Theorem 5.62. *Let V finite, $T \in \mathcal{L}(V)$. Show that T is diagonalizable $\iff$ the minimal polymonial of T is in form of $(z-\lambda_0)\cdots(z-\lambda_{n-1})$ where $\lambda_0, \cdots, \lambda_{n-1} \in F$ are distinct to each others.*

Proof.

- ($\Rightarrow$) Let $p(z) = (z-\lambda_0)\cdots(z-\lambda_{n-1})$ where $\lambda_0, \cdots, \lambda_{n-1}$ are distinct numbers in the diagonal of $\mathcal{M}(T)$. Then $p(T)v_i = (T-\lambda_0)\cdots(T-\lambda_i)(T-\lambda_{i+1})\cdots(T-\lambda_{n-1})v_i$, we can see $(T-\lambda_k)v = (\lambda_i - \lambda_k)v \neq 0$, therefore $(T-\lambda_i)(\lambda v) = 0$ where $\lambda v = (T-\lambda_{i+1})\cdots(T-\lambda_{n-1})v_i$. In breif, $T-\lambda_k$ eliminates all vector with eigenvalue λ_k in basis in vector $v = \lambda_0 v_0 + \cdots + \lambda_{n-1}v_{n-1}$. therefore $p(T)$ eliminates all v_k of a vector $v \in V$, thus $p(T) = 0$

  $p(T)v = p(T)(\lambda_0 v_0 + \cdots + \lambda_{n-1}v_{n-1}) = \lambda_0 p(T)v_0 + \cdots + \lambda_{n-1}p(T)v_{n-1} = 0 + \cdots + 0 = 0.$

- ($\Leftarrow$) Induction on n.

  Base($n=1$): $p(T) = (T - \lambda_0 I)$, then $T = \lambda_0 I$, which is diagonal matrix.

  Ind($n = n+1$): Consider $T - \lambda_{n+1-1}$ and define $U = \operatorname{range}(T - \lambda_n)$. It is easy to verify that U is invariant under T. Then $q(T\big|_U) = (T-\lambda_0)\cdots(T-\lambda_{n-1}) = 0$, therefore U is diagonalizable. Thus $U = E(\lambda_0, T\big|_U) \oplus \cdots \oplus E(\lambda_{n-1}, T\big|_U)$, note that $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim E(\lambda_n, T) + \dim U$, thus we need to show that $U + E(\lambda_n, T)$ is a direct sum, which is immediately holds by the property of eigenspace. Thus $V = U \oplus E(\lambda_n), T$, which is a direct sum of eigenspace while $\lambda_0, \cdots, \lambda_n$ are all distinct eigenvalue of T (since they are zeros of the minimal polymonial of T).

$\square$