1 Adjoint

Theorem 1.1. *Show that the hom-functor preserves limit, that is, for any $Y \in \mathcal{C}$ and diagram $\mathcal{D}$, we have:*

$$\mathrm{Lim}(\mathcal{C}(Y, \mathcal{D}_-)) \cong \mathcal{C}(Y, \mathrm{Lim}\,\mathcal{D})$$

Proof. The idea comes from *The Dao of FP* and nlab.

We may consider the cone with singleton set as vertex:

$$\begin{tikzcd}
 & 1 \arrow[d, "u"] \arrow[ddl, bend right, "\mathrm{const}_{p_i}"'] \arrow[ddr, bend left, "\mathrm{const}_{p_j}"] & \\
 & \mathrm{Lim}(\mathcal{C}(Y, \mathcal{D}_-)) \arrow[dl] \arrow[dr] & \\
\mathcal{C}(Y, \mathcal{D}_i) \arrow[rr, "\mathcal{C}(Y, \mathcal{D}_f)"] & & \mathcal{C}(Y, \mathcal{D}_j)
\end{tikzcd}$$

where const_{p_i} is the function that takes a morphism $p_i \in \mathcal{C}(Y, \mathcal{D}_i)$.

We know there is a one-to-one corresponding between u and the pair $\langle \mathrm{const}_{p_i}, \mathrm{const}_{p_j} \rangle$:

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, \mathcal{C}(Y, \mathcal{D}_-)) \cong \mathbf{Set}(1, \mathrm{Lim}(\mathcal{C}(Y, \mathcal{D}_-)))$$

or we can simplify the equation by defining $F_j = \mathcal{C}(Y, \mathcal{D}_j) : \mathcal{J} \to \mathbf{Set}$.

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, F) \cong \mathbf{Set}(1, \mathrm{Lim}\,F)$$

We may recall that the hom-set maps out from 1 is isomorphic to the target, that is:

$$\mathbf{Set}(1, \mathrm{Lim}\,F) \cong \mathrm{Lim}\,F$$

Similarly, the cone with 1 as vertex is a pair of selection of $\mathcal{C}(Y, \mathcal{D}_i)$, it forms a cone with Y as vertex:

$$\begin{tikzcd}
 & Y \arrow[dl, bend right, "p_i"'] \arrow[dr, bend left, "p_j"] & \\
\mathcal{D}_i \arrow[rr, "\mathcal{D}_f"] & & \mathcal{D}_j
\end{tikzcd}$$

the diagram is indeed commute since

$$\begin{tikzcd}
 & 1 \arrow[dl, bend right, "\mathrm{const}_{p_i}"'] \arrow[dr, bend left, "\mathrm{const}_{p_j}"] & \\
\mathcal{C}(Y, \mathcal{D}_i) \arrow[rr, "\mathcal{C}(Y, \mathcal{D}_f)"] & & \mathcal{C}(Y, \mathcal{D}_j)
\end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd}
 & 1 \arrow[dl, mapsto, bend right, "\mathrm{const}_{p_i}"'] \arrow[dr, mapsto, bend left, "\mathrm{const}_{p_j}"] & \\
p_i \arrow[rr, mapsto, "\mathcal{D}_f \circ -"] & & p_j
\end{tikzcd}$$

Also we can make a pair of selection of $\mathcal{C}(Y, \mathcal{D}_-)$ from a cone with Y as vertex.

Then the cone of $\mathcal{C}(Y, \mathcal{D}_-)$ with vertex 1, is isomorphic to the cone of $\mathcal{D}_-$ with vertex Y:

$$[\mathcal{J}, \mathbf{Set}](\Delta_1, F) \cong [\mathcal{J}, \mathcal{C}](\Delta_Y, D)$$

While the later one is naturally isomorphic to the limit of $\mathcal{J}_\bullet$:

$$[\mathcal{J}, \mathcal{C}](\Delta_Y, D) \cong \mathcal{C}(Y, \operatorname{Lim} D)$$

Finally, we have:

$$\begin{gathered}
\operatorname{Lim} \mathcal{C}(Y, \mathcal{D}_-) \\
\cong \\
\mathbf{Set}(1, \operatorname{Lim}(\mathcal{C}(Y, \mathcal{D}_-))) \\
\cong \\
[\mathcal{J}, \mathbf{Set}](\Delta_1, \mathcal{C}(Y, \mathcal{D}_-)) \\
\cong \\
[\mathcal{J}, \mathcal{C}](\Delta_Y, D) \\
\cong \\
\mathcal{C}(Y, \operatorname{Lim} D)
\end{gathered}$$

$\square$

Dually, we also have:

$$\mathrm{Lim}(\mathcal{C}(\mathcal{D}_{-}, Y)) \cong \mathcal{C}(\mathrm{Colim}\,\mathcal{D}, Y)$$

1.1 Unit and Counit

Suppose $L \dashv R$, that is, for any $a \in \mathcal{D}$ and $b \in \mathcal{C}$, we have:

$$\mathcal{C}(La, b) \cong \mathcal{D}(a, Rb)$$

If you keep Yoneda trick in mind, you may trying to replace b by La or a by Rb, which produce two morphism:

$$\eta_a : a \to R(La)$$
$$\epsilon_b : L(Rb) \to b$$

or we can regard them as natural transformations:

$$\eta : \mathrm{Id} \to R \circ L$$
$$\epsilon : L \circ R \to \mathrm{Id}$$

1.2 Back to Adjoint

What condition those unit and counit should hold such that we can construct a adjoint back? Let's see how we can construct a adjoint from unit and counit, suppose $f : La \to b$ a morphism, we need to provide a morphism $a \to Rb$. One obvious way is:

$$a \xrightarrow{\eta_a} RLa \xrightarrow{Rf} Rb$$

Similarly, for given $g : a \to Rb$, we have:

$$La \xrightarrow{Lg} LRb \xrightarrow{\epsilon_b} b$$

Since the adjoint is an isomorphism between hom-set, so if we send the morphism we mapped from f back to $\mathcal{C}(La, b)$, it should still be f:

$$\begin{tikzcd}
La \arrow[r, "f"] \arrow[d, "L(\eta_a)"'] & b \\
L(RLa) \arrow[r, "LRf"'] & L(Rb) \arrow[u, "\epsilon_b"']
\end{tikzcd}$$

or in symbol:

$$\epsilon_b \circ LRf \circ L\eta_a = f$$

Since ϵ is a natural transformation, we have this diagram commutes:

$$\begin{tikzcd}
LR(La) \arrow[r, "LR(f)"] \arrow[d, "\epsilon_{La}"'] & LR(b) \arrow[d, "\epsilon_b"] \\
La \arrow[r, "f"'] & b
\end{tikzcd}$$

Then we can rewrite the equation:

$$f \circ \epsilon_{La} \circ L\eta_a = f$$

or

$$f \circ (\epsilon \cdot L)_a \circ (L \cdot \eta)_a = f$$