x^{n+1} + r$ where $\deg r \leq n$, then $Dp = D(\lambda x^{n+1} + r) = D(\lambda x^{n+1}) + Dr$ where $\deg D(\lambda x^{n+1}) = n$ and $\deg Dr \leq n - 1$. So $D(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$ since: $\mathcal{P}_n(\mathbb{R}) \subseteq D(\mathcal{P}_{n+1}(\mathbb{R}))$ and $D(\lambda x^{n+1}) \in D(\mathcal{P}_{n+1}(\mathbb{R}))$, it is sufficient to span $\mathcal{P}_n(\mathbb{R})$.

□

Exercise 3.29. *For any $p \in \mathcal{P}(\mathbb{R})$, show that there is $q \in \mathcal{P}(\mathbb{R})$ such that $5q'' + 3q' = p$.*

Proof. We can rewrite the goal as $5DDq + 3Dq = p$ where $D(p) = p'$, then $5DDq + 3Dq = D(5Dq) + D(3q) = D(5Dq + 3q) = p$. We know D is surjective by the previous exercise, the goal is now showing that $5Dq + 3q = r$ where $Dr = p$. Then we continue rewrite the goal $5Dq + 3q = (5D)q + (3I)q = (5D + 3I)q = r$, we will show that $5D + 3I$ is surjective, we use the same method in previous exercise.

We denote $5D+3I$ by F, and induction on $n \in \mathbb{N}$ to show that $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$.

- Base: We should show that $F(\mathcal{P}_0(\mathbb{R})) = \mathcal{P}_0(\mathbb{R})$, for any $p \in \mathcal{P}_0(\mathbb{R})$, we have $Fp = 5Dp + 3p$, where $Dp = 0$ since $\deg p = 0$, so $Fp = 3p$, which means we have $1 \in F(\mathcal{P}_0(\mathbb{R}))$ since p is literally a number and $\frac{1}{3p} Fp = 1$.

- Induction: We have induction hypothesis: $F(\mathcal{P}_n(\mathbb{R})) = \mathcal{P}_n(\mathbb{R})$, and we want to show $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

  For any $p \in \mathcal{P}_{n+1}(\mathbb{R})$, we have $Fp = 5Dp + 3p$ where $\deg 5Dp = n$ and $\deg 3p = n + 1$, then we can eliminate $5Dp$ and every term in p with degree less then $n + 1$ since $\mathcal{P}_n(\mathbb{R}) \subseteq \operatorname{range} F$, then we get z^{n+1}, thus $F(\mathcal{P}_{n+1}(\mathbb{R})) = \mathcal{P}_{n+1}(\mathbb{R})$.

Therefore there is q such that $(5D+3I)q = r$ since $5D+3I$ is surjective.

Another solution from internet: Define $Tq = 5q'' + 3q'$, we can see for any $q \in \mathcal{P}(\mathbb{R})$ we have $\deg Tq = \deg q - 1$, so T is surjective. Then there is q such that $Tq = 5q'' + 3q' = p$. □

Exercise 3.30. *Suppose $\varphi \in \mathcal{L}(V, F)$ not zero, and $u \in V$ that $u \notin \operatorname{null}\varphi$, show that $V = \operatorname{null}\varphi \oplus \{\ au \mid a \in F\ \}$.*

Proof. We can see φ is surjective since $\varphi u \neq 0$, then for any $i \in F$, we have $(i(\varphi u)^{-1})\varphi u = i$.

For any $v \in V$, since φ is surjective (in a particular way), so we have $a\varphi u$ such that $a\varphi u = \varphi v$, then $\varphi(au - v) = 0$ so $au - v \in \operatorname{null}\varphi$. That means $(-1)(au - v) + au = v$ where $(-1)(au - v) \in \operatorname{null}\varphi$ and $au \in \{\ au \mid a \in F\ \}$, so $V = \operatorname{null}\varphi + \{\ au \mid a \in F\ \}$.

Then $\operatorname{null}\varphi \oplus \{\ au \mid a \in F\ \}$ since $u \notin \operatorname{null}\varphi$. □

Exercise 3.31. *Suppose V is finite ($\dim V > 1$), show that if $\varphi : \mathcal{L}(V) \to F$ is a linear mapping with property $\varphi(ST) = \varphi(S)\varphi(T)$ for any $S, T \in \mathcal{L}(V)$, show that $\varphi = 0$.*

Proof. Consider $\operatorname{null}\varphi$, since $\dim V > 1$ while $\dim F = 1$, so φ cannot be injective, therefore $\operatorname{null}\varphi \neq \{0\}$.

For any non-zero $S \in \operatorname{null}\varphi$ and $T \in \mathcal{L}(V)$, we have $\varphi(ST) = \varphi(S)\varphi(T) = 0 = \varphi(T)\varphi(S) = \varphi(TS)$ since $S \in \operatorname{null}\varphi$, thus $ST \in \operatorname{null}\varphi$. We show that $\operatorname{null}\varphi$ is an ideal of $\mathcal{L}(V)$, recall that the property of $\mathcal{L}(V)$, the only ideal of $\mathcal{L}(V)$ is $\{0\}$ and $LT(V)$, so $\operatorname{null}\varphi = \mathcal{L}(V)$, which means $\varphi = 0$. □

Exercise 3.32. *Let V, W are vector spaces and $T \in \mathcal{L}(V, W)$, define $T_C : V_C \to W_C$:*

$$T_C(u + iv) = Tu + iTv$$

for any $u, v \in V$.

1. *Show that T_C is a (complex) linear mapping from V_C to W_C.*
2. *Show that T_C is injective $\iff$ T is injective.*
3. *Show that $\operatorname{range} T_C = W_C \iff \operatorname{range} T = W$.*

Proof.

1. For any $u, v, s, t \in V$ $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
&T((u+iv)+(s+it)) \\
=&T(u+s+i(v+t)) \\
=&T(u+s)+iT(v+t) \\
=&Tu+Ts+iTv+iTt \\
=&T(u+iv)+T(s+it)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda T(u+iv) \\
=&\lambda(Tu+iTv) \\
=&\lambda Tu+\lambda iTv \\
=&T(\lambda u)+iT(\lambda v) \\
=&T(\lambda u+i(\lambda v)) \\
=&T(\lambda u+i\lambda v) \\
=&T(\lambda(u+iv))
\end{aligned}$$

I believe these are trivial, so the future me should be able to prove these without any effort. $\square$

Exercise 3.4. *Suppose $D(p) = p' : \mathcal{L}(\mathcal{P}_3(\mathbb{R})) \to \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$. Find a basis of $\mathcal{L}(\mathcal{P}_3(\mathbb{R}))$ and a basis of $\mathcal{L}(\mathcal{P}_2(\mathbb{R}))$, such that $\mathcal{M}(D)$ about these basis is:*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Proof. Consider $x, x^2, x^3, 1$ the basis of $\mathcal{P}_3(\mathbb{R})$ and $1, x, 2x^2$. $\square$

Exercise 3.5. *Suppose V and W are finite and $T \in \mathcal{L}(V, W)$. Show that there are basis of V and W respectively, such that $\mathcal{M}(T$, those basis$)$ is all zero except 1 at $k, k \quad (1 \leq k \leq \dim \operatorname{range} T)$.*

Proof. Consider the basis $w_0, \cdots, w_{k-1}$ of range T and the basis $w_0, \cdots, w_{m-1}$ of W which expands from $w_0, \cdots, w_{k-1}$. Then there must be $v_0, \cdots, v_{k-1}$ such that $Tv_i = w_i$ for all $0 \leq i < k$, we know $v_0, \cdots, v_{k-1}$ is linear independent since $w_0, \cdots, w_{k-1}$ is linear independent, so we can expand it to a basis of V, say $v_0, \cdots, v_{n-1}$.

We claim that $\mathcal{M}(T, v_0, \cdots, v_{n-1}, w_0, \cdots, w_{m-1})$ is a matrix with all zero but 1 at $k, k \quad (1 \leq k < \operatorname{range} T)$. For any $\lambda_0 v_0 + \cdots + \lambda_{n-1} v_{n-1} \in V$, we have $T(\lambda_0 v_0 + \cdots + \lambda_{n-1} v_{n-1}) = \lambda_0 w_0 + \cdots + \lambda_{k-1} w_{k-1}$, note that all v_i where $i \geq k$ disappear, since they maps to 0. Therefore $\mathcal{M}(T)$ is all zero but 1 at k, k (since $\lambda_i w_i$ in the last equation). $\square$

Exercise 3.6. *Show that* $-^T : F^{m,n} \to F^{n,m}$ *is a linear mapping.*

Proof. Trivial, sorry. $\square$

Exercise 3.7. *Show that* $(AB)^T = B^T A^T$.

Proof. Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix, then for any $i \in [1, m]$ and $j \in [1, p]$, we have $(AB)^T_{i,j} = (AB)_{j,i} = \sum_{r=1}^{n} A_{j,r} B_{r,i} = \sum_{r=1}^{n} B^T_{i,r} A^T_{r,j} = (B^T A^T)_{i,j}$. $\square$

Exercise 3.8. *Let A a $m \times n$ matrix, show that the rank of A is 1 $\iff$ there is $c_0, \cdots, c_{m-1} \in F^m$ and $d_0, \cdots, d_{n-1} \in F^n$ such that $A_{j,k} = c_j d_k$ for all $j = 0, \ldots, m-1$ and $k = 0, \ldots, n-1$.*

Proof. The right hand side is actually the external product of vectors, that is vw^T.

($\Rightarrow$) is easy since we can use the theorem that any $m \times n$ matrix A can be expressed by CR where C is a $m \times r$ matrix, R is a $r \times n$ matrix, r is the rank of A. In this case, $r = 1$, so C and R are just vectors.

($\Leftarrow$) is also easy since other column is a scalar multiple of the first column, therefore the rank of A is 1. $\square$

Exercise 3.9. *Let $T \in \mathcal{L}(V)$, $u_0, \cdots, u_{n-1}$ and $v_0, \cdots, v_{n-1}$ are the bases of V, show that the following statements are equivalent:*

1. *T is injective*
2. *The columns of $\mathcal{M}(T)$ is linear independent*
3. *The columns of $\mathcal{M}(T)$ spans $F^{n,1}$*
4. *The lines of $\mathcal{M}(T)$ is linear independent*
5. *The lines of $\mathcal{M}(T)$ spans $F^{1,n}$*

Proof. (2), (3) are obviously equivalent and (4), (5) too.

Although I want to make an arrow loop, but the arrow between (1) and (4), (5) is too hard, so I will show that (1) $\iff$ (2), (3) and (2), (3) $\iff$ (4), (5).

- ($\Rightarrow$) Let $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1} = [0, \dots, 0]$, then $T(\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}) = 0$, so λ_i are 0 since T is injective, which means $\operatorname{null} T = \{0\}$.

  ($\Leftarrow$) For any $T(\lambda_0 u_0 + \cdots + \lambda_{n-1} u_{n-1}) = 0$, we have the linear combination of v_i is 0 where the coefficients come from $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1}$ (w_i are the columns of $\mathcal{M}(T)$), therefore the coefficients are all 0 since v_i is linear independent, thus $\lambda_0 w_0 + \cdots + \lambda_{n-1} w_{n-1} = 0$, which means λ_i are all 0 since w_i is linear independent.

- For any matrix, its line rank is equal to its column rank, so columns independent $\iff$ lines independent.

□

Exercise 3.4. *Let V a finite vector space with* $\dim V > 1$*, show that* $S = \{ T \text{ is singular} \mid T \in \mathcal{L}(V) \}$ *is* ***NOT*** *a subspace of* $\mathcal{L}(V)$.

Proof. If S is a subspace of $\mathcal{L}(V)$, then it is an ideal of $\mathcal{L}(V)$ since for any $A \in S$ and $B \in \mathcal{L}(V)$, AB and BA are singular, therefore $AB, BA \in S$. However, we know the only two ideals of $\mathcal{L}(V)$ is $\{0\}$ and $\mathcal{L}(V)$, none of them is S. □

Exercise 3.11. *Let V finite vector space, and $S, T \in \mathcal{L}(V)$, show that*

$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible}$$

Proof.

- ($\Rightarrow$) Suppose $STW = WST = I$, then $S(TW) = (TW)S = I$ since $\dim V = \dim V$, therefore $S^{-1} = TW$, also $(WS)T = T(WS) = I$ since $\dim V = \dim V$, therefore $T^{-1} = WS$.

- ($\Leftarrow$) Trivial.

□

Exercise 3.12. *Let V finite vector space, and $S, T, U \in \mathcal{L}(V)$ such that $STU = I$, Show that $T^{-1} = US$.*

Proof. Since $STU = I$ we know U is invertible (since STU is invertible), then $ST = U^{-1}$. Since U^{-1} is invertible, we know S and T are invertible therefore $T = S^{-1}U^{-1}$ and $T^{-1} = US$. $\square$

Exercise 3.13. *Show that the conclusion of previous exercise can be false if V is not finite.*

Proof. Let $S(x_0, x_1, \dots) = (x_1, \dots)$ the backward-shift mapping and $U(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ the forward-shift mapping and $T = I$ the identity mapping.

We have $SU = I$ and $US \neq I$, T is clearly inveritble with $T^{-1} = I$, but we know $US \neq I$, so $T^{-1} = US \neq I$.

In fact, this also disprove the infinite version of 3.11 since SU is invertible but neither S nor U is invertible. $\square$

Exercise 3.17. *Let V a finite vector space, $S \in \mathcal{L}(V)$, define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}(T) = ST$, show that:*

1. $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$
2. $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$

Proof. Since $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$, we know $\dim \mathcal{L}(V) = \dim \operatorname{null} \mathcal{A} + \dim \operatorname{range} \mathcal{A}$, also, $\dim \mathcal{L}(V) = (\dim V)^2$ and $\dim V = \dim \operatorname{null} S + \dim \operatorname{range} S$. Therefore we have $\dim \operatorname{null} A + \dim \operatorname{range} A = (\dim V)(\dim \operatorname{null} S + \dim \operatorname{range} S)$, which means we only need to prove one of (1) and (2).

We will show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$. We found that $\dim \mathcal{L}(V, \operatorname{null} S) = (\dim V)(\dim \operatorname{null} S)$, so it would be nice if $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$. For any $T \in \operatorname{null} \mathcal{A}$, we have $ST = 0$, which means $\operatorname{range} T \subseteq \operatorname{null} S$, therefore $T \in \mathcal{L}(V, \operatorname{null} S)$. For any $T \in \mathcal{L}(V, \operatorname{null} S)$, we have $ST = 0$ since $\operatorname{range} T \subseteq \operatorname{null} S$, so $T \in \operatorname{null} \mathcal{A}$, therefore $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$, thus $\dim \operatorname{null} A = (\dim V)(\dim \operatorname{null} S)$. $\square$

Exercise 3.18. *Show that V and $\mathcal{L}(F, V)$ are isomorphic.*

Proof. This can be proven by $\dim V = \dim \mathcal{L}(F, V) = 1(\dim V)$, but we can find $\varphi(v) = x \mapsto xv$ an isomorphism. For any $T \in \mathcal{L}(F, V)$, T is determined by $T(1)$. $\square$

Exercise 3.1. *Let $T : V \to W$, the graph of T is a subset of $V \times W$ such that*

$$\textit{graph of } T = \{\ (v, Tv) \mid v \in V\ \}$$

.

Show that T is a linear mapping $\iff$ the graph of T is a subspace.

Proof.

- $(0, T0) \in$ graph of T. $(v, Tv) + (w + Tw) = (v + w, Tv + Tw) = (v + w, T(v + w))$. $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v))$
- $(v, Tv) + (w + Tw) = (v + w, T(v + w))$ since the graph of T is a subspace, therefore $Tv + Tw = T(v + w)$. Similarly, $\lambda Tv = T(\lambda v)$.

□

Exercise 3.3. *Let V_i are vector spaces, show that $\mathcal{L}(V_0 \times \cdots \times V_{m-1}, W) \simeq \mathcal{L}(V_0, W) \times \cdots \times \mathcal{L}(V_{m-1}, W)$.*

Proof. This can be proven by $A \times B$ is a categorical product, so we will show that for any A, B are vector spaces, $A \times B$ is a product.

In order to show that $A \times B$ is a product, or more specificly, $A \times B$ equipped with linear mappings

$$\pi_0(a, b) = a$$
$$\pi_1(a, b) = b$$

is a product, we have to show that for any C, $s \in \mathcal{L}(C, A)$ and $t \in \mathcal{L}(C, B)$, there is a unique $u \in \mathcal{L}(C, A \times B)$ such that $s = \pi_0 \circ u$ and $t = \pi_1 \circ u$.

Define $u(c) = (sc, tc) : C \to A \times B$, we will show that u is a linear mapping.

- For all $v, w \in C$, $u(v) + u(w) = (sv, tv) + (sw, tw) = (sv + sw, tv + tw) = (s(v + w), t(v + w)) = u(v + w)$
- For all $c \in C$ and $\lambda \in F$, $\lambda u(c) = \lambda(sc, tc) = (\lambda sc, \lambda tc) = (s(\lambda c), t(\lambda c)) = u(\lambda c)$.

Then we can see $\pi_0(u(c)) = \pi_0(sc, tc) = sc$ and $\pi_1(u(c)) = \pi_1(sc, tc) = tc$. Now we have to show that u is unique (which is trivial, I don't want to prove this, sorry). □

Exercise 3.5. *Let m a positive number, define $V^m = \underbrace{V \times \cdots \times V}_{m}$, show that $V^m \simeq \mathcal{L}(F^m, V)$.*

Proof. Define $\varphi(v_0, \cdots, v_{m-1}) = i_0, \cdots, i_{m-1} \mapsto i_0 v_0 + \cdots + i_{m-1} v_{m-1}$ which accept a list of vector and a list of coefficients then produce a linear combination.

For any $T \in \mathcal{L}(F^m, V)$, T is completely determined by $T(1, \cdots, 1) = v_0 + \cdots + v_{m-1}$, therefore $\varphi(v_0, \cdots, v_{m-1}) = T$ and thus φ is surjective.

For any $(v_0, \cdots, v_{m-1}), (w_0, \cdots, w_{m-1}) \in V^m$ such that $\varphi(v_0, \cdots, v_{m-1}) = \varphi(w_0, \cdots, w_{m-1})$, then $w_0 = \varphi(v_0, \cdots, v_{m-1})(1, 0, \cdots, 0) = \varphi(v_0, \cdots, v_{m-1})(1, 0, \cdots, 0) = v_0$, same for other v_i and w_i, so $(v_0, \cdots, v_{m-1}) = (w_0, \cdots, w_{m-1})$, therefore φ is injective. □

Exercise 3.6. *Let $v, x \in V$ and $U, W \subseteq V$ are subspaces such that $v + U = x + W$. Show that $U = W$.*

Proof. We know $v = x + w_0$ for some $w_0 \in W$ since $v + U = x + W$ and $v \in v + U$, then for any $u \in U$, we have $v + u = x + w$ for some $w \in W$, then $(x + w_0) + u = x + w$ therefore $u = x + w - x - w_0 = w - w_0 \in W$ thus $U \subseteq W$. Similarly $W \subseteq U$. □

Exercise 3.7. *Let $U = \{ (x, y, z) \in R^3 \mid 2x + 3y + 5z = 0 \}$ and $A \subseteq R^3$. Show that A is a translate of U (that is $A = a + U$) $\iff$ there is c such that $A = \{ (x, y, z) \in R^3 \mid 2x + 3y + 5z = c \}$.*

Proof.

- $(\Rightarrow)$ For any $(a_0, a_1, a_2) + (x, y, z) \in a + U$, we have $2(a_0 + x) + 3(a_1 + y) + 5(a_2 + z) = 2a_0 + 3a_1 + 5a_2$, therefore $c = 2a_0 + 3a_1 + 5a_2$.

- $(\Leftarrow)$ We can see 2, 3 and 5 are coprime to each other, therefore there is $2a_0 + 3a_1 + 5a_2 = 1$ (I am not sure if this is true in generalized case, I just extends the theorem "$as + bt = 1 \iff$ a coprime to b" to three elements case without checking), in this case we have $2(1) + 3(-2) + 5(1) = 1$, then for any $2x + 3y + 5z = c$, we have $2x + 3y + 5z = 2(ca_0) + 3(ca_1) + 5(ca_2)$, then $2(x - ca_0) + 3(y - ca_1) + 5(z - ca_2) = 0$, therefore $A = ((-c)(a_0, a_1, a_2)) + U$.

□

Exercise 3.8. *Let $T \in \mathcal{L}(V, W)$ and $c \in W$, show that $\{ v \in V \mid Tv = c \}$ is an empty set or a translate of* null T*. Then explain why the solutions of a system of linear equations is either an empty set or a translate of some subspace of F^n.*

Proof. Let $Ta = c$ for some $a \in V$, if no such a, then $\{ v \in V \mid Tv = c \} = \varnothing$. We claim $\{ v \in V \mid Tv = c \} = a + \operatorname{null} T$. For any $v \in V$ such that $Tv = c$, then $v = a+v-a$ and $T(v-a) = Tv-Ta = c-c = 0$, therefore $v-a \in \operatorname{null} T$, thus $v \in a + \operatorname{null} T$. In another direction, for any $a + v \in a + \operatorname{null} T$, we have $T(a + v) = Ta + Tv = c + 0 = c$. □

Exercise 3.9. *Let $A \subseteq V$ a non-empty subset. Show that A is a translate of some subspace of $V \iff \lambda v + (1 - \lambda)w \in A$ for any $v, w \in A$ and $\lambda \in F$.*

Proof.

- ($\Rightarrow$) Suppose $A = a + U$ for some subspace $U \subseteq V$.
- ($\Leftarrow$) Let $w \in A$, we will show that $(-w) + A$ is a subspace of V.

  For any $a - w, b - w \in (-w) + A$, we need to show that $a - w + b - w = (a + b - w) - w \in (-w) + A$ or equivalently $a + b - w \in A$. We found that the property $\lambda v + (1 - \lambda)w \in A$ gives us the ability to construct something like $v - w$. Since $2v + (1-2)w = 2v - w$, we just let $w = v + a$ then $2v - (v + a) = v - a$. Therefore, we let $\lambda = 2$, $v = a + b$ and $w = a + b + w$, and now $2(a + b) - (a + b + w) = a + b - w \in A$, so $a + b - w - w \in (-w) + A$.

  For any $a - w \in (-w) + A$ and $\lambda \in F$, we need to show that $\lambda(a - w) \in (-w) + A$. $\lambda(a - w) = \lambda a - \lambda w = \lambda a - (\lambda - 1)w - w$. We let $\lambda = (-1)(\lambda - 1) = (1 - \lambda)$, $v = w$ and $w = a$ in $\lambda v + (1 - \lambda w) \in A$, then $(1 - \lambda)w + (1 - (1 - \lambda))a = (-1)(\lambda - 1)w + \lambda a = \lambda a - (\lambda - 1)w \in A$, therefore $\lambda a - (\lambda - 1)w - w = \lambda a - \lambda w \in (-w) + A$.

  Therefore $(-w) + A$ is a subapce of V and $w + (-w) + A$ is a translate.

□

Exercise 3.10. *Let $A = a + U$ and $B = b + W$ where $a, b \in V$, $U, W \subseteq V$ are subspaces. Show that $A \cap B$ is either a translate of some subspace of V or an empty space.*

Proof. Suppose $A \cap B \neq \varnothing$, we claim that $A \cap B$ is a translate of $U \cap W$, more specifically, for any $a + u_0 = b + w_0 \in A \cap B$, we claim that $A \cap B = (a + u_0) + U \cap W$.

For any $u = w \in U \cap W$, we have $(a + u_0) + u = a + (u_0 + u) \in a + U$, similarly, we have $(b + w_0) + w = b + (w_0 + w) \in b + W$, therefore $(a + u_0) + (U \cap W)\ subseteq A \cap B$.

For any $a + u = b + w \in A \cap B$, we have $a + u - (a + u_0) = u - u_0 \in U$ and $b + w - (b + w_0) = w - w_0 \in W$, therefore $A \cap B \subseteq (a + u_0) + (U \cap W)$. □

Exercise 3.12. *Let $v_0, \cdots, v_{m-1} \in V$ and*

$$A = \{ \ \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1} \mid \lambda_i \in F \text{ and } \lambda_0 + \cdots + \lambda_i = 1 \ \}$$

1. *Show that A is a translate of a subspace of V.*
2. *If B a translate of a subspace of V such that $v_0, \cdots, v_{m-1} \in B$, show that $A \subseteq B$.*
3. *Base on (1), show that the dimension of such subspace is less then m.*

Proof.

- If A is a translate of a subspace of V, say B, then for any $a \in A$, we have $A = a + B$. Therefore $B = (-a) + A$, we may pick $a = v_0$, we find that for any $b \in B$, it is in form $(-1)(v_0) + \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1}$ where $\lambda_0 + \cdots + \lambda_{m-1} = 1$, which implies $(-1) + \lambda_0 + \cdots + \lambda_{m-1} = 0$. Then we claim $B = \{ \ \lambda_0 v_0 + \cdots + \lambda_{m-1} v_{m-1} \mid \lambda_0 + \cdots + \lambda_{m-1} = 0 \ \}$ is a subspace and $A = v_0 s + B$.
-

$\square$

Exercise 3.16. *Let $\varphi \in \mathcal{L}(V, F)$ where $\varphi \neq 0$, show that* $\dim(V/(\operatorname{null} \varphi)) = 1$.

Proof. For any non-zero $v + \operatorname{null} \varphi, w + \operatorname{null} \varphi \in V/(\operatorname{null} \varphi)$ (existence is guaranteed since $\varphi \neq 0$), since $\varphi(w) \in F$, then there is some λ such that $\lambda \varphi(w) = \varphi(v)$ cause $\varphi(v)$ and $\varphi(w)$ are non-zero, then $\varphi(\lambda w) = \varphi(v)$, which means $v + \operatorname{null} T = (\lambda w) + \operatorname{null} T$, therefore $\dim(V/\operatorname{null} \varphi)$ cause any two (non-zero) vectors are linear dependent. $\square$

Exercise 3.17. *Let $U \subseteq V$ a subspace such that* $\dim(V/U) = 1$. *Show that there is $\varphi \in \mathcal{L}(V, F)$ such that* $\operatorname{null} \varphi = U$.

Proof. We know there is an isomorphism $i \in \mathcal{L}(V/U, F)$ since $\dim(V/U) = \dim F = 1$, then $\varphi = i \circ \pi$ where $\pi \in \mathcal{L}(V, V/U)$. Since i is injective, $\operatorname{null} \varphi = \operatorname{null} \pi = U$. $\square$

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