Theorem 5.39. *Let $T \in \mathcal{L}(V)$ and $v_0, \cdots, v_{n-1}$ a basis of V, then the following statements are equivalent to each others.*

- *$\mathcal{M}(T)$ about $v_0, \cdots, v_{n-1}$ is upper-triangular matrix*
- *For any $k = 1, \cdots, n$, $\operatorname{span}(v_0, \cdots, v_{k-1})$ is invariant under T.*
- *For any $k = 1, \cdots, n$, $Tv_{k-1} \in \operatorname{span}(v_0, \cdots, v_{k-1})$.*

Proof.

- (1) $\Rightarrow$ (2) Induction on k. In breif, first k columns are in $\operatorname{span}(v_0, \cdots, v_{k-1})$, therefore $\operatorname{span}(v_0, \cdots, v_{k-1})$ is invariant under T.
- (2) $\Rightarrow$ (3) Trivial, $v_{k-1} \in \operatorname{span}(v_0, \cdots, v_{k-1})$ which is invariant under T.
- (3) $\Rightarrow$ (1) Basically the definition of upper-triangular matrix, $Tv_{k-1} \in \operatorname{span}(v_0, \cdots, v_{k-1})$ means the k-th column of $\mathcal{M}(T)$ consists of first k number (the coefficients of Tv_{k-1}) and 0s.

$\square$

Theorem 5.40. *Let $T \in \mathcal{L}(V)$ and $v_0, \cdots, v_{n-1}$ a basis of V, such that $\mathcal{M}(T)$ is upper-triangular matrix, and $\lambda_0, \cdots, \lambda_{n-1}$ are the numbers of its diagonal. Show that*

$$(T - \lambda_0 I) \cdots (T - \lambda_{n-1} I) = 0$$

Proof. All numbers since i of $(T - \lambda_i I)v$ are 0 if numbers after i of v are 0. Thus $(T - \lambda_0 I) \cdots (T - \lambda_{n-1} I)$ makes $n-1$-th number 0, then $n-2$-th number and so on. $\square$

Theorem 5.41. *Let $T \in \mathcal{L}(V)$ and $\mathcal{M}(T)$ about some basis of V is upper-triangular matrix. Show that the eigenvalues of T are the numbers in the diagonal.*

Proof. Let $v_0, \cdots, v_{n-1}$ a basis of V and $\mathcal{M}(T)$ about this basis is upper-triangular matrix. For λ_i where $i = 0, \cdots, n-1$, we will show that $T - \lambda_i I$ is not invertible.

We will see first i columns of $T - \lambda_i I$ is linear dependent, since they have at most $i - 1$ non-zero numbers while the list they form has length i.

Another part of proof follows the book. Let $q(z) = (z - \lambda_0) \cdots (z - \lambda_{n-1})$, then $q(T) = 0$ by 5.40, thus q is polynomial multiple of the minimal polynomial of T, thus any zero of the minimal polynomial of T is also a zero of q, which means it belongs to the list $\lambda_0, \cdots, \lambda_{n-1}$. $\square$

Theorem 5.44. *Let V finite and $T \in \mathcal{L}(V)$. Show that $\mathcal{M}(T)$ is upper-triangular matrix about some basis of $V \iff$ the minimal polynomial of T is in form of $(z-\lambda_0)\cdots(z-\lambda_{n-1})$ where $\lambda_i \in F$.*

Proof. This proof comes from the book.

The ($\Rightarrow$) part follows theorem 5.41, $(z-\lambda_0)\cdots(z-\lambda_{m-1})$ ($\lambda_0, \cdots, \lambda_{m-1}$ are the numbers in the diagonal) is polynomial multiple of the minimal polynomial of T, then the minimal polynomial of T must in a similar form.

For ($\Leftarrow$), we will induction on n.

- Base($n = 0$), the minimal polynomial of T is in form $(z-\lambda_0)$, thus $T = \lambda_0 I$.

- Ind($n = n+1$), the minimal polynomial of T is in form $p(z) = (z - \lambda_0)\cdots(z-\lambda_{n+1-1})$. Consider $T - \lambda_n I$, there is non-zero $v \in V$ such that $(T-\lambda_n I)v = 0$ since λ_n is a zero of p therefore an eigenvalue of T. We define $U = \text{range}(T-\lambda_n I)$, consider $q(z) = (z-\lambda_0)\cdots(z-\lambda_{n-1})$ and then $q(T\big|_U) = 0$, recall that $U = \text{range}(T-\lambda_n I)$, therefore for any $v \in U$, there is u such that $(T-\lambda_n I)u = v$. Thus $q(T\big|_U)u = q(T)(T-\lambda_n I)v = p(T)v = 0$ where $u = (T-\lambda_n I)v$ and $u, v \in U$. Thus the matrix of $T\big|_U$ is upper-triangular.

  Then we consider $u_0, \cdots, u_{k-1}$ a basis of U, we will expand $u_0, \cdots, u_{k-1}$ to a basis of V, say $u_0, \cdots, u_{k-1} v_0, \cdots, v_{m-1}$. Then for any v_i where i, we have $Tv_i = Tv_i - \lambda_n v_i + \lambda_n v_i = (T-\lambda_n I)v_i + \lambda_n v_i$, where $(T-\lambda_n)v_i \in U$ (recall the definition of U), thus $Tv_i \in \text{span}(u_0, \cdots, u_{k-1}, v_0, \cdots, v_i)$, then by 5.39, we know $\mathcal{M}(T)$ is an upper-triangular matrix about the basis $u_0, \cdots, u_{k-1}, v_0, \cdots, v_{m-1}$.

  One may confused that the "length" of u_i is greater than the size of $\mathcal{M}(T\big|_U) = \dim U$ (thus it won't be a square matrix but tall and thin), however, these two things are unrelated, a matrix only represents how to combine the basis, and doesn't care what the basis looks like.

$\square$