1 Pullback

Theorem 1.1. *Suppose we have two joined commuting squares like:*

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\ X \arrow[r, "h"] & Y \arrow[r, "j"] & Z \end{tikzcd}$$

Then:

1. *The outer rectangle is a pullback square if two inner squares are pullback squares.*

2. *The inner-left square is a pullback square if the ouer rectangle and the inner-right square are pullback squares.*

Proof.

1. For any (A, a, b) such that $j \circ h \circ a = n \circ b$, then there is a unique $u : A \to M$ such that $h \circ a = m \circ u$ and $b = g \circ u$. Then there is a unique $v : A \to L$ such that $l \circ a = v$ and $f \circ v = u$, which makes (A, a, b) against to the outer rectangle commutes.

$$\begin{tikzcd} A \arrow[dr, dashed, "v"] \arrow[drr, dashed, "u"] \arrow[drrr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & & \\ & L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\ & X \arrow[r, "h"] & Y \arrow[r, "j"] & Z \end{tikzcd}$$

2. For any (A, a, b) such that $h \circ a = m \circ b$, consider the inner-right pullback,

then we have a unique $u : A \to M$ such that the diagram commutes:

$$\begin{tikzcd}
A \arrow[ddr, bend right, "a"'] \arrow[drr, bend left, "b"] \arrow[drrr, bend left, "g\circ b"] \arrow[drr, dashed, "u"] & & & \\
& L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\
& X \arrow[r, "h"] & Y \arrow[r, "j"] & Z
\end{tikzcd}$$

However, if we replace u with b, we have $g \circ b = g \circ b$ and $h \circ a = m \circ b$, that means b can do u's job, but we know u is unique, so $b = u$. Now consider the outer pullback, we have a unique $v : A \to L$ such that the diagram commutes:

$$\begin{tikzcd}
A \arrow[ddr, bend right, "a"'] \arrow[dr, dashed, "v"] \arrow[drr, dashed, "u=b"] \arrow[drrr, bend left, "g\circ b"] & & & \\
& L \arrow[r, "f"] \arrow[d, "l"] & M \arrow[r, "g"] \arrow[d, "m"] & N \arrow[d, "n"] \\
& X \arrow[r, "h"] & Y \arrow[r, "j"] & Z
\end{tikzcd}$$

That is, $l \circ v = a$ and $g \circ f \circ v = g \circ b$, we claim that v is the unique factorization from $(A, a, u = b)$ to (L, l, f). It is obvious that $l \circ v = a$, we need to show $f \circ v = u = b$. We may use the trick we just used, we can see that $g \circ f \circ v = g \circ u$ and $m \circ f \circ v = h \circ l \circ v = h \circ a$. So $f \circ v$ can do b's job, so $f \circ v = b$.

For any arrow $w : A \to L$ such that $l \circ a = w$ and $f \circ w = b$, then we have also $g \circ f \circ w = g \circ b$, which implies w is the unique arrow from $A \to L$ such that the outer diagram commutes, so $w = v$.

$\square$

Theorem 1.2. *A pullback square for the corner $D_1 \to V \leftarrow D_2$ is a product of $D_1 \to V$ and $V \leftarrow D_2$ in the slice category $\mathcal{C}/V$.*

Proof. Suppose (L, f, g) is the pullback of such corner, then we first need to show that there is an arrow $l : L \to V$ such that $s \circ f = l$ (therefore a morphism from (L, l) to (D_1, s)) and $t \circ g = l$ (a morphism from (L, l) to (D_2, t)).

Since (L, f, g) makes the pullback square commutes, we know $s \circ f = t \circ g$, therefore we let $l = s \circ f$ (or equivalently $t \circ g$).

We need to show that $((L, l), f, g)$ forms a product of (D_1, s) and (D_2, t), consider any $((A, a), b, c)$ where $a : A \to V$ such that $s \circ b = a$ and $t \circ c = a$. Just like l for L, a is redundant, so we may omit it. Now, the diagram looks like:

$$\begin{tikzcd}
A \arrow[dr, dashed, "!u"] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "b"'] \arrow[ddrr, "a"] & & \\
& L \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, "l"] & D_2 \arrow[d, "t"] \\
& D_1 \arrow[r, "s"] & V
\end{tikzcd}$$

Since (L, f, g) is a pullback, we know there is a unique $u : A \to L$ such that two triangle commutes. However, we must first show that u is an arrow from (A, a) to (L, l), that is, $l \circ u = a$. It is easy to see that $l \circ u = s \circ f \circ u = s \circ b = a$. □

Theorem 1.3. *if a category has all binary products and all equalizers for every pair of parallel arrows, then it has a pullback for any corners.*

Proof. Suppose $X \to Z \leftarrow Y$ a corner, then consider the product $X \times Y$:

$$\begin{tikzcd}
X \times Y \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & Y \arrow[d, "t"] \\
X \arrow[r, "s"] & Z
\end{tikzcd}$$

Now, consider the equalizer for the parallel arrows $t \circ \pi_1$ and $s \circ \pi_0$:

$$\begin{tikzcd}
E \arrow[r, "e"] & X \times Y \arrow[r, shift left, "s \circ \pi_0"] \arrow[r, shift right, "t \circ \pi_1"'] & Z
\end{tikzcd}$$

We claim $(E, \pi_0 \circ e, \pi_1 \circ e)$ is a pullback of such corner. For any (F, f, g) such that the outer diagram commutes:

$$
\begin{tikzcd}
F \arrow[rrrd, bend left, "b"] \arrow[dddr, bend right, "a"'] & & & \\
& E \arrow[rr, "\pi_1 \circ e"] \arrow[dd, "\pi_0 \circ e"'] \arrow[dr, dashed, "e"] & & Y \arrow[dd, "t"] \\
& & X \times Y \arrow[ur, "\pi_1"] \arrow[dl, "\pi_0"] & \\
& X \arrow[rr, "s"] & & Z
\end{tikzcd}
$$

it is easy to see that there is a unique arrow $u : F \to X \times Y$ such that $\pi_0 \circ u = a$ and $\pi_1 \circ u = b$ since $X \times Y$ is a product. Then there is another unique arrow $v : F \to E$ such that $e \circ v = u$ since E is a equalizer.

$$
\begin{tikzcd}
F \arrow[rrrd, bend left, "b"] \arrow[dddr, bend right, "a"'] \arrow[dr, dashed, "v"] \arrow[ddrr, dashed, bend left, "u"] & & & \\
& E \arrow[rr, "\pi_1 \circ e"] \arrow[dd, "\pi_0 \circ e"'] \arrow[dr, dashed, "e"] & & Y \arrow[dd, "t"] \\
& & X \times Y \arrow[ur, "\pi_1"] \arrow[dl, "\pi_0"] & \\
& X \arrow[rr, "s"] & & Z
\end{tikzcd}
$$

Obviously, (commute) $\pi_0 \circ e \circ v = \pi_0 \circ u = a$ and $\pi_1 \circ e \circ v = \pi_1 \circ u = b$. (unique) If an arrow $w : F \to E$ can do the job, then $e \circ w : F \to X \times Y$ is another factorization from F to the product $X \times Y$, so $e \circ w = u$, but that means w is also a factorization from F to the equalizer E, which means $v = w$.

So $(E, \pi_0 \circ e, \pi_1 \circ e)$ is a pullback of such corner. $\square$

Theorem 1.4. *If a category has a terminal object and has a pullback for every corner, then it has all binary product.*

Theorem 1.5. *If a category has a terminal object and has a pullback for every corner, then it has a equalizer for every parallel arrwos.*

Proof. Suppose $s, t : X \to Y$ are parallel arrows, then the following diagram commutes:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\langle s,t \rangle"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\
& Y \times Y \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & Y \\
& Y &
\end{tikzcd}$$

Note that we have $Y \times Y$ since this category has all binary products. Then consider this corner:

$$\begin{tikzcd}
& Y \arrow[d, "\langle 1_Y, 1_Y \rangle"'] \\
X \arrow[r, "\langle s,t \rangle"] & Y \times Y
\end{tikzcd}$$

We have an object E, $e : E \to X$ and $f : E \to Y$ such that the square commutes:

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "e"'] & Y \arrow[d, "\langle 1_Y, 1_Y \rangle"'] \\
X \arrow[r, "\langle s,t \rangle"] & Y \times Y
\end{tikzcd}$$

(Proof comes from textbook until here)

We can see that $\pi_0 \circ \langle s, t \rangle \circ e = s \circ e$ while $\pi_0 \circ \langle 1_Y, 1_Y \rangle \circ f = 1_Y \circ f = f$, therefore $s \circ e = f$, similarly $t \circ e = f$, so $s \circ e = t \circ e$. We claim E is the equalizer for the parallel arrows $s, t : X \to Y$. For any (F, e') such that $s \circ e' = t \circ e'$, then we have a unique arrow $u : F \to E$ such that this diagram commutes:

$$
\begin{tikzcd}
F \arrow[dr, dashed, "u"] \arrow[drr, bend left, "s\circ e'=t\circ e'"] \arrow[ddr, bend right, "e'"'] & & \\
& E \arrow[r, "f"] \arrow[d, "e"'] & Y \arrow[d, "\langle 1_Y,1_Y\rangle"'] \\
& X \arrow[r, "\langle s,t\rangle"] & Y\times Y
\end{tikzcd}
$$

where $e \circ u = e'$. Suppose $v : F \to E$ where $e \circ v = e'$, then $f \circ v = s \circ e \circ v = s \circ e'$. $\square$