Implementation of TLC (Tiny Lambda Calculus)

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Contents

2015 级弘毅班编译原理课程设计第 6 次编程作业 (the typing of TLC)

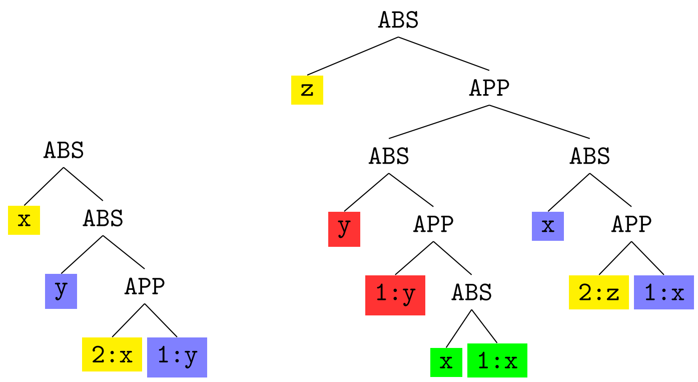
1 Introduction

Our goal is the effective implementation of the programming language TLC (Tiny Lambda Calculus) by using the closure.
Lambda calculus is a formal system in mathematical logic and computer science for expressing computation by way of variable binding and substitution (see `https://en.wikipedia.org/wiki/Lambda_calculus`).
It is computation model of Functional Programming (see L. Paulson's lecture lambda.pdf).

1.1 Specification of the language LAMBDA

the syntax of TLC can be described as:

```
lines : lines decl
      | decl
      ;
decl : LET ID '=' expr ';'
     | expr ';'
```



(a) `@x.@y.x y` (b) `@z.(@y.y(@x.x))(@x.z x)`

Figure 1: AST with binding deepth (the first number of ID node)

```
      ;
expr  : INT
      | ID
      | IF expr THEN expr ELSE expr FI
      | '(' expr ')'
      | '@' ID '.' expr
      | expr expr
      ;
```

where `@x.M` is the abstraction (instead of "λ" in lambda calculus for input). `M N` is the application. and the conditional construct is specially added for the lazy evaluation of the conditional lambda terms. the application is left associative. and the precedence from low to high is: conditional construct, abstraction and application.
see `lexer.l` and `grammar.y` in detail.

1.2 Abstract syntax trees

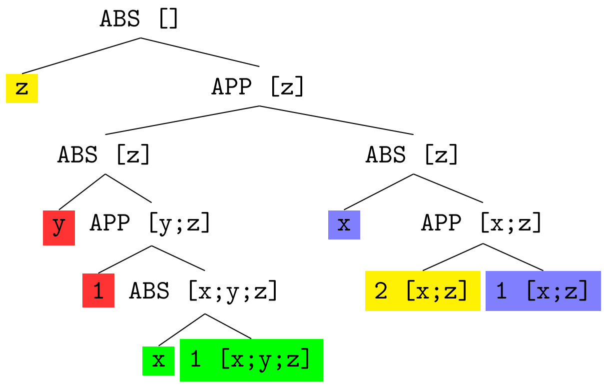
We use De Brujin index for the AST, it will replace the binding variable by the *binding depth*. Ex. `@x.@y.x` is `@x.@y.2`, `@z.(@y.y(@x.x))(@x.z x)` is `@z.(@y.1(@x.1))(@x.2 1)` (see Figure 1). It will be the key to access the closure environment in the implementation. the free occurence of variable is strictly forbidden in TLC.

```
typedef enum {CONST=1, VAR=2, COND=3, ABS=4, APP=5} Node_kind;

typedef struct Ast {
  Node_kind kind;
  int value; /* for CONST and De Brujin index */
  struct Ast  *lchild, /*  for variable name and
                           abstraction variable
                         & apply function  body*/
    *rchild; /* for abstraction body  and app argument*/
  struct Ast *cond; /* for condition */
} AST;
```

1.3 Binding Deepth

to find the binding deepth, we use the static stack `char *name_env[MAX_ENV]` with the cursor `int current` (`tree.c`) to store the abstraction level. each time enter AST with `ABS` node, we push the abstraction name in the stack, increase `current` for the next, and popup by decreasing `current` after leave the abstraction body. each time a variable encountered in the abstraction body, `find_deepth()` will return the number of the deepth in stack when first occurrence is found, see Figure 2.



(a) `@z.(@y.y(@x.x))(@x.z x)`

Figure 2: Binding deepth

```
int find_deepth(char *name)
{
  int i = current - 1;
  while (i + 1) {
    if (strcmp(name, name_env[i]) == 0) return current - i ;
    i--;
  }
  printf("id %s is unbound!\n", name);
  exit (1);
}
```

1.4 Primitive operations

`char *name_env[]` will also store the name of the declaration. so when the following statement is parsed:

```
let I = @x.x;
```

I will stored in `name_env[current]`. and we also store the AST of `@x.x` in the global `AST *ast_env[MAX_ENV]` (all defined in `grammar.y`) for the further uses (typing).
to support the arithmetic operations, `name_env[]` is prestored the following prefined functions:

```
char *name_env[MAX_ENV] = {"+", "-", "*", "/", "=", "<"};
```

to the above binary operators work correctly in λ-calculus, its should interpret as `@x.@y.op x y`, that is prefix notations! so we will write `+ (* 2 3) 4` instead of `2 * 3 + 4`.
the binding deepth is also the key to access the function defined in the declaration. so when `I` is declared, the `name_env[]` and `ast_env[]` will be

```
name_env[MAX_ENV] = {"+", "-", "*", "/", "=", "<", "I"}
ast_env[MAX_ENV] = {NULL, NULL, NULL, NULL, NULL, NULL, @x.(1:x)}
/* AST of operators is not needed for typing */
```

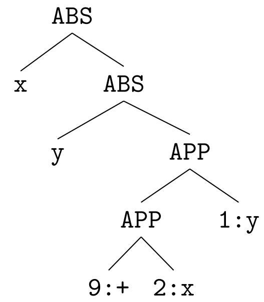
if we declare `PLUS` by input:

```
let PLUS = @x.@y. + x y;
```

the parser will generate the `(@x.(@y.(((+:9)(x:2))(y:1))))`. see Figure 3. In fact, after the parser enter the abstraction body `+ x y`, `name_env[]` will be:

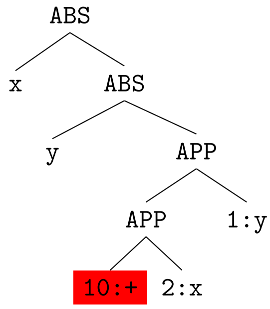
```
name_env[MAX_ENV] = {"+", "-", "*", "/", "=", "<", "I", "x", "y"}
```

so `find_deepth("+")` will return `9`, `find_deepth("x") = 2`, and `find_deepth("y") = 1`. after finish parsing, `name_env[]` changed to:



(a) @x.@y.+ x y

Figure 3: AST of PLUS



(a) @x.@y.+ x y

Figure 4: AST of PLUS2

```
name_env[MAX_ENV] = {"+", "-", "*", "/", "=", "<", "I", "PLUS"}
```

if we continue define PLUS2 by input:

```
let PLUS = @x.@y + x 2;
```

the parser will generate the (@x.(@y.(((+:10)(x:2))(y:1)))). please remark that the binding deepth of + changed to 10 (see Figure 4). this is because the parsing of PLUS2 is based with the new stack top "PLUS" of name_env[], the the relative place of "+" is increased by 1.
after PLUS2, name_env[] changed to:

```
name_env[MAX_ENV] = {"+", "-", "*", "/", "=", "<", "I", "PLUS", "PLUS2"}
```

the operators "+", "-", ... must be scanned as normal ID with their binding deepth. but "=" is also used as a single character token in the declaration like "let I = ...". we use a global int is_decl (defined in grammar.y) to tell the lexer if "=" should return '=' or ID, and add a middle action in the decl production to active is_decl:

```
decl : LET {is_decl = 1; } ID '=' expr ';' {...}
```

deactive each time before return '=' in lexer.l:

```
"=" {
      char *id;
      if (is_decl) {is_decl = 0; return '=';}
      id = (char *) smalloc(yyleng + 1);
      strcpy(id, yytext);
      yylval = make_string(id);
      return ID;
}
```

1.5 output

We use the LaTeX graphic system tikz/pgf (`https://sourceforge.net/projects/pgf/`) and tikz-qtree (`https://ctan.org/pkg/tikz-qtree`) to illustrate AST. `printtree(AST *)` transforms the AST to LaTeX commands and store it in the file `expr.tex` which is the included file of `exptree.tex`. "`pdflatex exptree.tex`" generates the pdf of the AST (see exptree.pdf).

2 Typing

Well-typed programs cannot go wrong — Robin Milner

(see `https://en.wikipedia.org/wiki/Type_safety`)

2.1 Syntax-directed type synthesiser

As we know, if we admit the "`x x`" in lambda term, this will cause the Russell's paradox (see L. Paulson's lecture " Foundation of Functional Programming" (PP. 23). to avoid this paradox, we can annotate each lambda term with a type, and if such type can't be established, the term will be rejected.

as example, if "`x x`" the first "`x`" should be a function type of form "`A -> B`" (`A` and `B` are any sets, we call them *type variables*) if it can be applied by an argument, the second "`x`" must be the type of the domain of first `x`, that is `A`. (see `https://en.wikipedia.org/wiki/Simply_typed_lambda_calculus` or Pierce's Book "Types and Programming Languages", Ch. 9: Simply Typed Lambda-Calculus, in our `compiler_cd` directory). So the type constraint is the equation of type:

```
A =  A -> B
```

where `A` is *type variable* in the type set defined recursively as:

1. type constant `int` is a type.
2. `X`, `Y`, `Z`, . . . , the alphabets of *type variable* are the type.
3. if `A` and `B` are any type, then `A -> B` is a type. (so `X -> X`, `X -> Y`, `(X -> Y) -> Z`, . . . , are types).

because type variable `A` appears in right side of the above equation, we have not solution for it.

but the lambda term `(@x.x)(@x.x)` can be type as `X -> X`. the first `(@x.x)` (denoted by `alpha`) can be type as `(A -> A)` and the second `(@x.x)` (denoted by `beta`) can be typed as `(B -> B)`. for the term "`alpha beta`" have sense, the term `alpha` must have type function with domain `B -> B`, so the type equation is:

```
A =  B -> B
```

so `A = C -> C` and `B = C` is the solution.

and the *domain* type (left side of the arrow) of `alpha` is `C -> C`, and the type of `(@x.x)(@x.x)` is the type of the *range* (right side of the arrow) of `alpha`, so `C -> C`. Remark the type variable may be changed to any other type.

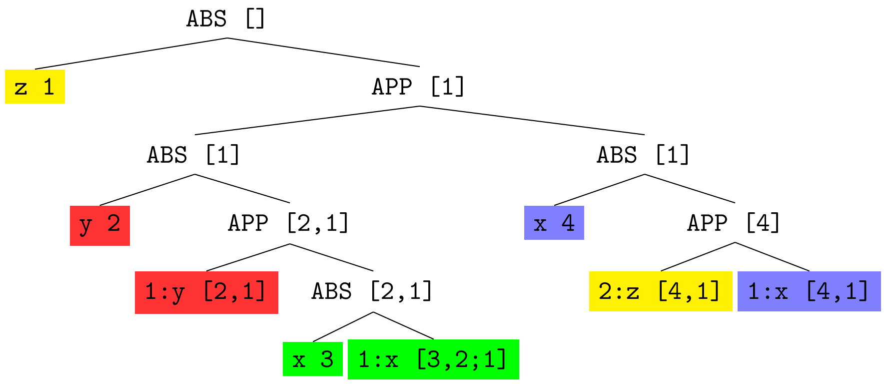
the above equation has infinite solution like:

```
A = (D -> D) -> (D -> D) and B = D -> D
A = ((D -> D) -> D) -> ((D -> D) -> D) and B = ((D -> D) -> D)
......
```

but all the above solution can be obtained by the substitution of `C` in the first solution. the first solution is so called *most general*.

the difference of the above 2 term `(x x)` and `(@x.x)(@x.x)` is the 2 occurrence of `x` in the first one are the same `x`. but the second are not the same.

Our goal is establish the most general type of any given lambda term if it has, or announce the type error if not. the method is *syntax-directed type synthesis*.



(a) `@z.(@y.y(@x.x))(@x.z x)`

Figure 5: Binding deepth for retrieve the type

2.2 step by step of type synthesis

the type structure is defined in `type.h` as:

```
typedef enum { Typevar = 1,  Arrow = 2, Int = 3 } Type_kind;
  /* for type tree node */

typedef struct type {
  int index;  /* for coding type variable */
  Type_kind kind;
  struct type * left, *right;
} Type;

typedef Type * Type_ptr;

typedef struct type_env{
  int redirect;   /* for unification use */
  Type_ptr type;   /* pointer to the type tree structure */
} Type_env;

typedef Type_env * Type_env_ptr;

extern Type_ptr global_type_env[MAX_ENV];
  /* like name_env[] and ast_env[], global_type_env[]
     will store the type for the declared lambda term */
```

1. each type variable is coded with a unique index as its type, and the each node of type tree is also coding with this index.
2. store each type tree node in the typing environment `Type_env type_env[MAXNODE]` (see `type.c`), and any type of the index `i` will store in `type_env[i]`.
3. with the De Brujin index 1.2, a lambda variable in the abstraction is represented by the binding deepth. to access the corresponding type variable, we can use a stack, like build the AST. each time enter an abstraction body, the corresponding abstraction variable is push to stack, the type of the variable encountered in the body, is just the `n`-th element from the top of the stack, where `n` is the binding deepth. in Figure 5, the preorder tree traversal of AST generate a new type variable (indexed from `1` to `4` for each abstraction, and push it in the stack when enter the abstraction body. for the `x` in the subterm `@x.x`, the binding deepth is `1`, so the first element (that's `3`) from the top stack `[3,2,1]` (stack top is on the **left**) is the corresponding type. for the `z` in the subterm `@x.z x`, the binding deepth is `2`, so the second element (that's `1`) from the top stack `[4,1]` is the corresponding type.

   the stack is implemented by a dynamic list:

```
typedef struct varlist {
char *var_name;
struct varlist * next;
} Var_list;
typedef Var_list * Var_list_ptr;
```

4. normally, we should initialze the stack with all elements in `global_type_env[]` of predefined lambda term. it's very heady do to it, if the predefined terms are too many. we split the stack into two parts, the all abstraction of the current term in the dynamic list `Var_list_ptr abs`, and all predefined term in `global_type_env[]` with the top `current - 1` (see 1.3).

in the next, an entry `type_env[n]` is denoted as `n(redirect, type)`, `type` is either `n` (type variable), either `int` (type constant), or `(x -> y)` (arrow), like

```
1(1, (3 -> 4))
```

Example of `type_env[]`

```
type_env[] = [0(0,int), 1(1,(3 -> 4)), 2(2,2), 3(2,3), 4(4,4)]
```

where `3(2,3)` means type variable `3` is *redirect* to `2`.

we deonote `M |== T` if the lambda term `M` has the most general type `T`.

the global environments are setting as:

```
name_env[] = {"+", "-", "*", "/", "=", "<"}
global_type_env[] = [0(int->(int->int)), 1(int->(int->int)), 2(int->(int->int)),
                     3(int->(int->int)), 4(int->(int->int)), 5(int->(int->int))]
current = 6  /* stack top + 1 */
```

the type constant `int` always has the index `0`.

Each time typing a lambda term, `type_env[]` will be set to

```
type_env[] = [0("",0,0,int)]
nindex = 1 /* next index for type variable */
```

its can be done by

```
void init_type_env()
{
  int i = 0;

  type_env[0] = &inttype_entry;
  type_env[0] -> type = & inttype;

  while (i < INIT_POS) {
    new_env();
    global_type_env[i] =
      storetype(make_arrowtype( &inttype, make_arrowtype(&inttype, &inttype)));
    /*  int -> (int -> int) */
    i++;
  }
  return;
}
```

We will use a serie of the examples to stepwise the typing processus by the **recursive tree traversal** of AST.

2.2.1 Example 1: `@x.@y.x y` (`MN`: `M` is a type variable)

Like computing the binding deepth, the **preorder tree traversal** of AST find the abstraction stack for each AST node. the following **postorder traversal** type the lambda term. here is the postorder

step 1

```
top = 8        /* the stack top index = current + lenght(abs) */
abs: [2,1]     /* the stack of abstraction */
type_env[] = [0(0,int), 1(1,1), 2(2,2)]
(x:2) |== 1    /* obtain by get 2-th from top of abs */
```

setp 2

```
top = 8:
abs: [2,1]
type_env[] = [0(0,int), 1(1,1), 2(2,2)]
(y:1) |== 2   /* obtain by get 1-th from top of abs */
```

step 3

```
top = 8:
abs: [2,(3 -> 4)]
type_env[] = [0(0,int), 1(1,(3 -> 4)), 2(2,2), 3(2,3), 4(4,4)]
((x:2)(y:1)) |== 4
```

if the term `x y` has sense, the `x` must have the arrow type, but isn't the case. fortunately, `x` is a type variable, it can be change to any type. function `get_instance(int index)` will do that:

```
Type_ptr get_instance(Type_ptr type_tree)
{
  Type_ptr p = final_type(type_tree);
     /* p must be a no redirect type, see below */
  p -> kind = Arrow;
  p -> left = make_vartype(0, 0);
  p -> right = make_vartype(0, 0);
  return p;
}
```

it rewrites the type of the index `1` to an arrow type. and generate 2 type variables of `index` `3` and `4`. after `get_instance(1)`, `x` changed to type `(3 -> 4)`. and `y` is of type `2`. to the application `x y` have the sense, the domain type `2` of `x` and type `3` of `y` **must be** the same. We say they should *unified*. we can do it by changing `3` to `2`. the problem is that all typing envrionment which refers `3` must be changed to `2`. that is hard job (search all `type_env[]`, replacing the index of `3` with `2`, just like update the primary key in the database, you must alter all foreign keys that refer the updated primary key)! To simply this tedious job, we just redirect the entry of `3` in `type_env[]` from itself (`3`) to `2`. so we have `3(2,3)` in `type_env[]` where the first component change from `3` to `2` (points to `2`). this work can be done by the **side-effect** function `unify_leaf()` (it change the global `type_env[]`!).

```
void unify_leaf(Type_ptr t1, Type_ptr t2)
{
  int index1 =  (t1 -> index);
  int index2 =  (t2 -> index);

  if (index1 != index2) {
    type_env[index1] -> redirect = index2;
  }
  return;
}
```

a type variable `n(m, t)` in `type_env[]` is called *final* iff `n == m`, ifnot we call it *non-final*. `int final_type()` will across the redirect chain to get the final type. be careful, each time we access the type node which is not-final, retrieve the final type.

```
int final_index (int index)
{
  int i = index;
  if (type_env[i] == NULL) return -1;
  if ( type_env[i] -> type -> kind == Arrow )
    return i;
  if (i == (type_env[i] -> redirect))
    return i;
  return final_index(type_env[i] -> redirect );
}

/* return final type node for a giving Typevar node */
Type_ptr final_type(Type_ptr t)
{
  int i;
  i = final_index(t -> index);
  if (i == -1) return NULL;
  return type_env[i] -> type;
}
```

step 4

```
top = 7:
abs: [(3 -> 4)]
type_env[] = [0(0,int), 1(1,(3 -> 4)), 2(2,2), 3(2,3), 4(4,4), 5(5,(2 -> 4))]
(@y.((x:2)(y:1))) |== (2 -> 4)
```

`y` is of type `2`, and `x y` is of type `4`, the abstraction of `@y.x y` will be typed as an arrow type with a new generated index `5`.

step 5

```
top = 6:
abs: []
type_env[] = [0(0,int), 1(1,(3 -> 4)), 2(2,2), 3(2,3), 4(4,4), 5(5,(2 -> 4)),
              6(6,((3 -> 4) -> (2 -> 4)))]
(@x.(@y.((x:2)(y:1)))) |== ((3 -> 4) -> (2 -> 4))
```

`printtype(Type_ptr)` will recode the final type variable to `A - Z`, so `2` as `A`, `4` as `B`, then output

```
(@x.(@y.((x:2)(y:1)))) |==  ((A -> B) -> (A -> B))
```

step 6

`printtype(Type_ptr)` will recode the final type variable to `A - Z`, so `2` as `A`, `4` as `B`, then output

```
(@x.(@y.((x:2)(y:1)))) |==  ((A -> B) -> (A -> B))
```

step 7

after the typing processus finished, all type that we dynamically allocated must be freed. because every type we maked has an index in `type_env[]`, so we can free all of them easily without any memory leak by

```
void new_env(void)
{
  int i;
  for (i = 1; i < nindex; i++) {
    sfree(type_env[i] -> type);
    sfree(type_env[i]);
    /* redefine free as sfree (in emalloc.c) for
       gprofile the call frequences of free() */
```

```
  }
  for (i = 0; i < order; i++)
    index_order [i] = 0;

  nindex = 1;
  order = 1;
  step = 0;
}
```

2.2.2 Example 2: `@x.(@x.@y.x y) x` (`M` is arrow and `N` is type variable)

the first 5 steps is the same as Example 1 with all indexes increased by `1`.
step 5

```
top = 7:
abs: [1]
type_env[] = [0(0,int), 1(1,1), 2(2,(4 -> 5)), 3(3,3), 4(3,4), 5(5,5), 6(6,(3 -> 5)),
              7(7,((4 -> 5) -> (3 -> 5)))]
(@x.(@y.((x:2)(y:1)))) |== ((4 -> 5) -> (3 -> 5))
```

step 6

```
top = 7:
abs: [1]
type_env[] = [0(0,int), 1(1,1), 2(2,(4 -> 5)), 3(3,3), 4(3,4), 5(5,5), 6(6,(3 -> 5)),
              7(7,((4 -> 5) -> (3 -> 5)))]
(x:1) |== 1
```

step 7

```
top = 7:
abs: [1]
type_env[] = [0(0,int), 1(2,1), 2(2,(3 -> 5)), 3(3,3), 4(3,4), 5(5,5), 6(6,(3 -> 5)),
              7(7,((3 -> 5) -> (3 -> 5)))]
((@x.(@y.((x:2)(y:1))))(x:1)) |== (3 -> 5)
```

in this time, the function `((@x.(@y.((x:2)(y:1))))` of the application is already an arrow type `((4 -> 5) -> (3 -> 5))` where `(4 -> 5)` is of the index `2`, the argument is type variable `1`, we can just redirect `1` to unify the domain type of the arrow and the argument type. so we have the side-effect `1(2,1)` in `type_env[]`

```
void unify_leaf_arrow(Type_ptr leaf, Type_ptr t)
{
  int index = leaf -> index;
  type_env[index] -> redirect = t -> index;
  return;
}
```

step 8

```
top = 6:
abs: []
type_env[] = [0(0,int), 1(2,1), 2(2,(3 -> 5)), 3(3,3), 4(3,4), 5(5,5), 6(6,(3 -> 5)),
              7(7,((3 -> 5) -> (3 -> 5))), 8(8,(1 -> (3 -> 5)))]
(@x.((@x.(@y.((x:2)(y:1))))(x:1))) |== (1 -> (3 -> 5))
```

remark that the domain type `1` of `8(8,(1 -> (3 -> 5)))` is non-final, the redirect final type is `2(2,(3 -> 5))`. so

step 9

```
(@x.((@x.(@y.((x:2)(y:1))))(x:1))) |== ((A -> B) -> (A -> B))
```

2.2.3 Example 3: `(@x.@y.x y)(@x.x)` (`M` and `N` are both arrow)

the first 5 steps is the same as Example 1.

step 5

```
top = 6:
abs: []
type_env[] = [0(0,int), 1(1,(3 -> 4)), 2(2,2), 3(2,3), 4(4,4), 5(5,(2 -> 4)),
              6(6,((3 -> 4) -> (2 -> 4)))]
(@x.(@y.((x:2)(y:1)))) |== ((3 -> 4) -> (2 -> 4))
```

step 6

```
top = 7:
abs: [7]
type_env[] = [0(0,int), 1(1,(3 -> 4)), 2(2,2), 3(2,3), 4(4,4), 5(5,(2 -> 4)),
              6(6,((3 -> 4) -> (2 -> 4))), 7(7,7)]
(x:1) |== 7
```

step 7

```
top = 6:
abs: []
type_env[] = [0(0,int), 1(1,(3 -> 4)), 2(2,2), 3(2,3), 4(4,4), 5(5,(2 -> 4)),
              6(6,((3 -> 4) -> (2 -> 4))), 7(7,7), 8(8,(7 -> 7))]
(@x.(x:1)) |== (7 -> 7)
```

step 8

```
top = 6:
abs: []
type_env[] = [0(0,int), 1(1,(2 -> 4)), 2(7,2), 3(2,3), 4(7,4), 5(5,(2 -> 4)),
              6(6,((2 -> 4) -> (2 -> 4))), 7(7,7), 8(8,(7 -> 7))]
((@x.(@y.((x:2)(y:1))))(@x.(x:1))) |== (2 -> 4)
```

in this time, the argument of function is also function `(7 -> 7)` which should be unified with `(2 -> 4)`. the unification can be done by unify both domain type and range type of the arrow. `unify(2, 7)` is the case `unify_leaf()` which redirect 2 to 7. `unify(4, 7)` redirect 4 to 7 also.

step 9

```
((@x.(@y.((x:2)(y:1))))(@x.(x:1))) |== (A -> A)
```

2.2.4 Example 4: `@x.x x`

step 1

```
top = 7:
abs: [1]
type_env[] = [0(0,int), 1(1,1)]
(x:1) |== 1
```

step 2

```
top = 7:
abs: [1]
type_env[] = [0(0,int), 1(1,1)]
(x:1) |== 1
```

step 3

```
top = 7:
abs: [(2 -> 3)]
type_env[] = [0(0,int), 1(1,(2 -> 3)), 2(2,2), 3(3,3)]
((x:1)(x:1)) |== NULL
type A and type (A -> B) can't be unified!
```

the function type is a type variable `1`. `get_instance(1)` rewrite to an arrow. the side-effect change the argument type (second `x`) to the same arrow. `unify(2, 1)` is the case `unify_leaf_arrow(2, 1)`, but if we redirect `2` to `1`. but `2` is in type `1(1, (2 -> 3))`. it's not typable this will also cause the cyclic chain of redirection. it is strictly forbidden. `is_occur_node(2, (2 -> 3))` checks if such case. if the occurrence take place, report typing error and return `NULL`.

```
int is_occur_node(int index, Type_ptr type_tree)
{
  int i = index;
  if (type_tree == NULL) return 1;

  switch (type_tree -> kind) {
  case Typevar:
    return  type_env[type_tree -> index] -> redirect == i;
  case Arrow:
    /* left and right may be not final!!! */
    return is_occur_node (i, final_type(type_tree -> left)) ||
      is_occur_node(i, final_type(type_tree -> right));
  case Int:
    return 0;
  }
}
```

2.2.5 Example 5: `let MY = @x.@y.x y;`

the first 5 steps is the same as Example 1.

<u>step 5</u>

```
top = 6:
abs: []
type_env[] = [0(0,int), 1(1,(3 -> 4)), 2(2,2), 3(2,3), 4(4,4), 5(5,(2 -> 4)),
              6(6,((3 -> 4) -> (2 -> 4)))]
(@x.(@y.((x:2)(y:1)))) |== ((3 -> 4) -> (2 -> 4))
```

<u>step 6</u>

We will store the name `MY` in `name_env[6]` and the AST `(@x.(@y.((x:2)(y:1))))` in `ast_env[6]`, and the type in `global_type_env[6]`. if the next lambda term referes the `MY`, we should restore the type of `MY` in new `type_env[]`. `storetype(Type_ptr t)` will reindex the arrow type with `0` and the final type variable from `1` to `n` if the type tree has `n` different leaves.

```
/* generate type_env independant  type tree */
Type_ptr storetype(Type_ptr tree)
{
  if (tree == NULL) return;
  switch ( tree -> kind ) {
  case Int:  return &inttype;
  case Typevar: {
    int i = final_index(tree -> index);
    Type_ptr t = type_env[i] -> type;
    switch (t -> kind) {
    case Int:   return &inttype;
    case Arrow:
```

```
      tree -> left = t -> left;
      tree -> right = t -> right;
      break;
    default: {
      int offset = find_index(i); /* reindex the type variable */
      Type_ptr tmp;
      if (offset == 0) {
        return &inttype;
      }
      tmp = (Type_ptr) smalloc(sizeof(Type));
      tmp -> index = offset;
      tmp -> kind = Typevar;
      tmp -> left = tmp -> right = NULL;
      return tmp;
    }
    }
  }
  }
  {
    Type_ptr tmp = (Type_ptr) smalloc(sizeof(Type));
    tmp -> index = 0;
    tmp -> kind = Arrow;
    tmp -> left = storetype(tree -> left);
    tmp -> right = storetype(tree -> right);
    return tmp;
    }
}
```

so `global_type_env[6] = (1 -> 2) -> (1 -> 2)`, and the cursor `currrent` for the next abstraction or definition is increased to 7.

2.2.6 Example 6: `@x.MY x`

it is the same of Example 2, but the subterm `(@x.@y.x y)` is predefined. we should access the correct place in `global_type_env[6]` and restore the type of `MY` in the new `type_env[]`.

step 1

```
top = 8:
abs: [1]
type_env[] = [0(0,int), 1(1,1), 2(2,2), 3(3,3), 4(4,(3 -> 2)), 5(5,(3 -> 2)),
              6(6,((3 -> 2) -> (3 -> 2)))]
(MY:2) |== ((3 -> 2) -> (3 -> 2))
```

`MY` is of binding deepth 2 which is greater than the stack length `abs`, so it is a predefined name. because the `glolal_env[current - 1]` is the extension of the stack, we can get it by `top - 2`. `get_n_th([1], 2, 8)` will restore the type of `MY` as `6(6,((3 -> 2) -> (3 -> 2)))` with new series of indexes from 2 to 6.

```
Type_ptr get_n_th_from_global(int i)
{
  /* if is the fixed-point combinator, we will
     assign it with the type (A -> A) -> A.
     Z, Y and rec is defined in library.txt */
  if (strcmp(name_env[i], "Z") == 0 ||
      strcmp(name_env[i], "Y") == 0 ||
      strcmp(name_env[i], "rec") == 0) {
    return make_rec_type();
  }
  return restoretype(global_type_env[i]);
```

```
  /* restoretype will reindex the type variable in new type_env[] */
}

/* pos is the current top of name_env[] */
/* n is the binding deepth of the lambda variable */
Type_ptr get_n_th(Var_list_ptr list, int n, int pos)
{
  int i = 0;
  while (i != n - 1 && list != NULL) {
    list = list -> next;
    i++;
  }

  /* is an abstraction */
  if (i == n - 1  && list != NULL)
    return list -> type_var;

  /* is a predefined name */
  if ((pos - n) >= 0)
    return get_n_th_from_global(pos - n);

  printf("wrong access global type env\n");
  exit (1);
}
```

the following steps are as Example 2.

<u>step 2</u>

```
top = 8:
abs: [1]
type_env[] = [0(0,int), 1(1,1), 2(2,2), 3(3,3), 4(4,(3 -> 2)), 5(5,(3 -> 2)),
              6(6,((3 -> 2) -> (3 -> 2)))]
(x:1) |== 1
```

<u>step 3</u>

```
top = 8:
abs: [1]
type_env[] = [0(0,int), 1(5,1), 2(2,2), 3(3,3), 4(4,(3 -> 2)), 5(5,(3 -> 2)),
              6(6,((3 -> 2) -> (3 -> 2)))]
((MY:2)(x:1)) |== (3 -> 2)
```

<u>step 4</u>

```
top = 7:
abs: []
type_env[] = [0(0,int), 1(5,1), 2(2,2), 3(3,3), 4(4,(3 -> 2)), 5(5,(3 -> 2)),
              6(6,((3 -> 2) -> (3 -> 2))), 7(7,(1 -> (3 -> 2)))]
(@x.((MY:2)(x:1))) |== (1 -> (3 -> 2))
```

<u>step 5</u>

```
(@x.((MY:2)(x:1))) |== ((A -> B) -> (A -> B))
```

2.2.7 Example 7: recursions

the fixed-point combinator (`https://en.wikipedia.org/wiki/Fixed-point_combinator`) can be predefined as

```
let Y=@f.(@x.f(x x))(@x.f(x x));
let Z=@f.(@x.f(@y.(x x)y))(@x.f(@y.(x x)y));
```

so we can input:

```
let fact = (Z (@f.@n. (if (= n 0) then 1 else (* n (f (- n 1))) fi)));
```

it will return

```
((Z:1)(@f.(@n.if(((=:7)(n:1))0)then1else(((*:9)(n:1))((f:2)(((-:10)(n:1))1))))))
|= (int -> int)
```

2.3 Church encoding and Type

In mathematics, Church encoding is a means of representing data and operators in the lambda calculus. The data and operators form a mathematical structure which is embedded in the lambda calculus. The Church numerals are a representation of the natural numbers using lambda notation. The method is named for Alonzo Church, who first encoded data in the lambda calculus this way (https://en.wikipedia.org/wiki/Church_encoding). You can see this encoding in `library.txt`.

1. Church numerals:

```
ZERO |== (A -> (B -> B))
ONE |== ((A -> B) -> (A -> B))
TWO |== ((A -> A) -> (A -> A))
......
FIVE |== ((A -> A) -> (A -> A))
```

2. Arithmetic operation:

```
ADD |== (((((A -> B) -> (C -> A)) -> ((A -> B) -> (C -> B)))
        -> (D -> E)) -> (D -> E))
SUB |== (A -> ((((((B -> (C -> B)) -> D) -> ((E -> (D -> F)) -> F)) ->
        (((G -> (G -> H)) -> H) -> ((I -> (J -> J)) -> K))) -> ((D -> E) ->
        (G -> K))) -> (A -> L)) -> L))
MULT |== (((A -> B) -> ((C -> (D -> D)) -> E)) -> (((((F -> G) -> (H -> F))
         -> ((F -> G) -> (H -> G))) -> (A -> B)) -> E))
PRED |== ((((A -> (B -> A)) -> C) -> ((D -> (C -> E)) -> E)) -> (((F -> (F -> G)) -> G)
         -> ((H -> (I -> I)) -> J))) -> ((C -> D) -> (F -> J)))
```

3. Booleans

```
(TRUE:65) |== (A -> (B -> A))
(FALSE:64) |== (A -> (B -> B))
IF |== ((A -> (B -> C)) -> (A -> (B -> C)))
OR |== (((A -> (B -> A)) -> (C -> D)) -> (C -> D)
AND |== ((A -> ((B -> (C -> C)) -> D)) -> (A -> D))
NOT |== (((A -> (B -> B)) -> ((C -> (D -> C)) -> E)) -> E)
GE |== ((((((A -> (B -> A)) -> C) -> ((D -> (C -> E)) -> E)) -> (((F -> (F -> G)) -> G)
       -> ((H -> (I -> I)) -> J))) -> ((C -> D) -> (F -> J))) ->
       (K -> ((L -> (M -> (N -> N))) -> ((O -> (P -> O)) -> Q)))) -> (K -> Q))
LE |== (A -> ((((((B -> (C -> B)) -> D) -> ((E -> (D -> F)) -> F)) ->
       (((G -> (G -> H)) -> H) -> ((I -> (J -> J)) -> K))) -> ((D -> E) -> (G -> K))) ->
       (A -> ((L -> (M -> (N -> N))) -> ((O -> (P -> O)) -> Q)))) -> Q))
EQ |== NULL
let LEQ = @m.@n.ISZERO (SUB m n);
LEQ |== (A -> ((((((B -> (C -> B)) -> D) -> ((E -> (D -> F)) -> F)) ->
        (((G -> (G -> H)) -> H) -> ((I -> (J -> J)) -> K))) -> ((D -> E) -> (G -> K)))
        -> (A -> ((L -> (M -> (N -> N))) -> ((O -> (P -> O)) -> Q)))) -> Q))
let EQ1 = @m.@n. AND (LEQ m n) (LEQ n m);
EQ1 |== NULL
```

`EQ` and `EQ1` can't be typed!

4. Recursions

```
Y |== NULL
Z |== NULL
FACT |== NULL
SUM |== NULL
DIV |= NULL
fact |== (int -> int)
ACK |== ((((((A -> B) -> (A -> B)) -> C) -> (((((A -> B) -> (A -> B)) -> C) ->
        (C -> D)) -> D)) -> ((((E -> F) -> (F -> G)) ->
        ((E -> F) -> (E -> G))) -> H)) -> H)
```

`FACT` and `SUM` coding with Church numerals can't be typed, even added the axiom `Y |== (A -> A) -> A`. `fact` with lazy `if` is typable under the axiom. Ackermann function (https://en.wikipedia.org/wiki/Ackermann_function) is typable.

3 TODO

3.1 Unification algorithm

refer the unification algorithm of Dragon book (PP. 397) finish the side-effect unification algorithm

```
/* return 1 if unified; return 0 ifnot */
int unify(Type_ptr t1, Type_ptr t2)
{
  t1 =  simply(t1);
  t2 =  simply(t2);
  if (t1 == NULL || t2 == NULL) {
    printf("null type occur! typing error!\n");
    return 0;
  }
  switch (t1 -> kind) {
  case Int: {
     /* todo */
  }
  case Typevar: {
    /* todo */
  }
  case Arrow: {
    /* todo */
  }
  }
  return 1;
}
```

3.2 the semantic rules of type synthesis

each `AST T` has 3 attributes:

1. `T.top : = current + abstraction deepth`
2. `T.abs`: the stack of the type of the abstraction.
3. `T.type`: type

the typing will work with the global `type_env[]`, `name_env`, `global_type_env[]`, `nindex`, `current`...

AST	semantic rules
`ROOT`	`ROOT.abs = [ ] /* empty stack */` `ROOT.top = current`
`T = CONST n`	`T.type = int`
`T = VAR (n:x)`	`T.type = get_nth(T.abs, n, T.top)`

`T = ABS (x, T1)`	`x.type = make_vartype()`
	`T1.abs = add_list(x.type, T.abs)`
	`T.type = make_arrow(x.type, T1.type)`
	`T1.top = T.top + 1`

`T = COND(T1, T2, T3)`	`T1.abs = T.abs  T1.top = T.top`
	`T2.abs = T.abs  T2.top = T.top`
	`T3.abs = T.abs  T3.top = T.top`
	`if (T1.type == int && unify(T2.type, T3.type))`
	`  T.type = T2.type`
	`else T.type = NULL`

`T = APP (T1, T2)`	`/* todo */` `/* you should included this SDD in the file type.c */`

3.3 Typing

the attributes `T.abs` and `T.top` are L-attributed. `T.type` is S-attributed. so we can solve them with recursive tree traversal. Implement the following typing function.

```
Type_ptr typing (Var_list_ptr abs, AST *t, int top)
{
  Type_ptr tmp; /* for store the return type */

  if (t == NULL) return NULL;

  switch (t -> kind) {
  case CONST: return make_inttype();
  case VAR: {
    tmp = get_n_th(abs, t -> value, top);
    break;
  }

  case ABS:  {
    /* todo */
  }

  case COND: {
    /* todo */
  }
  case APP: {
   /* todo */
  }
  }
  if (yyin == stdin) {
    printf("typing step %d and top = %d:\n", ++ step, top);
    print_abs(abs);
    print_env();
    print_expression(t, stdout);
    printf(" |== "); print_type_debug(tmp); printf("\n");
  }
  free_list(abs);
  return tmp;
}
```

Be careful: every recursive call of `typing()` should pass the copy of `abs` list by call `Var_list_ptr list_copy (Var_list_pt)`.

3.4 Memory leaks

You program should have no memory leaks! you can test it by input multiple lines:

```
@m.m(@f.@n.n f(f(@f.@x.f x)))(@n.@f.@x.n f (f x));
```

then `gprof ./lambda`. the difference of `smalloc` and `sfree` should be the same like:

ONE INPUT of the above lambda term:

```
 %   cumulative   self              self     total
time   seconds   seconds    calls  Ts/call  Ts/call  name
 0.00      0.00     0.00      602     0.00     0.00  print_type_debug
 0.00      0.00     0.00      268     0.00     0.00  smalloc
 0.00      0.00     0.00      226     0.00     0.00  sfree
 ......
```

TWO INPUT of the above lambda term:

```
 %   cumulative   self              self     total
time   seconds   seconds    calls  Ts/call  Ts/call  name
 0.00      0.00     0.00     1204     0.00     0.00  print_type_debug
 0.00      0.00     0.00      470     0.00     0.00  smalloc
 0.00      0.00     0.00      428     0.00     0.00  sfree
 ......
```

please send your `type.c` as attached file to mailto:hfwang@whu.edu.cn?subject=ID(05) where the ID is your student id number.

–hfwang November 16, 2017