ME-552 Term project

Missile Dynamics

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1 Introduction

Compiling relevant literature on missile dynamics and control models poses challenges, particularly in scenarios where well-established design parameters are not readily available for academic investigation. In 1984, a seminal technical report authored by Philip N. Jenkins, the Director of the US Army Missile Laboratory, comprehensively outlined diverse dynamic models, providing detailed analyses [1]. This report stands as a valuable academic resource for gaining insights into the nuanced aspects of missile dynamics. Followed by [2, 3], missile control and guidance procedures were discussed widely. The fundamental challenge in understanding missile dynamics is the need to connect forces expressed in body-fixed frames (such as aerodynamic and thrust forces) with those expressed in the earth-fixed frame (such as gravitational forces) to determine the missile's acceleration from an inertial frame of reference, where Newton's first law can be applied [1, 3]. To describe the acceleration and any time derivative of the missile's position, it is necessary to be able to perform transformations from the body frame of reference to the inertial frame of reference and vice versa. To address this problem, one can employ either Euler angles or quaternions. These approaches utilize angular rates to derive coordinate transformations from the body coordinate frame to the inertial reference frame [1]. Elshafie M. & M. Talal used quarterions in [4] to provide a dynamic model for the high performance tactical missiles control. In this report, the Euler angles will be used as a replication of the work done in [1]. Simulations of different cases and of the dynamic system will be discussed in the results section.

Current research in the field of missile dynamics and control is increasingly integrating machine learning techniques, particularly in the context of air battle decision-making. In [10], an algorithm has been developed for missile maneuvering, employing a hierarchical proximal policy optimization (PPO) reinforcement learning approach. This method empowers a missile to concurrently guide towards a target and evade interception. Significant emphasis in ongoing research is placed on computational missile guidance, where deep learning algorithms are integrated to address and enhance guidance problems in missile systems [11].

2 Missile Equations of Motion

2.1 The missile is a rigid body

The missile is a rigid body [3]. Thus, the translation and rotation equations of a rigid body encapsulate the dynamic behavior of the missile in terms of translation and rotation. In concise mathematical expressions, the translation and rotation of a rigid body can be articulated as follows:

Translation:

$$\mathbf{F} = m\mathbf{a} \tag{1}$$

Rotation:

$$\mathbf{M} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{V}) = \dot{\mathbf{H}} \tag{2}$$

Where:

$\mathbf{F}$	is the net Forces on the system,
m	is the mass of the system,
$\mathbf{a}$	is the acceleration,
$\mathbf{M}$	is the net Moments on the system,
$\mathbf{r}$	is the position vector,
$\mathbf{V}$	is the velocity vector,
$\mathbf{H}$	is the angular momentum.

2.2 Forces acting on a missile

The forces acting on the missile can be classified into three primary groups [1]:

1. **Thrust:** The assessment of thrust involves utilizing established testing and design methodologies. Static firings are conducted to obtain thrust versus time profiles for the engine.

2. **Aerodynamic:** Aerodynamic forces are accurately evaluated through standard procedures, including wind tunnel measurements to determine the missile's aerodynamic characteristics.

3. **Gravitational:** Gravitational forces are calculated based on a comprehensive understanding of the missile's positional relationship with respect to the Earth.

In order to develop the equations of motion for a missile, it is essential to employ at least two coordinate systems:

1. **Inertial Frame:** This frame serves as the basis for applying Newton's second law. It provides a reference frame fixed in space, where the motion equations can be formulated accurately.

2. **Body-Fixed Frame:** The body-fixed frame is crucial for describing the thrust and aerodynamic forces acting on the missile. This frame moves with the missile, allowing for a detailed representation of the forces influencing its motion.

2.3 Coordinate Systems

It should be noted here that the coordinate system used in the present development aligns with the one employed in aircraft applications [1]. The formulation involves defining four orthogonal axes systems to derive the pertinent equations governing the motion of vehicles, be it aircraft or missiles. These axes systems are delineated as follows:

1. The inertial frame, which remains fixed in space and upholds the validity of Newton's Laws of Motion.
2. An Earth-centered frame that co-rotates with the Earth.
3. An Earth-surface frame, parallel to the Earth's surface, with its origin located at the center of gravity (cg) of the vehicle in the north, east, and down directions.
4. Conventional body axes are chosen to represent the vehicle, with the frame's center positioned at the vehicle's cg. The components of this frame are oriented forward, out of the right wing, and downward.

In the context of ballistic missiles, two alternative coordinate systems find common use [3]. These coordinate systems are:

1. **Launch Centered Inertial:** This system remains inertially fixed and is centered at the launch site at the moment of launch. Within this system, the x-axis is conventionally aligned with the horizontal plane and in the direction of launch, the positive z-axis extends vertically, and the y-axis completes the right-handed coordinate system.
2. **Launch Centered Earth-Fixed:** This represents an Earth-fixed coordinate system with the same orientation as the inertial coordinate system mentioned previously. This system offers advantages in gimbaled inertial platforms, as there is no need to eliminate the Earth rate torquing signal from the gyroscopes during launch.

In this work, a Launch Centered Earth-Fixed coordinate frame (X_e, Y_e, Z_e) is employed as the chosen inertial reference frame (see Figure 1). This coordinate system undergoes translational motion synchronized with the Earth's motion. Importantly, it maintains a non-rotational characteristic, signifying that while the origin of the coordinate system co-moves with the movement of Earth, the coordinate frame itself does not experience rotation in relation to the ostensibly "fixed" stars. When it comes to the body-fixed frame (X_b, Y_b, Z_b), it is typically considered fixed to the missile itself (Figure 2, 3).

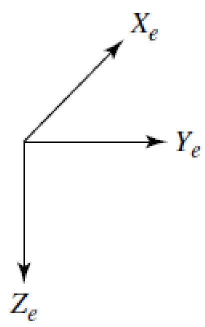

Figure 1: Inertial Frame Chosen for this work

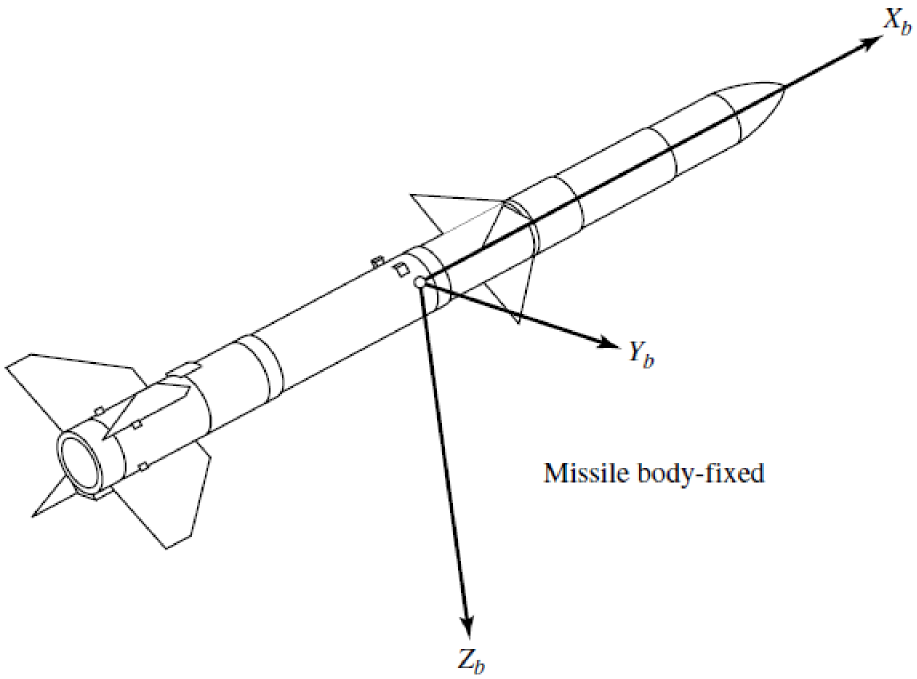

Figure 2: Body-Fixed Frame Chosen for this work [3]

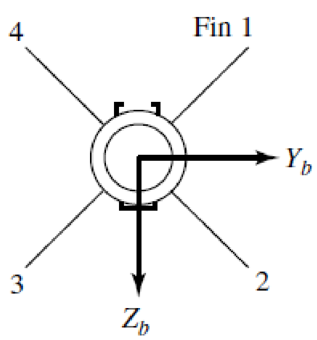

Figure 3: Rear View [3]

The transformation from the inertial frame to the body frame will be done by the three Euler angles ψ, θ and ϕ.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{\psi\theta\phi} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \quad (3)$$

2.4 Missile Motion Parameters

The missile possesses six degrees of freedom (6-DOF), consisting of three translations and three rotations along and about the missile axes (X_b, Y_b, Z_b). These motions, illustrated in Figure 4, involve translations denoted by (u, v, w) and rotations represented by (P, Q, R). The effect side of the missile equations of motion are explained in this section.

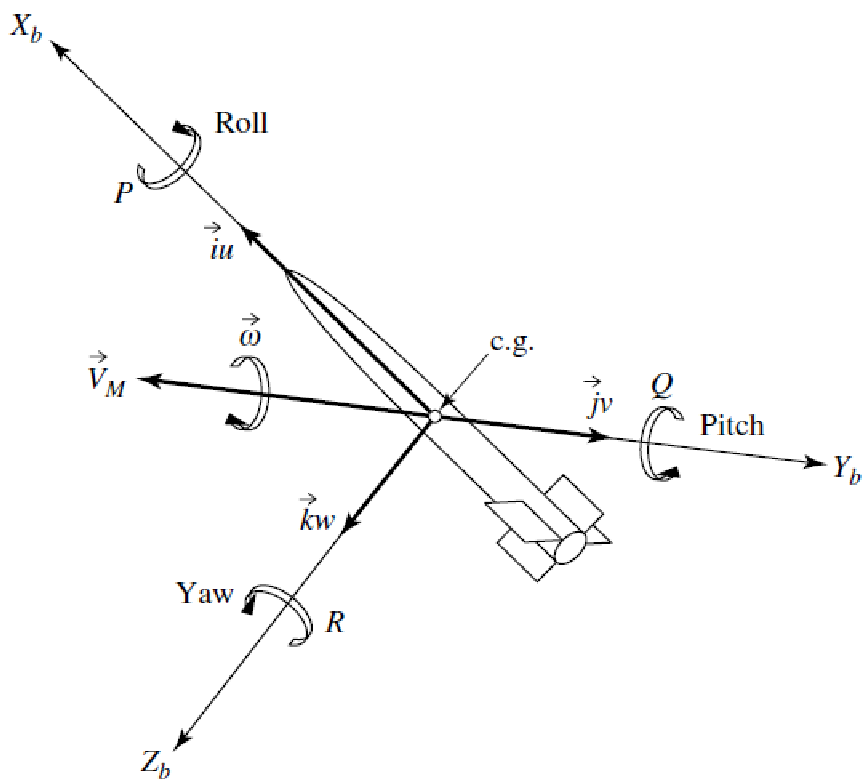

Figure 4: DOF of Missile [3]

2.4.1 Translational Motion Parameters

The Velocity $\mathbf{V}$ can be expressed in terms of the missile body translational rates (u, v, and w) as:

$$\mathbf{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4}$$

And so, the acceleration $\mathbf{a}$ is:

$$\mathbf{a} = [\frac{d\mathbf{V}}{dt}]_{\text{ body}} + \boldsymbol{\omega} \times \mathbf{V} \tag{5}$$

Where $\boldsymbol{\omega}$ is the angular velocity of the body.

The angular velocity vector can be written in terms of the missile body rates (P, Q, and R) as:

$$\boldsymbol{\omega} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \tag{6}$$

So,

$$\mathbf{a} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} + \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ u & v & w \end{bmatrix} = \begin{bmatrix} \dot{u} + Qw - Rv \\ \dot{v} + Ru - Pw \\ \dot{w} + Pv - Qu \end{bmatrix} \tag{7}$$

2.4.2 Rotational Motion Parameters

The angular momentum $\mathbf{H}$ is given by:

$$\mathbf{H} = \mathbf{I} \cdot \boldsymbol{\omega} \tag{8}$$

where $\mathbf{I}$ is the inertia tensor (a 3×3 matrix). Note that the elements of $\mathbf{I}$ are not usually constant except when given in a body-fixed coordinate system [1]. Thus, the use of a body coordinate system simplifies the application of this equation. The inertia tensor can be written in matrix form as:

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \tag{9}$$

Thus,

$$\mathbf{H} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} I_{xx}P + I_{xy}Q + I_{xz}R \\ I_{yx}P + I_{yy}Q + I_{yz}R \\ I_{zx}P + I_{zy}Q + I_{zz}R \end{bmatrix} \tag{10}$$

The rate of change of angular momentum $\dot{\mathbf{H}}$ is given by:

$$\dot{\mathbf{H}} = \frac{\partial \mathbf{H}}{\partial t} + \omega \times \mathbf{H} \tag{11}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \mathbf{I} \cdot \dot{\boldsymbol{\omega}} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xy}\dot{Q} + I_{xz}\dot{R} \\ I_{yx}\dot{P} + I_{yy}\dot{Q} + I_{yz}\dot{R} \\ I_{zx}\dot{P} + I_{zy}\dot{Q} + I_{zz}\dot{R} \end{bmatrix} \tag{12}$$

$$\omega \times \mathbf{H} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P & Q & R \\ H_x & H_y & H_z \end{bmatrix} = \begin{bmatrix} QH_z - RH_y \\ RH_x - PH_z \\ PH_y - QH_x \end{bmatrix} \tag{13}$$

So,

$$\dot{\mathbf{H}} = \begin{bmatrix} L \\ M \\ Q \end{bmatrix} \begin{bmatrix} I_{xx}\dot{P} + I_{xy}(\dot{Q} - PR) + I_{xz}(\dot{R} + PQ) + (I_{zz} - I_{yy})QR + I_{yz}(Q^2 - R^2) \\ I_{yy}\dot{Q} + I_{yz}(\dot{R} - PQ) + I_{xy}(\dot{P} + QR) + (I_{xx} - I_{zz})PR + I_{xz}(R^2 - P^2) \\ I_{zz}\dot{R} + I_{xz}(\dot{P} - QR) + I_{yz}(\dot{Q} + PR) + (I_{yy} - I_{xx})PQ + I_{xy}(P^2 - Q^2) \end{bmatrix} \tag{14}$$

Assuming that the product of inertia terms are zero is a good assumption in many cases. Though, for missiles having two planes of symmetry (Figure 5) we may assume that $I_{xy} = I_{yz} = 0$ only, while $I_{xz} \neq 0$. Thus, $\dot{\mathbf{H}}$ could be simplified to:

$$\dot{\mathbf{H}} = \begin{bmatrix} I_{xx}\dot{P} + (I_{zz} - I_{yy})QR + I_{xz}(\dot{R} + PQ) \\ I_{yy}\dot{Q} + (I_{xx} - I_{zz})PR + I_{xz}(R^2 - P^2) \\ I_{zz}\dot{R} + (I_{yy} - I_{xx})PQ + I_{xz}(\dot{P} - QR) \end{bmatrix} \tag{15}$$

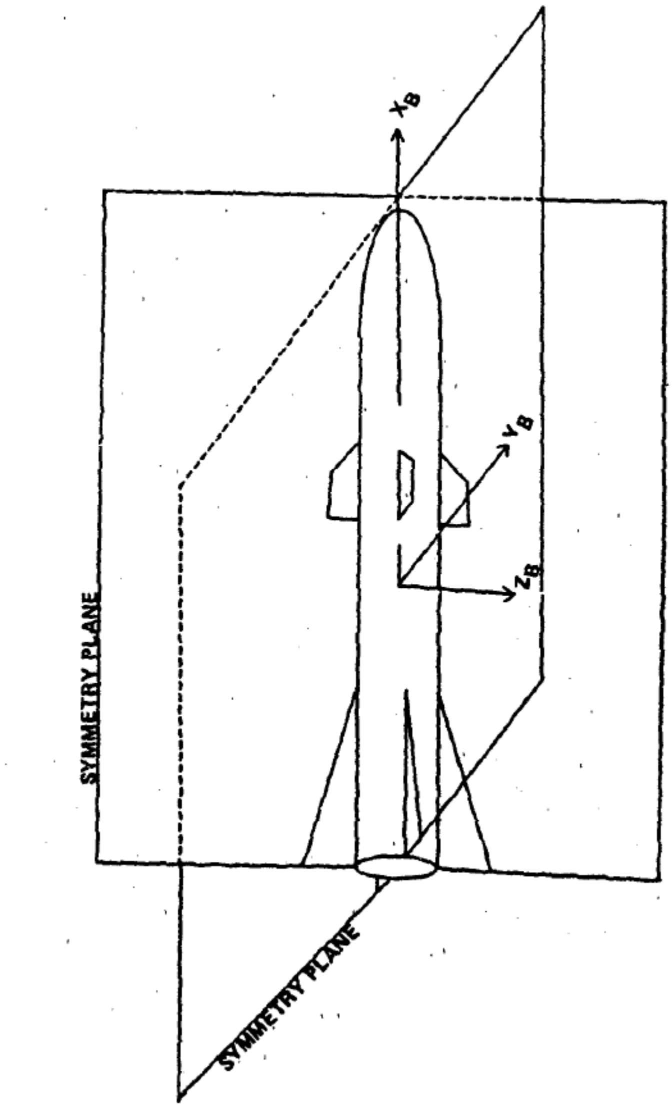

Figure 5: Symmetry Planes of Missile [1]

Tables 1 and 2 summarizes the motion parameters of the missile.

Axis	Direction	Name	Linear Velocity	Angular Displacement	Angular Rates
OX	Forward	Roll	u	ϕ	P (Roll Rate)
OY	Right Wing	Pitch	v	θ	Q (Pitch Rate)
OZ	Downward	Yaw	w	ψ	R (Yaw Rate)

Table 1: Axis Definitions

Moment Axis	Moment of Inertia	Product of Inertia	Moment Designation
OX	I_{xx}	$I_{xy} = 0$	L (Rolling Moment)
OY	I_{yy}	$I_{yz} = 0$	M (Pitching Moment)
OZ	I_{zz}	$I_{xz} \neq 0$	N (Yawing Moment)

Table 2: Moment Designations

2.5 Missile Aerodynamic Effects

The cause side of the equations of motion is explained in this section. In this work, it is assumed the utilization of a skid-to-turn technique for the missile, as this method is prevalent in a majority of surface-to-air and air-to-air missile applications. However, it's important to note that both aerodynamics and rigid-body dynamics exhibit high non-linearity. For a more comprehensive exploration of these forces, one can refer to [5]. Generally, the magnitude of forces and moments acting on an air vehicle is influenced by the combined effects of numerous variables. Table 3 is a summery for the the aerodynamic effects on a missile.

Table 3: Missile Aerodynamic Moments, Coordinates, and Velocity Components

Parameter	Roll Axis (X_b)	Pitch Axis (Y_b)	Yaw Axis (Z_b)
Angular Rates	P	Q	R
Velocity Components	u	v	w
Aerodynamic Force Components	F_X	F_Y	F_Z
Aerodynamic Force Coefficients	C_D	C_Y	C_L
Aerodynamic Moment Coefficients	C_l	C_m	C_n

2.5.1 Aerodynamic Forces

The fundamental aerodynamic forces for missiles executing skidding (yaw) maneuvers for turning, are represented through dimensionless coefficients, flight dynamic pressure, and a reference area. Figure 2 illustrates these forces, and their calculation is expressed as follows:

$$Drag : D = C_D q S \tag{16}$$

$$Lift : L = C_L q S \tag{17}$$

$$SideForce : F_Y = C_Y q S \tag{18}$$

Where:

C_D	is the coefficient of drag in the wind axis system,
C_L	is the coefficient of lift in the wind axis system,
C_Y	is the side force coefficient,
q	is the free-stream dynamic pressure at a point far from the airfoil $= \frac{1}{2}\rho V^2$
S	is the reference area, usually the area of one of the airfoils,
V	is the free-stream velocity.
ρ	is the atmospheric density,

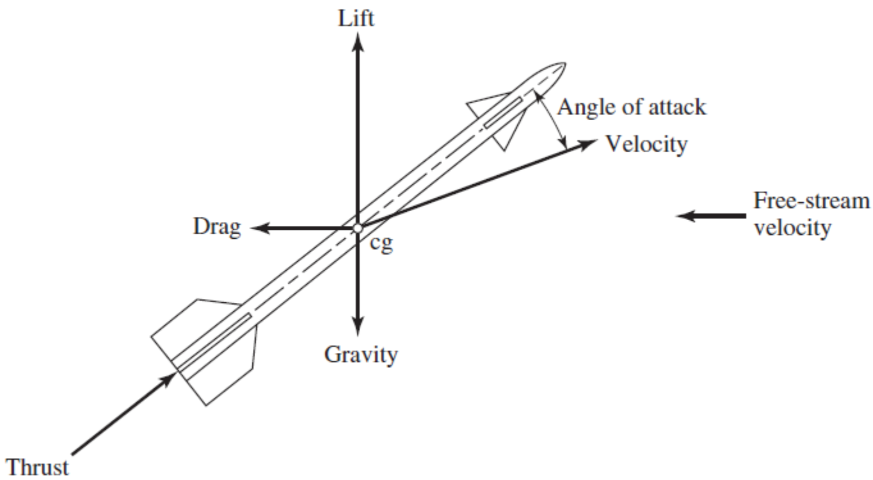

Figure 6: Aerodynamic forces and thrust acting on a missile [3]

The atmospheric density can be expressed by different models. In this work the troposphere model [8] is used. The coefficients of aerodynamic forces, C_L, C_D, and C_Y, are commonly defined in the wind-axis system (see Figure 7), aligned with the free-stream direction. However, as the aerodynamic force components are required in the body-fixed coordinate system for the equations of motion, it becomes necessary to express these coefficients with respect to the angle of attack α and side-slip angle β. In the body-fixed axis system, denoted as C_{Xb}, C_{Yb}, and C_{Zb}, these coefficients can be determined through wind tunnel experiments [1, 3]. Here, C_{Xb}, C_{Yb}, and C_{Zb} are expressed as a matrix product of the coefficients C_D, C_Y, and C_L through the transformation matrix.

$$\begin{aligned}
\begin{bmatrix} C_{Xb} \\ C_{Yb} \\ C_{Zb} \end{bmatrix} &= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -C_D \\ C_Y \\ -C_L \end{bmatrix} \\
&= \begin{bmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \begin{bmatrix} -C_D \\ C_Y \\ -C_L \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} C_{Xb} \\ C_{Yb} \\ C_{Zb} \end{bmatrix} = \begin{bmatrix} -C_D \cos\alpha\cos\beta - C_Y \cos\alpha\sin\beta + C_L \sin\alpha \\ -C_D \sin\beta + C_Y \cos\beta \\ -C_D \sin\alpha\cos\beta - C_Y \sin\alpha\sin\beta - C_L \cos\alpha \end{bmatrix}. \tag{19}$$

Where,

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{w}{u}\right) \\ \beta &= \sin^{-1}\left(\frac{v}{V_M}\right) \\ \text{where } V_M &= \left(u^2+v^2+w^2\right)^{1/2}. \end{aligned} \tag{20}$$

And for small angles of attack and side slip angles:

$$\begin{aligned} \alpha &= \frac{w}{u} \\ \beta &= \frac{v}{u} \end{aligned} \tag{21}$$

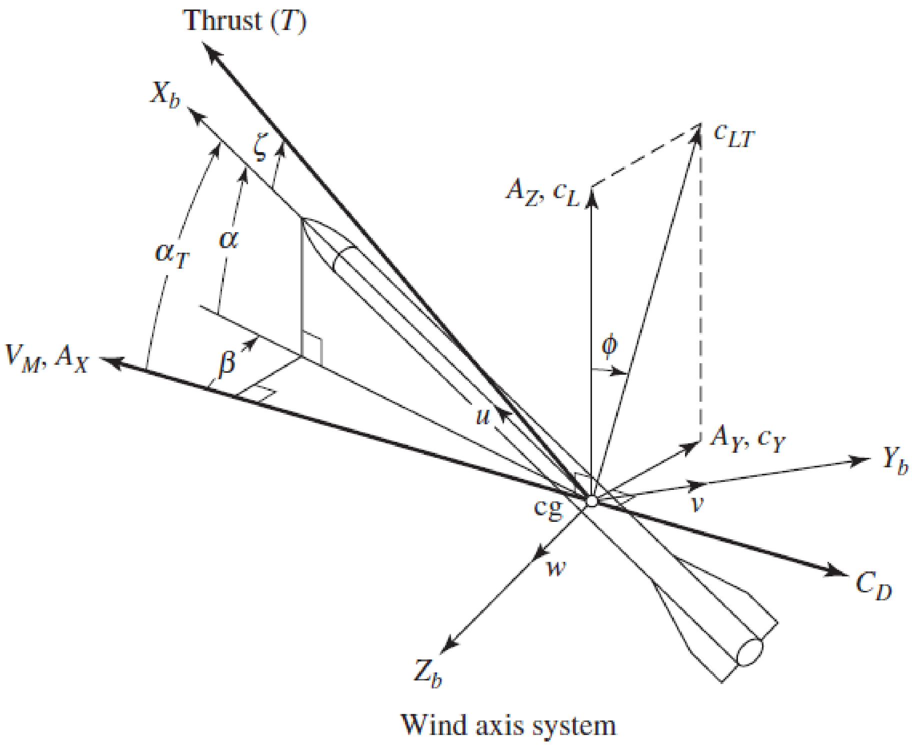

Figure 7: Aerodynamic forces: wind axes [3]

2.5.2 Aerodynamic Moments

Similar to the discussion on aerodynamic forces, the analysis of moments on a missile involves separating the effects caused by the distribution of aerodynamic loads and the thrust force that does not act through the center of gravity. In particular, the moment resulting from the force acting at a distance from the origin can be categorized into three components aligned with the missile's body reference axes. These components include the pitching moment, rolling moment, and yawing moment.

$$RollingMoment(L) : L = Cl \cdot q \cdot S \cdot l \tag{22}$$

$$PitchingMoment(M) : M = Cm \cdot q \cdot S \cdot l \tag{23}$$

$$YawingMoment(N) : N = Cn \cdot q \cdot S \cdot l \tag{24}$$

Where:

C_l	aerodynamic moment coefficients in roll
C_m	aerodynamic moment coefficients in pitch
C_n	aerodynamic moment coefficients in yaw
l	is the the mean diameter

Note that the detailed explanation and derivation of the Aerodynamic Forces & Moment Coefficients is not that simple, and are further investigated in [3] and [5]. The Missile constants and coefficient values when needed one could return to [6] and [7].

2.6 Final Representation of the equations of motion

Combining the out comes of the last two sections, namely Missile Motion Parameters, and Missile Aerodynamic Effects, the translational and rotational equations of motion could be expressed as follows.

2.6.1 Translation Equations

$$\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m\mathbf{a}$$

$$\begin{bmatrix} T_{\text{Thrust}} - x_{mg} - F_{Aero_x} \\ y_{mg} - F_{Aero_y} \\ z_{mg} - F_{Aero_z} \end{bmatrix} = m \begin{bmatrix} \dot{u} + Qw - Rv \\ \dot{v} + Ru - Pw \\ \dot{w} + Pv - Qu \end{bmatrix}$$

And by re-arrangement:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} T_{\text{Thrust}} - x_{mg} - F_{Aero_x} + Rv - Qw \\ y_{mg} - F_{Aero_y} + Pw - Ru \\ z_{mg} - F_{Aero_z} + Qu - Pv \end{bmatrix}$$

where,

$$\mathbf{F_g} = \begin{bmatrix} x_m g \\ y_m g \\ z_m g \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \tag{25}$$

$$\mathbf{F_g} = \begin{bmatrix} -mg\sin\theta \\ mg\sin\phi\cos\theta \\ mg\cos\phi\cos\theta \end{bmatrix}$$

and,

$$\mathbf{F_{Aero}} = qS \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix}$$

So,

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = 1/m \begin{bmatrix} T_{\text{Thrust}} \\ 0 \\ 0 \end{bmatrix} + g \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix} + qS/m \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{26}$$

and,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = R^T_{\psi\theta\phi} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{27}$$

2.6.2 Rotation Equations

$$\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \dot{\mathbf{H}}$$

$$\begin{bmatrix} M_{Aero_x} \\ M_{Aero_y} \\ M_{Aero_z} \end{bmatrix} = \mathbf{I} \begin{bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{bmatrix} + \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \times (\mathbf{I} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix})$$

By re-arrangement:

$$\begin{bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{bmatrix} = \mathbf{I}^{-1} \left[\begin{bmatrix} M_{Aero_x} \\ M_{Aero_y} \\ M_{Aero_z} \end{bmatrix} - \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \times (\mathbf{I} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix}) \right]$$

where,

$$\mathbf{M}_{\text{Aero}} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} = qSl \begin{bmatrix} C_l \\ C_m \\ C_n \end{bmatrix}$$

So,

$$\begin{bmatrix} \dot{P} \\ \dot{Q} \\ \dot{R} \end{bmatrix} = \mathbf{I}^{-1} \left[qSl \begin{bmatrix} C_l \\ C_m \\ C_n \end{bmatrix} - \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \mathbf{I} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \right] \tag{28}$$

and,

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta\sin\phi + \dot{\theta}\cos\phi \\ \dot{\psi}\cos\theta\cos\phi - \dot{\theta}\sin\phi \end{bmatrix}$$

By re-arrangement:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\theta & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \tag{29}$$

The final 12 equations of motion could be written as:

$$\begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix} = 1/m\begin{bmatrix}T_{\text{Thrust}}\\ 0\\ 0\end{bmatrix} + g\begin{bmatrix}-\sin\theta\\ \sin\phi\cos\theta\\ \cos\phi\cos\theta\end{bmatrix} + qS/m\begin{bmatrix}C_x\\ C_y\\ C_z\end{bmatrix} + \begin{bmatrix}0 & -R & Q\\ R & 0 & -P\\ -Q & P & 0\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi\\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi\\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}$$

$$\begin{bmatrix}\dot{P}\\ \dot{Q}\\ \dot{R}\end{bmatrix} = \mathbf{I}^{-1}\left[qSl\begin{bmatrix}C_l\\ C_m\\ C_n\end{bmatrix} - \begin{bmatrix}0 & -R & Q\\ R & 0 & -P\\ -Q & P & 0\end{bmatrix}\mathbf{I}\begin{bmatrix}P\\ Q\\ R\end{bmatrix}\right]$$

$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\theta & -\sin\phi\\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta\end{bmatrix}\begin{bmatrix}P\\ Q\\ R\end{bmatrix}$$

3 Simulations

The motion equations governing a missile's behavior can be simplified by removing the aerodynamic effects. This involves eliminating angular rates and momentum equations, as aerodynamics is the primary source of rotation. The objective is to analyze various simplifications to validate the system. It is excepted that the missile will have the hardest time with the pitching moments and stability [3]. The ultimate goal is to conduct the base for future work that should include a controlled model, ensuring a thorough understanding of the missile's dynamics.

3.1 Simple 2-D Projectile Motion (Mortar)

A special case of the missile motion would be the projectile motion. One famous application of projectile motion would be the famous widely-used Mortars [9]. In this simplification, the Aerodynamic effects are cancelled out as zeros, all states are initially equal to zero, except for θ and u which are the angle of projection, and the Velocity in the x-body frame. It's important to note that regardless of whether we consider the mortar as a rigid body or a particle, it will not be acted upon by any moments in the absence of aerodynamic effects.

The θ should be given a definition in this case, as well as Q. Since there are no moments affecting the system, the rate of pitch angle cannot be driven from the moment equation, but rather by finding $\dot{\theta}$ and mapping it to Q through the suitable transformation.

$$\begin{aligned} \theta &= \tan^{-1} \dot{z}/\dot{x}, \\ \tan\theta &= \dot{z}/\dot{x} \end{aligned} \tag{30}$$

$$\begin{aligned} \dot{\theta}\sec^2\theta &= \frac{\ddot{z}\dot{x} - \ddot{x}\dot{z}}{\dot{x}^2} \\ \dot{\theta} &= \frac{\ddot{z}\dot{x} - \ddot{x}\dot{z}}{\dot{x}^2}\cos^2\theta \end{aligned} \tag{31}$$

$$Q = \dot{\theta}, \phi = \psi = 0 \tag{32}$$

For this system, 2 scenarios are considered: 1 unforced in which there is no thrust, and the other simulates a particular thrust profile. The forced system motion is initiated by employing a thrust profile characterized by a gradual linear increase during a specific burnout time, reaching a maximum thrust value, maintaining a constant thrust for a defined duration, and concluding with a gradual linear decrease to zero thrust (Figure 9). The simulation yielded results consistent with expectations as shown in plot of Figure 8 and 10.

$$F_{\text{thrust}}(t) = \begin{cases} F_{\text{thrust, max}}\,(t/t_{\text{burnout}}) & \text{if } t < t_{\text{burnout}}, \\ F_{\text{thrust, max}} & \text{if } t_{\text{burnout}} \le t < t_{\text{off}}, \\ F_{\text{thrust, max}} \max\left(0, 1-[\,t - t_{\text{off}}\,]/r_f\right) & \text{if } t \ge t_{\text{off}}. \end{cases} \tag{33}$$

Where, r_f is the rate of thrust decay, t_{burnout} is the time until the whole fuel is burned, and the F_{thrust} reaches its maximum value. t_{off} is the time at which the thrust starts to decay reaching zero.

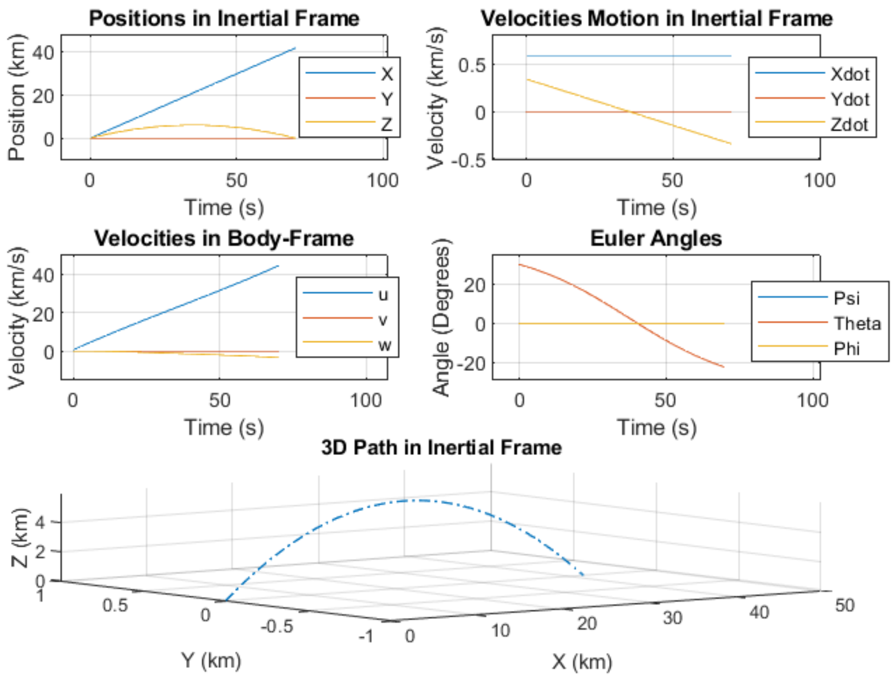

Figure 8: 2-D Un-Forced Projectile Test case

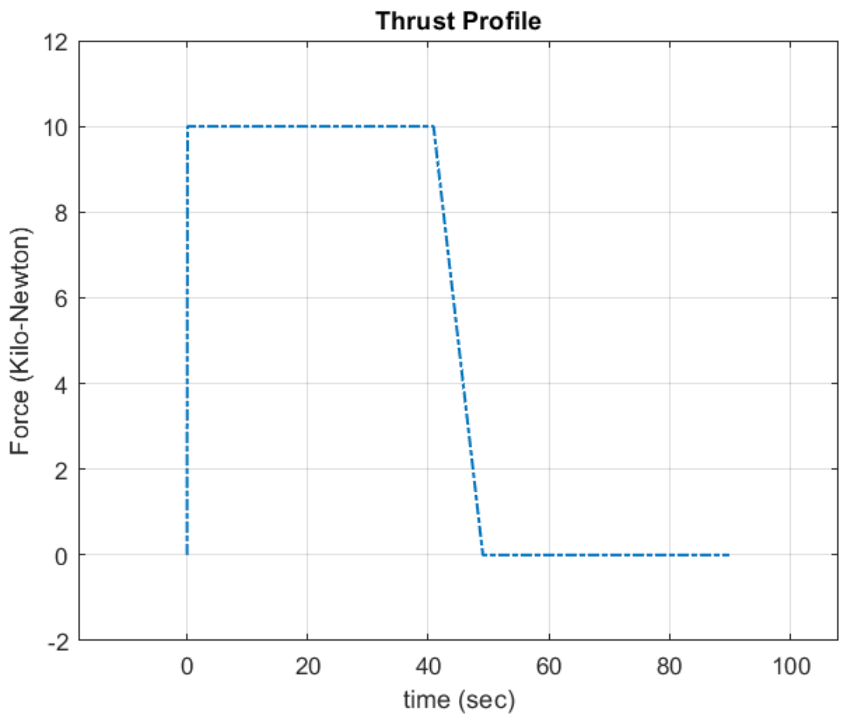

Figure 9: Thrust Profile

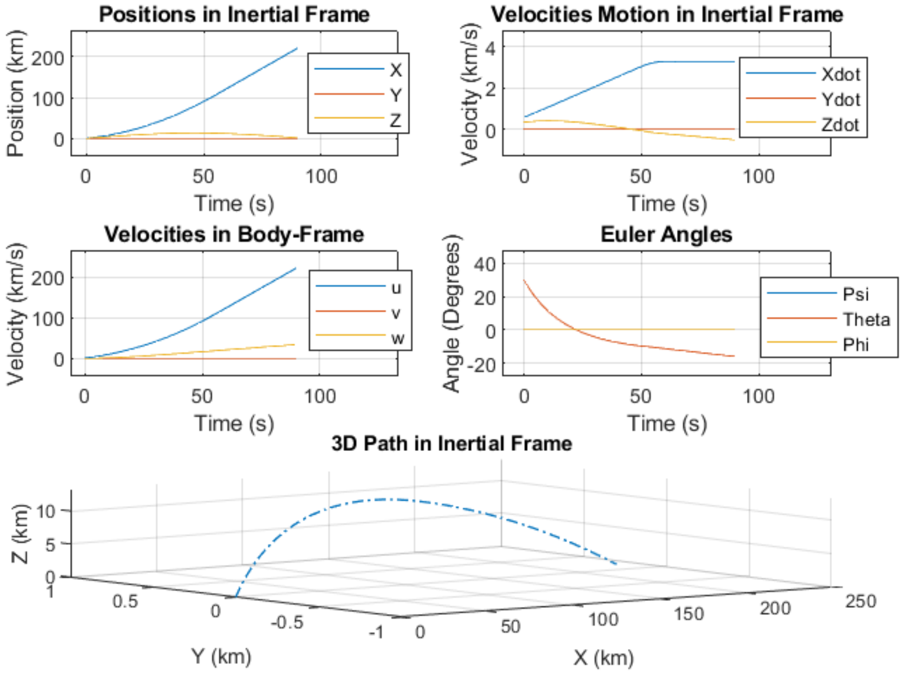

Figure 10: 2-D Projectile Forced by Thrust Test case

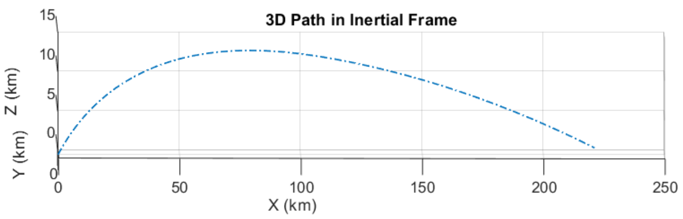

Figure 11: 2-D Projectile Forced by Thrust Test case

The path depicted in figure 8 closely resembles the conventional 2-D model for projectile motion with an initial velocity. An intriguing observation lies in the behavior of the pitch angle θ. Initially, it exhibits a gradual decrease until reaching zero at the peak height of the path. Subsequently, the angle undergoes a gradual clockwise increase, eventually reaching a negative value equivalent to the initial angle when it returns to its initial height. Following that point, the missile or projectile undergoes a counterclockwise rotation to eventually settle back at $\theta = 0$. The impact of thrust in figure 10 can be concisely described as the loss of path symmetry (see Figure 11). The forced motion during the ascending portion of the trajectory reduces the time required to reach the maximum height. Consequently, this alteration in the trajectory results in a quicker attainment of an instantaneous zero for θ.

3.2 3-DOF Missile (No lateral motion, no yaw, no roll)

This simplification allows the missile under examination to rotate around a single axis, in addition to translation along the X and Z axes. As in the previous scenario, all initial conditions are configured to be zero, with the exception of an initial pitch angle $\theta = 10$ degrees and x-body velocity $u = 2$ Mach, simulating the idea of a surface-to-air TM X-1025 missile. The missile parameters were retrieved from [7]. According to the reference [3], oscillations are expected in the missile's motion due to aerodynamic moments. This challenge can be mitigated by designing for the missile's damping moments, which are crucial for stabilizing the missile and preventing oscillations. Missile designers prioritize addressing these damping moments to ensure the overall stability of the system. Without the intervention of a controller to counteract these oscillations by applying a compensatory moment through the missile's control surface, we anticipate the occurrence of pitching instability.

Figures 12 and 13 shows the unforced case scenarios.

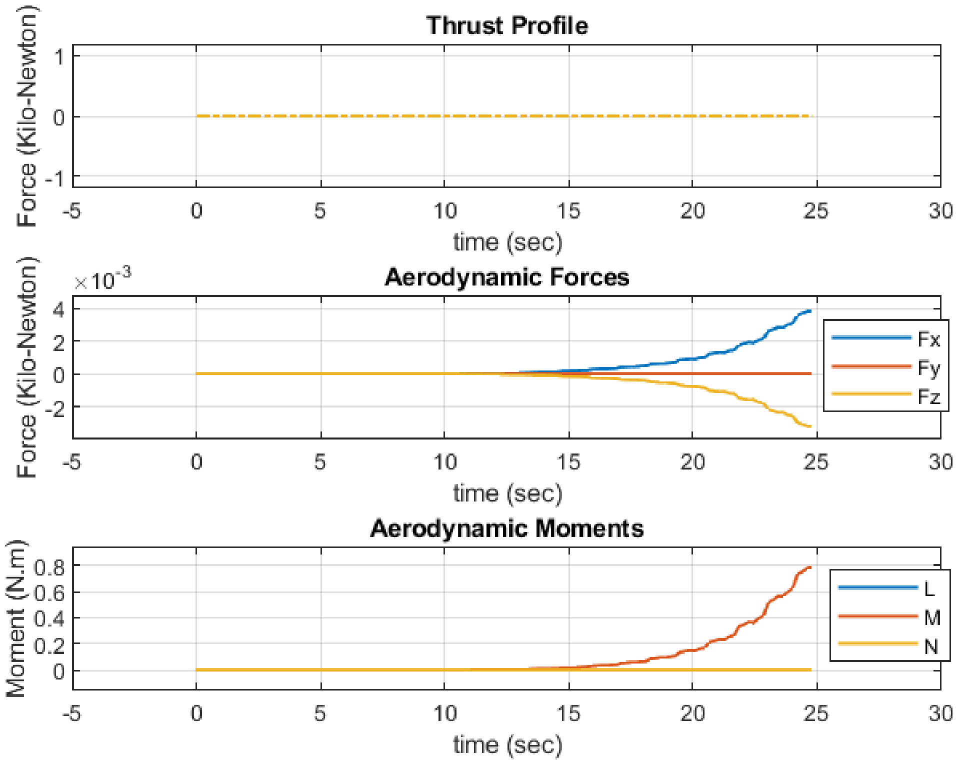

Figure 12: Un-Forced flight: Thrust, Aerodynamic Forces & Aerodynamic Moments (3-DOF)

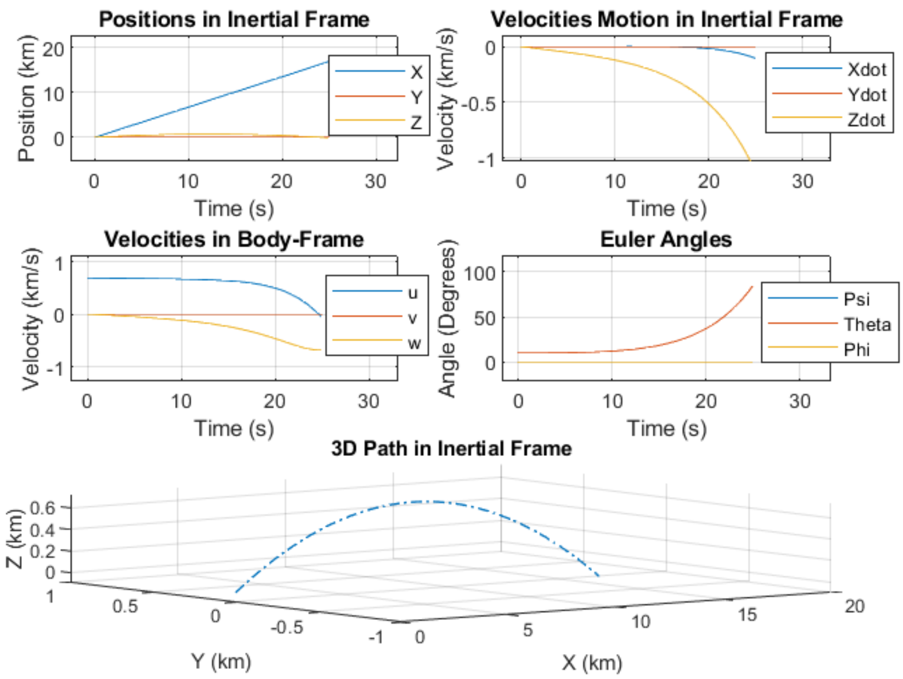

Figure 13: 3-DOF Missile Un-Forced Test Case

An intersting insight arises when contrasting the unforced scenario with the forced ones depicted in Figures 14 and 15. The persistence of an opposite moment effect highlights the effect of aerodynamic moments, primarily influenced by flight speed (Mach number) and angle of attack.

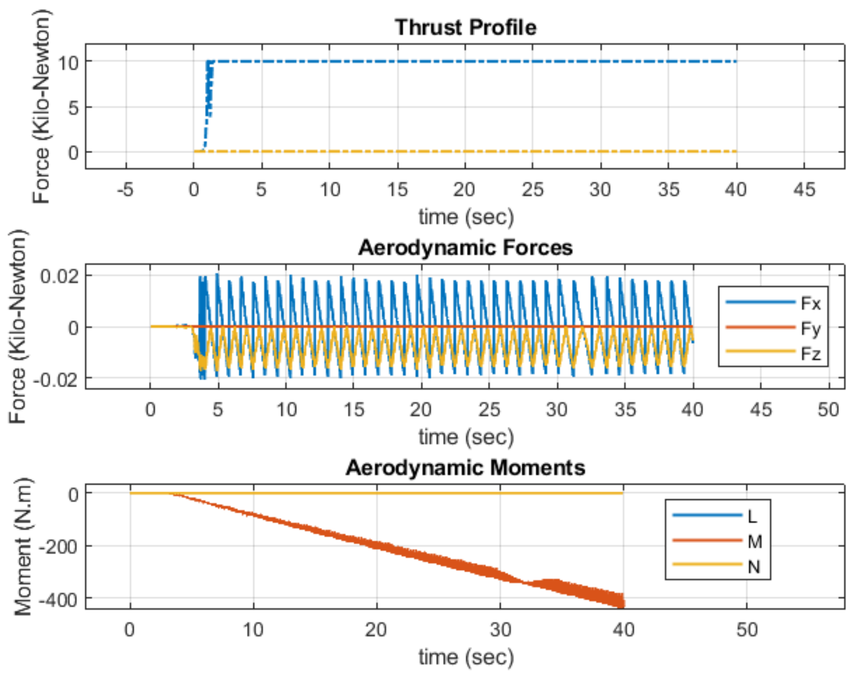

Figure 14: Forced flight: Thrust, Aerodynamic Forces & Aerodynamic Moments (3-DOF)

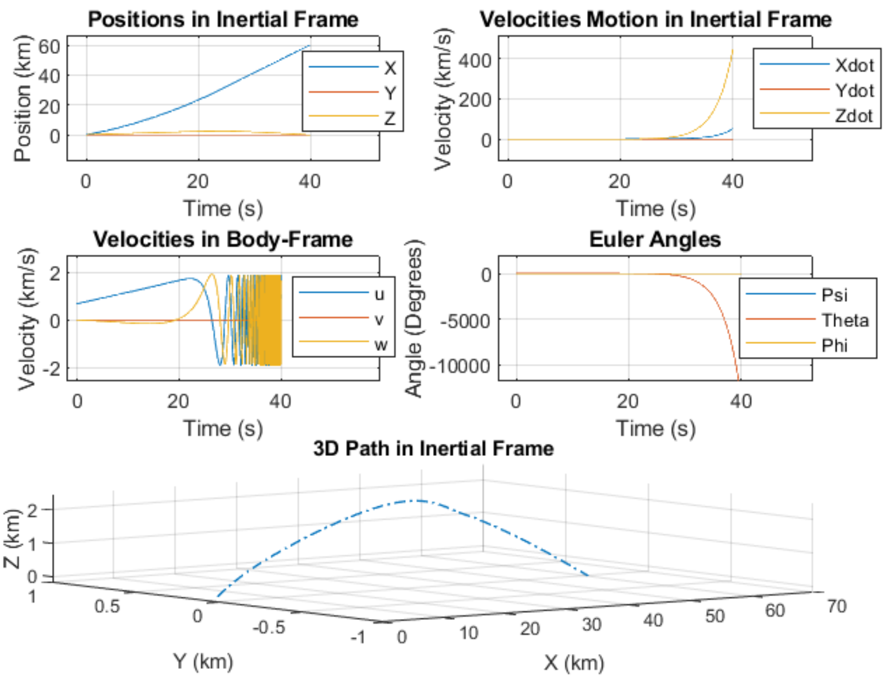

Figure 15: 3-DOF Missile Forced Test Case

Several conclusions can be drawn from this scenario. Firstly, it appears that aerodynamic lift plays a minor role in missile flight, as evidenced by the limited impact on the path. This aligns with the observation that the small size of missile fin designs doesn't contribute significantly to lift production, emphasizing their primary function for navigation. Additionally, the negligible drag associated with a small missile diameter allows for efficient utilization of thrust. Furthermore, the aggressive pitching moment of the missile is notable. This suggests that controlling the pitch moment of a missile is challenging, making it impractical to fire a missile with pitch moment control. Overall, these observations shed light on the the importance of factors such as thrust, size, and aerodynamics in shaping the missile's behavior.

3.3 6-DOF Missile

In this scenario, there is no simplification, and we consider all the aspects of the missile's motion. The initial conditions for this scenario closely parallel those of the 3-DOF missile, differing only in the specification of the yaw angle *psi*, where it is set to 10 degrees. It's noted that both roll and yaw exist and have an impact on the missile. Figures 16 & 17 and 19 & 20 show the results for forced and un-forced cases respectively.

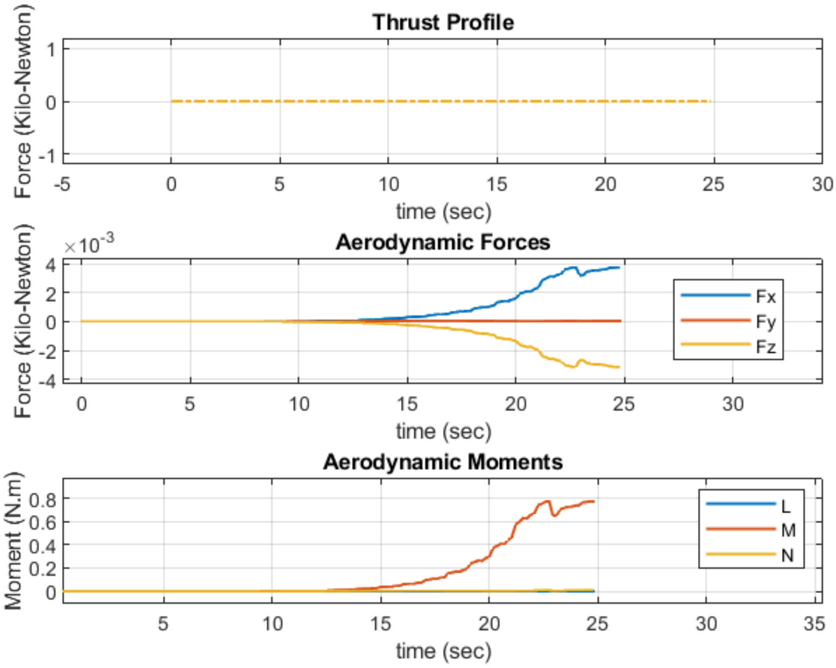

Figure 16: Un-Forced Flight: Thrust, Aerodynamic Forces & Aerodynamic Moments (6-DOF)

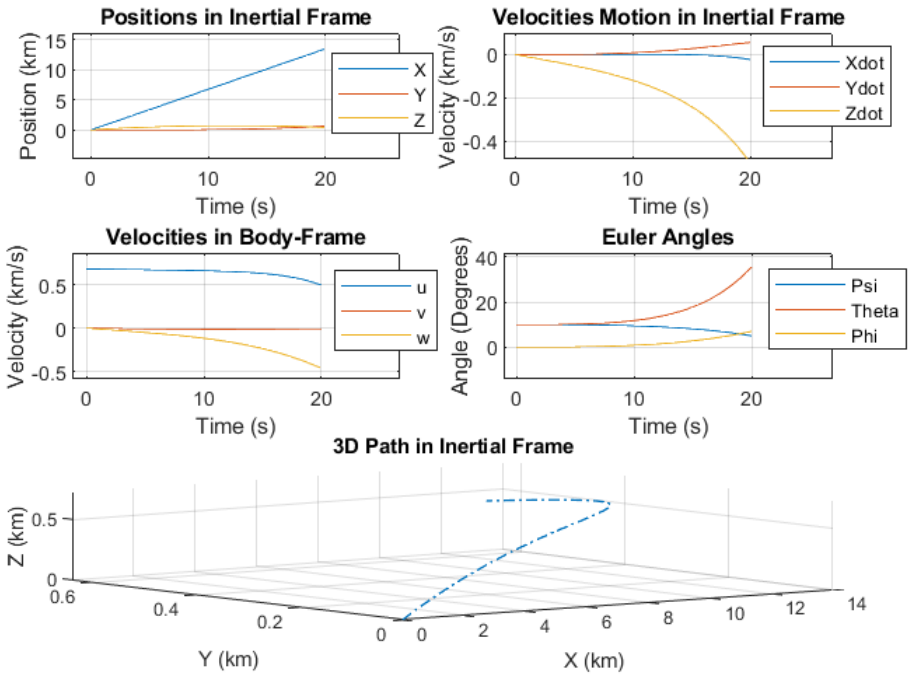

Figure 17: Un-Forced Flight: 6-DOF Missile Test Case

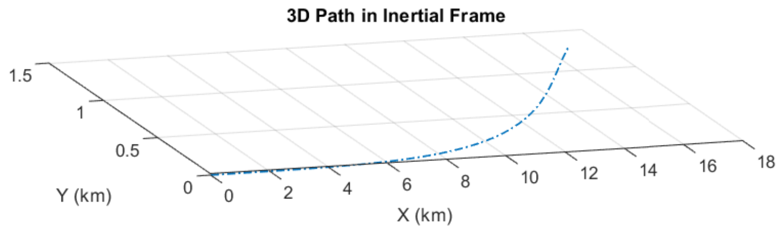

Figure 18: Un-Forced Flight: Projection of the Path

The findings from the unforced 6-DOF missile scenario further reinforce the conclusions drawn from the 3-DOF scenario. The simulations confirm that both lift and drag exert negligible influence on the missile's trajectory, aligning with the earlier observations. Notably, the pitch moment remains aggressive in this extended scenario. An additional insight emerges from the shorter simulation duration of 25 seconds before the pitch angle divergence occurs. This observation underscores the critical point that the aggressive pitch moment not only persists but also dominates over other moments of roll and yaw. This further emphasizes the challenges associated with pitch control and highlights the unique characteristics of missile dynamics.

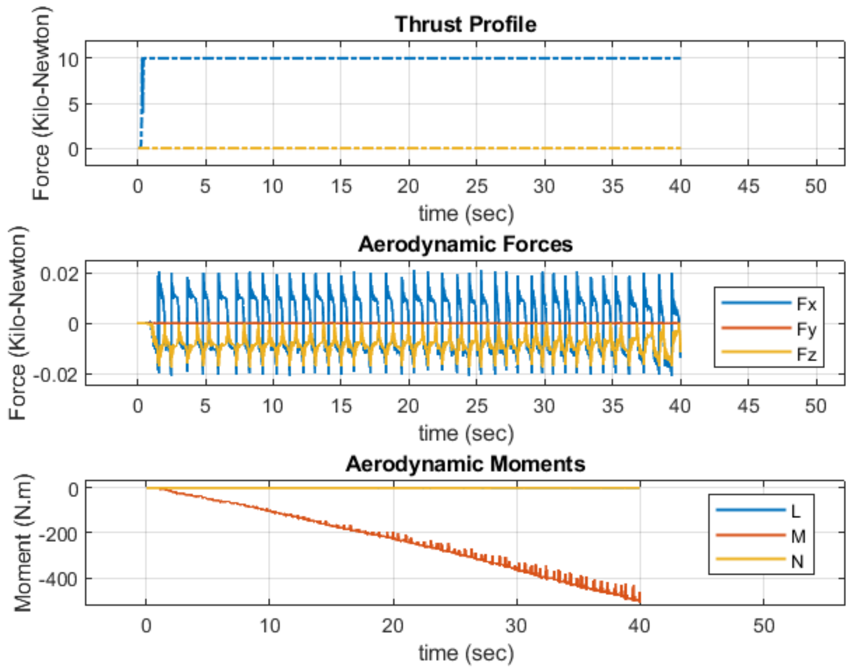

Figure 19: Forced Flight: Thrust, Aerodynamic Forces & Aerodynamic Moments (6-DOF)

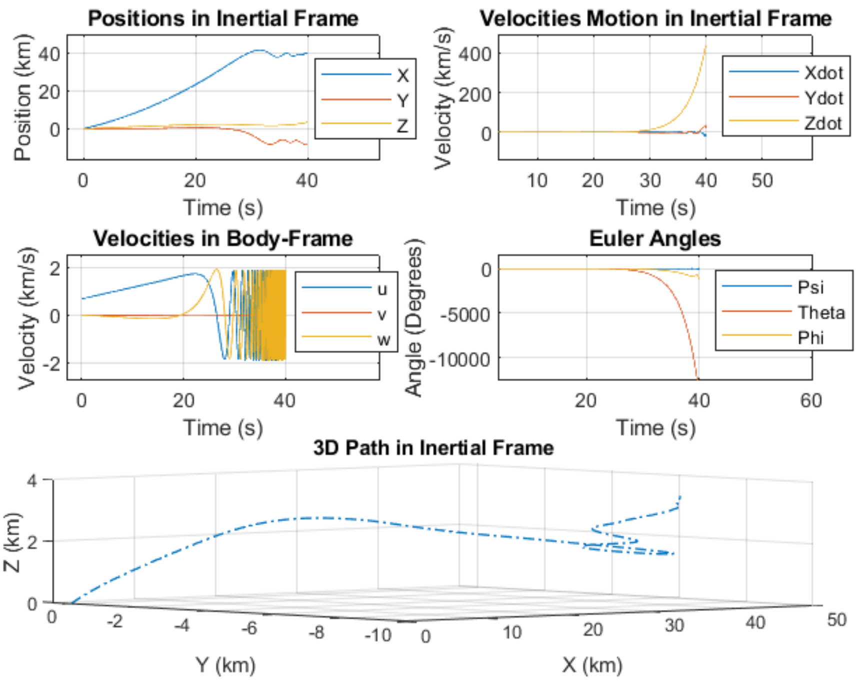

Figure 20: Forced Flight: 6-DOF Missile Test Case

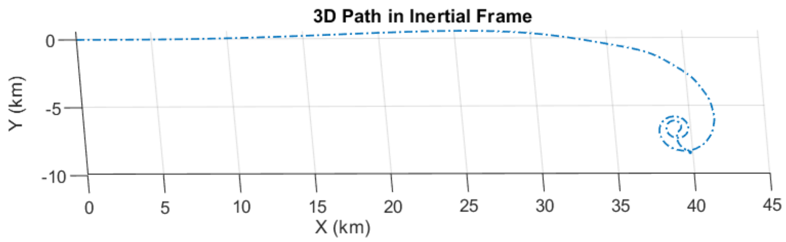

Figure 21: Forced Flight: Projection of the Path

Finally, in the forced 6-DOF scenario, the influence of the three rotational movements is evident. Despite the persistence of the dominant pitching moment, both roll and yaw moments now play a direct and discernible role in shaping the missile's trajectory. It's crucial to highlight that, given the accuracy requirements of missile applications - typically requiring a miss distance of only a few centimeters or meters - the design of the controller demands careful attention before tackling tracking and pursuit challenges.

4 Conclusion

Compiling literature on missile dynamics and control models presents challenges, especially in the absence of well-established design parameters for academic exploration. Understanding missile dynamics requires addressing the connection between forces in body-fixed and earth-fixed frames. Coordinate transformations using Euler angles or quaternions are essential for describing acceleration and derivatives of missile position.

The analysis considers a rigid body missile with six degrees of freedom, encompassing translations and rotations along and about missile axes. Equations of motion are derived, incorporating launch-centered Earth-fixed and body-fixed frames. Aerodynamic effects, expressed in terms of coefficients, are introduced, and moments from both aerodynamics and thrust forces are examined.

In the special case of 2-D projectile motion, aerodynamic effects are canceled. The 3-DOF missile case considers rotation around a single axis and translation along X and Z axes. Oscillations due to aerodynamic moments emphasize the importance of designing for damping moments. In the comprehensive 6-DOF missile scenario, no simplifications are made, considering all aspects of missile motion. Once again, it was evident that successful missile flight requires attention to control surface parameters and the design of an effective controller. In all cases, both forced and unforced flights were examined, and the behavior aligned closely with expectations; pitch instability was prevalent.

The complexity of missile dynamics necessitates a comprehensive understanding of translational and rotational motions, aerodynamic effects, and interconnected forces and moments. Future work should focus on refining the model, incorporating more precise stability derivatives, and adding some control action to the missile dynamics.

Research in missile dynamics and control is witnessing a notable integration of machine learning methodologies, particularly in the realm of air battle decision-making. Ongoing research places substantial emphasis on computational missile guidance, incorporating deep learning algorithms to address and enhance guidance and maneuvering challenges in missile systems.

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