Definition 6.2. *A **inner product** of a vector space V is a function that maps $u, v \in V$ to $\langle u, v\rangle \in F$, and it satisfies:*

- *Positivity:* $\langle v, v\rangle \geq 0$.
- *Definiteness:* $\langle v, v\rangle = 0 \iff v = 0$.
- *Additivity:* $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
- *Homogeneity:* $\langle \lambda v, w\rangle = \lambda\langle v, w\rangle$.
- *Conjugate Symmetry:* $\langle u, v\rangle = \overline{\langle v, u\rangle}$

Definition 6.4. *A vector space equipped with an inner product is called an **inner product space**.*

We assume vector spaces V, W are inner product space for the rest of chapter.

Theorem 6.6. *Properties of inner product:*

- *Let $v \in V$, then $\langle -, v\rangle$ is a linear map $V \to F$.*
- *For any $v \in V$, we have $\langle 0, v\rangle = \langle v, 0\rangle = 0$.*
- *For any $u, v, w \in V$, $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$.*
- *For any $v, w \in V$ and $\lambda \in F$, $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$.*

Proof.

- Trivial from the definition.
- First, $\langle 0, v\rangle = 0$ since $\langle -, v\rangle$ is a linear map, thus maps 0 to 0. Then $\langle v, 0\rangle = \overline{\langle 0, v\rangle} = \overline{0} = 0$.
- $\langle u, v + w\rangle = \overline{\langle v + w, u\rangle} = \overline{\langle v, u\rangle + \langle w, u\rangle} = \overline{\langle v, u\rangle} + \overline{\langle w, u\rangle} = \langle u, v\rangle + \langle u, w\rangle$.
- $\langle u, \lambda v\rangle = \overline{\langle \lambda v, u\rangle} = \overline{\lambda\langle v, u\rangle} = \overline{\lambda}\overline{\langle v, u\rangle} = \overline{\lambda}\langle u, v\rangle$

$\square$

Definition 6.7. *For any $v \in V$, the **norm** of v is denoted by $\|v\|$, and is defined by:*

$$\|v\| = \sqrt{\langle v, v\rangle}$$

Theorem 6.9. *Let $v \in V$,*

- $\|v\| = 0 \iff v = 0.$
- *For any $\lambda \in F$, $\|\lambda v\| = |\lambda| \, \|v\|$.*

Proof.

- Trivial by the definition of inner product.
- $\|\lambda v\| = \langle \lambda v, \lambda v\rangle = \sqrt{\lambda\overline{\lambda}}\langle v, v\rangle = |\lambda| \, \|v\|$.

□

Definition 6.10. *Let $u, v \in V$, u and v are **orthogonal** $\iff \langle u, v\rangle = 0$*

Theorem 6.11.

- 0 *is orthogonal to any $v \in V$.*
- 0 *is the only vector that orthogonal to itself.*

Proof. Both trivial by the definition, (1) is equivalent to $\langle 0, v\rangle = 0$ and (2) is equivalent to $\langle v, v\rangle = 0 \iff v = 0$. □

Theorem 6.12 (勾股定理)**.** *Let $u, v \in V$, if u is orthogonal to v, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.*

Proof.

$$\begin{aligned}
\|u + w\|^2 &= \langle u + w, u + w\rangle \\
&= \langle u, u + w\rangle + \langle w, u + w\rangle \\
&= \langle u, u\rangle + \langle u, w\rangle + \langle w, u\rangle + \langle w, w\rangle \\
&= \langle u, u\rangle + \langle w, w\rangle
\end{aligned}$$

The last equation is by $\langle u, w\rangle = \langle w, u\rangle = 0$ cause u, w are orthogonal. □

Theorem 6.13 (One Orthogonal Factorization)**.** *Let $u, v \in V$ and $v \neq 0$. Let $c = \dfrac{\langle u, v\rangle}{\|v\|^2}$ and $w = u - cv$, then $u = cv + w$ and $w \perp v$.*

Proof. 嗯算。 □